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SOME RESULTS AND CONJECTURES ON FINITE GROUPS ACTING ON HOMOLOGY SPHERES

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Abstract. This is a note based on a talk given in the Workshop on geometry and topology of 3-manifolds, Novosibirsk, 22-26 August 2005. We consider the class of finite groups, which admit arbitrary, i.e. not necessarily free actions on integer and mod 2 homology spheres, with an emphasis on the 3- and 4-dimensional cases. We recall some classical results and present some recent progress as well as new results, open problems and the emerging conjectural picture of the situation.

We are interested in the class of finite groups, and in particular in finite nonsolvable and simple groups, which admit actions on integer and mod 2 homology spheres (arbitrary, i.e. not necessarily free actions), with an emphasis on the 3- and 4-dimensional case. We present some classical results, some recent progress as well as new results, open problems and the emerging conjectural picture of the situation.

1. **Basic problem**. Which finite groups G admit orientation-preserving smooth actions on certain classes of manifolds: spheres S^n, integer homology spheres, mod 2 homology spheres (i.e., homology with coefficients in the integers $\mathbb{Z}_2$ mod 2).

We consider only orientation-preserving, faithful, but not necessarily free actions (in general, the free case is classical, the main new results presented concern nonfree actions). Particular emphasis will be on dimension three. We note that every finite group admits a free action on a rational homology 3-sphere [4]. Also, any finite group admits a faithful orthogonal action on a sphere (by choosing a linear faithful representation); on the other hand, the classes of groups admitting free actions on integer or mod 2 homology spheres are very restricted.

The most important single case is that of the 3-sphere. If an action of a finite group G on S^3 is nonfree then, by Thurston's orbifold geometrization theorem, it is

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conjugate to a finite subgroup of the orthogonal group SO(4). If the action of G is free then, as part of the spherical case of the 3-manifold geometrization conjecture, G is again conjugate to a subgroup of SO(4); an approach to a proof of this has recently been given by Perelman.

2. **Finite subgroups of SO(4)**. There is an isomorphism $\mathrm{SO}(4) \cong S^3 \times_{\mathbb{Z}_2} S^3$ of SO(4) with the central product of two copies of the unit quaternions S^3 (the direct product with identified centers $\mathbb{Z}_2 = \pm 1$). Given an element (q_1, q_2) in $S^3 \times_{\mathbb{Z}_2} S^3$, it acts on S^3 by the orthogonal map $x \to q_1 x q_2^{-1}$. Also, there is a 2-fold covering of Lie groups $S^3 \to SO(3)$, with kernel $\mathbb{Z}_2 = \pm 1$; an element $q \in S^3$ acts orthogonally on S^3 by $x \to qxq^{-1}$, fixing $\pm 1 \in S^3$ and hence mapping $S^2 \subset S^3$ to itself.

The finite subgroups of SO(3) are the *polyhedral groups* which are cyclic $\mathbb{Z}_n$, dihedral $\mathbb{D}_{2n}$, tetrahedral $\mathbb{A}_4$, octahedral $\mathbb{S}_4$ or dodecahedral $\mathbb{A}_5$. The finite subgroups of S^3 are their preimages is S^3 which are the *binary polyhedral groups*: these are cyclic, binary dihedral (or generalized quaternion) $\mathbb{D}^*_{2n}$, binary tetrahedral $\mathbb{A}^*_4$, binary octahedral $\mathbb{S}^*_4$ or binary dodecahedral $\mathbb{A}^*_5$. Hence, the finite subgroups of SO(4) $\cong S^3 \times_{\mathbb{Z}_2} S^3$ are exactly the subgroups of the central products $A_1 \times_{\mathbb{Z}_2} A_2$ where A_1 and A_2 are binary polyhedral groups.

The most interesting example of a finite subgroup of SO(4) is the central product $A^*_5 \times_{\mathbb{Z}_2} A^*_5$ of two binary dodecahedral groups. This is the orientation-preserving symmetry group of the 4-dimensional regular 120-cell, or the lift of an isometry group $\mathbb{A}_5$ of the Poincaré sphere $S^3/\mathbb{A}^*_5$ to its universal covering S^3 (tesselated by 120 spherical dodecahedral with dihedral angles $2\pi/3$). Note that $A^*_5 \times_{\mathbb{Z}_2} A^*_5$ contains $\mathbb{A}_5$ as diagonal subgroup, and also a subgroup $\mathbb{A}_5 \times \mathbb{Z}_2$.

3. Results. It is a general principle in finite transformation groups to compare general actions on spheres and homology spheres with orthogonal actions on spheres. The results depend strongly on the type of finite groups considered:

- cyclic groups of prime power order: among the first important results there is the fixed point theory of Smith which implies that the fixed point set of an action of a cyclic group of prime power order on a homology sphere is again a homology sphere (see e.g. [2]);

- finite p-group and abelian groups: by [6] and [5], any finite p-group and any finite abelian group which acts on a homology sphere admits also an orthogonal action on a sphere of the same dimension (and with the same dimension function for the fixed point sets of subgroups); this remains no longer true for more general classes of groups:

- solvable groups: by [11], some of the Milnor groups $Q(8a, b, c)$ ([12]) admit a free action on a homology 3-sphere but no orthogonal action on S^3; the recent results of Perelman imply that they do not act freely on S^3;

- nonsolvable groups: the linear groups $SL(2, p)$ admit free actions on spheres ([8]) but no orthogonal free actions, for primes $p > 5$.

In the 3-dimensional case, the class of finite nonsolvable groups acting on homology 3-spheres is close to the class of finite nonsolvable subgroups of SO(4):

Theorem 1 ([9]). *A finite nonsolvable group of orientation-preserving diffeomorphisms of an integer homology 3-sphere is isomorphic to one of the following groups:*

$$\mathbb{A}_5, \quad \mathbb{A}_5 \times \mathbb{Z}_2, \quad \mathbb{A}^*_5 \times_{\mathbb{Z}_2} C, \quad \mathbb{A}^*_5 \times_{\mathbb{Z}_2} \mathbb{A}^*_5$$

where C is solvable with a unique involution. Each involution of $\mathbb{A}_5$ has nonempty fixed point set, and each of the groups $\mathbb{A}_5^*$ and C acts freely.

Hence, for a complete classification, it would remain to show that C is one of the polyhedral groups: then the class of nonsolvable groups acting on a homology 3-sphere would coincide with the class of finite nonsolvable subgroups of SO(4); however, at present we cannot exclude the Milnor groups $Q(8a, b, c)$ (see [16]).

In the following, we restrict further to the case of finite *simple* and *quasisimple groups*. We note that a finite (nonabelian) simple group does not act freely on a mod 2 homology sphere (each such group has a subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$ which does not act freely); the corresponding candidates for free actions are the quasisimple groups; by definition, these are the perfect central extensions of simple groups (see [14]). Examples are the special linear groups $SL(2, q)$, with center $\pm I$ of order two and central quotient $SL(2, q)/\pm I$ isomorphic to the projective linear or linear fractional group $PSL(2, q)$ which is simple ($q = p^n$ denotes an odd prime power greater than three).

Corollary 1. *a) The only finite simple group acting on a homology 3-sphere is the dodecahedral group $\mathbb{A}_5 \cong PSL(2, 5)$.*

b) The only finite perfect or quasisimple group acting freely on a homology 3-sphere is the binary dodecahedral group $\mathbb{A}_5^* \cong SL(2, 5)$.

The second result on free actions is classical. If a finite group acts freely on a homology sphere then it has periodic cohomology (and at most one involution by [12]), and these groups have been classified by Zassenhaus in the solvable and Suzuki in the nonsolvable case (see e.g. [1]). Passing to dimension four, the following holds.

Theorem 2 ([10])**.** *A finite simple group acting on a homology 4-sphere is isomorphic to $\mathbb{A}_5 \cong PSL(2, 5)$ or $\mathbb{A}_6 \cong PSL(2, 9)$.*

In dimension 5, there occur the simple groups $\mathbb{A}_5$, $\mathbb{A}_6$, $\mathbb{A}_7$ and its subgroup $PSL(2, 7)$, and the unitary group $U_4(2)$ (a subgroup of index two in the Weyl group of type E_6 which has a 6-dimensional integer linear presentation); we suppose that these are all but do not have a proof at present.

The situation for mod 2 homology spheres is quite different.

Theorem 3 ([9])**.** *A finite simple group acting on a $\mathbb{Z}_2$-homology 3-sphere is isomorphic to $PSL(2, q)$, for an odd prime power $q > 3$.*

We do not have a classification but examples for many small values of q (see [17]); it seems likely that all groups $PSL(2, q)$ occur. For free actions, the following holds (see [8, 13]).

Theorem 4. *a) A quasisimple group acts freely on a $\mathbb{Z}_2$-homology sphere if and only if it is isomorphic to $SL(2, q)$, for an odd prime power $q > 3$, or to the unique perfect central extension $\hat{\mathbb{A}}_7$ of $\mathbb{A}_7$, with center of order two.*

b) A perfect or quasisimple group acts freely on a homology sphere if and only if it is isomorphic to $SL(2, p)$, for a prime $p > 3$.

The existence of free actions in a) follows from [13]; it is likely that all groups $SL(2, q)$ in a) occur already in dimension three (and this might follow from the high-dimensional surgery methods applied in [13], dropping the condition of simply-connectivity of the resulting manifolds). In order to state a generalization of a) for arbitrary nonsolvable groups, we introduce some notation.

For a generator ω of the multiplicative group of the finite field with $q = p^n$ elements, consider the matrix

$$y = \begin{pmatrix} 0 & -1 \\ \omega & 0 \end{pmatrix}$$

in $GL(2,q)$; note that y induces, by conjugation, an automorphism of order two of $SL(2,q)$. Let

$$TL(2,q) \ = < SL(2,q), Y \mid Y^2 = -I,\ Y^{-1}gY = y^{-1}gy \ \text{ for } \ g \in SL(2,q) >$$

which is a 2-fold extension of $SL(2,q)$ with a unique involution $-I$; this involution generates the center of $TL(2,q)$, with factor group isomorphic to $PGL(2,q)$ (see also [1, chapter IV.6],[8]).

For a finite group G, we denote by $\mathcal{O}(G)$ the maximal normal subgroup of odd order of G [13, p.392]. As a consequence of some deep results from finite group theory and finite transformation groups, the finite nonsolvable groups admitting free actions on $\mathbb{Z}_2$-homology spheres can be characterized as follows.

Theorem 5. *A finite nonsolvable group G admits a free action on a $\mathbb{Z}_2$-homology sphere if and only if $G/\mathcal{O}(G)$ contains a normal subgroup G_0, of odd index with cyclic factor group, isomorphic to*

$$SL(2,q), \ \ TL(2,q) \ \ \textit{or} \ \ \hat{\mathbb{A}}_7,$$

for an odd prime power $q > 3$; also, the unique involution of G_0 induces the trivial outer automorphism of $\mathcal{O}(G)$.

These groups are exactly the nonsolvable groups which have generalized quaternion Sylow 2-subgroups (or equivalently, 2-period four), and a unique involution. Which of these groups admit actions already in dimension three, i.e. on a $\mathbb{Z}_2$-homology 3-sphere? We note that the Milnor groups $Q(8a,b,c)$, of period four, admit free orthogonal actions on S^7 but that, by [7], various of these groups do not admit an action on a $\mathbb{Z}_2$-homology 3-sphere.

Proof of Theorem 5. Let G be a finite group acting freely on a $\mathbb{Z}_2$-homology sphere. We can assume that a Sylow 2-subgroup S of G is nontrivial (otherwise, by the Feit-Thompson theorem, G is solvable). By [2, p.148,Theorem 8.1], the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ does not act freely on a $\mathbb{Z}_2$-homology sphere, so G and S have no subgroups $\mathbb{Z}_2 \times \mathbb{Z}_2$. Since the center of the finite 2-group S is nontrivial and contains an involution, it follows that S has a unique involution. By a theorem of Burnside [3, p.99, Theorem 4.3], S is either cyclic or a generalized quaternion group. If S is cyclic, by [14, chapter 5.2, Corollary 2] the group G is solvable. So we can assume that S is a generalized quaternion group. Now by Theorem 8.7 and its proof in [14, chapter 6], the group G is either solvable, or $G/\mathcal{O}(G)$ has a normal subgroup G_0 isomorphic to $\mathrm{SL}(2,q)$, $\mathrm{TL}(2,q)$ or $\hat{\mathbb{A}}_7$, of odd index and with cyclic factor group (the only central extension of $PSL(2,q)$, with center of order two and no subgroups $\mathbb{Z}_2 \times \mathbb{Z}_2$ (or equivalently, with quaternion Sylow 2-subgroups), is $SL(2,q)$; also, the only 2-fold extension of $SL(2,q)$, with central quotient $PGL(2,q)$ and without subgroups $\mathbb{Z}_2 \times \mathbb{Z}_2$, is $TL(2,q)$). By [12], G has a unique involution (representing the unique involution of G_0); this involution is central in G and hence acts trivially on $\mathcal{O}(G)$.

Conversely, suppose that G is of the form described in Theorem 1. Then the Sylow 2-subgroups of G are generalized quaternion groups (in particular, G is "2-periodic"). Since the unique involution of G_0 induces the trivial outer automorphism of $\mathcal{O}(G)$, G has a unique involution. Now by [13, Theorem A], G acts freely on a simply connected $\mathbb{Z}_2$-homology sphere (of a dimension determined by the 2-period of G: this is a mod 2 version of the solution of the spherical space from problem solved in [8] where the existence of free actions on spheres of periodic groups with a unique involution is established). This finishes the proof of Theorem 5.

4. Conjectural picture of the situation.

Integer homology spheres. In each dimension, there are only finitely many groups $PSL(2, q)$ (and, more generally, only finitely many finite nonabelian simple groups) acting on an integer homology sphere. What is the minimal dimension such that $PSL(2, q)$ acts?

REMARK. The minimal dimension of an orthogonal action of $PSL(2, q)$ on a sphere is $(q-1)/2$ if $q \equiv 1$ mod 4, and $q-2$ if $q \equiv 3$ mod 4, and it is likely that the minimal dimensions for arbitrary and for orthogonal actions coincide. The minimal possible dimension for a free action of $SL(2, p)$ is $lcm(4, p-1) - 1$ where the least common multiple $lcm(4, p-1)$ is the cohomological period of $SL(2, p)$, and the existence of free actions on spheres is proved in [8]. For $p > 5$, these actions are necessarily non-orthogonal: by a result of Zassenhaus, the only finite perfect group acting orthogonally and freely on a homology sphere is the binary dodecahedral group $\mathbb{A}_5^* \cong SL(2, 5)$ (see e.g. [15]).

Mod 2 homology spheres. Let q denote an odd prime power.

a) All groups $PSL(2, q)$ admit an action on a $\mathbb{Z}_2$-homology 3-sphere.

b) All groups $SL(2, q)$ admit a free action on a $\mathbb{Z}_2$-homology 3-sphere.

As noted above, b) might follow from the high-dimensional surgery methods employed in [13]. We have explicit examples of actions of groups $PSL(2, q)$ on $\mathbb{Z}_2$-homology 3-spheres for many small values of q [17]. We found it more difficult to construct such examples for free actions of the groups $SL(2, q)$, in fact except for the ubiquitous $\mathbb{A}_5^* \cong SL(2, 5)$ we found such examples so far only for the groups $SL(2, 7)$ and $SL(2, 9)$:

Theorem 6. *The groups $SL(2, 7)$ and $SL(2, 9)$ admit free actions on hyperbolic $\mathbb{Z}_2$-homology 3-spheres. In particular, the 4-fold cyclic branched covering of the figure-8 knot admits a regular $SL(2, 7)$-covering which is a hyperbolic $\mathbb{Z}_2$-homology 3-sphere. Similarly, the 4-fold cyclic branched covering of the knot 5_2 admits a regular $SL(2, 9)$-covering which is a hyperbolic $\mathbb{Z}_2$-homology 3-sphere.*

The examples, and in particular the homology of the manifolds, were obtained with the support of the group theory package GAP.

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