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COMPLETELY REGULAR CODES IN THE n-DIMENSIONAL RECTANGULAR GRID

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Abstract. It is proved that two sequences of the intersection array of an arbitrary completely regular code in the n-dimensional rectangular grid are monotonic. It is shown that the minimal distance of an arbitrary completely regular code is at most 4 and the covering radius of an irreducible completely regular code in the grid is at most $2n$.

Keywords: n-dimensional rectangular grid, completely regular code, intersection array, covering radius, perfect coloring

1. Introduction

A vertex coloring of a graph is perfect if any two vertices of the same color "see" the same number of vertices of any fixed color. If in addition the vertices are colored by distance from some initial set of vertices then the coloring is distance regular and this set is a completely regular code. These notions are closely related with distance regular graphs. In fact, in a distance regular graph a distance coloring with respect to an arbitrary vertex is perfect and parameters of the corresponding distance regular coloring do not depend on the choice of the vertex. But this property does not hold for the graph of the n-dimensional rectangular grid in case $n > 1$. Completely regular codes in distance regular graphs are extensively investigated.

In [9], it was conjectured that two sequences of the intersection array of a completely regular code in a distance regular graph are monotonic. In [7], it was shown that this conjecture is not true in general, but is true for most of the classical

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graphs. First, we prove (Theorem 1) that the conjecture is true for the n-dimensional rectangular grid. Second, we obtain an upper bound $2n$ for the covering radius of an arbitrary completely regular code in the n-dimensional rectangular grid and show that this bound is attainable (Theorem 3). Third, we prove that the upper bound for the minimal code distance of such a code is equal to 4 (Theorem 4). Earlier the results of this paper were presented in part in [3].

The complete classification of perfect colorings of the 2-dimensional rectangular grid into 2, 3 and up to 9 colors can be found in [4], [11] and [8] respectively. All feasible parameters of distance regular colorings of the 2-dimensional rectangular grid were described in [2]. Completely regular codes of infinite hexagonal and triangular grids were investigated in [1] and [12] respectively. Parameters of perfect colorings with two colors of infinite circulant graphs were studied in [5, 6, 10].

Let us pass to precise definitions. An *r-coloring* of the vertices of a graph is a function φ over the graph vertices with values in the set $\{0, 1, \dots, r-1\}$ and it can be presented as a partition $\Phi = \{\Phi_0, \Phi_1, \dots, \Phi_{r-1}\}$ of the graph vertices, where

$$\Phi_i = \{\mathbf{x} \;:\; \varphi(\mathbf{x}) = i\},\ i = 0, 1, \dots, r-1.$$

An r-coloring is *perfect* (in other terms, the corresponding partition is *equitable*) with the *parameter matrix* $P = (p_{ij})_{r\times r}$ if any vertex of the color i has exactly p_{ij} adjacent vertices of color j for all $i, j \in \{0, 1, ..., r-1\}$.

A vertex set D is the *completely regular code* if the distance coloring

$$\Phi = (\Phi_0 = D, \Phi_1, \dots, \Phi_{r-1}), \quad \Phi_i = \{\mathbf{x} \;:\; \rho(D, \mathbf{x}) = i\},\ i = 0, \dots, r-1,$$

with respect to D is perfect (here $\rho(\ ,\ )$ denotes the graph distance), then $r-1$ is the *covering radius* of D and $\min\{\rho(\mathbf{x}, \mathbf{y}) \;:\; \mathbf{x}, \mathbf{y} \in D\}$ is the *code distance* of D. The code Φ_{r-1} is also completely regular. A vertex of the color i, $0 \le i \le r-1$, "sees" vertices of colors $i-1, i$ and $i+1$ only and then the parameter matrix P of the partition Φ is three-diagonal. We denote its nonzero elements as follows:

$$\begin{bmatrix} a_0 & b_0 & & & & \\ c_1 & a_1 & b_1 & & & \\ & c_2 & a_2 & b_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & c_{r-2} & a_{r-2} & b_{r-2} \\ & & & & c_{r-1} & a_{r-1} \end{bmatrix},$$

$a_i = p_{i,i}$ $(i = 0, 1, \dots, r-1)$ – the *inner degree* of the i-th color;
$b_i = p_{i,i+1}$ $(i = 0, 1, \dots, r-2)$ – the *upper degree* of the i-th color;
$c_i = p_{i,i-1}$ $(i = 1, 2, \dots, r-1)$ – the *lower degree* of the i-th color.
In these terms, any vertex of color i "sees" c_i vertices of color $i-1$, a_i vertices of color i and b_i vertices of color $i+1$. We will say that the color i has the *degree triple* (c_i, a_i, b_i) and will write the parameter matrix as follows:

$$[a_0, b_0 | c_1, a_1, b_1 | \dots | c_i, a_i, b_i | \dots | c_{r-1}, a_{r-1}].$$

Only distance colorings with respect to a completely regular code will be studied further, so the parameter matrices will be three-diagonal and all their nonzero elements are contained in degree triples.

The ordered pair $(b_0, \dots, b_{r-2}; c_1, \dots, c_{r-1})$ of sequences of upper and lower degrees is called as the *intersection array* of the completely regular code D.

The *n-dimensional rectangular grid* is the graph G_n with the vertex set

$$\mathbb{Z}^n = \{\mathbf{x} = (x_1, \dots, x_n) \ : \ x_i \in \mathbb{Z},\ i = 1, \dots, n\}$$

and with the edge set

$$\{(\mathbf{x}, \mathbf{y}) \ : \ \sum_{i=1}^{n} |x_i - y_i| = 1\}.$$

Then the graph distance between vertices $\mathbf{x}$ and $\mathbf{y}$ can be written as $\rho(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} |x_i - y_i|$. Let $\mathbf{e}^i \in \mathbb{Z}^n,\ i = 1, \dots, n,$ be the $(0,1)$-vector with the unique one at the i-th position.

Further in the paper D denotes a completely regular code in the n-dimensional rectangular grid, $\varphi = \varphi_D$ denotes the distance coloring

$$\varphi(\mathbf{x}) = \rho(D, \mathbf{x}), \quad \mathbf{x} \in \mathbb{Z}^n,$$

with respect to D and

$$\Phi = (\Phi_0 = D, \Phi_1, \dots, \Phi_{r-1}), \quad \Phi_i = \{\mathbf{x} \in \mathbb{Z}^n \ : \ \varphi(\mathbf{x}) = i\},\ i = 0, \dots, r-1,$$

denotes the corresponding partition of $\mathbb{Z}^n$ by colors. For an arbitrary vertex $\mathbf{x} \in \mathbb{Z}^n$ let us introduce the following sets of unit vectors:

$$\begin{aligned} A_\varphi(\mathbf{x}) &= \{\mathbf{y} - \mathbf{x} \ : \ \varphi(\mathbf{y}) = \varphi(\mathbf{x}),\ \rho(\mathbf{x}, \mathbf{y}) = 1\}, \\ B_\varphi(\mathbf{x}) &= \{\mathbf{y} - \mathbf{x} \ : \ \varphi(\mathbf{y}) = \varphi(\mathbf{x}) + 1,\ \rho(\mathbf{x}, \mathbf{y}) = 1\}, \\ C_\varphi(\mathbf{x}) &= \{\mathbf{y} - \mathbf{x} \ : \ \varphi(\mathbf{y}) = \varphi(\mathbf{x}) - 1,\ \rho(\mathbf{x}, \mathbf{y}) = 1\}. \end{aligned}$$

We will omit the subscript φ if the coloring φ is clear from the context. We will call vectors from the sets $A_\varphi(\mathbf{x}), B_\varphi(\mathbf{x}), C_\varphi(\mathbf{x})$ as *inner, upper* and *lower directions* of the vertex $\mathbf{x}$ with respect to φ. Obviously,

$$|A_\varphi(\mathbf{x})| = a_i, \quad |B_\varphi(\mathbf{x})| = b_i, \quad |C_\varphi(\mathbf{x})| = c_i,$$
$$A_\varphi(\mathbf{x}) \cup B_\varphi(\mathbf{x}) \cup C_\varphi(\mathbf{x}) = \{\pm \mathbf{e}^i : \quad i = 1, \dots, n\}.$$

For any set L of directions, we denote $-L = \{-l \ : \ l \in L\}$.

We say that two colorings φ and ψ are *equivalent* if ψ can be obtained from φ by some graph automorphism and some color renumbering. In particular, for a distance regular coloring φ, the coloring ψ with the inverse order of colors is equivalent and

$$(1) \qquad B_\varphi(\mathbf{x}) = C_\psi(\mathbf{x}), \quad C_\varphi(\mathbf{x}) = B_\psi(\mathbf{x}), \quad A_\varphi(\mathbf{x}) = A_\psi(\mathbf{x}), \quad \mathbf{x} \in \mathbb{Z}^n.$$

We say that two codes D_1 and D_2 are *equivalent* if D_2 can be obtained from D_1 by some graph automorphism.

2. Reducible colorings

For an arbitrary r, there exist only three nonequivalent completely regular codes in the 1-dimensional grid G_1 with the covering radius $r-1$:

$$\{2(r-1)t \ : \ t \in \mathbb{Z}\},$$
$$\{(2r-1)t \ : \ t \in \mathbb{Z}\},$$
$$\{2rt,\ 2rt-1 \ : \ t \in \mathbb{Z}\}.$$

The corresponding distance colorings are periodical, here theirs periods are presented as sequences of colors together with the parameter matrices:

$$0, 1, 2, \dots, r-2, r-1, r-2, \dots, 1; \qquad [02|101|101|\dots|101|20],$$

$$0,0,1,2,\ldots,r-2,r-1,r-2,\ldots,1; \qquad [11|101|101|\ldots|101|20],$$

$$0,0,1,2,\ldots,r-2,r-1,r-1,r-2,\ldots,1; \quad [11|101|101|\ldots|101|11].$$

A code D in the n-dimensional rectangular grid G_n is called *reducible* if there exists a completely regular code D' of G_1 and $\delta_1,\ldots,\delta_n \in \{0,1,-1\}$ such that

$$D = \{\mathbf{x} = (x_1,x_2,\ldots,x_n) \ : (\delta_1 x_1 + \delta_2 x_2 + \ldots + \delta_n x_n) \in D'\}\,. \tag{2}$$

The parameter matrix is referred to as *reducible* if it admits a reducible coloring φ_D. Obviously, each color of the distance regular coloring with respect to the reducible completely regular code D is an union of sets

$$\{\mathbf{x} = (x_1,x_2,\ldots,x_n) \ : (\delta_1 x_1 + \delta_2 x_2 + \ldots + \delta_n x_n) = const\}\,.$$

So, the corresponding coloring φ_D of the n-dimensional grid is also called *reducible* and it means there exists an r-coloring φ' of G_1 and $\delta_1,\ldots,\delta_n \in \{0,1,-1\}$ such that for any $(x_1,x_2,\ldots,x_n) \in \mathbb{Z}^n$,

$$\varphi(x_1,x_2,\ldots,x_n) = \varphi_1\left(\delta_1 x_1 + \delta_2 x_2 + \ldots + \delta_n x_n\right). \tag{3}$$

If t denotes the number of nonzero coefficients $\delta_k, 1 \le k \le n$, then an arbitrary vertex of the color $i, 1 \le i \le r-2$, "sees" precisely t vertices of the color $i-1$ and precisely t vertices of the color $i+1$. As a result, we obtain

Lemma 1. *Let P be an arbitrary reducible matrix of a distance regular coloring of G_n. Then there exists $t \in \{0,1,\ldots,n\}$ such that*

$$P = [2n-\varepsilon_1 q, \varepsilon_1 q \mid q, 2n-2q, q \mid \ \ldots \ |q, n-2q, q \mid \varepsilon_2 q, 2n-\varepsilon_2 q]\,,$$

where $\varepsilon_1,\varepsilon_2 \in \{1,2\}$ (the colorings with $(\varepsilon_1,\varepsilon_2) = (1,2)$ and $(2,1)$ are equivalent).

3. Upper and lower degrees

Throughout this section $r \ge 3$ and a fixed distance regular r-coloring of G_n is denoted by φ . We are going to prove the monotonicity of the upper degrees (and the lower degrees) of the coloring φ.

Lemma 2. *Let $\mathbf{x}$ and $\mathbf{y}$ be two adjacent vertices and $\varphi(\mathbf{y}) = \varphi(\mathbf{x})+1$. Then*

$$C(\mathbf{x}) \subseteq C(\mathbf{y}), \quad B(\mathbf{x}) \supseteq B(\mathbf{y}).$$

Proof. If $\varphi(\mathbf{x}) = 0$ then $\emptyset = C(\mathbf{x}) \subseteq C(\mathbf{y})$.

Now $i = \varphi(\mathbf{x}) \ge 1$ and then $C(\mathbf{x}) \ne \emptyset$. Let us take $d \in C(\mathbf{x})$. If $d = \mathbf{x}-\mathbf{y}$ then obviously $d \in C(\mathbf{y})$. Then consider the case $d \ne \mathbf{x}-\mathbf{y}$. The color of the vertex $\mathbf{x}+d$ equals $i-1$. The color of the vertex $\mathbf{y}+d$ is equal to i, because $\varphi(\mathbf{y}) = i+1$ and the vertex $\mathbf{x}+d = (\mathbf{y}+d)+(\mathbf{x}-\mathbf{y})$ has the color $i-1$. Then $d \in C(\mathbf{y})$.

Finally, $C(\mathbf{x}) \subseteq C(\mathbf{y})$. Further, (1) gives $B(\mathbf{x}) \supseteq B(\mathbf{y})$. □

Lemma 2 immediately gives us the monotonicity of the lower degrees and the upper degrees of an arbitrary distance regular r-coloring of G_n:

Theorem 1. *Let D be an arbitrary completely regular code in the n-dimensional rectangular grid with the intersection array $(b_0,\ldots,b_{r-2};c_1,\ldots,c_{r-1})$. Then*

$$c_1 \le \ldots \le c_{r-2} \le c_{r-1},$$

$$b_0 \ge b_1 \ge \ldots \ge b_{r-2}.$$

It follows from Theorem 1 that there exist two colors $I = I(\varphi)$ and $J = J(\varphi)$ such that

$$I(\varphi) = \max\{i \ : \ c_{i+1} \geq b_{i+1}\}, \tag{4}$$

$$J(\varphi) = \min\{i \ : \ c_{i-1} \leq b_{i-1}\}. \tag{5}$$

Thus, all colors of a distance regular coloring φ are partitioned into three segments:
$\{0, \ldots, I(\varphi)\} \neq \emptyset$, here $c_i < b_i$ for any $i, 0 < i \leq I(\varphi)$;
$\{I(\varphi)+1, \ldots, J(\varphi)-1\}$, here $c_i = b_i$ for any $i, I(\varphi) \leq i \leq J(\varphi)$;
$\{J(\varphi), \ldots, r-1\} \neq \emptyset$, here $c_i > b_i$ for any $i, J(\varphi) \leq i < r-1$.

Lower degrees in the first segment and upper degrees in the last segment are considered in the following

Lemma 3. *a) If $i \leq I(\varphi)$ then $c_i \neq c_{i+1}$. b) If $i \geq J(\varphi)$ then $b_i \neq b_{i-1}$.*

Proof. a) Let us suppose $i \leq I(\varphi)$ and $c_i = c_{i+1}$. We take an arbitrary vertex $\mathbf{x}$ of the color i and an arbitrary direction $d \in B(\mathbf{x})$. It means that $-d \in C(\mathbf{x}+d)$. By assumption, $c_i = c_{i+1}$, then it follows from Lemma 2 that $-d \in C(\mathbf{x})$. Hence, $-B(\mathbf{x}) \subseteq C(\mathbf{x})$ and $b_i \leq c_i$ in contrary to the choice of i. Therefore a) is true. Now (1) gives b). □

4. Colors with the same degree triple

Lemma 3 establishes that only the degree triple of form $(t, 2n-2t, t)$ can be repeated in the intersection matrix of a completely regular code. Everywhere in this section φ denotes an arbitrary distance regular coloring, moreover, we suppose $r \geq 4$ and $J(\varphi) > I(\varphi)+2$; i.e., the different colors $I(\varphi)+1$ and $J(\varphi)-1$ have the same degree triple.

Lemma 4. *Let the colors i and $i+1$ have the same degree triple. Then for any two adjacent vertices $\mathbf{x}$ and $\mathbf{y}$ of colors i and $i+1$, respectively, we have*

$$C(\mathbf{x}) = C(\mathbf{y}) = -B(\mathbf{x}) = -B(\mathbf{y}), \tag{6}$$

$$A(\mathbf{x}) = A(\mathbf{y}) = -A(\mathbf{x}) = -A(\mathbf{y}). \tag{7}$$

Proof. Equalities $C(\mathbf{x}) = C(\mathbf{y})$ and $B(\mathbf{x}) = B(\mathbf{y})$ follow from Lemma 2. Further, we put the direction $d \in B(\mathbf{x})$. It means that $-d \in C(\mathbf{x}+d)$. Then $-d \in C(\mathbf{x})$ by Lemma 2 provided $c_i = c_{i+1}$. Hence $-B(\mathbf{x}) \subseteq C(\mathbf{x})$. But we have $b_i = c_i$, so $-B(\mathbf{x}) = C(\mathbf{x})$. Finally, (7) follows from (6). □

We emphasize that according to Lemma 4, two opposite directions, d and $-d$, belong or do not belong to the set $A(\mathbf{x})$ of inner directions of a vertex $\mathbf{x}$ of a color i, $i = I(\varphi)+1, \ldots J(\varphi)-1$, simultaneously.

For any set $V \subseteq \mathbb{Z}^n$ of vertices $G(V)$ denotes the subgraph of the n-dimensional rectangular grid generated by V. Let V_i^{i+1} be the vertex set of an arbitrary connected component of the graph $G(\Phi_i \cup \Phi_{i+1})$.

Lemma 5. *Let the colors i and $i+1$ have the same degree triples. Then for any two vertices $\mathbf{x}, \mathbf{y} \in V_i^{i+1}$ the equalities (6) and (7) hold.*

Proof. It is sufficient to prove (6) and (7) for two adjacent vertices of the same color i or $i+1$, without loss of generality of the color i. Let $\mathbf{x}, \mathbf{y} \in \Phi_i$ be two adjacent vertices. Then $\mathbf{x}-\mathbf{y} \in A(\mathbf{y})$ and by Lemma 4 also $\mathbf{y}-\mathbf{x} \in A(\mathbf{y})$, whence we have

$$2\mathbf{y}-\mathbf{x} = \mathbf{y}+(\mathbf{y}-\mathbf{x}) \in \Phi_i. \tag{8}$$

Suppose (7) is not true for the vertices $\mathbf{x}$ and $\mathbf{y}$. Then there exists $d \in A(\mathbf{x})\backslash A(\mathbf{y})$. In this case $d \in B(\mathbf{y}) \cup C(\mathbf{y})$. First let $d \in B(\mathbf{y})$. Then $\mathbf{y}+d \in \Phi_{i+1}$ and $\mathbf{x}-\mathbf{y} = (\mathbf{x}+d)-(\mathbf{y}+d) \in C(\mathbf{y}+d)$. By Lemma 4, $\mathbf{y}-\mathbf{x} \in B(\mathbf{y}+d)$ from which we have

$$2\mathbf{y}-\mathbf{x}+d = (\mathbf{y}+d)+(\mathbf{y}-\mathbf{x}) \in \Phi_{i+2}$$

that contradicts (8). It remains to consider the case $d \in C(\mathbf{y})$. Here $-d \in B(\mathbf{y})$ and analogously we obtain the contradiction. Thus, (7) holds for the vertices $\mathbf{x}$ and $\mathbf{y}$.

The equation (6) follows from (7) and Lemma 4. □

Now we will recognize that an arbitrary connected component V_i^{i+1} of the graph $G(\Phi_i \cup \Phi_{i+1})$, $I(\varphi) < i < J(\varphi)$, consists of two hyperplanes. Let $\gamma \in \mathbb{Z}$ and $\delta = (\delta_1, \delta_2, \ldots, \delta_n) \in \{0, 1, -1\}^n$. Let us denote

$$(9) \qquad M(\delta,\gamma) = \{\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n \ : \ \delta_1 x_1 + \delta_2 x_2 + \ldots + \delta_n x_n = \gamma\}.$$

Lemma 6. *Let the colors i and $i+1$ have the same degree triple. Then there exist integer γ and $(0,1,-1)$-valued vector $\delta = (\delta_1, \ldots, \delta_n)$ such that*

$$\Phi_{i+\varepsilon} \cap V_i^{i+1} = M(\delta, \gamma+\varepsilon), \qquad \varepsilon \in \{0,1\}.$$

Proof. Let $0 \le t \le n$, the repeated degree triple be $(t, 2n-2t, t)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in V_i^{i+1}$ be the vertex of color i. Lemma 5 follows that

$$B(\mathbf{v}) = \{\mathbf{e}^1, \ldots, \mathbf{e}^s, -\mathbf{e}^{s+1}, \ldots, -\mathbf{e}^t\},$$

$$C(\mathbf{v}) = \{-\mathbf{e}^1, \ldots, -\mathbf{e}^s, \mathbf{e}^{s+1}, \ldots, \mathbf{e}^t\},$$

$$A(\mathbf{v}) = \{\pm\mathbf{e}^{t+1}, \ldots, \pm\mathbf{e}^n\},$$

up to the numbering of unit vectors. Then let us put:

$$\delta_1 = \ldots = \delta_s = 1,$$

$$\delta_{s+1} = \ldots = \delta_t = -1,$$

$$\delta_{t+1} = \ldots = \delta_n = 0,$$

$$\gamma = v_1 + \ldots + v_s - v_{s+1} - \ldots - v_t.$$

Under this choice of constants, $\mathbf{v} \in M(\delta,\gamma)$ obviously. For an arbitrary vertex $\mathbf{x} \in V_i^{i+1}$, one can easily check by induction on distance between $\mathbf{x}$ and $\mathbf{v}$ that $\mathbf{x} \in M(\delta,\gamma)$ in case $\mathbf{x} \in \Phi_i$ and $\mathbf{x} \in M(\delta,\gamma+1)$ in case $\mathbf{x} \in \Phi_{i+1}$. □

Theorem 2. *Let $\varphi : \mathbb{Z}^n \longrightarrow \{0, 1, \ldots, r-1\}$ be an arbitrary distance regular coloring, $2 \le i < j \le r-2$, the colors i and j have the same degree triple. Then the degree triples coincide for all colors from 1 to $r-2$ and the coloring is reducible.*

Proof. We will show that any color consists of hyperplanes of form (9). All colors from i to j, in particular, the colors i and $i+1$ have the same degree triple. By Lemma 6, there exists $\gamma \in \mathbb{Z}$ and $\delta \in \{0,1,-1\}^n$ such that $M(\delta,\gamma+\varepsilon) \subseteq \Phi_{i+\varepsilon}$, $\varepsilon \in \{0,1\}$. Then for any $k \in \{0, \ldots, r-1\}$ by induction on $|k-i|$ one can easily check that $M(\delta,\gamma+k-i) \subseteq \Phi_k$ because the coloring is distance regular. In particular, for the initial and the last colors it holds $M(\delta,\gamma-i) \subseteq \Phi_0$ and $M(\delta,\gamma+r-i-1) \subseteq \Phi_{r-1}$. By distance regularity of the coloring, $M(\delta,\gamma-i-1) \subseteq \Phi_1$ or $M(\delta,\gamma-i-1) \subseteq \Phi_0$ and $M(\delta,\gamma+r-i) \subseteq \Phi_{r-2}$ or $M(\delta,\gamma+r-i) \subseteq \Phi_{r-1}$.

Hence, by distance regularity of the coloring, for each $\gamma' \in \mathbb{Z}$ all vertices of $M(\delta, \gamma')$ have the same color which depends on $\gamma' = \delta_1 x_1 + \delta_2 x_2 + \ldots + \delta_n x_n$ only. It means that there exists a distance regular coloring φ' of the graph G_1 such that

$$\varphi(x_1, x_2, \ldots, x_n) = \varphi'\left(\delta_1 x_1 + \delta_2 x_2 + \ldots + \delta_n x_n\right),$$

i.e., the coloring φ of G_n is distance regular. □

5. Minimal distance and covering radius

The Hamming graph H_N consists of the following vertex set and edge set:

$$\mathbf{F}^N = \{\alpha = (\alpha_1, \ldots, \alpha_N) \ : \ \alpha_k \in \{0,1\}, k = 1, \ldots, N\},$$

$$\{(\alpha, \beta) \ : \ \alpha, \beta \in \mathbf{F}^N, \ \sum_{k=1}^{N} |\alpha_k - \beta_k| = 1\}.$$

The Hamming graph H_{2n} is covered by the graph G_n of n-dimensional rectangular grid and the covering mapping is $g : \mathbb{Z}^n \to \mathbf{F}^{2n}$, such that for every $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n$

$$g(x_1, \ldots, x_n) = (g_0(x_1 \bmod 4), \ldots, g_0(x_n \bmod 4)),$$
$$g_0(0) = 00, \ g_0(1) = 01, \ g_0(2) = 11, \ g_0(3) = 10.$$

Actually, this mapping preserves the adjacency: if two vertices $\mathbf{x}, \mathbf{y}$ of the n-dimensional rectangular grid differ in exactly one position, then clearly the vertices $g(\mathbf{x}), g(\mathbf{y})$ of the $2n$-dimensional Hamming graph H_{2n} also differ in exactly one position.

Let ψ be a coloring of the Hamming graph H_{2n}. Define the coloring φ_ψ of the n-dimensional rectangular grid G_n as follows:

$$\varphi_\psi(x_1, \ldots, x_n) = \psi\left(g(x_1, \ \ldots \ , x_n)\right), \quad (x_1, \ldots x_n) \in \mathbb{Z}^n. \tag{10}$$

Therefore the colorings ψ and φ_ψ are perfect (respectively, distance regular) simultaneously and have the same parameter matrix. We take as ψ the distance coloring of H_{2n} with respect to the all-zero vertex:

$$\psi(\alpha) = wt(\alpha), \quad \alpha \in \mathbf{F}^{2n}, \tag{11}$$

where $wt(\alpha) = \sum_{i=1}^{2n} \alpha_i$ is the Hamming weight of the vertex α. In this case ψ distance regular $(2n+1)$-coloring with the parameter matrix

$$[0, 2n|1, 0, 2n-1| \ldots |i, 0, 2n-i| \ldots |2n-1, 0, 1|2n, 0].$$

Then φ_ψ is also the distance regular $(2n+1)$-coloring with the same parameter matrix, moreover, it is not reducible because its parameter matrix is not reducible.

Finally, we can state the main theorem.

Theorem 3. *For an arbitrary irreducible distance regular r-coloring of n-dimensional rectangular grid, it holds $r \leq 2n+1$. An irreducible distance regular $(2n+1)$-coloring exists.*

Proof. Let φ be an irreducible distance regular coloring of the n-dimensional rectangular grid. By Theorem 2, every two colors have different degree triples. This means that $J(\varphi) - I(\varphi) \leq 2$. By definition (4) of $I(\varphi)$, we have $c_{I(\varphi)} < b_{I(\varphi)}$, and also we know that $c_{I(\varphi)} + b_{I(\varphi)} \leq 2n$. It follows from Lemma 3 that $\{c_1, \ldots, c_{I(\varphi)}\}$ are pairwise different. Then $I(\varphi) + 1 \leq n$. Analogously, we obtain that $J(\varphi) \geq r - n$. Finally, $r = (I(\varphi)+1) + (J(\varphi) - I(\varphi) - 1) + (r - J(\varphi)) \leq n + 1 + n = 2n + 1$.

The coloring φ_ϕ defined by (10), (11) gives us the example of the irreducible $(2n+1)$-coloring. $\square$

Let us rewrite Theorem 3 in terms of completely regular codes:

Corollary 1. *The covering radius of an arbitrary completely regular code in the n-dimensional rectangular grid is at most $2n$.*

We also can obtain the upper bound for the minimal distance of the completely regular code in n-dimensional rectangular grid.

Theorem 4. *The minimal distance of an arbitrary completely regular code in n-dimensional rectangular grid is at most 4.*

Proof. Let D be an arbitrary completely regular code with minimal distance $d = d(D) \geq 5$ in the graph G_n, and $\mathbf{x} \in D$. Let us consider the corresponding distance regular r-coloring $\Phi = (\Phi_0 = D, \dots, \Phi_{r-1})$. Here $r-1$ is equal to the covering radius of the code D, and $r-1 \geq 2$ as far as $d \geq 5$. Then any vertex of the sphere $S_2(\mathbf{x})$ (of radius 2 centered in the vertex $\mathbf{x}$) is of the color 2 and has ajacent vertices of color 1 only in the sphere $S_1(\mathbf{x})$ (of radius 1 centered in the vertex $\mathbf{x}$).

We consider the vertices $\mathbf{x}+2\mathbf{e}^1$ and $\mathbf{x}+\mathbf{e}^1+\mathbf{e}^2$ in the sphere $S_2(\mathbf{x})$. Theirs sets of lower directions are $C(\mathbf{x}+2\mathbf{e}^1) = \{\mathbf{e}^1\}$ and $C(\mathbf{x}+\mathbf{e}^1+\mathbf{e}^2) = \{\mathbf{e}^1, \mathbf{e}^2\}$ and have different cardinalities. This lead us to the contradiction. $\square$

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