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THE VOLUME OF A HYPERBOLIC ANTIPODAL OCTAHEDRON

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Abstract. We consider the hyperbolic antipodal octahedron. It is an octahedron with antipodal symmetry in the hyperbolic space $\mathbb{H}^3$. We establish necessary and sufficient conditions for the existence of such an octahedron in $\mathbb{H}^3$. By dividing the octahedron into appropriate tetrahedra we obtain an explicit integral formula for the volume of the hyperbolic antipodal octahedron.

Keywords: hyperbolic octahedron, hyperbolic volume, antipodal symmetry, hyperbolic tetrahedron, integral formula.

1. Introduction

Hyperbolic structures on 3-manifolds is an important topic in low dimensional topology. The theory was quickly developed in the seventies due to W. P. Thurston, who obtained several important results. He showed that many compact 3-dimensional manifolds admitted a unique hyperbolic structure. His genius and some years of fruitful researching led him to a proposal of a geometrization conjecture in 1982. That implies several other conjectures, such as the Poincare conjecture and Thurston's ellipticalization conjecture. He offered a proof of the geometrization conjecture for the case of Haken manifolds, using the Andreev's theorem [1] as a fundamental tool. Thurston's work made a revolution in the world of low-dimensional topology. In the early 2000s G. Perelman gave a proof of the full Thurston geometrization conjecture. H. Poincare is the first to obtain a construction of hyperbolic manifolds by gluing paired faces of a finite collection of 3-dimensional hyperbolic finite polyhedra. A remarkable example of this construction

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is the Seifert–Weber space which is obtained by gluing opposite faces of a regular dodecahedron. A hyperbolic manifold, constructed this way, has finite volume and the volume is equal to the sum of volumes of hyperbolic polyhedra. The Mostow-Prasad rigidity theorem [2, 3] states that the hyperbolic structure of a hyperbolic 3–manifold of finite volume is uniquely determined by its homotopy type. So a geometric invariant such as the volume can be used to define new topological invariants. The relevance of the problem of computing hyperbolic volumes of a polyhedron comes from the rigidity theorem of G. D. Mostow. The researchers C. Petronio, D. Heard, E. Pervova classified (in [4] and [5]) the compact orientable 3-manifolds that one can obtain by gluing together in pairs the faces of the octahedron and then removing open regular neighbourhoods of the singular points. There are 132 manifolds out of this collection that are hyperbolic. The past works [6, 7, 8] by N. Abrosimov et al. give some volume formulas for octahedra with different types of symmetries. Our aim for this work is to give an explicit integral formula for the volume of a type of hyperbolic octahedron, which has the antipodal symmetry.

2. Preliminary

We give some notions about hyperbolic tetrahedron, which will be used in the main part of the present paper.

Definition 1. *A hyperbolic tetrahedron is a convex hull of four distinct points in the three dimensional hyperbolic space* $\mathbb{H}^3$. *These points are called vertices of the tetrahedron.*

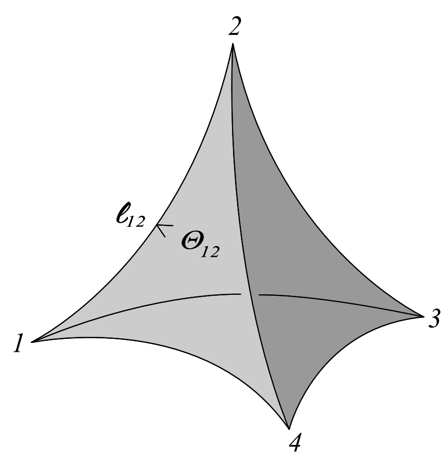

Fig. 1. A hyperbolic tetrahedron

Let us denote the tetrahedron by T and its vertices by numbers $1, 2, 3$ and 4 (see Fig. 1). Then denote by ℓ_{ij} the length of the edge connecting i-th and j-th vertices. Let θ_{ij} be the dihedral angle along the corresponding edge.

A *Gram matrix* $G(T)$ of tetrahedron T is defined as $G(T) =$

$$\langle -\cos\theta_{ij}\rangle_{i,j=1,2,3,4} = \begin{pmatrix} 1 & -\cos\theta_{12} & -\cos\theta_{13} & -\cos\theta_{14} \\ -\cos\theta_{12} & 1 & -\cos\theta_{23} & -\cos\theta_{24} \\ -\cos\theta_{13} & -\cos\theta_{23} & 1 & -\cos\theta_{34} \\ -\cos\theta_{14} & -\cos\theta_{24} & -\cos\theta_{34} & 1 \end{pmatrix},$$

where $-\cos\theta_{ii}$ for $i = 1, ..., 4$ are assumed to be equal to 1.

An *Edge matrix* $E(T)$ is formed by hyperbolic cosines of the edge lengths and defined as follows

$$E(T) = \langle \operatorname{ch}\ell_{ij}\rangle_{i,j=1,2,3,4} = \begin{pmatrix} 1 & \operatorname{ch}\ell_{12} & \operatorname{ch}\ell_{13} & \operatorname{ch}\ell_{14} \\ \operatorname{ch}\ell_{12} & 1 & \operatorname{ch}\ell_{23} & \operatorname{ch}\ell_{24} \\ \operatorname{ch}\ell_{13} & \operatorname{ch}\ell_{23} & 1 & \operatorname{ch}\ell_{34} \\ \operatorname{ch}\ell_{14} & \operatorname{ch}\ell_{24} & \operatorname{ch}\ell_{34} & 1 \end{pmatrix},$$

where $\ell_{ii} = 0$ and $\operatorname{ch}\ell_{ii} = 1$.

It is known that a hyperbolic tetrahedron T can be uniquely determined up to isometry either by the Gram matrix $G(T)$ or the edge matrix $E(T)$ (see, e.g., [9]). This is unlikely to Euclidean case, where the edge matrix defines a tetrahedron up to isometry, but the Gram matrix defines a tetrahedron only up to similarity. The notion of similarity has no place in the hyperbolic geometry.

There exist a few different formulas for the volume of an arbitrary hyperbolic tetrahedron in terms of dihedral angles. The most significant integral formula have been obtained by G. Sforza [10, 11] and another one is due to D. A. Derevnin, A. D. Mednykh [12]. Here we present the closed integral formula by G. Sforza for the volume of a hyperbolic tetrahedron.

Theorem 1 (G. Sforza, 1907)**.** *Let T be a compact hyperbolic tetrahedron given by the Gram matrix $G = G(T)$. We assume that all the dihedral angles are fixed exept θ_{34} which is formal variable. Then the volume $V = V(T)$ is given by the formula*

$$Vol\,(T) = \frac{1}{4}\int\limits_{t_0}^{\theta_{34}} \log\frac{c_{34}(t) - \sqrt{-\det\,G(t)}\sin t}{c_{34}(t) + \sqrt{-\det\,G(t)}\sin t}dt,$$

where t_0 is a suitable root of the equation $\det G(t) = 0$, c_{34} is $(3,4)$-cofactor of the matrix G, and $c_{34}(t), G(t)$ are functions in one variable θ_{34} denoted by t.

An analog of Sforza's formula for the volume of an arbitrary hyperbolic tetrahedron in terms of the edge matrix was given in [13] by N. Abrosimov and the author.

Theorem 2 (N. Abrosimov, B. Vuong, 2021)**.** *Let T be a compact hyperbolic tetrahedron given by its edge matrix E and $c_{ij} = (-1)^{i+j}E_{ij}$ is ij-cofactor of E. We assume that all the edge lengths are fixed exept ℓ_{34} which varies. Then the volume $V = V(T)$ is given by the formula*

$$V = \frac{1}{2}\int\limits_{\ell_1}^{\ell_{34}}\Big[\frac{-t}{\sqrt{-\Delta^3}}\Big(\frac{c_{14}(c_{11}c_{23} - c_{12}c_{13})}{c_{11}} + \frac{c_{24}(c_{13}c_{22} - c_{12}c_{23})}{c_{22}}\Big) - \frac{\operatorname{sh} t}{\sqrt{-\Delta}}$$
$$\times\Big(\frac{\ell_{24}\operatorname{sh}\ell_{24}c_{14} + \ell_{14}\operatorname{sh}\ell_{23}c_{13}}{c_{11}} + \frac{\ell_{13}\operatorname{sh}\ell_{13}c_{23} + \ell_{23}\operatorname{sh}\ell_{14}c_{24}}{c_{22}} + \ell_{12}\operatorname{sh}\ell_{12}\Big)\Big]dt,$$

where cofactors c_{ij} and edge matrix determinant $\Delta = \det E$ are functions in one variable ℓ_{34} denoted by t. The lower limit of integration ℓ_1 is defined by expression

$$\begin{aligned}\operatorname{ch}\ell_1 = \operatorname{ch}\ell_{13}\operatorname{ch}\ell_{14} - \operatorname{csch}^2\ell_{12}\Big[&(\operatorname{ch}\ell_{13}\operatorname{ch}\ell_{12} - \operatorname{ch}\ell_{23})(\operatorname{ch}\ell_{14}\operatorname{ch}\ell_{12} - \operatorname{ch}\ell_{24})\\ &+ \sqrt{(\operatorname{ch}\ell_{23} - \operatorname{ch}(\ell_{13}+\ell_{12}))(\operatorname{ch}\ell_{23} - \operatorname{ch}(\ell_{13}-\ell_{12}))}\\ &\times \sqrt{(\operatorname{ch}\ell_{24} - \operatorname{ch}(\ell_{14}+\ell_{12}))(\operatorname{ch}\ell_{24} - \operatorname{ch}(\ell_{14}-\ell_{12}))}\Big].\end{aligned}$$

We recall a property of a compact hyperbolic tetrahedron, defined by its edge-lengths (see Theorem 3.2 in [13]), for using in volume computation of a compact hyperbolic antipodal octahedron in next section of present paper.

Theorem 3 (N. Abrosimov, B. Vuong, 2021)**.** *Let E be the edge matrix of a compact hyperbolic tetrahedron T. Then the following conditions hold:*

$$c_{ii} > 0, \tag{1}$$

$$detE < 0, \tag{2}$$

$$\cos\theta_{5-i,5-j} = \frac{-c_{ij}}{\sqrt{c_{ii}\cdot c_{jj}}}, \tag{3}$$

where $i,j \in \{1,2,3,4\}$, $c_{ij} = (-1)^{i+j}E_{ij}$ is ij-cofactor of edge matrix E and $\theta_{5-i,5-j}$ is the dihedral angle along the edge $\ell_{5-i,5-j}$ which is opposite to ℓ_{ij}

3. Hyperbolic antipodal octahedron

Definition 2. *An antipodal transformation with respect to a point C in three dimensional hyperbolic space $\mathbb{H}^3$ is a map $A : \mathbb{H}^3 \to \mathbb{H}^3$. The map A is defined in such a way, that for every point $x \in \mathbb{H}^3$ the point C is the midpoint of the geodesic line segment between x and $A(x)$. The point C is symmetric center.*

Definition 3. *Antipodal symmetry in hyperbolic space $\mathbb{H}^3$ is symmetry with respect to an antipodal transformation.*

Definition 4. *Hyperbolic antipodal octahedron is a hyperbolic convex polyhedron, that has antipodal symmetry.*

In the Fig. 2 we describe an antipodal octahedron $\mathcal{O}(a,b,c,d,e,f)$, defined in term of it's edge lengths a,b,c,d,e,f, with the symmetric center C, six vertices $v_1, v_2, v_2, v_4, v_5, v_6$.

4. Existence criterion of an antipodal octahedron $\mathcal{O}$ in $\mathbb{H}^3$

Consider a hyperbolic antipodal octahedron $\mathcal{O}(a,b,c,d,e,f)$, defined in term of it's edge lengths a,b,c,d,e,f, with the symmetric center C, six vertices v_1, v_2, v_2, v_4, v_5, v_6 (see Fig. 2). The antipodal symmetry leads us to study medians in a hyperbolic triangle, that is crucial for the existence conditions of the antipodal octahedron $\mathcal{O}$. We have the following lemma.

Lemma 1. *Let $\Delta(u,v,w)$ be a compact hyperbolic triangle, defined by its side lengths u, v and w. Let m_u, m_v, m_w be the lengths of medians, corresponding to the*

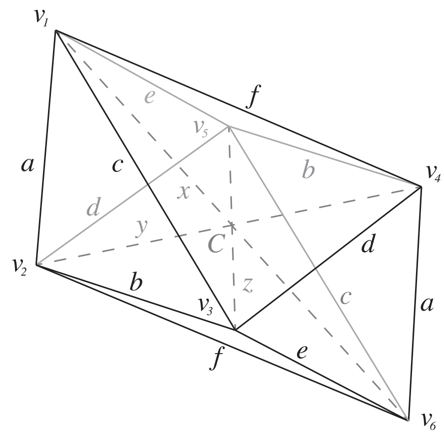

FIG. 2. An antipodal octahedron $\mathcal{O}(a, b, c, d, e, f)$

side u, v, w respectively. Then the following equations are hold in triangle Δ.

$$\operatorname{ch} m_w = \frac{\operatorname{ch} u + \operatorname{ch} v}{2 \operatorname{ch} \frac{w}{2}};$$
$$\operatorname{ch} m_u = \frac{\operatorname{ch} v + \operatorname{ch} w}{2 \operatorname{ch} \frac{u}{2}};$$
$$\operatorname{ch} m_v = \frac{\operatorname{ch} u + \operatorname{ch} w}{2 \operatorname{ch} \frac{v}{2}}.$$

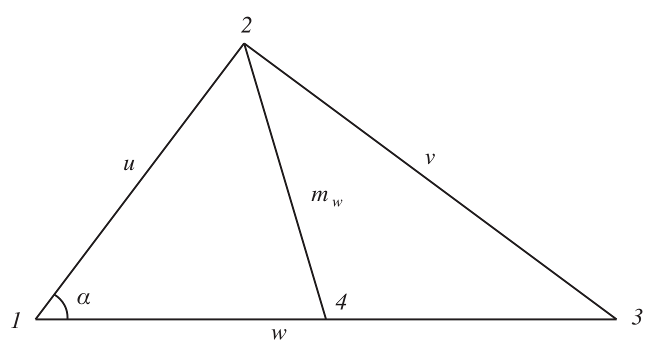

FIG. 3. Compact hyperbolic triangle $\Delta(u, v, w)$.

Proof. Consider a compact hyperbolic triangle $\Delta(u, v, w)$ (see Fig. 3), by the cosine rule we have

$$\cos \alpha = \frac{\operatorname{ch} u \operatorname{ch} w - \operatorname{ch} v}{\operatorname{sh} u \operatorname{sh} w}. \tag{4}$$

Now in the triangle 124 we also have by the cosine rule,

$$\cos\alpha = \frac{\operatorname{ch} u \operatorname{ch} \frac{w}{2} - \operatorname{ch} m_w}{\operatorname{sh} u \operatorname{sh} \frac{w}{2}}. \tag{5}$$

From (4), (5) we obtain the relation

$$\frac{\operatorname{ch} u \operatorname{ch} w - \operatorname{ch} v}{\operatorname{sh} u \operatorname{sh} w} = \frac{\operatorname{ch} u \operatorname{ch} \frac{w}{2} - \operatorname{ch} m_w}{\operatorname{sh} u \operatorname{sh} \frac{w}{2}}.$$

After simple transformations, we have

$$\operatorname{ch} m_w = \frac{\operatorname{ch} u + \operatorname{ch} v}{2 \operatorname{ch} \frac{w}{2}}.$$

Analogously, the following relations are true

$$\operatorname{ch} m_u = \frac{\operatorname{ch} v + \operatorname{ch} w}{2 \operatorname{ch} \frac{u}{2}};$$
$$\operatorname{ch} m_v = \frac{\operatorname{ch} u + \operatorname{ch} w}{2 \operatorname{ch} \frac{v}{2}}.$$

□

Theorem 4. *A compact hyperbolic antipodal octahedron $\mathcal{O}(a,b,c,d,e,f)$ with edge lengths a,b,c,d,e,f. exists in $\mathbb{H}^3$ if and only if the following inequalities hold*

$$\begin{aligned}
&(\operatorname{ch} c + \operatorname{ch} e)(\operatorname{ch} b + \operatorname{ch} d) > 2(\operatorname{ch} a + \operatorname{ch} f);\\
&(\operatorname{ch} a + \operatorname{ch} f)(\operatorname{ch} b + \operatorname{ch} d) > 2(\operatorname{ch} c + \operatorname{ch} e);\\
&(\operatorname{ch} c + \operatorname{ch} e)(\operatorname{ch} a + \operatorname{ch} f) > 2(\operatorname{ch} b + \operatorname{ch} d);\\
&a + b > c > |a - b|;\\
&d + f > c > |d - f|;\\
&C + S \geq \sqrt{\frac{(\operatorname{ch} a + \operatorname{ch} f)(\operatorname{ch} b + \operatorname{ch} d)}{2(\operatorname{ch} c + \operatorname{ch} e)}} \geq C - S,\\
&C = \operatorname{ch} b \operatorname{ch} d - \operatorname{csch}^2 c(\operatorname{ch} b \operatorname{ch} c - \operatorname{ch} a)(\operatorname{ch} d \operatorname{ch} c - \operatorname{ch} f);\\
&S = \operatorname{csch}^2 c \sqrt{(\operatorname{ch} a - \operatorname{ch}(b + c))(\operatorname{ch} a - \operatorname{ch}(b - c))}\\
&\quad \times \sqrt{(\operatorname{ch} f - \operatorname{ch}(d + c))(\operatorname{ch} f - \operatorname{ch}(d - c))}.
\end{aligned}$$

Proof. Consider a compact hyperbolic antipodal octahedron $\mathcal{O}(a,b,c,d,e,f)$ (see Fig. 2), with vertices $v_i, i = 1,...,6$. Due to antipodal symmetry, this octahedron is completely defined by three points as vertices and one point as the center of symmetry. Let v_1, v_2, v_3 be three vertices of a compact hyperbolic antipodal octahedron and C its center of symmetry. Then the remaining vertices v_6, v_4, v_5 of the octahedron $\mathcal{O}$ are symmetric to v_1, v_2, v_3 through the symmetric center C, respectively. This means that there is a one-to-one correspondence between compact hyperbolic tetrahedra $\mathcal{T} = Cv_1v_2v_3$ of general form and compact hyperbolic antipodal octahedra $\mathcal{O}(a,b,c,d,e,f)$. Denote by x, y, z the distance from C to v_1, v_2, v_3 respectively. Let us show explicitly that if the antipodal octahedron $\mathcal{O}$ has edge lengths a,b,c,d,e,f. Then the tetrahedron $\mathcal{T}$ with edge lengths a,b,c,x,y,z is uniquely defined depending on a,b,c,d,e,f.

Consider the triangle $v_1v_2v_4$, by Lemma 1 we obtain the equation

$$\operatorname{ch} x = \frac{\operatorname{ch} a + \operatorname{ch} f}{2 \operatorname{ch} y}. \tag{6}$$

Analogously, in the triangle $v_1 v_3 v_5$, by Lemma 1 the equation is hold

$$\operatorname{ch} x = \frac{\operatorname{ch} c + \operatorname{ch} e}{2 \operatorname{ch} z}. \tag{7}$$

According to Lemma 1 for the triangle $v_2 v_3 v_5$, we have the equation

$$\operatorname{ch} y = \frac{\operatorname{ch} b + \operatorname{ch} d}{2 \operatorname{ch} z}. \tag{8}$$

We solve the system of equations (6, 7, 8) with respect to $\operatorname{ch} x$, $\operatorname{ch} y$ и $\operatorname{ch} z$. Since $\operatorname{ch} x$, $\operatorname{ch} y$, and $\operatorname{ch} z$ are always positive numbers, choosing positive roots, we get

$$\operatorname{ch} x = \sqrt{\frac{(\operatorname{ch} a + \operatorname{ch} f)(\operatorname{ch} c + \operatorname{ch} e)}{2(\operatorname{ch} b + \operatorname{ch} d)}}; \tag{9}$$

$$\operatorname{ch} y = \sqrt{\frac{(\operatorname{ch} b + \operatorname{ch} d)(\operatorname{ch} a + \operatorname{ch} f)}{2(\operatorname{ch} c + \operatorname{ch} e)}}; \tag{10}$$

$$\operatorname{ch} z = \sqrt{\frac{(\operatorname{ch} c + \operatorname{ch} e)(\operatorname{ch} b + \operatorname{ch} d)}{2(\operatorname{ch} a + \operatorname{ch} f)}}. \tag{11}$$

Thus, the octahedron $\mathcal{O}$ exists in the hyperbolic space $\mathbb{H}^3$ if and only if the tetrahedron $\mathcal{T}$ exists in the hyperbolic space $\mathbb{H}^3$. This means that for given edge lengths a, b, c, d, e, f of the octahedron $\mathcal{O}$, the values $\operatorname{ch} x, \operatorname{ch} y, \operatorname{ch} z$ should be defined as positive numbers greater than 1 or equivalently

$$\begin{aligned}
(\operatorname{ch} a + \operatorname{ch} f)(\operatorname{ch} c + \operatorname{ch} e) &> 2(\operatorname{ch} b + \operatorname{ch} d);\\
(\operatorname{ch} b + \operatorname{ch} d)(\operatorname{ch} a + \operatorname{ch} f) &> 2(\operatorname{ch} c + \operatorname{ch} e);\\
(\operatorname{ch} c + \operatorname{ch} e)(\operatorname{ch} b + \operatorname{ch} d) &> 2(\operatorname{ch} a + \operatorname{ch} f).
\end{aligned}$$

That gives us necessary conditions for the existence of the octahedron. For complement to sufficiency we have to fill in the conditions for the existence of the tetrahedron $\mathcal{T}$ itself. In turn the tetrahedron $\mathcal{T}$ exists if and only if the tetrahedron $\mathcal{T}' = v_1 v_2 v_3 v_4$ exists. For that we refer to the Theorem 2.1 of the work [13], then we have following inequalities as the existence conditions of the tetrahedron $\mathcal{T}'$.

$$\begin{aligned}
a + b &> c > |a - b|;\\
d + f &> c > |d - f|;\\
C + S &\geq \operatorname{ch} y \geq C - S, \quad \text{where}
\end{aligned}$$

$$\operatorname{ch} y = \sqrt{\frac{(\operatorname{ch} a + \operatorname{ch} f)(\operatorname{ch} b + \operatorname{ch} d)}{2(\operatorname{ch} c + \operatorname{ch} e)}};$$
$$C = \operatorname{ch} b \operatorname{ch} d - \operatorname{csch}^2 c(\operatorname{ch} b \operatorname{ch} c - \operatorname{ch} a)(\operatorname{ch} d \operatorname{ch} c - \operatorname{ch} f);$$
$$S = \operatorname{csch}^2 c\sqrt{(\operatorname{ch} a - \operatorname{ch}(b+c))(\operatorname{ch} a - \operatorname{ch}(b-c))}$$
$$\times \sqrt{(\operatorname{ch} f - \operatorname{ch}(d+c))(\operatorname{ch} f - \operatorname{ch}(d-c))}.$$

That completes the proof of the Theorem. □

5. Volume formula for the octahedron $\mathcal{O}$ in $\mathbb{H}^3$

Lemma 2. *Let $\mathcal{O}(a,b,c,d,e,f)$ be a compact hyperbolic antipodal octahedron given by edge lengths a,b,c,d,e,f (see Fig. 2). Denote by $\theta_a, \theta_{b_1}, \theta_{c_1}, \theta_{d_1}, \theta_{e_1}, \theta_{z_1}$ the dihedral angles of the tetrahedron $T_1(v_1,v_2,v_3,v_5)$ along edges a,b,c,d,e,z. And by $\theta_{b_2}, \theta_{c_2}, \theta_{d_2}, \theta_{e_2}, \theta_f, \theta_{z_2}$ we denote the dihedral angles of the tetrahedron $T_1(v_1,v_3,v_4,v_5)$ along edges b,c,d,e,f,z respectively. Then following equations hold*

$$\cos\theta_a = \frac{-c_{43}}{\sqrt{c_{44}\cdot c_{33}}},\ \cos\theta_{c_1} = \frac{-c_{42}}{\sqrt{c_{44}\cdot c_{22}}},\ \cos\theta_{e_1} = \frac{-c_{41}}{\sqrt{c_{44}\cdot c_{11}}},$$
$$\cos\theta_{b_1} = \frac{-c_{32}}{\sqrt{c_{33}\cdot c_{22}}},\ \cos\theta_{d_1} = \frac{-c_{31}}{\sqrt{c_{33}\cdot c_{11}}},\ \cos\theta_{z_1} = \frac{-c_{21}}{\sqrt{c_{22}\cdot c_{11}}},$$
$$\cos\theta_f = \frac{-c'_{43}}{\sqrt{c'_{44}\cdot c'_{33}}},\ \cos\theta_{d_2} = \frac{-c'_{42}}{\sqrt{c'_{44}\cdot c'_{22}}},\ \cos\theta_{b_2} = \frac{-c'_{41}}{\sqrt{c'_{44}\cdot c'_{11}}},$$
$$\cos\theta_{c_2} = \frac{-c'_{32}}{\sqrt{c'_{33}\cdot c'_{22}}},\ \cos\theta_{e_2} = \frac{-c'_{31}}{\sqrt{c'_{33}\cdot c'_{11}}},\ \cos\theta_{z_2} = \frac{-c'_{21}}{\sqrt{c'_{22}\cdot c'_{11}}}.$$

where the cofactors c_{ij}, c'_{ij} of the edge matrices E_1, E_2 correspondingly and

$$\operatorname{ch} z = \sqrt{\frac{(\operatorname{ch} c + \operatorname{ch} e)(\operatorname{ch} b + \operatorname{ch} d)}{2(\operatorname{ch} a + \operatorname{ch} f)}}$$

$$E_1 = \begin{pmatrix} 1 & \operatorname{ch} a & \operatorname{ch} c & \operatorname{ch} e \\ \operatorname{ch} a & 1 & \operatorname{ch} b & \operatorname{ch} d \\ \operatorname{ch} c & \operatorname{ch} b & 1 & \operatorname{ch} z \\ \operatorname{ch} e & \operatorname{ch} d & \operatorname{ch} z & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & \operatorname{ch} f & \operatorname{ch} d & \operatorname{ch} b \\ \operatorname{ch} f & 1 & \operatorname{ch} c & \operatorname{ch} e \\ \operatorname{ch} d & \operatorname{ch} c & 1 & \operatorname{ch} z \\ \operatorname{ch} b & \operatorname{ch} e & \operatorname{ch} z & 1 \end{pmatrix}.$$

Proof. Consider a compact hyperbolic antipodal octahedron $\mathcal{O}(a,b,c,d,e,f)$ given by edge lengths a,b,c,d,e,f (see Fig. 2). Connecting the vertices v_3 and v_5 of the antipodal octahedron $\mathcal{O}$ along the diagonal z. From the formula (11) we have

$$\operatorname{ch} z = \sqrt{\frac{(\operatorname{ch} c + \operatorname{ch} e)(\operatorname{ch} b + \operatorname{ch} d)}{2(\operatorname{ch} a + \operatorname{ch} f)}}.$$

Thus, the equations come immediately from relation (3) of the Theorem 3. □

Theorem 5. *Let $\mathcal{O}(a,b,c,d,e,f)$ be a compact hyperbolic antipodal octahedron given by edge lengths a,b,c,d,e,f. Then the volume $V = V(\mathcal{O})$ is defined by*

$$Vol\,(T) = \frac{1}{2}\int\limits_{t_1}^{\theta_{z_1}} \log \frac{c_{34}(x) - \sqrt{-\det\, G_1(x)}\sin x}{c_{34}(x) + \sqrt{-\det\, G_1(x)}\sin x}dx + \frac{1}{2}\int\limits_{t_2}^{\theta_{z_2}} \log \frac{c'_{34}(y) - \sqrt{-\det\, G_2(y)}\sin y}{c'_{34}(y) + \sqrt{-\det\, G_2(y)}\sin y}dy,$$

where t_1, t_2 *are suitable roots of the equations* $\det G_1(x) = 0$ *and* $\det G_2(y) = 0$*;* c_{34}, c'_{34} *are* $(3,4)$*-cofactors of the matrices* G_1, G_2 *respectively,*

$$G_1 = \begin{pmatrix} 1 & -\cos\theta_a & -\cos\theta_{c_1} & -\cos\theta_{e_1} \\ -\cos\theta_a & 1 & -\cos\theta_{b_1} & -\cos\theta_{d_1} \\ -\cos\theta_{c_1} & -\cos\theta_{b_1} & 1 & -\cos x \\ -\cos\theta_{e_1} & -\cos\theta_{d_1} & -\cos x & 1 \end{pmatrix},$$

$$G_2 = \begin{pmatrix} 1 & -\cos\theta_f & -\cos\theta_{d_2} & -\cos\theta_{b_2} \\ -\cos\theta_f & 1 & -\cos\theta_{c_2} & -\cos\theta_{e_2} \\ -\cos\theta_{d_2} & -\cos\theta_{c_2} & 1 & -\cos y \\ -\cos\theta_{b_2} & -\cos\theta_{e_2} & -\cos y & 1 \end{pmatrix}.$$

Also the integration limits θ_{z_1} *and* θ_{z_2} *and the elements of matrices* G_1, G_2 *are as in Lemma 2.*

Proof. Consider a compact hyperbolic antipodal octahedron $\mathcal{O}(a,b,c,d,e,f)$ given by edge lengths a,b,c,d,e,f (see Fig. 2). As shown in the figure we can split the octahedron into four tetrahedra of two types. The first type contains the tetrahedron $T_1 = v_1v_2v_3v_5$ and tetrahderon $v_3v_4v_5v_6$, defined by the edge lengths a,b,c,d,e,z. The other type contains the tetrahedron $T_2 = v_1v_3v_4v_5$ and the tetrahedron $v_2v_3v_5v_6$, defined by the edge lengths b,c,d,e,f,z. From the formula (11) we can defined z as soon as we have values for a,b,c,d,e,f. So the two tetrahedra T_1, T_2 are well defined by the edge-lengths of the octahedron $\mathcal{O}$. We can compute their dihedral angles as in Lemma 2, which correspond to fixed edge-lengths of $\mathcal{O}$.

Thus, we assert that the volume of the antipodal octahedron $\mathcal{O}$ is equal to twice the sum of volumes of the tetrahedron T_1 and the tetrahedron T_2. Applying Theorem 1 for each type of tetrahedron, we obtain the volume formula of the antipodal octahedron $\mathcal{O}$, presented in the formulation of the present theorem. □

Remark. We can compute the volume of the compact hyperbolic antipodal octahedron $\mathcal{O}(a,b,c,d,e,f)$ in term of its edge-lengths by using Theorem 2. We suppose this as an exercise for the reader.

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