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REIDEMEISTER CLASSES IN WREATH PRODUCTS OF ABELIAN GROUPS

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Abstract. Among restricted wreath products $G \wr \mathbb{Z}^k$, where G is a finite abelian group, we find three large classes of groups admitting an automorphism φ with finite Reidemeister number $R(\varphi)$ (number of φ-twisted conjugacy classes). In other words, groups from these classes do not have the R_∞ property.

Moreover, we prove that if φ is a finite order automorphism of $G \wr \mathbb{Z}^k$ with $R(\varphi) < \infty$, then $R(\varphi)$ is equal to the number of fixed points of the map $[\rho] \mapsto [\rho \circ \varphi]$ defined on the set of equivalence classes of finite dimensional irreducible unitary representations of $G \wr \mathbb{Z}^k$.

Keywords: Reidemeister number, twisted conjugacy class, Burnside-Frobenius theorem, unitary dual, finite-dimensional representation.

1. Introduction

Suppose, Γ is a group and $\phi : \Gamma \to \Gamma$ is an endomorphism. Two elements $x, y \in \Gamma$ are *ϕ-conjugate* or *twisted conjugate,* if and only if there exists an element $g \in \Gamma$ such that

$$y = gx\phi(g^{-1}).$$

The corresponding classes are called *Reidemeister* or *twisted conjugacy* classes. The number $R(\phi)$ of them is called the *Reidemeister number* of ϕ.

The study of Reidemeister numbers is an important problem related with topological dynamics, number theory and representation theory (see [4]). One of the main problems in the field is to prove or disprove the so-called TBFT (a conjecture

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about the twisted Burnside-Frobenius theory (or theorem)), which has numerous important consequence for Reidemeister zeta function and for other problems in topological dynamics (see a more extended discussion in [14]). Namely the problem is to identify $R(\varphi)$ (when $R(\varphi) < \infty$) in a natural way with the number of fixed points of the induced map $\widehat{\varphi}$ of an appropriate dual object. In the initial formulation of the conjecture [6], the dual object was the unitary dual $\widehat{\Gamma}$ and $\widehat{\varphi} : [\rho] \mapsto [\rho \circ \varphi]$. The conjecture about TBFT was proved in many cases, but failed for an example in [13], which led to the new formulation: TBFT_f, where $\widehat{\Gamma}$ was replaced by its finite-dimensional part, which is evidently invariant under $\widehat{\varphi}$. This is the version, which we will study in this paper for a class of groups. In [14] an example of a group that has neither TBFT nor TBFT_f was presented. The most general proved cases of TBFT_f are the case of polycyclic-by-finite groups [11] and the case of nilpotent torsion-free groups of finite Prüfer rank [10].

Another important problem in the field is to localize the class of groups, where one can consider the TBFT conjecture, i.e. where automorphisms with $R(\varphi) < \infty$ do exist. The opposite case is called the R_∞ property. It has some topological consequences itself (see e.g. [18]). A part of recent results about Reidemeister clases and R_∞ can be found in [1, 3, 7, 12, 21, 23, 26, 27] (see also an overview in [9]).

We consider the following restricted wreath product $G \wr \mathbb{Z}^k = \Sigma \rtimes_\alpha \mathbb{Z}^k$, where G is a finite abelian group, Σ denotes $\oplus_{x\in\mathbb{Z}^k} G_x$, and $\alpha(x)(g_y) = g_{x+y}$. Here g_x is $g \in G \cong G_x$.

The R_∞ property was completely studied for $k = 1$ in [17], for $G = \mathbb{Z}_p$ with a prime p and arbitrary k in [27], for $G = \mathbb{Z}_m$ and arbitrary k in [15]. In all these cases the TBFT_f was proved.

The complexity of the study increases drastically when we move from $k = 1$ to $k > 1$, because $\mathbb{Z}$ has only one non-trivial automorphism, namely $m \mapsto -m$, in contrast with $\mathbb{Z}^k$, where automorphisms may have infinite orbits.

The groups under consideration can be viewed as generalized lamplighter groups. For a generalization of the lamplighter group in other directions, the twisted conjugacy was considered in [25], [24], and other papers.

In the present paper, we prove (Theorem 3.1) that the groups under consideration do not have the R_∞ property in the following three cases:

1) all prime-power components of G for 2 and 3 have multiplicity at least 2;
2) there is no prime-power components for 2 and k is even;
3) all prime-power components of G for 2 have multiplicity at least 2 and $k = 4s$ for some s.

To prove this, we construct corresponding examples, and all of them have a finite order. This motivates us to prove the TBFT_f for all groups of the form $G \wr \mathbb{Z}^k$ and their automorphisms of finite order (Corollary 4.2). The proof is based on a description of Reidemeister classes of φ as cylindrical sets (Theorem 4.1).

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2. Preliminaries

We start from some general statements about Reidemeister classes of extensions. Suppose, a normal subgroup H of G is φ-invariant under an automorphism $\varphi : G \to$

G and $p: G \to G/H$ is the natural projection. Then φ induces automorphisms $\varphi': H \to H$ and $\widetilde{\varphi}: G/H \to G/H$.

Definition 2.1. Denote by $\mathrm{Fix}(\varphi)$ the set $\{g \in G \colon \varphi(g) = g\}$, i.e. $\mathrm{Fix}(\varphi)$ is the subgroup of G, formed by φ-fixed elements.

We will use the notation $\tau_g(x) = gxg^{-1}$ for an inner automorphism as well as for its restriction on a normal subgroup.

The following important properties were obtained in [6, 16], see also [11, 18].

Theorem 2.2. *For G, H, φ, φ', and $\widetilde{\varphi}$ as above, we have the following.*

1. Epimorphity: *the projection $G \to G/H$ maps Reidemeister classes of φ onto Reidemeister classes of $\widetilde{\varphi}$, in particular $R(\widetilde{\varphi}) \le R(\varphi)$;*
2. Estimation by fixed elements: *if $|\mathrm{Fix}(\widetilde{\varphi})| = n$, then $R(\varphi') \le R(\varphi) \cdot n$;*
3. Fixed elements-free case: *if $\mathrm{Fix}(\widetilde{\varphi}) = \{e\}$, then each Reidemeister class of φ' is an intersection of the appropriate Reidemeister class of φ and H;*
4. Summation: *if $\mathrm{Fix}(\widetilde{\varphi}) = \{e\}$, then $R(\varphi) = \sum_{j=1}^{R} R(\tau_{g_j} \circ \varphi')$, where $g_1, \dots, g_R$ are some elements of G such that $p(g_1), \dots, p(g_R)$ are representatives of all Reidemeister classes of $\widetilde{\varphi}$, $R = R(\widetilde{\varphi})$.*

Also we will need the following statement [19] (Lemma 4 and the step (2) in the proof of Theorem A'):

Lemma 2.3. *Suppose, Γ is a residually finite group and $\varphi: \Gamma \to \Gamma$ is an automorphism of finite order with $R(\varphi) < \infty$. Then $|\mathrm{Fix}(\varphi)| < \infty$.*

One can find in [19] an estimation for $|\mathrm{Fix}(\varphi)|$, but we will not use it.

Passing to a semidirect product $\Sigma \rtimes_\alpha \mathbb{Z}^k$, we have by [2] that a couple of automorphisms $\varphi': \Sigma \to \Sigma$ and $\overline{\varphi}: \Sigma \rtimes_\alpha \mathbb{Z}^k/\Sigma \cong \mathbb{Z}^k \to \mathbb{Z}^k \cong \Sigma \rtimes_\alpha \mathbb{Z}^k/\Sigma$ define an automorphism φ of $\Sigma \rtimes_\alpha \mathbb{Z}^k$ (not unique) if and only if

$$\varphi'(\alpha(m)(h)) = \alpha(\overline{\varphi}(g))(\varphi'(h)), \qquad h \in \Sigma, \quad m \in \mathbb{Z}^k. \tag{1}$$

Since Σ is abelian, by [2, p. 207] the mapping φ_1 defined as φ' on Σ and by $\overline{\varphi}$ on $\mathbb{Z}^k \subset \Sigma \rtimes \mathbb{Z}^k$ is still an automorphism. Moreover, from the following commutative diagrams

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Sigma & \longrightarrow & \Sigma \rtimes \mathbb{Z}^k & \longrightarrow & \mathbb{Z}^k & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi'} & & {\scriptstyle \varphi}\downarrow\;\downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \overline{\varphi}} & & \\
0 & \longrightarrow & \Sigma & \longrightarrow & \Sigma \rtimes \mathbb{Z}^k & \longrightarrow & \mathbb{Z}^k & \longrightarrow & 0
\end{array}$$

we have $R(\varphi) = R(\varphi_1)$. Indeed, if $R(\overline{\varphi}) = \infty$, then $R(\varphi) = R(\varphi_1) = \infty$. If $R(\overline{\varphi}) < \infty$ then $C_{\overline{\varphi}} = \{0\}$ and by Theorem 2.2

$$R(\varphi) = \sum_{\text{representatives } m \in \mathbb{Z}^k \text{of Reidemeister classes of } \overline{\varphi}} R(\tau_m \circ \varphi') = R(\varphi_1).$$

So, without loss of generality in the R_∞ questions (not in Section 4) we will assume

$$\mathbb{Z}^k \subset A \wr \mathbb{Z}^k \text{ is } \varphi\text{-invariant and } \varphi|_{\mathbb{Z}^k} = \overline{\varphi}. \tag{2}$$

This was discussed briefly in [17, Lemma 3.5] in a particular case.

Lemma 2.4. *An authomorphism $\varphi: G \wr \mathbb{Z}^k \to G \wr \mathbb{Z}^k$ has $R(\varphi) < \infty$ if and only if $R(\overline{\varphi}) < \infty$ and $R(\tau_m \circ \varphi') < \infty$ for any $m \in \mathbb{Z}^k$ (in fact, it is sufficient to verify this for representatives of Reidemeister classes of $\overline{\varphi}$).*

Proof. Suppose, $R(\varphi) < \infty$. By Theorem 2.2, we have $R(\overline{\varphi}) < \infty$. Then by Lemma 2.3, we obtain $|\operatorname{Fix}(\overline{\varphi})| < \infty$ (in fact, $|\operatorname{Fix}(\overline{\varphi})| = 1$, because an automorphism of $\mathbb{Z}^k$ can not have finitely many fixed elements except of 0). So, by Theorem 2.2, $R(\varphi') < \infty$. Considering $\tau_z \circ \varphi$, which has $R(\tau_z \circ \varphi) = R(\varphi) < \infty$, instead of φ, we obtain in the same way that $R(\tau_z \circ \varphi') < \infty$.

Conversely, having $|\operatorname{Fix}(\overline{\varphi})| = 1$, one can apply the summation formula from Theorem 2.2. □

Lemma 2.5. *Suppose, $\overline{\varphi} : \mathbb{Z}^k \to \mathbb{Z}^k$ is an automorphism and $F : G \to G$ is an automorphism. Then φ' defined by*

$$\varphi'(a_0) = (Fa)_0, \qquad \varphi'(a_x) = (Fa)_{\overline{\varphi}(x)} \tag{3}$$

satisfies (1) and so defines an automorphism of $G \wr \mathbb{Z}^k$.

Evidently the subgroups $\oplus G_x$, where x runs over an orbit of $\overline{\varphi}$, are φ'-invariant summands of Σ.

Proof. It is sufficient to prove (1) on generating elements of the form a_x. Then for any $z \in \mathbb{Z}^k$,

$$\varphi'(\alpha(z)a_x) = \varphi'(a_{x+z}) = (Fa)_{\overline{\varphi}(x+z)} = \alpha(\overline{\varphi}(z))(Fa)_{\overline{\varphi}(z)} = \alpha(\overline{\varphi'}(z))\varphi'(a_x)$$

and (1) is fulfilled. The first equality in (3) is in fact a particular case of the second one. □

It is not difficult to prove (see [5]) that, for $\overline{\varphi} : \mathbb{Z}^k \to \mathbb{Z}^k$ defined by a matrix M, one has

$$R(\overline{\varphi}) = \#\operatorname{Coker}(\operatorname{Id} - \overline{\varphi}) = |\det(E - M)|, \tag{4}$$

if $R(\overline{\varphi}) < \infty$, and $|\det(E - M)| = 0$ otherwise.

3. Some classes of wreath products without R_∞ property

Theorem 3.1. *Suppose, the prime-power decomposition of G is $\oplus_i (\mathbb{Z}_{(p_i)^{r_i}})^{d_i}$. Then under each of the following conditions the corresponding wreath products $G \wr \mathbb{Z}^k$ admit an automorphism φ with $R(\varphi) < \infty$, i.e. do not have the property R_∞:*

- **Case 1):** *for all $p_i = 2$ and $p_i = 3$, we have $d_i \geq 2$ (and is arbitrary for primes > 3);*
- **Case 2):** *there is no $p_i = 2$ and k is even;*
- **Case 3):** *for all $p_i = 2$, we have $d_i \geq 2$ and $k = 4s$ for some s.*

Proof. In each of these cases we will take an automorphism $\overline{\varphi} : \mathbb{Z}^k \to \mathbb{Z}^k$ with $R(\overline{\varphi}) < \infty$ (in fact, of finite order) and define $\varphi' : \Sigma \to \Sigma$ with appropriate properties in accordance with Lemmas 2.4 and 2.5.

Case 1). In this case we can take $\overline{\varphi} = -\operatorname{Id} : \mathbb{Z}^k \to \mathbb{Z}^k$ and construct φ similarly to [17]. More specifically, note that $R(\overline{\varphi}) = 2^k$ and define $\varphi' : \Sigma \to \Sigma$ in the following way. The subgroups $G_x \oplus G_{-x}$ will be invariant subgroups of φ' and we define

$$\varphi' : G_x \oplus G_{-x} \to G_x \oplus G_{-x} \text{ as } \begin{pmatrix} 0 & \Psi \\ \Psi & 0 \end{pmatrix},$$

where $\Psi : G \to G$ is defined as a direct sum of blocks of the following types:

$$F_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} : (\mathbb{Z}_q)^2 \to (\mathbb{Z}_q)^2, \quad F_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} : (\mathbb{Z}_q)^3 \to (\mathbb{Z}_q)^3, \tag{5}$$

where q are some $(p_i)^{r_i}$ and for each summand $\left(\mathbb{Z}_{(p_i)^{r_i}}\right)^{d_i}$ of G ($d_i \geq 2$, $p_i = 2$ or $p_i = 3$) we have s summands F_2, if $d_i = 2s$, or $s-1$ summands F_2 and one summand F_3, if $d_i = 2s+1$. For the remaining summands (i.e. for $p_i > 3$) we do not need to group summands in the above way and we can consider $F_1 : \mathbb{Z}_q \to \mathbb{Z}_q$, $1 \mapsto m(q)$ where $q = (p_i)^{r_i}$. This $m = m(q)$ should be taken in such a way that

$$m^2 \text{ and } 1-m^2 \text{ are invertible in } \mathbb{Z}_q. \tag{6}$$

This can be done for $p_i > 3$: one can take $m = 2$ (and impossible for $p_i = 2$ or 3).

By Lemma 2.5, we defined an automorphism φ of $G \wr \mathbb{Z}^k$ in this way (one may assume (2) to have a unique φ).

We claim that $R(\tau_{z_i} \circ \varphi') = R(\alpha(z_i) \circ \varphi') = 1$, $i = 1, \dots, 2^k$. Consequently, by Theorem 2.2, $R(\varphi) = R(\overline{\varphi}) = 2^k < \infty$. So we need to prove that $\mathrm{Id}_\Sigma - \alpha(z_i) \circ \varphi'$ is an epimorphism, because, for abelian groups, this is evidently the same as $R(\alpha(z_i) \circ \varphi') = 1$. This homomorphism has a decomposition of Σ into invariant subgroups $G_x \oplus G_{-x+z_i}$, because $\alpha(z_i) : G_{-x} \to G_{-x+z_i}$, $\varphi' : G_{-x+z_i} \to G_{x-z_i}$ and $\alpha(z_i) : G_{x-z_i} \to G_x$. Note that the subgroups G_x and G_{-x+z_i} coincide if $z_i = 2x$ (this corresponds to the case of G_0 for φ'). Thus it is sufficient to verify the epimorphity for each $G_x \oplus G_{-x+z_i}$ and for the exceptional case. Passing to summands of G, it is sufficient to verify the epimorphity of

$$\begin{pmatrix} -E & F_2 \\ F_2 & -E \end{pmatrix}, \quad \begin{pmatrix} -E & F_3 \\ F_3 & -E \end{pmatrix} \text{ and } \begin{pmatrix} -E & F_1 \\ F_1 & -E \end{pmatrix} = \begin{pmatrix} -1 & m \\ m & -1 \end{pmatrix}.$$

The first two are isomorphisms with the explicit inverses

$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} -2 & 0 & 1 & 1 & 1 & -1 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 1 \\ 1 & 1 & -1 & -2 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 1 \\ -1 & 0 & 1 & 1 & 1 & -1 \end{pmatrix}.$$

For the third one the invertibility follows from (6). For the exceptional case we formally do not need to verify the epimorphity, because it can add only a finite number to $R(\varphi')$, but we wish to prove our (more strong) claim (this will be helpful for TBFT). So we have to prove, that

$$F_2 - E, \qquad F_3 - E, \qquad m - 1$$

are epimorphisms. This can be done immediately: $\det(F_2 - E) = 1 \mod 2$, $\det(F_3 - E) = 1 \mod 2$, and $1 - m^2 = (1-m)(1+m)$.

Case 2). Now consider the case of even $k = 2t$ and G without 2-subgroup. In this case the construction starts as in [27]: we take $\overline{\varphi} : \mathbb{Z}^{2t} \to \mathbb{Z}^{2t}$ to be the direct sum of t copies of

$$\mathbb{Z}^2 \to \mathbb{Z}^2, \quad \begin{pmatrix} u \\ v \end{pmatrix} \mapsto M \begin{pmatrix} u \\ v \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

Then M generates a subgroup of $GL(2, \mathbb{Z})$, which is isomorphic to $\mathbb{Z}_3$ (see, [22, p. 179]). All orbits of M have length 3 (except of the trivial one) and the corresponding Reidemeister number is equal to $\det(E - M) = 3$. Similarly for $\overline{\varphi}$: the length of any orbit is 3 (except of the zero) and $R(\overline{\varphi}) = 3^t$. Also

$$M^2 + M + E = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{7}$$

Now define φ' as a direct sum of actions for $\mathbb{Z}_q$, $q = (p_i)^{r_i}$, $p_i \geq 3$.

For $p_i \geq 3$ choose $m = m_i$ such that

$$m^3 \text{ and } 1 - m^3 \text{ are invertible in } \mathbb{Z}_q. \tag{8}$$

This can be done for $p_i \geq 3$: one can take $m = 3$ for $p_i = 7$ and $m = 2$ in the remaining cases (and impossible for $p_i = 2$). Define $\varphi'(a_0) = (ma)_0$ and $\varphi'(a_x) = (ma)_{\overline{\varphi}(g)}$, where $a \in \mathbb{Z}_q \subset G$. So, the corresponding subgroup $\oplus_{g\in\mathbb{Z}^k}(\mathbb{Z}_q)_g \subset \Sigma$ is φ'-invariant and decomposed into infinitely many invariant summands $(\mathbb{Z}_q)_g \oplus (\mathbb{Z}_q)_{\overline{\varphi}(g)} \oplus (\mathbb{Z}_q)_{\overline{\varphi}^2(g)}$ isomorphic to $(\mathbb{Z}_q)^3$ (over generic orbits of $\overline{\varphi}$) and one summand $(\mathbb{Z}_q)_0$ (over the trivial orbit). Then the corresponding restrictions of φ' and $1 - \varphi'$ can be written as multiplication by

$$\begin{pmatrix} 0 & 0 & m \\ m & 0 & 0 \\ 0 & m & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & -m \\ -m & 1 & 0 \\ 0 & -m & 1 \end{pmatrix}, \quad \text{and } m, \quad 1 - m,$$

respectively. The three-dimensional mappings are isomorphisms by (8). Since an element ℓ is not invertible in $\mathbb{Z}_{(p_i)^{r_i}}$ if and only if $\ell = u \cdot p_i$, the invertibility of one-dimensional mappings follows from (8) and the factorization $1 - m^3 = (1 - m)(1 + m + m^2)$. (This construction gives a more explicit presentation of a part of proof of [27, Theorem 4.1])

For $\tau_z \circ \varphi'$ we have

$$(\tau_z \circ \varphi')(g_x) = (mg)_{\overline{\varphi}(x)+z}, \quad (\tau_z \circ \varphi')(g_{\overline{\varphi}(x)+z}) = (mg)_{\overline{\varphi}^2(x)+\overline{\varphi}z+z},$$

$$(\tau_z \circ \varphi')g_{\overline{\varphi}^2(x)+\overline{\varphi}z+z} = (mg)_{\overline{\varphi}^3(x)+\overline{\varphi}^2z+\overline{\varphi}z+z} = (mg)_x,$$

because $\overline{\varphi}^3(x) = x$ and $\overline{\varphi}^2 z + \overline{\varphi} z + z = 0$ by (7). So $\tau_z \circ \varphi'$ has the same matrices as φ', but on new invariant summands $(\mathbb{Z}_q)_x \oplus (\mathbb{Z}_q)_{\overline{\varphi}(x)+z} \oplus (\mathbb{Z}_q)_{\overline{\varphi}^2(x)+\overline{\varphi}z+z}$. Similarly for the exceptional orbit. This completes the proof of this case.

Case 3): when $d_i > 1$ for $p_i = 2$ and $k = 4s$. Using the cyclotomic polynomial we can define (similarly to the above M) an element of order 5 in $GL(4, \mathbb{Z})$

$$M_4 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

(see e.g. [20] for an elementary introduction). For any $k = 4s$, let $M \in GL(k, \mathbb{Z})$ be the direct sum of s copies of M_4. Let $\overline{\varphi} : \mathbb{Z}^k \to \mathbb{Z}^k$ be defined by M. One can calculate

$$\det(M_4 - E) = 5, \qquad \det(M - E) = 5^s.$$

Hence, by (4), $R(\overline{\varphi}) = 5^s < \infty$. The length of any non-trivial orbit is 5, hence an *odd* number.

Similarly to M, one can verify that

$$(M_4)^4 + (M_4)^3 + (M_4)^2 + M_4 + E = 0. \tag{9}$$

This can be also deduced from the fact that the characteristic polynomial of the "companion matrix" of a polynomial p is just p.

For p-power components $\mathbb{Z}_{p^r}$ with $p > 2$, we define φ' (as above) by $a_0 \mapsto (p - 1)a_0$. Then, for an orbit $u, \overline{\varphi}u, \ldots, \overline{\varphi}^\gamma u$, we need to verify (for finiteness of $R(\varphi')$) that $(p - 1)^\gamma$ as a homomorphism $\mathbb{Z}_{p^r} \to \mathbb{Z}_{p^r}$ has no non-trivial fixed elements, i.e.

$(p-1)^\gamma \not\equiv 1 \mod p$. This is fulfilled because, for an odd γ, $(p-1)^\gamma - 1 \equiv -2 \not\equiv 0 \mod p$.

For 2-power components $\mathbb{Z}_{2^i} \oplus \mathbb{Z}_{2^i}$, we define φ' by $a_0 \mapsto F_2 a_0$ (as in (5)). Then, for an orbit $u, \overline{\varphi}u, \dots, \overline{\varphi}^\gamma u$, we need to verify that $(F_2)^\gamma$ as a homomorphism $\mathbb{Z}_{2^i} \oplus \mathbb{Z}_{2^i} \to \mathbb{Z}_{2^i} \oplus \mathbb{Z}_{2^i}$ has no non-trivial fixed elements. Here we need to use not only the fact that γ is odd, but its more specific form: $\gamma = 5$. In particular it can not be divided by 3 =order of $F_2 \mod 2$. Hence $(F_2)^\gamma = (F_2)^5 = (F_2)^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \mod 2$. It has no non-trivial fixed elements $\mod 2^i$ for any i.

For 2-power components $\mathbb{Z}_{2^i} \oplus \mathbb{Z}_{2^i} \oplus \mathbb{Z}_{2^i}$, we define φ' by $a_0 \mapsto F_3 a_0$ (as in (5)). Then, for an orbit of $\overline{\varphi}$ of length γ, we need to verify that $(F_3)^\gamma$ as a homomorphism $\mathbb{Z}_{2^i} \oplus \mathbb{Z}_{2^i} \oplus \mathbb{Z}_{2^i} \to \mathbb{Z}_{2^i} \oplus \mathbb{Z}_{2^i} \oplus \mathbb{Z}_{2^i}$ has no non-trivial fixed elements. One can verify, for $i = 1$, i.e. for $2^i = 2$, that the order of F_3 is relatively prime with 5, namely it is equal to 7. Moreover, $(F_3)^j$, $j = 1, \dots, 6$, has no non-trivial fixed elements. The absence of non-trivial fixed elements is equivalent to $\det((F_3)^j - E) \not\equiv 0 \mod 2$. Then $\det((F_3)^j - E) \not\equiv 0 \mod 2^i$. Hence, for $i > 1$ these automorphisms still have no non-trivial fixed elements. The elements $(F_3)^{7u}$, $u = 1, 2, \dots$, typically are not $E \mod 2^i$, but in any case $7u \neq 5$, for any u. In fact, we are interested only in properties of $(F_3)^5$.

Collecting together these homomorphisms defined on the summands, we obtain as in the first two cases, φ' with the desired properties. It remains only to verify the epimorphity of $\tau_z \circ \varphi'$. This can be done quite similarly to the end of Case 2) with the help of (9). □

4. Twisted Burnside-Frobenius Theorem

Theorem 4.1. *Let G be a finite abelian group, and $\varphi \in \operatorname{Aut}(G \wr \mathbb{Z}^k)$ be a finite order automorphism. Then $R(\varphi') \in \{1, \infty\}$.*

Corollary 4.2. *Let G be a finite abelian group, and $\varphi \in \operatorname{Aut}(G \wr \mathbb{Z}^k)$ be a finite order automorphism. Then φ has the TBFT$_f$ property.*

Proof of Corollary. By Lemma 2.4, $R(\varphi) < \infty$ implies $R(\varphi') < \infty$. Hence, by Theorem 4.1, $R(\varphi') = 1$. Considering $\tau_z \circ \varphi$ instead of φ from the very beginning, we see that $R(\tau_z \circ \varphi') = 1$, for any $z \in \mathbb{Z}^k$. Thus, by Theorem 2.2, Reidemeister classes $\{g\}_\varphi$ of φ are pull-backs of Reidemeister classes $\{z\}_{\overline{\varphi}}$ of $\overline{\varphi}$ under the natural projection $\pi : G \wr \mathbb{Z}^k \to \mathbb{Z}^k$, i.e. $\{g\}_\varphi = \pi^{-1}(\{\pi(g)\}_{\overline{\varphi}})$. So, if classes of $\overline{\varphi}$ are separated by an epimorphism $f : \mathbb{Z}^k \to F$ onto a finite group F, then classes of φ are separated by $f \circ \pi$. It remains to use the equivalence between TBFT$_f$ and separability of Reidemeister classes in the case of finite Reidemeister number (see [11] and [8]). □

Remark 4.3. In particular, this covers all automorphisms, which were considered in [27]. Indeed, it was proved there, that all orbits are finite and their length is equal to the length of orbits of $\overline{\varphi}$. But the structure of $\mathbb{Z}^k$ implies that $\overline{\varphi}$ has finite order (consider generators). Hence, φ' and φ are of finite order.

Proof of Theorem. Suppose, $R(\varphi') > 1$. Then there exists an element $\sigma \in \Sigma$ such that $\sigma \notin \operatorname{Im}(\operatorname{Id} - \varphi')$. Moreover, $\sigma \notin \operatorname{Im}(\operatorname{Id} - \varphi'_\sigma)$, where φ'_σ is the restriction of φ' onto the φ'-invariant subgroup Σ_σ generated by σ. In particular, $R(\varphi'_\sigma) > 1$. By the supposition Σ_σ is a finite group with generators $\sigma, \varphi'(\sigma), \dots, (\varphi')^s(\sigma)$ for some

s. Hence, φ'_σ has a nontrivial fixed element σ_0, $\varphi'_\sigma(\sigma_0) = \sigma_0$ and $\sigma_0 \neq 0$. For an element $m \in \mathbb{Z}^k$ consider the orbit

$$\alpha(m)\sigma_0, \quad \varphi'(\alpha(m)\sigma_0) = \alpha(\overline{\varphi}(m))\sigma_0, \ldots, \quad (\varphi')^t(\alpha(m)\sigma_0) = \alpha(\overline{\varphi}^t(m))\sigma_0,$$

and $\overline{\varphi}^{t+1}(m) = m$. Then $(\varphi')^{t+1}(\alpha(m)\sigma_0) = \alpha(m)\sigma_0$. Passing from m to nm, $n \in \mathbb{Z}$, $m \in \mathbb{Z}^k$, if necessary, we can assume that the supports in $\mathbb{Z}^k$ of σ_0, $\alpha(\overline{\varphi}^j(nm))\sigma_0$, $j = 0, \ldots, t$, do not intersect. Then $\sum_{j=1}^t \alpha(\overline{\varphi}^j(nm))\sigma_0$ is a fixed element of φ', which is distinct from 0 and σ_0. Increasing n "in sufficiently large steps" we obtain infinitely many distinct fixed elements in the same way. Then by Lemma 2.3, $R(\varphi') = \infty$. □

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