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ON RECOGNITION OF THE PROJECTIVE SPECIAL LINEAR GROUPS OVER BINARY FIELD

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ABSTRACT. The spectrum $\omega(G)$ of a finite group G is the set of element orders of G. Let L be the projective special linear group $L_n(2)$ with $n \geq 3$. First, for all $n \geq 3$ we establish that every finite group G with $\omega(G) = \omega(L)$ has a unique non-abelian composition factor and this factor is isomorphic to L. Second, for some special series of integers n we prove that L is recognizable by spectrum, i.e. every finite group G with $\omega(G) = \omega(L)$ is isomorphic to L.

INTRODUCTION

Throughout this paper, all groups are assumed to be finite and all simple groups are non-abelian. Some interesting problems in finite group theory are related to arithmetical characteristics of the group. For example for a group G we can consider the set $\pi(G)$ of prime divisors of $|G|$ and the set $\omega(G)$ of orders of all elements in G. We call this last set the *spectrum* of G, motivating it as follows.

We recall that for an element $A \in \mathrm{GL}(n, \mathbb{C})$, we have

$$\mathrm{Spec}(A) = \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of } A\}.$$

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We consider the regular representation of a finite group G over $\mathbb{C}$. Then G can be viewed as a subgroup of $GL(|G|, \mathbb{C})$ and we can consider

$$\mathrm{Spec}(G) = \bigcup_{g \in G} \mathrm{Spec}(g).$$

It can be easily seen that

$$\mathrm{Spec}(G) = \{\lambda \in \mathbb{C} : \lambda^m = 1 \ \text{ for } m \in \omega(G)\}.$$

Thus $\omega(G)$ and $\mathrm{Spec}(G)$ are uniquely determined one by the other and the definition of $\omega(G)$ as the spectrum of G is therefore consistent.

If Ω is a non-empty subset of the set of natural numbers, $h(\Omega)$ stands for the number of isomorphism classes of finite groups G with $\omega(G) = \Omega$ and put $h(G) = h(\omega(G))$. We say that G is *recognizable* (by spectrum) if $h(G) = 1$. The group G is *almost recognizable* (resp. *nonrecognizable*) if $1 < h(G) < \infty$ (resp. $h(G) = \infty$). A list of simple groups recognizable, almost recognizable or nonrecognizable by their spectrum is given in [15, 16].

In the present paper, we focus our attention on the projective special linear groups $L_n(2)$. We have good evidence that these groups are recognizable by their spectrum and therefore we put forward the following conjecture.

Conjecture. The projective special linear groups $L_n(2)$ are recognizable by their spectrum for all integers $n \geq 3$.

It has already been proved that the conjecture is true for $n \leq 8$ and $n = 11, 12$ (see [19, 20, 5, 6, 18, 17]). In [13] the conjecture is proved for the linear groups $L_p(2)$, where p is an odd prime such that 2 is a primitive root modulo p (note that this result implies recognizability of $L_{13}(2)$). In [7, 8] the groups $L_n(2^k)$, where $n = 2^m \geq 16$ and k is an arbitrary natural number, are shown to be recognizable; thus the conjecture also holds for $n = 2^m \geq 16$.

In this paper we first establish that for every $n \geq 3$ the projective special linear group $L = L_n(2)$ has the following property. If G is a finite group with the same spectrum as L, then G has a unique non-abelian composition factor and this factor is isomorphic to L; that is, L is *quasirecognizable* by spectrum.

Theorem 1. *The projective special linear group $L_n(2)$ is quasirecognizable by spectrum for all integers $n \geq 3$.*

Second, we prove the conjecture for some new series of integers n. In particular, we prove it for $n = 9, 10, 14, 15$. Thus Conjecture holds true for all $n < 17$.

Theorem 2. *Let p be a prime such that 2 is a primitive root modulo p and m be a natural number such that $2^m - 1 \geq p$. The projective special linear group $L_n(2)$ is recognizable by spectrum for $n = 2^m + p - 1$. If, in addition, 3 does not divide $p - 1$, then the projective special linear group $L_n(2)$ is recognizable by spectrum for $n = 2^m + p + 2$ and $n = p + 3$.*

1. Preliminaries

Our notation is standard. If n is a natural number, π is a set of primes, then by $\pi(n)$ we denote the set of all prime divisors of n, and by n_π we denote the maximal divisor t of n such that $\pi(t) \subseteq \pi$. Note that for a finite group G, $\pi(G) = \pi(|G|)$ by definition. For a set of integers X, by $\operatorname{lcm} X$ we denote the least common multiple of elements from X. By $[x]$ we denote the integer part of x, i. e., the greatest integer that is less than or equal to x.

The spectrum $\omega(G)$ of a group G determines the prime graph (or Gruenberg — Kegel graph) $GK(G)$ whose vertex set is $\pi(G)$ and two vertices p and q are adjacent if and only if $pq \in \omega(G)$. Denote by $s(G)$ the number of connected components of $GK(G)$.

Suppose that S is a simple non-abelian group with $s(S) > 1$ other than $L_4(3)$, $U_4(3)$, and $S_4(3)$, and G is a finite group with $\omega(G) = \omega(S)$. As follows from the Gruenberg-Kegel theorem on groups with disconnected prime graphs [23] and the main result of [1], the group G has a unique non-abelian composition factor H and $s(H) \geq s(G)$, in particular $s(H) > 1$. Simple groups H with $s(H) > 1$ were classified in [23] and [11]. So this classification may be used in proving that $H \simeq S$.

By [11], we have

$$s(L_n(2)) = \begin{cases} 1 & \text{if } n \neq p,\, p+1; \\ \\ 2 & \text{if } n = p \text{ or } p+1, \end{cases}$$

where $p > 3$ is a prime. Thus the class of linear groups over field of order 2 to which above technique can be applied is quite restricted. A similar situation arises for several families of simple groups. Recently a number of papers appeared, concerning the structure of G with $\omega(G) = \omega(S)$ under weaker conditions on the source group S. First, it is shown that G is generally insoluble.

Lemma 1 ([12, Theorem 2])**.** *Let S be a finite non-abelian simple group other than $L_4(3)$, $U_4(3)$, $S_4(3)$, and Alt_{10}. Suppose that G is a finite group with $\omega(G) = \omega(S)$. Then G is insoluble.*

More constructive result, generalizing in a certain way the Gruenberg-Kegel theorem, was obtained in [21]. The set of vertices of a graph is called independent if vertices of this set are pairwise nonadjacent. Following [21], we denote by $\rho(G)$ (by $\rho(r, G)$ where $r \in \pi(G)$) some independent set in $GK(G)$ (containing r) with maximal number of vertices. Moreover, we define the independence number $t(G)$ of G as $|\rho(G)|$ and the r-independence number $t(r, G)$ of G as $|\rho(r, G)|$.

Lemma 2 ([21])**.** *Let G be a finite group satisfying two conditions:*

(a) *there exist three primes in $\pi(G)$ which are pairwise nonadjacent in $GK(G)$, that is $t(G) \geq 3$;*

(b) *there exists an odd prime in $\pi(G)$ which is nonadjacent to prime 2 in $GK(G)$, that is $t(2, G) \geq 2$.*

Then there exists a finite non-abelian simple group S such that $S \leq \overline{G} = G/K \leq$ $\mathrm{Aut}(S)$ for maximal normal soluble subgroup K of G. Furthermore, $t(S) \geq t(G) - 1$ and one of the following statements holds:

(1) *$S \simeq \mathrm{Alt}_7$ or $L_2(q)$ for some odd q and $t(S) = t(2, S) = 3$.*

(2) *For every prime p in $\pi(G)$ nonadjacent to 2 in $GK(G)$ the Sylow p-subgroup of G is isomorphic to the Sylow p-subgroup of S. In particular, $t(2, S) \geq t(2, G)$.*

Remark that Condition (a) in the statement of above theorem may be replaced by a weaker condition that G is insoluble (see [21]). The information about values of independence and 2-independence numbers of finite simple groups obtained in [22] together with this remark imply the following corollary of Lemma 2.

Lemma 3 ([22, Corollary 7.2]). *Let S be a finite non-abelian simple group other than $L_3(3)$, $U_3(3)$, $S_4(3)$, Alt_{10} and Alt_n with n satisfying $\{r \mid n-3 \leq r \leq n, r \text{ is prime}\} = \varnothing$. Suppose that G is a finite group with $\omega(G) = \omega(S)$. Then the conclusion of Lemma 2 holds true for G.*

Above results were applied to the recognition problem in [7, 8], where a series of linear groups with connected prime graph were proved to be recognizable.

The following number-theoretic result is of fundamental importance for investigations of the prime graph structure of the finite simple groups of Lie type.

Lemma 4 (Zsigmondy[24]). *Let q and m be natural numbers greater than 1. There exists a prime divisor r of $q^m - 1$ such that r does not divide $q^i - 1$ for all $i < m$, except for the following cases:*
(a) $m = 6$ *and* $q = 2$;
(b) $m = 2$ *and* $q = 2^l - 1$ *for some natural number* l.

Such a prime r is called a *primitive prime divisor* of $q^m - 1$. If q is fixed, we denote by r_m any primitive prime divisor of $q^m - 1$ (obviously, $q^m - 1$ can have more than one primitive prime divisor). It is also convenient to use the following notation. If q is a natural number, r is an odd prime and $(q, r) = 1$, then by $e(r, q)$ we denote the smallest natural number m such that $q^m \equiv 1 \pmod r$. Thus for a primitive prime divisor r of $q^m - 1$ we have $e(r, q) = m$.

The last lemma describes the spectrum of $L_n(2)$.

Lemma 5 ([13, Lemma 1]). *Let $n = \sum_{i=1}^{N} k_i d_i$, where $k_1, k_2, \ldots, k_N, d_1, \ldots, d_N$ are natural numbers and $n \geq 3$. Let $e = \operatorname{lcm}\{2^{d_1} - 1, 2^{d_2} - 1, \ldots, 2^{d_N} - 1\}$ and m be the smallest integer with $2^m \geq \max\{k_1, k_2, \ldots, k_N\}$. Then $2^m e \in \omega(L_n(2))$. Moreover, every element of $\omega(L_n(2))$ is a divisor of a such product.*

2. Proof of Quasirecognizability for $L_n(2)$

In this paragraph we establish Theorem 1. Since $L_n(2)$ where $n \leq 8$ or $n = 11, 12, 13$ are proved to be recognizable we can assume that either $n = 9, 10$ or $n \geq 14$.

We consider the classical groups of Lie type and denote them according to [3]. Sometimes we use notations $A_l^\varepsilon(q)$, $D_l^\varepsilon(q)$, and $E_6^\varepsilon(q)$, where $\varepsilon \in \{+, -\}$ and $A_l^+(q) = A_l(q)$, $A_n^-(q) = {}^2A_l(q)$, $D_l^+(q) = D_l(q)$, $D_l^-(q) = {}^2D_l(q)$, $E_6^+(q) = E_6(q)$, $E_6^-(q) = {}^2E_6(q)$. We denote the alternating group of degree l by Alt_l to avoid confusing with groups of type A_l.

Let $L = L_n(2) = A_{n-1}(2)$ where $n \geq 9$. By [22, §8] we have $\rho(2, L) = \{2, r_n, r_{n-1}\}$, $t(2, L) = 3$, $t(L) = [(n-1)/2] = 4$ for $n = 9, 10$ and $t(L) = [(n+1)/2] \geq 7$ for $n \geq 14$. Furthermore, Lemma 5 implies that all elements of $\omega(L)$ do not exceed $2^n - 1$.

Let G be a finite group with $\omega(G) = \omega(L)$ and K be the maximal normal soluble subgroup of G. By Lemma 2 there is a finite non-abelian simple group S such that $S \leq \overline{G} = G/K \leq \operatorname{Aut}(S)$. Moreover $t(S) \geq t(G) - 1$ and either $r_n, r_{n-1} \in \pi(S)$ or $S \simeq Alt_7$, $A_1(q)$ where q is odd.

(1) First we consider the exceptions. Let $S \simeq Alt_7$ or $S \simeq A_1(q)$ where $q = p^k > 3$ is odd. Since $t(S) = 3$ it follows that $t(L) \leq 4$ and therefore $n = 9, 10$. By the criterion of adjacency from [22], primes $r_5 = 31$, $r_7 = 127$, $r_8 = 17$, and $r_9 = 73$ are pairwise nonadjacent in $GK(G)$. As follows from [21, Proposition 3], at least

three of these numbers belong to $\pi(S)$. Thus $S \not\simeq Alt_7$. Put $\rho = \{r_5, r_7, r_8, r_9\}$ and $\rho' = \rho \cap \pi(S)$. Since ρ' is an independent set of $GK(S)$ with maximal number of vertices, results of [22, Propositions 2.1,3.1,4.1] give that $\rho' = \{p, r_1', r_2'\}$ where r_1' divides $q-1$ and r_2' divides $q+1$. Thus $p \in \rho'$ and $\rho' \setminus \{p\} \subseteq \pi(q^2-1)$. Therefore $\pi(q^2-1) \cap \rho$ contains two elements. On the other hand, $q = p^k$ must satisfy the inequality $(q+1)/2 \leq 2^{10}-1$, otherwise $\omega(S) \not\subseteq \omega(L)$. Hence if $p = 17$ or 31 then $k \leq 2$; if $p = 73$ or 127 then $k = 1$. We calculate that $\pi(q^2-1) \subseteq \{2, 3, 5, 7, 13, 29, 37\}$ for all possibilities of q. Therefore $\pi(q^2-1) \cap \rho = \varnothing$; a contradiction.

Thus the second statement of Lemma 2 holds and therefore r_n and r_{n-1} divide $|S|$. Moreover, r_n and r_{n-1} are nonadjacent to 2 in $GK(S)$. Therefore $t(2, S) \geq 3$. The simple groups satisfying this condition are described in [22], and we consider them consequently.

(2) S is a sporadic group. Since $r_n, r_{n-1} \in \pi(S)$ there must be two odd primes p_1 and p_2 in $\rho(2, S)$ such that $e(p_1, 2)/e(p_2, 2) = n/(n+1)$. It is false when $n \geq 7$ and $n \neq 11$ (see [22, Table 2] or [4]).

(3) $S \simeq Alt_{n'}$. There are two odd primes among numbers $n', n'-1, n'-2, n'-3$; these are r_n and r_{n-1}. By [2, Proposition 7] we have $4 \cdot r_{n-2} \notin \omega(L)$, although $2 \cdot r_{n-2} \in \omega(L)$. Suppose that r_{n-2} divides the order of S. Since S does not contain an element of order $4 \cdot r_{n-2}$, it follows that $n' \geq r_{n-2} \geq n'-5$. Thus, there are three odd primes among six consecutive numbers $n', \dots, n'-5$, which implies $n' = 7$ or $n' = 8$. Hence either $n' = 7, 8$ or $r_{n-2} \in \pi(K)$.

If $n' = 7, 8$ we proceed as in (1). If $n' \geq 9$ and $r_{n-2} \in \pi(K)$ we obtain a contradiction by literally repeating the arguments from the part of [7, § 2] which concerns the alternating groups.

To consider the simple groups of Lie type, it is convenient to separate the case when $n = 9, 10$ from other cases. First we suppose that $n \geq 14$. Then S must satisfy $t(2, S) \geq 3$ and $t(S) \geq t(G) - 1 \geq 6$. We obtain such groups from [22, Tables 4–9].

(4) S is a group of Lie type over field of order $q = p^k$, p is odd.

Let $S \simeq E_8(q), E_7(q)$ or $E_6^{\varepsilon}(q)$. If $S \simeq E_8(q)$ then $t(S) = 11$ therefore $n \leq 24$. Since q^8-1 must be less than or equal to $2^{24}-1$, we have that $q \leq 8$. Thus $q = 3, 5$ or 7. If $S \simeq E_7(q)$ then $t(S) = 7$ therefore $n \leq 16$. Since $(q^7-1)/2 \leq 2^{16}-1$, we have that $q = 3, 5$. If $S \simeq E_6^{\varepsilon}(q)$ then $t(S) \leq 6$. Therefore $n \leq 14$ and $(q^6 - \varepsilon 1)/(3, q - \varepsilon 1) \leq 2^{14} - 1$, whence $q = 3, 5$. Whatever group S we consider, either a primitive prime divisor r_9' of $q^9 - 1$ or a primitive prime divisor r_{18}' of $q^{18}-1$ belongs to $\pi(S) \subseteq \pi(L)$. Suppose that $r_9' \in \pi(L)$. For each $q \in \{3, 5, 7\}$ we calculate r_9' and establish that $e(r_9', 2) \geq 36$. Hence the condition $r_9' \in \pi(L_n(2))$ implies $n \geq 36$, which contradicts to above inequality $n \leq 24$. The case $r_{18}' \in \omega(L)$ can be done similarly.

Let $S \simeq A_{n'-1}^{\varepsilon}(q)$, where $n_2' = (q - \varepsilon 1)_2 > 2$. The inequality $t(S) \geq t(G) - 1$ together with $t(S) = [(n'+1)/2]$, $t(G) = [(n+1)/2]$ implies $n' \geq n-3$. The group S contains an element of order $q^{n'-2} - 1$, and therefore so does L. Since every element of $\omega(L)$ does not exceed $2^n - 1$, we have $2^n - 1 \geq q^{n'-2} - 1$. On the other hand, $q^{n'-2} \geq q^{n-5} \geq 3^{n-5} > 2^n$ for all $n \geq 14$; a contradiction.

Let $S \simeq D_{n'}(q)$, where n' is odd and $q \equiv 5 \pmod 8$. The inequality $t(S) \geq t(G) - 1$ together with $t(S) = [(3n'+1)/4]$, $t(G) = [(n+1)/2]$ implies $n' \geq (2n-5)/3$. The group S contains an element of order $(q^{n'}-1)/4$ and therefore $(q^{n'}-1)/4 \leq 2^n -$

1. Whence $q^{n'} \leq 2^{n+2}$. This is impossible, since $q^{n'} \geq q^{(2n-5)/3} \geq 5^{(2n-5)/3} > 2^{n+2}$ for all $n \geq 14$.

Let $S \simeq {}^2D_{n'}(q)$, where n' is odd and $q \equiv 3 \pmod 8$. Since $t(S) = [(3n'+4)/4] = [(3n'+3)/4]$, it follows from $t(S) \geq t(G)-1$ that $n' \geq (2n-7)/3$. Since S contains an element of order $(q^{n'}+1)/4$, we have $(q^{n'}+1)/4 \leq 2^n - 1$. Whence $q^{n'} \leq 2^{n+2}$ and therefore $q^{(2n-7)/3} \leq 2^{n+2}$. The last inequality holds true only if $q = 3$ and $n \leq 100$.

Suppose $S \simeq {}^2D_{n'}(3)$ and $9 \leq n \leq 100$. Since $n' \geq 5$ the group S contains an element of order $(3^5+1)/4 = 61$. Therefore $61 \in \pi(L)$ and $n \geq e(61,2) = 60$. Since $n \geq 60$, we have $n' \geq (2n-7)/3 > 37$. Therefore S contains an element of order $r'_{36} = 757$. Since $757 \in \pi(L)$, we have $n \geq e(757,2) = 756$; a contradiction.

(5) S is a group of Lie type over field of order $q = 2^k$. Observe that S is not a simple Suzuki or Ree group, otherwise $t(S) < 6$.

Recall that r_n and r_{n-1} divide $|S|$. Put $e_n = e(r_n, 2^k)$ and $e_{n-1} = e(r_{n-1}, 2^k)$. Since r_n divides $2^{e_n k} - 1$ we have that n divides $e_n k$. By the same reason $n-1$ divides $e_{n-1}k$. Suppose that $e_n k > n$. Then prime r with $e(r,2) = e_n k$ divides the order of S and does not divide the order of L. Therefore $r \in \omega(S) \setminus \omega(G)$, which is impossible. Thus $e_n k = n$. Suppose that $e_{n-1}k > n-1$. Then $e_{n-1}k \geq 2(n-1) > n$ and the similar argumentation leads us to a contradiction. Thus $e_{n-1}k = n-1$.

If S is a classical group of Lie type other than L then we obtain a contradiction by literally repeating the arguments from the part of [7, § 2] which concerns the corresponding groups.

Let $S \simeq E_8(2^k)$. By [22, Proposition 3.2] an odd prime r is nonadjacent to 2 in $GK(S)$ if and only if $e(r, 2^k) \in \{15, 20, 24, 30\}$. Therefore $e_n, e_{n-1} \in \{15, 20, 24, 30\}$. On the other hand, $e_n/e_{n-1} = n/(n-1)$. These two conditions imply $n \leq 6$; a contradiction.

We consider the groups $E_7(2^k)$, $E_6^{\varepsilon}(2^k)$, $F_4(2^k)$, and $G_2(2^k)$ in the similar way. Namely, by solving the equation $e_n/e_{n-1} = n/(n-1)$ for each group, we find that all solutions are less than 14.

(6) Now we suppose that $L = L_9(2)$ or $L = L_{10}(2)$. Since $\omega(S) \subseteq \omega(L)$ and $r_9 = 73 \in \pi(S)$, we have $\{73\} \subseteq \pi(S) \subseteq \pi(L_{10}(2)) = \{2,3,5,7,11,17,31,73,127\}$.

Let S be a classical group of Lie type of rank n' (or $n'-1$ if S of type A^{ε}) over field of order $q = p^k$ where $p \in \pi(L_{10}(2))$. If $p = 2$ and $S \not\simeq L$ we obtain a contradiction as in (5). So we can assume that p is odd. In view of conditions $t(2,S) \geq 3$ and $t(S) \geq 3$ we have $n' \geq 4$. Therefore $q = 3, 5, 7, 9$, or 11, otherwise there is an element in S of order greater than $2^{10}-1$. On the other hand, either a primitive prime divisor of q^4-1 or a primitive prime divisor of q^4+1 belongs to $\omega(S)$. It follows that $q = 3$ or 7. Since $73 \in \pi(S)$ and $e(73,7) = 24$, $e(73,3) = 12$, we infer that $n' \geq 12$ for groups of type A^{ε} and $n' \geq 6$ for other classical groups. Thus q^5+1 divides $|S|$ and therefore a primitive prime divisor of $q^{10}-1$ belongs to $\pi(S)$. But primitive prime divisors of $7^{10}-1$ and $3^{10}-1$ do not lie in $\pi(L_{10}(2))$.

Let S be an exceptional group of Lie type over field of order q.

If q is odd, then S can be isomorphic to $E_8(q)$, $E_7(q)$, $E_6^{\varepsilon}(q)$, $G_2(q)$, or ${}^2G_2(3^{2k+1})$. The first three types of groups have been considered in (4) without using the assumption that $n \geq 14$.

Let $S \simeq G_2(q)$, $q = p^k$ is odd. If $q > 31$ then $2^{10} - 1 < q^2 + q + 1 \in \omega(S)$. Thus we can assume that $q \leq 31$. If $n = 9$ then $17 \in \pi(S)$ and so $17 \in \pi(p(q^6-1))$. If $p = 17$ then $307 = 17^2+17+1 \in \pi(S) \setminus \pi(L)$. Thus $q^6 \equiv 1 \pmod{17}$, whence $q^2 \equiv 1$

(mod 17) and therefore $q \equiv \pm 1 \pmod{17}$. This implies $q \geq 33$; a contradiction. If $n = 10$ then $11 \in \pi(S)$ and so $11 \in \pi(p(q^6-1))$. If $p = 11$ then prime divisor 19 of 11^2+11+1 lies in $\pi(S) \setminus \pi(L)$. Therefore $q \equiv \pm 1 \pmod{11}$. Thus either $q \geq 32$ or $q = 23$. Since $q \leq 31$, it follows that $q = 23$. Since $73 \notin \pi(G_2(23))$, we have a contradiction.

Let $S \simeq {}^2G_2(q)$, where $q = 3^{2k+1}$. It follows from $73 \in \pi(S)$ that 73 divides $q^6 - 1 = 3^{6(2k+1)} - 1$. Therefore $e(73,3) = 12$ divides $6(2k+1)$; a contradiction.

If q is even and S is not a Suzuki or Ree group, we use a technique described in (5). Solving the equation $e_n/e_{n-1} = n/(n-1)$ we find that $S \simeq E_6(q)$ and $n = 9$. Since $13 \in \omega(S) \setminus \omega(L)$, we have a contradiction.

Let $S \simeq {}^2B_2(q)$, where $q = 2^{2k+1} > 2$. If $k > 4$ then $2^{10} - 1 < q - 1 \in \omega(S)$, so we can assume that $k \leq 4$. Since $r_n, r_{n-1} \in \pi(S)$, we have that r_n, r_{n-1} divide $q^4 - 1 = 2^{4(2k+1)} - 1$ and therefore $n, n-1$ divide $4(2k+1)$; which contradicts to inequalities $n(n-1) \geq 72$ and $4(2k+1) \leq 36$.

Let $S \simeq {}^2F_4(q)$, $q = 2^{2k+1} > 2$. Again we can assume that $k \leq 4$. Since $r_n, r_{n-1} \in \pi(S)$, we have that r_n, r_{n-1} divide $q^6 - 1 = 2^{6(2k+1)} - 1$ and therefore $n, n-1$ divide $6(2k+1)$; which contradicts to inequalities $n(n-1) \geq 72$ and $6(2k+1) \leq 54$.

Thus $S \simeq L$ and Theorem 1 is proved.

3. Proof of Theorem 2

Let G be a finite group with $\omega(G) = \omega(L)$ and K be the maximal normal soluble subgroup of G. We conclude from Theorem 1 that $\overline{G} = G/K$ is an almost simple group with unique non-abelian composition factor isomorphic to L. Thus we can assume that $L \leq \overline{G} \leq \mathrm{Aut}(L)$.

Suppose that $\overline{G} \neq L$. Since $\mathrm{Out}(L) = 2$, we infer that $\overline{G} = \mathrm{Aut}(L) = L\langle\gamma\rangle$ where γ is a graph automorphism. Consider the centralizer $C_L(\gamma)$ of γ in L. If n is odd then $C_L(\gamma)$ contains a subgroup isomorphic to $B_{(n-1)/2}(2)$. Therefore $r_{n-1} \cdot 2 \in \omega(\overline{G}) \subseteq \omega(G)$; a contradiction. If n is even then $C_L(\gamma)$ contains a subgroup isomorphic to $C_{n/2}(2)$. Therefore $r_n \cdot 2 \in \omega(\overline{G}) \subseteq \omega(G)$; a contradiction. Thus $\overline{G} = L$.

Suppose that $K \neq 1$. Then there exists a prime r such that $O^r(K) \neq K$. Denote by $\widetilde{G}$ and $\widetilde{K}$ the factor groups $G/O^r(K)$ and $K/O^r(K)$ respectively. The group $\widetilde{K}$ is a nontrivial r-group. Let $\Phi(\widetilde{K})$ be the Frattini subgroup of $\widetilde{K}$. Denote by $\widehat{G}$ and $\widehat{K}$ the factor groups $\widetilde{G}/\Phi(\widetilde{K})$ and $\widetilde{K}/\Phi(\widetilde{K})$ respectively. Since $G/K \simeq \widehat{G}/\widehat{K}$, it is sufficient to proof that $\omega(\widehat{G}) \not\subseteq \omega(L)$. Therefore we may assume that $G = \widehat{G}$ and $K = \widehat{K}$ is a nontrivial elementary abelian r-group.

Suppose that $C = C_G(K) \neq K$. Since C is normal in G and L is simple, C/K contains L. Therefore $r \cdot \omega(L) \subseteq \omega(C) \subseteq \omega(G) = \omega(L)$. However by [7, Lemma 4(3)] there is $r' \in \pi(L)$ such that $r \cdot r' \notin \omega(L)$. Therefore $r \cdot r' \in r \cdot \omega(L) \setminus \omega(L)$; a contradiction. Thus $C = K$ and L acts faithfully on K.

Thus we can apply results concerning orders of elements arising when a group acts faithfully on an elementary abelian group.

Lemma 6 ([14, Lemma 1]). *Let G be a finite group, $K \lhd G$, and G/K be a Frobenius group with kernel F and cyclic complement C. If $(|F|,|K|) = 1$ and F does not lie in $KC_G(K)/K$, then $r \cdot |C| \in \omega(G)$ for some prime divisor r of $|K|$.*

Lemma 7 ([7, Lemma 5])**.** *Let L be a finite simple group $L_n(q)$, $d = (q-1, n)$.*

(1) *If there exists a primitive prime divisor r of $q^n - 1$, then L contains a Frobenius subgroup with kernel of order r and cyclic complement of order n;*

(2) *L contains a Frobenius subgroup with kernel of order q^{n-1} and cyclic complement of order $\frac{q^{n-1}-1}{d}$.*

Suppose that $r \neq 2$. Then we consider the Frobenius subgroup F of L from Lemma 7(2). Applying Lemma 6 to the preimage of F in G we obtain that $r \cdot (2^{n-1} - 1) \in \omega(G)$. On the other hand, Lemma 5 implies that $r \cdot (2^{n-1} - 1) \notin \omega(L)$; a contradiction.

Thus we can assume that $r = 2$. Observe that the above argumentation does not require a special form of n, as declared in the statement of the Theorem. This form is crucial when K is an elementary abelian 2-group. More precisely, we obtain the following statement.

Proposition 1. *Let $L = L_n(2)$, $n \geq 3$. If a finite group G is a minimal counterexample to the assertion: $\omega(G) = \omega(L) \Rightarrow G \simeq L$, then G is isomorphic to an extension $K \cdot L$ where K is an elementary abelian 2-group on which L acts faithfully.*

Now we fix our attention on groups $L_n(2)$ with n satisfying the conditions of Theorem 2.

Lemma 8. *Let p be a prime such that 2 is a primitive root modulo p and $n = 2^m + p - 1$, $m \geq 1$. If $L = L_n(2)$, then $2^{m+1}p \notin \omega(L)$.*

Proof. Suppose that $2^{m+1}p \in \omega(L)$. By Lemma 5 there exist natural numbers $k_1, \ldots, k_N$ and $d_1, \ldots, d_N$ with $\sum_{i=1}^{N} k_i d_i = n$ satisfying two conditions: (a) $e = \mathrm{lcm}\{2^{d_1} - 1, \ldots, 2^{d_N} - 1\}$ is divisible by p; (b) the smallest integer l with $2^l \geq \max\{k_1, \ldots, k_N\}$ is greater than or equal to $m+1$. Since p divides e, it follows that p divides $2^{d_i} - 1$ for some d_i. By hypothesis, 2 is a primitive root modulo p, therefore d_i is divisible by $p - 1$. On the other hand, from $l \geq m + 1$ we deduce that there exists j such that $k_j > 2^m$. If $i = j$ then $n \geq k_i d_i > 2^m(p-1) \geq 2^m + p - 1 = n$; a contradiction. If $i \neq j$ then $n \geq k_j + d_i > 2^m + p - 1 = n$; a contradiction. The lemma is proved. □

Lemma 9. *Let p be a prime such that 2 is a primitive root modulo p, $n = 2^m + p - 1$, $2^m - 1 \geq p$ and $L = L_n(2)$. Suppose that K is an elementary abelian 2-group on which L acts faithfully. Then there exists an element of order $2^{m+1}p$ in KL and $\omega(KL) \neq \omega(L)$.*

Proof. The group L contains two subgroups $A \simeq L_p(2)$ and $B \simeq L_{n-p}(2)$ such that $A \times B$ is a subgroup of L. By Lemma 7(1) there is a Frobenius subgroup $F = \langle x, y \rangle$ of A with $|x| = p$, $|y| = r_p$, where r_p is a primitive prime divisor of $2^p - 1$. The group F acts on $M = [K, y]$ in such a way that $C_M(y) = 1$ and $C_M(x) \neq 1$. In particular,

$$K_0 = C_K(x) \not\leq C_K(y). \qquad (*)$$

It is easy to see that $C_L(x) = \langle x \rangle \times N$ where $N \simeq L_{n-p+1}(2) = L_{2^m}(2)$ and N acts on K_0. If this action is not faithful then N centralizes K_0 and hence $C_L(x)$ centralizes K_0. It is obvious that $B \leq N$ contains a subgroup F^z which is a conjugate of F in L. Since $|C_K(x^z)| = |K_0|$ and $K_0 \leq C_K(N) \leq C_K(x^z)$, we see that $C_K(x^z) = K_0$ and hence $C_L(x^z)$ centralizes K_0. Since $y \in C_L(x^z)$, then y

centralizes K_0. This contradicts $(*)$. So N acts faithfully on K_0. By Lemma 7(1), there exists a Frobenius subgroup in N of type $r_{2^m} : 2^m$. By Lemma 6 we have $2^{m+1} \in \omega(K_0N)$. Hence there is an element of order $2^{m+1}p$ in KL. On the other hand, by Lemma 8 there is no element of order $2^{m+1}p$ in L, thus concluding the proof. □

Lemma 10. *Let p be a prime such that 2 is a primitive root modulo p, 3 does not divide $p-1$, $n = 2^m + p - 1$, $2^m - 1 \geq p$ or $n = p$ and $L = L_{n+3}(2)$. Let K be an elementary abelian 2-group on which L acts faithfully. Then $\omega(KL) \neq \omega(L)$.*

Proof. Using [9] or [10] it is easy to verify that every element of order 7 from $L_5(2)$ centralizes some nontrivial element in every irreducible $L_5(2)$-module over a field of characteristic 2 and so the same is true for every $L_5(2)$-module over a field of characteristic 2. If x is an element of order 7 of L contained in a subgroup isomorphic to $L_3(2)$, then its centralizer $K_0 = C_K(x)$ in K is not trivial. It is easy to see that $C_L(x) = \langle x\rangle \times N$ where $N \simeq L_n(2)$ and N acts on K_0. If this action is not faithful, then N centralizes K_0. At first, assume that 3 does not divide n. Using Lemma 5 and arguments as in proof of Lemma 8, we obtain that L does not contain an element of order $2 \cdot 7 \cdot r_n$ where r_n is a primitive prime divisor of $2^n - 1$. On the other hand, since N centralizes K_0, there exists an element of order $2 \cdot r_n$ in K_0N, which implies that there is an element of order $2 \cdot 7 \cdot r_n$ in KL. So $\omega(KL) \neq \omega(L)$ and the lemma is proved in this case. If 3 divides n then using Lemma 5 we obtain that $2 \cdot 7 \cdot r_{n-1} \notin \omega(L)$. But K_0N contains an element of order $2 \cdot r_{n-1}$, and so KL contains an element of order $2 \cdot 7 \cdot r_{n-1}$. Thus $\omega(KL) \neq \omega(L)$ again.

Therefore we can suppose that N acts on K_0 faithfully. We first suppose that $n = 2^m + p - 1$. By Lemma 9 there is an element of order $p \cdot 2^{m+1}$ in K_0N which implies that there is an element of order $7 \cdot p \cdot 2^{m+1}$ in KL. Suppose that $7 \cdot p \cdot 2^{m+1} \in \omega(L_{n+3}(2))$. Since 7 is a primitive prime divisor of $2^3 - 1$, p is a primitive prime divisor of $2^{p-1} - 1$, and 3 does not divide $p - 1$, by Lemma 5 we have that $n + 3 > 2^m + (p - 1) + 3 = n + 3$. Thus $7 \cdot p \cdot 2^{m+1} \notin \omega(L)$.

Suppose now that $n = p + 3$. Then by Lemma 6 and Lemma 7, there exists an element of order $p \cdot 2$ in K_0N. Therefore $7 \cdot p \cdot 2 \in \omega(KL)$. Suppose that $7 \cdot p \cdot 2 \in \omega(L_{n+3})$. Since 7 is a primitive prime divisor of $2^3 - 1$, p is a primitive prime divisor of $2^{p-1} - 1$, and 3 does not divide $p - 1$, by Lemma 5 we have that $n + 3 \geq 2 + (p - 1) + 3 = n + 4$, a contradiction. The Lemma is thus proved. □

Applying Lemmas 9 and 10 we establish that $K = 1$ and therefore $L = G$. Theorem 2 is proved.

Remark. Since 2 is a primitive root modulo p for $p = 3, 5, 11$, Theorem 2 yields that groups $L_n(2)$ are recognizable for $n = 4+(3-1)+3 = 9$, $n = 8+(3-1) = 10$, $n = 11 + 3 = 14$, and $n = 8 + (5 - 1) + 3 = 15$. Together with previous results it implies that groups $L_n(2)$ are recognizable by spectrum for all $n < 17$.

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