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ASYMPTOTIC MODELING OF CURVILINEAR NARROW INCLUSIONS WITH ROUGH BOUNDARIES IN ELASTIC BODIES: CASE OF A SOFT INCLUSION

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Abstract. Within the framework of two-dimensional elasticity theory, a heterogeneous body with a narrow inclusion lying strictly inside the body is considered. It is assumed that the elastic properties of inclusion and its width depend on the small parameter $\delta > 0$. Moreover, we assume that the inclusion has a curvilinear rough boundary. We show that there exist three type of limiting problem as $\delta \to 0$: $p > 1$ – body with crack without interaction of its faces; $p = 1$ – body with crack with adhesive interaction of its faces; $p \in [0, 1)$ – homogeneous body (no crack).

Keywords: asymptotic analysis, inhomogeneous elastic body, narrow inclusion, curvilinear crack, interface conditions.

1. Introduction

Modern aerospace, automobile, and many other engineering structures require new materials with low weight and high strength. One of the ways to obtain such materials is construction of composites containing various kinds of thin small inclusions. The numerical solution of models of the inclusions is fraught with large computational costs due to small size compared to the structure. To overcome this difficulty approximation models are derived. In these models inclusions are replaced by some submanifolds of codimension 1 on which the interface conditions inherited from the original full model are properly set.

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In the present paper, within the framework of two-dimensional elasticity theory, a heterogeneous body with a narrow inclusion lying strictly inside the body is considered. It is assumed that the elastic properties of inclusion and its width depend on the small parameter $\delta > 0$: width is in direct ratio δ, while Young's modulus of the inclusion is proportional to δ^p, where p is a real number. We investigate the case $p \geq 0$. In this case such inclusion is called a soft one (see, e.g., [4, 9, 29]). The inclusion has a curved rough border. The latter means that a perturbation of the inclusion boundary depending on the parameter of δ is introduced. Mixed boundary conditions are prescribed on the external boundary of the body. The equilibrium problem is formulated as the problem of minimizing the energy functional in Sobolev's space H^1.

We present some methodology to derive models of elastic bodies for the crack problems. The methodology combines asymptotic analysis with variational properties of the equilibrium problem. Its major advantage over existing asymptotic approaches consists of the straightforward derivation of crack models for any non-negative value of parameter p simultaneously. It is shown that there exist three type of limiting problem as $\delta \to 0$: $p > 1$ – body with crack without interaction of its faces; $p = 1$ – body with crack with adhesive interaction of its faces (so called spring type condition); $p \in [0, 1)$ – homogeneous body (no crack).

Note recent studies [8, 24, 25] where the analogous technics is applied to some models of elastic bodies and plates. Papers [5, 7, 10, 14, 28] is devoted the investigation of thin inclusions problems including issues of numerical solutions, shape optimization, homogenization. Papers [8, 21], where case of hard inclusions (with $p < 0$) is studied for some models of Elasticity. At last, we mention several works on investigations of problems with thin structure (not only inclusions) in elastic bodies (see [6, 12, 13, 14, 20]).

Summarizing, in our consideration we take into account three key features, which, in our opinion, were not taken into account earlier simultaneously; namely,

– the inclusion is located strictly inside the body;
– the inclusion has a curvilinear boundary;
– the inclusion boundary is rough.

2. Statement of the problem

Let $\Omega \subset \mathbb{R}^2$ be a convex bounded domain with a Lipschitz boundary $\partial\Omega$; let Γ_N and Γ_D be parts of $\partial\Omega$ such that $\overline{\Gamma}_N \cup \overline{\Gamma}_D = \partial\Omega$, $\operatorname{meas}\Gamma_D > 0$. Let I_1 and I_2 be two intervals lying on the abscissa axis Oy_1 such that $\bar{I}_1 \subset I_2$, $\bar{I}_2 \subset I$, where I is an intersection of the domain Ω with Oy_1-axis.

Let us consider the Lipschitz functions φ, ψ_-, ψ_+ defined on the interval I such that:

1. $\psi_+ - \psi_- > 0$ on I_1;
2. $\psi_+ - \psi_- = 0$ on $I \setminus \bar{I}_2$;
3. $\varphi = 0$ on $I \setminus \bar{I}_2$;
4. a graph of the function φ is located strictly inside the domain Ω.

Let us fix a small parameter $\delta > 0$ and introduce the notations:

$$\Omega_\pm = \{(x_1, x_2) \in \Omega \ \mid \ \pm x_2 > \pm\varphi(x_1),\ x_1 \in I\},$$

$$\Omega_m^\delta = \{(y_1, y_2) \in \Omega \ \mid \ \varphi(y_1) - \delta\psi_-(y_1) < y_2 < \varphi(y_1) + \delta\psi_+(y_1),\ y_1 \in I_1\},$$

$$S_\pm^\delta = \{(y_1, y_2) \in \Omega \ \mid \ y_2 = \varphi(y_1) \pm \delta\psi_\pm(y_1),\ y_1 \in I_1\},$$

Note that for all small enough $\delta > 0$ the domain Ω_m^δ lies strictly inside Ω.

We assume that the domain Ω is an elastic inhomogeneous body, containing a narrow inclusion Ω_m^δ with width of order δ and with a rough boundary (roughness is characterized by functions $\psi_\pm$). By C^δ, C^0 we denote tensors characterizing material constants of the inclusion Ω_m^δ and an elastic matrix $\Omega \setminus \overline{\Omega_m^\delta}$, respectively, with usual symmetrical and elliptical properties (see, e.g., [15]). We prescribe homogeneous Dirichlet's conditions on a part Γ_D of the external boundary $\partial\Omega$ and Neumann's conditions on the rest part Γ_N of $\partial\Omega$. It means that the body Ω is clamped on Γ_D, whereas the forces $g \in L_2(\Gamma_N)$ is applied to Γ_N.

An equilibrium problem of the body Ω with the inclusion Ω_m^δ is formulated as a variational one in the framework of two-dimensional Elasticity. Let $u = (u_1, u_2)$ be a vector of displacements of the composite body, $\sigma(u) = (\sigma_{ij}(u))_{i,j=1,2}$ be the stress tensor, $\varepsilon(u) = (\varepsilon_{ij}(u))_{i,j=1,2}$ be the strain tensor, where

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \quad i, j = 1, 2,$$

$$\sigma(u) = C\varepsilon(u), \text{ where } C = \begin{cases} C^0 \text{ in } \Omega \setminus \overline{\Omega_m^\delta} \\ C^\delta \text{ in } \Omega_m^\delta. \end{cases}$$

Lower indices after comma denote the operation of differentiation with respect to corresponding coordinate; the summation over repeated indices is performed. Additionally, we assume that the inclusion Ω_m^δ is an isotropic and homogeneous body such that $C^\delta = \delta^p C^1$ with $p \geq 0$, where

$$C^1\varepsilon(u) = \lambda_m I \operatorname{tr} \varepsilon(u) + 2\mu_m \varepsilon(u),$$

with the associated Lamé coefficients given by λ_m and μ_m. We assume $3\lambda_m + 2\mu_m > 0$ and $\mu_m > 0$, in which case the tensor C^1 is positive definite. In general, the elastic matrix is anisotropic and homogeneous with elastic coefficients from $L^\infty_{loc}(\mathbb{R}^2)$.

Thus, the equilibrium problem is as follows: for a given $g \in L_2(\Gamma_N)$, find $u_\delta \in H^1_{\Gamma_D}(\Omega)$ satisfying the following variational equality:

$$\int\limits_\Omega \sigma(u_\delta) : \varepsilon(u)\, dy = \int\limits_{\Gamma_N} gu\, ds \tag{1}$$

for all kinematically admissible displacement functions $u \in H^1_{\Gamma_D}(\Omega)$, where

$$H^1_{\Gamma_D}(\Omega) = \{u \in H^1(\Omega)^2 \;\mid\; u = 0 \text{ a.e. on } \Gamma_D\}.$$

Let us rewrite problem (1) in equivalent form. First of all, we introduce an extended inclusion

$$\Omega_M^\delta = \{(y_1, y_2) \in \Omega \;\mid\; \varphi(y_1) - \delta\psi(y_1) < y_2 < \varphi(y_1) + \delta\psi_+(y_1),\; y_1 \in I_2\},$$

containing the inclusion Ω_m^δ, and put

$$T_\pm^\delta = \{(y_1, y_2) \in \Omega \;\mid\; y_2 = \varphi(y_1) \pm \delta\psi_\pm(y_1),\; y_1 \in I_2\},$$

containing curves $S_\pm^\delta$. Moreover, we define domains

$$\omega_m^\delta = \Omega_M^\delta \setminus \overline{\Omega_m^\delta}, \quad \Omega_e^\delta = \Omega \setminus \overline{\Omega_M^\delta}, \quad \Omega_\pm^\delta = \Omega_e^\delta \cap \Omega_\pm$$

We introduce a set

$$K^\delta = \{(v^+, v^-, v^m) \in H^1(\Omega_+^\delta) \times H^1(\Omega_-^\delta) \times H^1(\Omega_M^\delta) \;\mid$$
$$v^\pm = 0 \text{ a.e. on } \Gamma_D \cap \partial\Omega_\pm^\delta, \;\; v^\pm = v^m \text{ a.e. on } T_\pm^\delta, \; v^+ = v^- \text{ a.e. on } \partial\Omega_-^\delta \cap \partial\Omega_+^\delta\}.$$

Then problem (1) is rewritten as follows: find a triplet $(u_\delta^+, u_\delta^-, u_\delta^m) \in K^\delta$, satisfying a variational equality

$$(2)\quad \int\limits_{\Omega_+^\delta} C^0\varepsilon(u_\delta^+) : \varepsilon(u^+)\,dy + \int\limits_{\Omega_-^\delta} C^0\varepsilon(u_\delta^-) : \varepsilon(u^-)\,dy + \int\limits_{\omega_m^\delta} C^0\varepsilon(u_\delta^m) : \varepsilon(u^m)\,dy +$$
$$+ \delta^p \int\limits_{\Omega_M^\delta} C^1\varepsilon(u_\delta^m) : \varepsilon(u^m)\,dy = \int\limits_{\Gamma_N \cap \partial\Omega_+^\delta} gu^+\,ds + \int\limits_{\Gamma_N \cap \partial\Omega_-^\delta} gu^-\,ds$$

for all $(u^+, u^-, u^m) \in K^\delta$.

By the Lions-Stampacchia theorem, problem (2) has a unique solution $(u_\delta^+, u_\delta^-, u_\delta^m)$ for all $\delta > 0$, and the following relations

$$u_\delta|_{\Omega_\pm^\delta} = u_\delta^\pm, \quad u_\delta|_{\Omega_M^\delta} = u_\delta^m$$

hold.

3. Coordinate transformations

We put

$$\Omega_M = \{(z_1, z_2) \in \mathbb{R}^2 \mid \psi_-(z_1) < z_2 < \psi_+(z_1),\ z_1 \in I_2\},$$
$$\Omega_m = \{(z_1, z_2) \in \mathbb{R}^2 \mid \psi_-(z_1) < z_2 < \psi_+(z_1),\ z_1 \in I_1\},$$
$$T_\pm = \{(z_1, z_2) \in \mathbb{R}^2 \mid z_2 = \psi_\pm(z_1),\ z_1 \in I_2\},$$
$$S_\pm = \{(z_1, z_2) \in \mathbb{R}^2 \mid z_2 = \psi_\pm(z_1),\ z_1 \in I_1\},$$
$$T = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 = \varphi(x_1),\ x_1 \in I_2\},$$
$$S = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 = \varphi(x_1),\ x_1 \in I_1\},$$

and introduce coordinate transformation mapping domains $\Omega_\pm^\delta$ and Ω_M^δ into domains $\Omega_\pm$ and Ω_M, respectively. Let us take two convex open domains D_1 and D_2 such that $\overline{I}_2 \subset D_1$, $\overline{D}_1 \subset D_2$, $\overline{D}_2 \subset \Omega$, and consider the following domains:

$$D_i^\varphi = \{(z_1, z_2) \ \mid \ z_1 = y_1,\ z_2 = \varphi(y_1) + y_2,\ (y_1, y_2) \in D_i\},\ i = 1, 2.$$

As a result, we have the following inclusions: $\overline{T} \subset D_1^\varphi$, $\overline{D}_1^\varphi \subset D_2^\varphi$. Additionally, we also suppose that the domain D_2^φ lies strictly inside Ω.

Now we consider a cut-off function $\theta \in C^1(\overline{\Omega})$ such that

$$\theta = 1 \text{ in } D_1^\varphi; \quad 0 < \theta < 1 \text{ in } D_2^\varphi;\ \theta = 0 \text{ in } \Omega \setminus \overline{D}_2^\varphi.$$

In the domain $\Omega_\pm$ and Ω_M we define coordinate transformations as follows

$$(3)\qquad y_1 = x_1,\ y_2 = x_2 \pm \delta\psi_\pm(x_1)\theta(x_1, x_2), \quad (x_1, x_2) \in \Omega_\pm,\ (y_1, y_2) \in \Omega_\pm^\delta,$$

$$(4)\qquad y_1 = z_1,\ y_2 = \delta z_2 + \varphi(z_1), \quad (z_1, z_2) \in \Omega_M,\ (y_1, y_2) \in \Omega_M^\delta.$$

It is easy to see that transformations (3) and (4) are one-to-one. Note that transformation of the type (3)are widely used in shape optimization and crack theory (see, e.g., [11, 23, 16, 17, 27]).

Due to smoothness properties of functions φ and $\psi_\pm$, coordinate transformations (3) and (4) generate one-to-one mappings between spaces $H^1(\Omega_\pm)$, $H^1(\Omega_M)$ and

$H^1(\Omega^\delta_\pm)$, $H^1(\Omega^\delta_M)$, respectively (see [19], Chapter 2, Lemma 3.4). Preimage of K^δ in $H^1(\Omega_-)\times H^1(\Omega_+)\times H^1(\Omega_M)$ is a set

$$K = \{(v^+, v^-, v^m) \in H^1(\Omega_+)\times H^1(\Omega_-)\times H^1(\Omega_M) \quad | \quad v^\pm|_T = v^m|_{T_\pm}, \quad v^\pm = 0 \text{ a.e. on } \Gamma_D\cap\partial\Omega_\pm, \quad v^+ = v^- \text{ a.e. on } (\partial\Omega_-\cap\partial\Omega_+)\setminus\overline{T}\}.$$

Hereinafter, $v^\pm|_T = v^m|_{T_\pm}$ means that

$$v^\pm(x_1,\varphi(x_1)) = v^m(x_1,\psi_\pm(x_1)),\ x_1\in I_2.$$

After applying coordinate transformations (3) and (4) to variational equality (2), we conclude that transformed displacements

$$\tilde u^\pm_\delta(x_1,x_2) = u^\pm_\delta(x_1, x_2 \pm \delta\psi_\pm(x_1)\theta(x_1,x_2)), \tag{5}$$

$$\tilde u^m_\delta(z_1,z_2) = u^m_\delta(z_1, \delta z_1 + \varphi(z_1)) \tag{6}$$

satisfy the following variational equality:

$$A^\delta_+(\tilde u^+_\delta, v^+) + A^\delta_-(\tilde u^-_\delta, v^-) + \delta\int\limits_{\omega_m} C^0 e^\delta(\tilde u^m_\delta) : e^\delta(v^m)\,dz + {} + \delta^{p+1}\int\limits_{\Omega_m} C^1 e^\delta(\tilde u^m_\delta) : e^\delta(v^m)\,dz = \int\limits_{\partial\Omega^+\cap\Gamma_N} g v^+\,ds + \int\limits_{\partial\Omega^-\cap\Gamma_N} g v^-\,ds \tag{7}$$

for any triplet $(v^+, v^-, v^m)\in K$, where $\omega_m = \Omega_M\setminus\overline{\Omega_m}$,

$$A^\delta_\pm(u^\pm, v^\pm) = \int\limits_{\Omega_\pm} J^\pm_\delta C^0 E(\Psi^\pm_\delta; u^\pm) : E(\Psi^\pm_\delta; v^\pm)\,dx,$$

$$E_{ij}(\Psi^\pm_\delta; v) = 1/2\left(v_{i,k}(\Psi^\pm_\delta)_{kj} + v_{j,k}(\Psi^\pm_\delta)_{ki}\right),\ i,j = 1,2;$$

$$e^\delta(v^m) = \begin{pmatrix} v^m_{1,1} - \frac{\varphi'}{\delta}v^m_{1,2} & \frac{1}{2}\left(\frac{1}{\delta}v^m_{1,2} + v^m_{2,1} - \frac{\varphi'}{\delta}v^m_{2,2}\right) \\ \frac{1}{2}\left(\frac{1}{\delta}v^m_{1,2} + v^m_{2,1} - \frac{\varphi'}{\delta}v^m_{2,2}\right) & \frac{1}{\delta}v^m_{2,2} \end{pmatrix},$$

matrices $\Psi^\pm_\delta$ are Jacobi matrices of transformations (3), and $J^\pm_\delta$ is Jacobian of transformations (3).

Since the main goal of the our investigation is to find a limit problem as $\delta\to 0$, in the sequel it is convenient to use the following asymptotic expansion:

$$A^\delta_\pm(u,v) = \int\limits_{\Omega_\pm} C^0\varepsilon(u):\varepsilon(v)\,dx + \delta r_\pm(\delta,u,v), \tag{8}$$

where

$$|r_\pm(\delta,u,v)| \le c(\|u\|^2_{H^1(\Omega_\pm)} + \|v\|^2_{H^1(\Omega_\pm)}) \tag{9}$$

with a constant c independent of δ, u, and v.

4. Asymptotic analysis

We need to recall auxiliary lemmas from [24, 8].

Lemma 1. *For any function $\tilde{v} \in H^1(\Omega_M)$ the following holds:*

$$\|\tilde{v}\|^2_{L_2(\Omega_M)} \leq c\left(\|\tilde{v}_{,2}\|^2_{L_2(\Omega_M)} + \|\tilde{v}\|^2_{L_2(T_s)}\right), \quad s \in \{-,+\},$$

with a constant c independent of $\tilde{v}$.

Lemma 2. *For any small enough $\delta > 0$ and any triplet $(\tilde{v}^+, \tilde{v}^-, \tilde{v}^m) \in K$ the following inequality holds:*

$$(10) \quad c\Bigg(\|\tilde{v}^+\|^2_{H^1(\Omega_+)} + \|\tilde{v}^-\|^2_{H^1(\Omega_-)} + \delta\|\tilde{v}^m_{1,1} - \frac{\varphi'}{\delta}\tilde{v}^m_{1,2}\|^2_{L_2(\Omega_M)} + \\ + \frac{1}{\delta}\|\tilde{v}^m_{1,2}\|^2_{L_2(\Omega_M)} + \delta\|\tilde{v}^m_{2,1} - \frac{\varphi'}{\delta}\tilde{v}^m_{2,2}\|^2_{L_2(\Omega_M)} + \frac{1}{\delta}\|\tilde{v}^m_{2,2}\|^2_{L_2(\Omega_M)}\Bigg) \leq \\ \leq \int\limits_{\Omega_+} C^0\varepsilon(\tilde{v}^+) : \varepsilon(\tilde{v}^+)\,dx + \int\limits_{\Omega_-} C^0\varepsilon(\tilde{v}^-) : \varepsilon(\tilde{v}^-)\,dx + \\ + \delta\int\limits_{\omega_m} C^0 e^\delta(\tilde{v}^m) : e^\delta(\tilde{v}^m)\,dz + \delta\int\limits_{\Omega_m} C^1 e^\delta(\tilde{v}^m) : e^\delta(\tilde{v}^m)\,dz$$

with a constant c independent of δ and $(\tilde{v}^+, \tilde{v}^-, \tilde{v}^m)$.

Proof. Due to the fact that the domain Ω and the set K do not depend on δ, from the Korn inequality it follows existence a constant c independent of $v \in H^1_{\Gamma_D}(\Omega)$ and $\delta > 0$ such that

$$(11) \quad c(\|v^+\|^2_{H^1(\Omega^\delta_+)} + \|v^-\|^2_{H^1(\Omega^\delta_-)} + \|v^m\|^2_{H^1(\Omega^\delta_m)} \leq \int\limits_{\Omega} C\varepsilon(v) : \varepsilon(v)\,dy = \\ = \int\limits_{\Omega^\delta_+} C^0\varepsilon(v^+) : \varepsilon(v^+)\,dy + \int\limits_{\Omega^\delta_-} C^0\varepsilon(v^-) : \varepsilon(v^-)\,dy + \\ + \int\limits_{\omega^\delta_m} C^0\varepsilon(v^m) : \varepsilon(v^m)\,dy + \int\limits_{\Omega^\delta_m} C^1\varepsilon(v^m) : \varepsilon(v^m)\,dy.$$

Here v^+, v^-, and v^m are the restrictions of the function v onto domains Ω^δ_+, Ω^δ_-, and $\Omega^\delta_M = \Omega^\delta_m \cup \omega^\delta_m$, respectively.

Let us apply coordinate transformations (3) and (4) in integrals in (11). After that we use expansions (8) and the fact that after coordinate transformation (3) the following expansion hold:

$$\|v^\pm\|^2_{H^1(\Omega^\delta_\pm)} = \|\tilde{v}^\pm\|^2_{H^1(\Omega_\pm)} + \tilde{r}_\pm(\delta; \tilde{v}^\pm)$$

with an estimate

$$|\tilde{r}_\pm(\delta; v^\pm)| \leq c\delta\|\tilde{v}^\pm\|^2_{H^1(\Omega_\pm)},$$

where $\tilde{v}^+$, $\tilde{v}^-$, and $\tilde{v}^m$ are defined akin to (5) and (6). Thus, inequality (10) is valid for all small enough $\delta > 0$. $\square$

Theorem 1. *The solution* $(\tilde{u}_\delta^+, \tilde{u}_\delta^-, \tilde{u}_\delta^m)$ *to problem* (7) *satisfies the uniform estimates:*

$$\|\tilde{u}_\delta^+\|_{H^1(\Omega_+)^2} \le c, \quad \|\tilde{u}_\delta^-\|_{H^1(\Omega_-)^2} \le c, \tag{12}$$

$$\delta^{\frac{p+1}{2}} \|\tilde{u}_{\delta 1,1}^m - \frac{\varphi'}{\delta} \tilde{u}_{\delta 1,2}^m\|_{L_2(\Omega_m)} \le c, \; \delta^{\frac{p-1}{2}} \|\tilde{u}_{\delta 2,2}^m\|_{L_2(\Omega_m)} \le c, \tag{13}$$

$$\delta^{\frac{p-1}{2}} \|\tilde{u}_{\delta 1,2}^m\|_{L_2(\Omega_m)} \le c, \; \delta^{\frac{p+1}{2}} \|\tilde{u}_{\delta 2,1}^m - \frac{\varphi'}{\delta} \tilde{u}_{\delta 2,2}^m\|_{L_2(\Omega_m)} \le c, \tag{14}$$

$$\delta^{-\frac{1}{2}} \|\tilde{u}_{\delta 2,2}^m\|_{L_2(\omega_m)} \le c, \tag{15}$$

$$\delta^{-\frac{1}{2}} \|\tilde{u}_{\delta 1,2}^m\|_{L_2(\omega_m)} \le c. \tag{16}$$

Proof. First of all, let us substitute $(\tilde{u}_\delta^+, \tilde{u}_\delta^-, \tilde{u}_\delta^m)$ in (7). Taking into account (8) and (9) and the Korn inequality, we easily get estimates (12), (13), and (15).

To prove (14) we substitute $(\tilde{u}_\delta^+, \tilde{u}_\delta^-, \tilde{u}_\delta^m)$ in (10) and multiply the obtained inequality by δ^p. Due to estimates (12) and since $p \ge 0$, we have the following chain of inequalities:

$$\begin{aligned}
c\left(\delta^{p-1}\|\tilde{u}_{\delta 1,2}^m\|^2_{L_2(\Omega_m)} + \delta^{p+1}\|\tilde{u}_{\delta 2,1}^m - \frac{\varphi'}{\delta}\tilde{u}_{\delta 2,2}^m\|^2_{L_2(\Omega_m)}\right) \le \\
\le \delta^p \int\limits_{\Omega_+} C^0\varepsilon(\tilde{u}_\delta^+) : \varepsilon(\tilde{u}_\delta^+)\, dx + \delta^p \int\limits_{\Omega_-} C^0\varepsilon(\tilde{u}_\delta^-) : \varepsilon(\tilde{u}_\delta^-)\, dx + \\
+ \delta^{p+1} \int\limits_{\omega_m} C^0 e^\delta(\tilde{u}_\delta^m) : e^\delta(\tilde{u}_\delta^m)\, dz + \delta^{p+1} \int\limits_{\Omega_m} C^1 e^\delta(\tilde{u}_\delta^m) : e^\delta(\tilde{u}_\delta^m)\, dz \le \\
\le \int\limits_{\Omega_+} C^0\varepsilon(\tilde{u}_\delta^+) : \varepsilon(\tilde{u}_\delta^+)\, dx + \int\limits_{\Omega_-} C^0\varepsilon(\tilde{u}_\delta^-) : \varepsilon(\tilde{u}_\delta^-)\, dx + \\
+ \delta \int\limits_{\omega_m} C^0 e^\delta(\tilde{u}_\delta^m) : e^\delta(\tilde{u}_\delta^m)\, dz + \delta^{p+1} \int\limits_{\Omega_m} C^1 e^\delta(\tilde{u}_\delta^m) : e^\delta(\tilde{u}_\delta^m)\, dz = \\
= \int\limits_{\partial\Omega^+\cap\Gamma_N} g\tilde{u}_\delta^+\, ds + \int\limits_{\partial\Omega^-\cap\Gamma_N} g\tilde{u}_\delta^-\, ds - \delta r_\pm(\delta, \tilde{u}_\delta^\pm, \tilde{u}_\delta^\pm) \le c.
\end{aligned}$$

Estimate (16) is followed from (10) and (12)-(14). □

Below we consider the limiting cases separately depending on the positive parameter p.

4.1. **Case** $0 < p < 1$ **(homogeneous body).** From (12), (15), and (16), we conclude that there exists sequence still denoting by δ, functions $u^\pm \in H^1(\Omega_\pm)$, and $\sigma^1 \in L_2(\Omega_M)^4$, such that the following convergences

$$\tilde{u}_\delta^\pm \to u^\pm \;\text{ weakly in }\; H^1(\Omega_\pm),$$

$$\delta^{\frac{p+1}{2}} C^1 e^\delta(\tilde{u}_\delta^m) \to \sigma^1 \;\text{ weakly in }\; L_2(\Omega_m)^4,$$

$$\delta^{\frac{1}{2}} C^0 e^\delta(\tilde{u}_\delta^m) \to \sigma^1 \;\text{ weakly in }\; L_2(\omega_m)^4,$$

$$\tilde{u}_{\delta,2}^m \to u_{,2}^m = 0 \;\text{ strongly in }\; L_2(\Omega_M)^2 \tag{17}$$

take place.

From (17) it follows that u is independent of z_2. Moreover, the traces of the functions u^+ and u^+ on the curve T coincide with each other (see [26]), i.e.,

$$u^-|_T = u^+|_T. \tag{18}$$

Let us take a smooth function $v \in C^1(\overline{\Omega})$ and define functions

$$v^\pm = v|_{\Omega_\pm}, \quad v^m(z_1, z_2) = v(z_1, \varphi(z_1)).$$

Since v^m does not depend on z_2,

$$\delta^{\frac{p+1}{2}} e^\delta(v^m) \to 0 \ \text{ strongly in } \ L_2(\Omega_m)^4,$$

$$\delta^{\frac{1}{2}} e^\delta(v^m) \to 0 \ \text{ strongly in } \ L_2(\omega_m)^4.$$

After substituting (v^-, v^+, v^m) in (7) and passing to the limit as $\delta \to 0$ we have

$$\int\limits_{\Omega_+} C^0\varepsilon(u^+) : \varepsilon(v^+)\, dx + \int\limits_{\Omega_-} C^0\varepsilon(u^-) : \varepsilon(v^-)\, dx = \tag{19}$$

$$= \int\limits_{\partial\Omega^+\cup\Gamma_N} gv^+\, ds + \int\limits_{\partial\Omega^-\cup\Gamma_N} gv^-\, ds,$$

Due to (18), variational equality (19) can be rewritten in an equivalent form

$$\int\limits_{\Omega} C^0\varepsilon(u) : \varepsilon(v)\, dx = \int\limits_{\Gamma_N} gv\, ds \quad \forall v \in H^1_{\Gamma_D}(\Omega), \tag{20}$$

where $u \in H^1_{\Gamma_D}(\Omega)$ is defined as follows:

$$u(x_1, x_2) = \begin{cases} u^-(x_1, x_2), & (x_1, x_2) \in \Omega_-, \\ u^+(x_1, x_2), & (x_1, x_2) \in \Omega_+ \end{cases} \tag{21}$$

Problem (20) describes equilibria of an homogeneous body without any inclusions.

4.2. **Case $p = 1$ (crack with adhesion of its faces).** From Lemma 1 and estimates (12)-(16) it follows

$$\|\tilde{u}^m_\delta\|_{L_2(\Omega_M)^2} \le c,$$

uniformly with respect to δ.

Theorem 1 allows us to find a sequence still denoting by δ and functions $u^\pm \in H^1_{\Gamma_D}(\Omega_\pm)$, $u^m \in L_2(\Omega_M)^2$, $\sigma^1 \in L_2(\Omega_M)^4$ such that the following convergences

$$\tilde{u}^\pm_\delta \to u^\pm \ \text{ weakly in } \ H^1_{\Gamma_D}(\Omega_\pm),$$

$$\tilde{u}^m_\delta \to u^m \ \text{ weakly in } \ L_2(\Omega_m)^2,$$

$$\tilde{u}^m_{\delta,2} \to u^m_{,2} \ \text{ weakly in } \ L_2(\Omega_m)^2,$$

$$\delta\tilde{u}^m_{\delta,1} \to 0 \ \text{ weakly in } \ L_2(\Omega_m)^2,$$

$$\delta\tilde{u}^m_{\delta,1} - \varphi'\tilde{u}^m_{\delta,2} \to -\varphi' u^m_{,2} \ \text{ weakly in } \ L_2(\Omega_m)^2,$$

$$\delta^{\frac{1}{2}} C^1 e^\delta(\tilde{u}^m_\delta) \to \sigma^1 \ \text{ weakly in } \ L_2(\omega_m)^4$$

hold as $\delta \to 0$. Then we have

$$\delta C^1 e^\delta(\tilde{u}^m_\delta) \to \sigma^0(u^m) \ \text{ weakly in } \ L_2(\Omega_m)^4, \tag{22}$$

where

$$\sigma^0(u^m) = \begin{pmatrix} -(\lambda_m + 2\mu_m)\varphi' u^m_{1,2} + \lambda_m u^m_{2,2} & \mu_m(u^m_{1,2} - \varphi' u^m_{2,2}) \\ \mu_m(u^m_{1,2} - \varphi' u^m_{2,2}) & (\lambda_m + 2\mu_m)\tilde{u}^m_{2,2} - \lambda_m \varphi' u^m_{1,2} \end{pmatrix}.$$

Moreover, for any $v^m \in H^1(\Omega_m)^2$ we have

$$\delta e^\delta(v^m) \to e^0(v) \ \text{ strongly in } \ L_2(\Omega_m)^4, \tag{23}$$

where

$$e^0(v^m) = \begin{pmatrix} -\varphi' v^m_{1,2} & \frac{1}{2}(v^m_{1,2} - \varphi' v^m_{2,2}) \\ \frac{1}{2}(v^m_{1,2} - \varphi' v^m_{2,2}) & v^m_{2,2} \end{pmatrix}.$$

Let us investigate properties limit functions. Firstly, by the same reasons as the previous case, we have equality (18) on part $T \setminus \overline{S}$ of the curve T.

Then, take a function $v^m \in C_0^\infty(\Omega_m)^2$ and substitute a triplet $(0, 0, v^m) \in K$ in variational equality (7) as a test function. After passing to the limit as $\delta \to 0$, due to (22) and (23), we get

$$\int\limits_{\Omega_m} \sigma^0(u^m) : e^1(v^m)\, dz = 0,$$

which is valid for all $v^m \in C_0^\infty(\Omega_m)^2$. Then there exist functions $\alpha_1(z_1)$ and $\alpha_2(z_1)$ such that

$$-\varphi'(z_1)(\mu_m + \lambda_m)u^m_{1,2} + (\varphi'^2(z_1)\mu_m + (\lambda_m + 2\mu_m))u^m_{2,2} = \alpha_1(z_1) \ \text{ a.e. in } \ \Omega_m, \tag{24}$$

$$((\lambda_m + 2\mu_m)\varphi'^2(z_1) + \mu_m)u^m_{1,2} - \varphi'(z_1)(\mu_m + \lambda_m)u^m_{2,2} = \alpha_2(z_1) \ \text{ a.e. in } \ \Omega_m, \tag{25}$$

Integrating (24) and (25) with respect to z_2 from $\psi_-(z_1)$ to $\psi_+(z_1)$, we find

$$\alpha_1(z_1) = \frac{-\varphi'(z_1)(\mu_m + \lambda_m)[u_1] + (\varphi'^2(z_1)\mu_m + (\lambda_m + 2\mu_m))[u_2]}{\psi_+(z_1) - \psi_-(z_1)},$$

$$\alpha_2(z_1) = \frac{((\lambda_m + 2\mu_m)\varphi'^2(z_1) + \mu_m)[u_1] - \varphi'(z_1)(\mu_m + \lambda_m)[u_2]}{\psi_+(z_1) - \psi_-(z_1)},$$

where $[u_i] = u_i^+(z_1, \varphi(z_1)) - u_i^-(z_1, \varphi(z_1))$, $z_1 \in I_1$, $i = 1, 2$.

Note that system of algebraic equations (24), (25) for $u^m_{1,2}$ and $u^m_{2,2}$ is nondegenerate, because its determinant is equal to $-\mu_m(\lambda_m + 2\mu_m)((\varphi')^2 + 1)^2$ and, therefore, is non-zero. After obvious calculation, we find

$$u^m_{i,2}(z_1, z_2) = \frac{u_i^+(z_1, \varphi(z_1)) - u_i^-(z_1, \varphi(z_1))}{\psi_+(z_1) - \psi_-(z_1)}, \quad i = 1, 2.$$

Integrating this representation with respect to z_2 from $\psi_+(z_1)$ to z_2, we get

$$u_i^m(z_1, z_2) = \frac{u_i^+(z_1, \varphi(z_1)) - u_i^-(z_1, \varphi(z_1))}{\psi_+(z_1) - \psi_-(z_1)} z_2 + \\ + \frac{u_i^-(z_1, \varphi(z_1))\psi_+(z_1) - u_i^+(z_1, \varphi(z_1))\psi_-(z_1)}{\psi_+(z_1) - \psi_-(z_1)}, \tag{26}$$

$i = 1, 2$.

Let us find a variational problem for functions $u^\pm$, u^m. For this, we take an arbitrary function $v \in C^1(\Omega)^2 \cap H^{1,0}(\Omega \setminus \overline{S})^2$, where

$$H^{1,0}(\Omega \setminus \overline{S}) = \{v \in H^1(\Omega \setminus \overline{S}) \ \mid \ v = 0 \text{ a.e. on } \Gamma_D\},$$

and define a function $v^m = (v_1^m, v_2^m)$ as follows:

$$v_i^m(z_1,z_2) = \frac{v_i^+(z_1,\varphi(z_1)) - v_i^-(z_1,\varphi(z_1))}{\psi_+(z_1)-\psi_-(z_1)} z_2 + \\ + \frac{-v_i^+(z_1,\varphi(z_1))\psi^-(z_1) + v_i^-(z_1,\varphi(z_1))\psi_+(z_1)}{\psi_+(z_1)-\psi_-(z_1)} \tag{27}$$

in the domain Ω_m;

$$v_i^m(z_1,z_2) = v_i^-(z_1,\varphi(z_1)), \tag{28}$$

in the domain ω_m, where $v^\pm$ is a restriction of v onto $\Omega_\pm$, $i=1,2$. Since $v \in C^1(\Omega)^2 \cap H^{1,0}(\Omega \setminus \overline{S})^2$, we have

$$v^-(z_1,\varphi(z_1)) = v^+(z_1,\varphi(z_1)), \quad z_1 \in T \setminus \overline{S}$$

and the function v^m is defined correctly. Moreover,

$$\delta^{\frac{1}{2}} e^\delta(v^m) \to 0 \ \text{ strongly in } \ L_2(\omega_m)^4$$

as δ goes to 0.

Now substitute a triplet (v^+, v^-, v^m) in (7) as a test function. After passing to the limit as $\delta \to 0$, we get

$$\int\limits_{\Omega_+} C^0\varepsilon(u^+) : \varepsilon(v^+)\,dx + \int\limits_{\Omega_-} C^0\varepsilon(u^-) : \varepsilon(v^-)\,dx + \int\limits_{\Omega_m} \sigma^0(u^m) : e^1(v^m)\,dz = \\ = \int\limits_{\partial\Omega^+\cup\Gamma_N} gv^+\,ds + \int\limits_{\partial\Omega^-\cup\Gamma_N} gv^-\,ds. \tag{29}$$

Due to (26), (27), and (28) we can integrate the third term in (29) with respect to z_2. Finally, the function u^+ and u^- satisfy the following variational equality:

$$\int\limits_{\Omega_+} C^0\varepsilon(\tilde{u}^+) : \varepsilon(v^+)\,dx + \int\limits_{\Omega_-} C^0\varepsilon(\tilde{u}^-) : \varepsilon(v^-)\,dx + \int\limits_{S} \frac{A_\varphi[u]\cdot[v]}{\psi_+-\psi_-}\,ds = \\ = \int\limits_{\partial\Omega^+\cup\Gamma_N} gv^+\,ds + \int\limits_{\partial\Omega^-\cup\Gamma_N} gv^-\,ds \quad \forall v \in H^{1,0}(\Omega\setminus\overline{S}), \tag{30}$$

where v^- and v^+ are restrictions of v onto domains Ω_- and Ω_+, respectively;

$$A_\varphi = \frac{1}{(1+(\varphi')^2)^{\frac{1}{2}}} \begin{pmatrix} (\lambda_m+2\mu_m)\varphi'^2 + \mu_m & -(\lambda_m+\mu_m)\varphi' \\ -(\lambda_m+\mu_m)\varphi' & \lambda_m + 2\mu_m + \mu_m\varphi'^2 \end{pmatrix}.$$

Due to the Korn inequality and the fact that matrix A_φ is positive definite, the problem (30) has the unique solution u^+, u^-.

Let us rewrite problem (30) in an equivalent form. Since $u^+ = u^-$ a.e. on $(\partial\Omega_- \cap \partial\Omega_+) \setminus \overline{S}$, the function u defined by (21) belongs to the space $H^{1,0}(\Omega)$, and it is a unique solution of the following variational problem:

$$\int\limits_{\Omega} C^0\varepsilon(u) : \varepsilon(v)\,dx + \int\limits_{S} \frac{A_\varphi[u]\cdot[v]}{\psi_+-\psi_-}\,ds = \int\limits_{\Gamma_N} gv\,ds \ \ \forall v \in H^{1,0}(\Omega\setminus\overline{S}). \tag{31}$$

In the sense of distributions, problem (31) implies a solution of the following boundary value problem:

$$-\operatorname{div}\sigma(u) = 0, \quad \sigma(u) = C^0\varepsilon(u) \ \text{ in } \ \Omega\setminus\overline{S},$$

$$[\sigma(u)\nu] = 0 \text{ on } S,$$

$$\sigma(u)\nu = \frac{A_\varphi[u]}{\psi_+ - \psi_-} \text{ on } S,$$

$$u = 0 \text{ on } \Gamma_D, \quad \sigma(u)\nu = g \text{ on } \Gamma_N.$$

Note that the interfacial condition on S is generalized known ones because it takes into account simultaneously both curvature and roughness of the inclusion (see, e.g., [1, 2, 4, 22, 18]).

Let us analyze the interface condition in the differential formulation of the equilibrium problem. Define by u_ν and u_τ normal and tangential components of the displacement vector on the crack S; and denote by σ_ν and σ_τ the shear and normal components of the traction $\sigma\nu$ on S, i.e.,

$$u_\nu = u\nu = u_i\nu_i, \quad u_\tau = u\tau = u_i\tau_i,$$

$$\sigma_\nu = (\sigma\nu)\nu = \sigma_{ij}\nu_i\nu_j, \quad \sigma_\tau = (\sigma\nu)\tau = \sigma_{ij}\nu_i\tau_j,$$

where ν and τ are unit normal and tangent vectors to S, respectively,

$$\nu = \frac{(-\varphi',1)}{(1+(\varphi')^2)^{\frac{1}{2}}}, \quad \tau = \frac{(1,\varphi')}{(1+(\varphi')^2)^{\frac{1}{2}}}.$$

After simple calculations we get

$$A_\varphi[u] = (\lambda_m + 2\mu_m)[u_\nu]\nu + \mu_m[u_\tau]\tau.$$

Then the interface condition on the crack S can be rewritten as follows:

$$\sigma_\nu = (\lambda_m + 2\mu_m)\frac{[u_\nu]}{\psi_+ - \psi_-}, \quad \sigma_\tau = \mu_m\frac{[u_\tau]}{\psi_+ - \psi_-}.$$

It is seen the shear and normal components of the traction are proportional to the jump of normal and tangential components of the displacements, respectively. These conditions, known in interface models of Elasticity as spring type conditions, is derived for crack case.

4.3. **Case $p > 1$ (crack with stress-free faces).** Again from Lemma 1, estimates (13) and (14) we can conclude that there exists sequence still denoting by δ, and functions $u^\pm \in H^1(\Omega_\pm)$, $\sigma^0 \in L_2(\Omega_m)^4$, such that the following convergences

$$\tilde{u}^\pm \to u^\pm \ \text{ weakly in } \ H^1_{\Gamma_D}(\Omega_\pm),$$

$$\delta^{\frac{p+1}{2}} C^1 e^\delta(\tilde{u}^m_\delta) \to \sigma^0 \ \text{ weakly in } \ L_2(\Omega_m)^4$$

hold. Moreover, for any $v^m \in H^1(\Omega_m)$ we have

$$\delta^{\frac{p+1}{2}} e^\delta(v^m) \to 0 \ \text{ strongly in } \ L_2(\Omega_m)^4.$$

To find a problem for $u^\pm$, take $v \in C^1(\Omega)\cap H^{1,0}(\Omega\setminus\overline{S})$ and define in the domain Ω_M a function v^m by (27) and (28). Then we substitute a triplet (v^+, v^-, v^m) in

(7) as a test function. After passing to the limit as $\delta \to 0$, we have the following variational equality:

$$\int\limits_{\Omega_+} C^0\varepsilon(u^+) : \varepsilon(v^+)\,dx + \int\limits_{\Omega_-} C^0\varepsilon(u^-) : \varepsilon(v^-)\,dx = \tag{32}$$

$$= \int\limits_{\partial\Omega^+\cup\Gamma_N} gv^+\,ds + \int\limits_{\partial\Omega^-\cup\Gamma_N} gv^-\,ds.$$

As before, we define a function u by (21). Then variational equality (32) can be rewritten in the following for:

$$\int\limits_{\Omega\setminus\overline{S}} C^0\varepsilon(u) : \varepsilon(v)\,dx + \int\limits_{\Gamma_N} gv\,ds \quad \forall v \in H^{1,0}(\Omega\setminus\overline{S}). \tag{33}$$

It is known that (33) describes equilibria of an elastic body with the crack S. In the sense of distributions, the function u satisfies the following boundary value problem:

$$-\operatorname{div}\sigma(u) = 0, \quad \sigma(u) = C^0\varepsilon(u) \;\text{ in }\; \Omega\setminus\overline{S},$$

$$\sigma(u)\nu = 0 \text{ on } S.$$

$$u = 0 \text{ on } \Gamma_D, \quad \sigma(u)\nu = g \text{ on } \Gamma_N.$$

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