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# COLOURING LATTICE POINTS BY REAL NUMBERS

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Abstract. We establish a criterion for the existence of an $f$-colouring with a finite span of the $d$-dimensional lattice graph $\mathbb{Z}^d$.

Let $G$ be an arbitrary connected simple graph. By $d(u, v)$ we denote the graph distance between the vertices $u, v$ of $G$. By a *constraints function* we mean any non-increasing non-negative function $f : \mathbb{N} \to \mathbb{R}$. We say that a function $c : V(G) \to \mathbb{R}$ is a colouring of $G$ satisfying the constraints $f$, or, simply, an *$f$-colouring of $G$*, if for every two distinct vertices $u, v \in V(G)$ holds the inequality

$$|c(u) - c(v)| \geq f(d(u, v)).$$

The *span* of a colouring $c$ is defined as

$$sp(c) = \sup\{c(v) \mid v \in V(G)\} - \inf\{c(v) \mid v \in V(G)\}.$$

The span of a colouring can be either finite or infinite.

The problem of finding a colouring satisfying certain constraints with minimum span is widely used as a model for the problem of optimal assignment of frequencies to transmitters in a radiocommunication network (cf. [1]). In this context, the constraints function is usually integer-valued, with only finitely many nonzero values.

In this note we consider the $f$-colouring problem with an arbitrary real-valued constraints function $f$. The question that we address is: under what conditions does there exist an $f$-colouring of $G$ with a finite span. We shall find an exact criterion for existence of such colourings when $G$ is the $d$-dimensional lattice graph $\mathbb{Z}^d$. This problem was first stated in [2], and solved there for $d = 1$.

**Definition 1.** *The* $d$**-dimensional lattice graph**, $\mathbb{Z}^d$, *is an infinite, locally finite graph whose vertices are all integer points of the d-dimensional space; two vertices*

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*$x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ adjacent iff, for some $i$, $|x_i - y_i| = 1$, and $x_j = y_j$ for all $j \neq i$.*

Thus, the graph distance in $\mathbb{Z}^d$ coincides with the usual $l_1$-distance:

$$d(x, y) = \sum_{i=1}^{d} |x_i - y_i|.$$

**Theorem 1.** *Let $f$ be an arbitrary constraints function. The graph $Z^d$ has an $f$-colouring with a finite span if and only if $f$ is $O(n^{-d})$, that is, $f(n) < C \cdot n^{-d}$ for some constant $C$ and for all $n > 0$.*

PROOF. The necessity of this condition is easy. Suppose that $f$ is not $O(n^{-d})$. Then for every $C$ one can find $n = n(C)$ such that $f(n) > C \cdot n^{-d}$. Take any $L > 0$, let $C = Ld^d$ and $n = n(C)$. Consider the set $X = \{0, \ldots, n/d\}^d$ of all lattice points with all coordinates in $\{0, \ldots, \lfloor n/d \rfloor\}$. The distance between any two points in $X$ is at most $n$, therefore in any $f$-colouring $c$, every two values $c(x)$, $x \in X$, differ by at least $f(n)$. The set $X$ has more than $n/d$ points; so the total span of the colours $c(x)$, $x \in X$, is more than $f(n)n^d/d^d > L$. As $L$ was taken arbitrary, $f$-colouring with finite span cannot exist.

To prove that the condition is sufficient, it is enough to find an $f$-colouring of $\mathbb{Z}^d$ with finite span, when $f(n) = Cn^{-d}$ for some $C > 0$. Then, multiplying all values of this colouring by a suitable constant, we obtain a colouring for any constraints function which is $O(n^{-d})$.

We shall explicitly define a colouring $c : \mathbb{Z}^d \to [0, 1)$ of span 1, and then we shall demonstrate that it satisfies the constraints function $f(n) = Cn^{-d}$ for certain $C > 0$.

Let $a = 2^{1/(d+1)}$.

**Lemma 1.** *There exists a positive constant $C_0$ such that for every collection $(p_0, p_1, \ldots, p_d)$ of $d + 1$ integers, not all equal to 0, if $N \geq \max |p_i|$ then*

$$|p_0 + p_1 a + p_2 a^2 + \ldots + p_d a^d| > C_0 N^{-d}.$$

The proof of this lemma has a completely different flavour, so it will be postponed till the end of the note.

For every lattice point $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$ let

$$c(x) = \{ax_1 + a^2 x_2 + \ldots + a^d x_d\},$$

where by $\{z\}$ we denote the fractional part of $z$: $\{z\} = z - \lfloor z \rfloor$. Setting $x_0 = -\lfloor ax_1 + a^2 x_2 + \ldots + a^d x_d \rfloor$, we can write

$$c(x) = x_0 + ax_1 + a^2 x_2 + \ldots + a^d x_d.$$

Now, let $x = (x_1, \ldots, x_d)$, $y = (y_1, \ldots, y_d)$ be two distinct vertices at distance at most $n$. Let $p_i = x_i - y_i$, $i = 1, \ldots, d$, and let

$$p_0 = y_0 - x_0 = \lfloor ay_1 + \ldots + a^d y_d \rfloor - \lfloor ax_1 + \ldots + a^d x_d \rfloor,$$

so that $c(x) - c(y) = p_0 + p_1 a + \ldots + p_d a^d$. We have $|p_i| \leq n$ for $i = 1, \ldots, d$, and

$$|p_0| \leq 1 + |ap_1 + \ldots + ap_d| \leq 1 + 2dn \leq (2d + 1)n.$$

Therefore, by Lemma, we have

$$|c(x) - c(y)| > C_0((2d + 1)n)^{-d} = \frac{C_0}{(2d + 1)^d} n^{-d},$$

and the theorem is proved. □

**Proof of the Lemma.** Let $w$ be a primitive complex root of 1 of degree $d+1$; the numbers $w^0 = 1, w, w^2, \ldots, w^d$ being all $(d+1)$-st roots of 1. Also, the numbers $a, aw, aw^2, \ldots, aw^d$ are all $(d+1)$-st roots of 2, and therefore they are algebraic integers. Let

$$L = \prod_{k=0}^{d} \sum_{i=0}^{d} p_i w^{ki} a^i.$$

The number $L$ is an algebraic integer, and it is invariant under all automorphisms of the field $\mathbb{Q}[a, w]$ (which is the splitting field of the polynomial $x^{d+1} - 2$). Therefore $L$ is a rational integer. Since $L \neq 0$, we conclude that $|L| \geq 1$. Now,

$$p_0 + p_1 a + p_2 a^2 + \ldots + p_d a^d = L / \prod_{k=1}^{d} \sum_{i=0}^{d} p_i w^{ki} a^i.$$

The absolute value of every factor in this formula can be bounded from above by

$$|\sum_{i=0}^{d} p_i w^{ki} a^i| \leq \sum_{i=0}^{d} |p_i| a^i < 2(d+1)N.$$

Therefore

$$|p_0 + p_1 a + p_2 a^2 + \ldots + p_d a^d| > \frac{|L|}{(2d+2)^d} N^{-d} \geq (2d+2)^{-d} N^{-d},$$

which proves the lemma. □

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## References

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