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MINIMAL SEIFERT MANIFOLDS

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We call Seifert manifold "minimal" if it doesn't admit degree one maps onto other Seifert manifolds except itself. The natural question is "which Seifert manifolds are minimal?" This problem was studied in 1996 by C. Hayat-Legrand, S. Wang, H. Zieschang (see [1]). They introduced a set of Seifert manifolds which were supposed to be minimal but it remained unknown whether these manifolds admit degree one maps onto Poincare homology sphere S^3/P_{120}. We suggest a complete solution of this problem. The answer is: all the problem manifolds described by C. Hayat-Legrand, S. Wang and H. Zieschang have no degree one maps onto Poincare homology sphere. The solution is based on S. Matveev's algorithm of the calculation of the degrees. The idea of the algorithm may be described like that. Let $p : \tilde{M} \to M$ and $p_1 : \tilde{P} \to P$ be universal coverings; M and P are both equipped with cellular structures, having only one 3-cell, and the map $f : M \to P$ is cellular. A map $\tilde{f} : \tilde{M} \to \tilde{P}$ is a lifting of the map f. If β_M is the only 3-cell of the manifold M then we define "boundary cycle" $\partial\tilde{\beta}_M$ as a boundary of any pre-image $\tilde{\beta}_M$ of β_M in $\tilde{M}$. Then the induced map of 2-chains $\tilde{f}_* : C_2(\tilde{M}; Z) \to C_2(\tilde{P}; Z)$ maps the boundary cycle to a sum of boundary cycles in $\tilde{P}$. It may be shown that the number of summands in this sum is equal to the degree f. At last

$$\deg f \equiv \xi_P(\tilde{f}_*(\partial\tilde{\beta}_M)) \ (\mathrm{mod}\ |\pi_1(P)|),$$

where $\xi_P \in C^2(\tilde{M}; Z)$ is a functional which is equal to 1 on every boundary cycle of P. (See [2].)

The main difficulty of the calculation by this formula is the calculation of the induced map of 2-chains, knowing only the homomorphism $f_* : \pi_1(M) \to \pi_1(P)$. In the language of group theory this operation may be described as decomposing

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of a word in generators representing unit element of the fundamental group of P into a product of conjugated relations. If M and P are Seifert manifolds and we use the standard geometric representations (see [2]) of their fundamental groups then such words consist of two generators in some complicated sequence. Using the fact that any two closed curves in a solid torus both are some powers of one element of its fundamental group it became possible to change two generators to only one and to prove that the image of the boundary cycle has a periodic dependence on the parameters of the exceptional fibres of M. In fact the set of all possible degrees of maps from M to P is completely determined by the residues of the parameters of M some modulo which depends only on $|\pi_1(P)|$. This kind of periodicity was observed by S. Matveev during a computer experiment devoted to the calculations of the degrees of maps from problem manifolds to S^3/P_{120} (see [2]) and the corresponding theorem was proved in [3]. All the problem series of Seifert manifolds are infinite and the existence of such a periodicity provides us restricting ourselves by considering just a finite number of cases. So the answer of the question "do these manifold have a degree one maps onto the Poincare homology sphere?" was obtained using a computer program which realizes Matveev's algorithm. Perhaps it became possibly to prove the inexistence of degree one maps theoretically in two of four problem cases. Here they are.

- $M(S^2; (2^k\alpha_i, \beta_i), i = 1, 2, 3)$, $k > 1$ with some other conditions
  Result: we prove theoretically that all the maps of such manifolds onto Poincare homology sphere are even.
- homology spheres $M(S^2; (2\alpha_1, \beta_1); (3\alpha_2, \beta_2); (5\alpha_3, \beta_3))$, all α_i are odd and $\alpha_1\alpha_2\alpha_3 \neq \pm 1, \pm 49 (\mathrm{mod}\ 120)$
  Result: an explicit formula for the degree is obtained. It gives the same result as Matveev's hypothetic formula constructed in [2]. There are no degree one maps.
- $M(S^2; (2\alpha_i, \beta_i), i = 1, 2, 3)$, all α_i are odd, $3|\alpha_2$, $5|\alpha_3$ with some other conditions
  Result: all the possible degrees are not coprime with 120. It is proved with a computer experiment.
- $M(T^2; (\alpha, \pm 1)$, $k|\alpha$, $k = 3, 4, 5$
  Result: all the maps of this manifolds onto S^3/P_{120} have even degrees. It is proved with a computer experiment.

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