S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

http://semr.math.nsc.ru

Том 2, стр. 204–205 (2005) УДК 512.5

Краткие сообщения MSC 13A99

ON THE HURWITZ EXISTENCE PROBLEM

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Given a branched covering $\widetilde{\Sigma} \to \Sigma$ between closed connected surfaces, one can easily establish some relations between the Euler characteristic and orientability of $\widetilde{\Sigma}$ and Σ, the degree d of the covering, the number n of branching points, the number $\widetilde{n}$ of preimages of the branching points, and the local degrees d_{ij} at these points. A classical problem dating back to Hurwitz asks whether these necessary conditions are sufficient as well.

More precisely, let us call *branch datum* a 5-tuple $\left(\widetilde{\Sigma}, \Sigma, n, d, (d_{ij})\right)$, where $\widetilde{\Sigma}$ and Σ are closed connected surfaces, $n \geqslant 0$ and $d \geqslant 2$ are integers, and $(d_{ij})_{j=1,\ldots,m_i}$ is a partition of d, for $i = 1, \ldots, n$. A *compatible* branch datum is one that satisfies the following conditions:

(1) $\chi(\widetilde{\Sigma}) - \widetilde{n} = d \cdot (\chi(\Sigma) - n)$, where $\widetilde{n} = m_1 + \ldots + m_n$;
(2) $n \cdot d - \widetilde{n}$ is even;
(3) If Σ is orientable then $\widetilde{\Sigma}$ is also orientable;
(4) If Σ is non-orientable and d is odd then $\widetilde{\Sigma}$ is also non-orientable;
(5) If Σ is non-orientable but $\widetilde{\Sigma}$ is orientable then each partition $(d_{ij})_{j=1,\ldots,m_i}$ of d refines the partition $(d/2, d/2)$.

These conditions are easily seen to be necessary for the existence of a covering realizing the branch datum. The Hurwitz existence problem asks which compatible data are realizable. Those that are not are called *exceptional*.

Thanks to the work of many authors, the problem is now completely solved (in the affirmative) when $\chi(\Sigma) \geqslant 0$. The cases where the base surface is the sphere or the projective plane remain elusive, but the latter reduces to the former, and many

Pervova, E., Petronio, C., On the Hurwitz existence problem.

Supported by the INTAS YS fellowship 03-55-1423 and by the INTAS project "CalcoMet-GT" 03-51-3663.

Communicated by S.V. Matveev October 15, 2005, published October 18, 2005.

partial results are known. In this work we describe several new series of exceptional and, *vice versa*, realizable data, when the base surface is the sphere $\mathbb{S}$.

The classical approach to the problem was based on the analysis of the possible monodromies of the covering, *i.e.* of the possible representations of the fundamental group of Σ minus n points into $\mathfrak{S}_d$. We employed three alternative different techniques, one based on Grothendieck's dessins d'enfants, one involving the factorization of a covering through two non-trivial ones, and one establishing the equivalence of the Hurwitz problem with a geometric condition (the existence of certain families of graphs on $\widetilde{\Sigma}$). The former of the next two non-existence theorems is based on the dessins technique, the latter on the factorization technique:

Theorem 1.

- *If $d = kh$ and $p \geqslant 2$ the following branch datum is exceptional:*
  $$\big(\mathbb{S}, \mathbb{S}, 3, d, (k, \ldots, k), (kh_1, \ldots, kh_p), (h + p - 1, 1, \ldots, 1)\big);$$
- *If $d \geqslant 8$ is even there exists an exceptional branch datum with $n = 3$, $\Sigma = \mathbb{S}$ and $\widetilde{\Sigma} = \mathbb{T}$ (the torus),* e.g.
  $$\big(\mathbb{T}, \mathbb{S}, 3, d, (2, \ldots, 2), (5, 3, 2, \ldots, 2), (d/2, d/2)\big);$$
- *If $d \geqslant 6$ is even there exists an exceptional branch datum with $n = 4$ and $\Sigma = \widetilde{\Sigma} = \mathbb{S}$,* e.g.
  $$\big(\mathbb{S}, \mathbb{S}, 4, d, (2, \ldots, 2), (2, \ldots, 2), (2, 1, \ldots, 1), (d - 2, 1, 1)\big).$$

Theorem 2. *Suppose d and all d_{ij} for $i = 1, 2$ are even. If the branch datum $\big(\mathbb{S}, \mathbb{S}, 3, d, (d_{ij})\big)$ is realizable then (d_{3j}) refines the partition $(d/2, d/2)$.*

On the existence side, the factorization technique leads to the first one of the next three theorems, while the geometric technique leads to the last two.

Theorem 3. *A compatible branch datum $\big(\widetilde{\Sigma}, \mathbb{S}, 3, d, (d_{ij})\big)$ such that all d_{ij} are divisible by some odd $p \geqslant 3$ is realizable.*

Theorem 4. *If d is odd then every compatible branch datum of the form $\big(\mathbb{S}, \mathbb{S}, 3, d, (d - 2, 2), (d_{2j}), (d_{3j})\big)$ is realizable.*

Theorem 5. *With the single exception of $\big(\mathbb{T}, \mathbb{S}, 3, 6, (4, 2), (3, 3), (3, 3)\big)$, every compatible branch datum of the form $\big(\mathbb{T}, \mathbb{S}, 3, d, (d - 2, 2), (d_{2j}), (d_{3j})\big)$ is realizable.*

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