ON THE COMPLEXITY OF GRAPH MANIFOLDS

E. FOMINYKH AND M. OVCHINNIKOV

Abstract. We provide a new formula for an upper bound of the complexity of non-Seifert graph-manifolds obtained by gluing together two Seifert manifolds fibered over the disc with two exceptional fibers. This bound turns out to be sharp for many manifolds.

The Matveev’s complexity $c(M)$ of a compact 3-manifold $M$ is equal to $k$ if $M$ possesses an almost simple spine with $k$ vertices and has no almost simple spines with a smaller number of vertices [1]. In general, the problem of calculating $c(M)$ is very difficult. The exact value of the complexity is only known for a finite number of closed orientable irreducible manifolds [2], for the complements of the figure eight knot and its twin, as well as for all their finite coverings [3], and also for manifolds having special spines with one 2-cell [4]. To estimate $c(M)$ it suffices to construct an almost simple spine $P$ of $M$. The number of vertices of $P$ is an upper bound for the complexity. On one hand, an almost simple spine can be easily constructed from many presentations of the manifold [5]. On the other hand, as a rule, $c(M)$ is significantly less than such an upper bound.

Let $X$ be some infinite set of manifolds. We say that an integral nonnegative function $\tilde{c} : X \to \mathbb{Z}$ is a $k$-sharp complexity bound for $X$ if $c(M) \leq \tilde{c}(M)$ for all $M \in X$, and $c(M) = \tilde{c}(M)$ for all $M \in X$ with $c(M) \leq k$. First example of a $k$-sharp complexity bound was obtained by S. Matveev for lens spaces. Using a computer he composed a table of all closed orientable irreducible manifolds of complexity $\leq 6$. Analyzing the table he proved that a function $\tilde{c}(L_{p,q}) = S(p,q) - 3$ is 6-sharp [1], where $S(p,q)$ is the sum of all partial quotients in the expansion of $p/q$ as a regular continued fraction. Later M. Ovchinnikov and B. Martelli, C. Petronio extended the table to complexity 7 and 9, respectively, and verified that the function $S(p,q) - 3$ is 9-sharp [6]. Also Martelli and Petronio found a 9-sharp
The complexity bound for all Seifert manifolds [7]. This year S. Matveev and V. Tarakanov composed a table of all closed orientable irreducible manifolds of complexity \( \leq 12 \). Now we can state that these complexity bounds are 12-sharp.

Denote by \( G \) the set of all non-Seifert graph-manifolds obtained by gluing together two Seifert manifolds fibered over the disc with two exceptional fibers along some homeomorphism of their boundary tori. Note that each manifold \( M \in G \) can be presented in the form

\[
(D^2, (p_1, q_1), (p_2, q_2 - p_2)) \bigcup_{A} (D^2, (p_3, q_3), (p_4, q_4 - p_4)),
\]

where \( p_i > q_i > 0 \), \( 1 \leq i \leq 4 \), and \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is an integer matrix with determinant \((-1)\).

**Theorem 1.** The function

\[
\hat{c}(M) = \max \{S(|a| + |b|, |c| + |d|) - 2, 0\} - 2 + \sum_{i=1}^{4} S(p_i, q_i)
\]

is a 12-sharp complexity bound for \( G \).

**References**


Evgeny Fominykh and Mikhail Ovchinnikov
Chelyabinsk State University,
ul. Bratev Kashirinykh 129,
454021, Chelyabinsk, Russia
E-mail address: fominykh@csu.ru