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ALMOST n-ARY AND ALMOST n-ARITIZABLE THEORIES

S.V. SUDOPLATOV

Abstract. We study possibilities for almost n-ary and n-aritizable theories. Their dynamics both in general case, for ω-categorical theories, and with respect to operations for theories are described.

Keywords: elementary theory, almost n-ary theory, almost n-aritizable theory.

We continue to study arities of theories and of their expansions [1]. In the present paper we introduce natural notions of almost n-ary and almost n-aritizable theories, and describe their dynamics both in general case, for ω-categorical theories, and with respect to operations for theories.

1. Preliminaries

Recall a series of notions related to arities and aritizabilities of theories.

Definition [2]. A theory T is said to be Δ-*based*, where Δ is some set of formulae without parameters, if any formula of T is equivalent in T to a Boolean combination of formulae in Δ.

For Δ-based theories T, it is also said that T has *quantifier elimination* or *quantifier reduction* up to Δ.

Definition [2, 3]. Let Δ be a set of formulae of a theory T, and $p(\bar{x})$ a type of T lying in $S(T)$. The type $p(\bar{x})$ is said to be Δ-*based* if $p(\bar{x})$ is isolated by a set of formulas $\varphi^\delta \in p$, where $\varphi \in \Delta$, $\delta \in \{0, 1\}$.

The following lemma, being a corollary of Compactness Theorem, noticed in [2].

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Lemma 1. *A theory T is Δ-based if and only if, for any tuple $\bar{a}$ of any (some) weakly saturated model of T, the type* $\mathrm{tp}(\bar{a})$ *is Δ-based.*

Definition [1]. An elementary theory T is called *unary*, or 1-*ary*, if any T-formula $\varphi(\overline{x})$ is T-equivalent to a Boolean combination of T-formulas, each of which is of one free variable, and of formulas of form $x \approx y$.

For a natural number $n \geq 1$, a formula $\varphi(\overline{x})$ of a theory T is called *n-ary*, or an *n-formula*, if $\varphi(\overline{x})$ is T-equivalent to a Boolean combination of T-formulas, each of which is of n free variables.

For a natural number $n \geq 2$, an elementary theory T is called *n-ary*, or an *n-theory*, if any T-formula $\varphi(\overline{x})$ is n-ary.

A theory T is called *binary* if T is 2-ary, it is called *ternary* if T is 3-ary, etc.

We will admit the case $n = 0$ for n-formulae $\varphi(\overline{x})$. In such a case $\varphi(\overline{x})$ is just T-equivalent to a sentence $\forall\overline{x}\varphi(\overline{x})$.

If T is a theory such that T is n-ary and not $(n-1)$-ary then the value n is called the arity of T and it is denoted by $\mathrm{ar}(T)$. If T does not have any arity we put $\mathrm{ar}(T) = \infty$.

Similarly, for a formula φ of a theory T we denote by $\mathrm{ar}_T(\varphi)$ the natural value n if φ is n-ary and not $(n-1)$-ary. If a theory T is fixed we write $\mathrm{ar}(\varphi)$ instead of $\mathrm{ar}_T(\varphi)$.

Clearly, $\mathrm{ar}(\varphi) \leq |\mathrm{FV}(\varphi)|$, where $\mathrm{FV}(\varphi)$ is the set of free variables of formula φ.

The following example illustrates the notions above, and it will be used below.

Example 1. Recall [4, 5, 6] that a *circular*, or *cyclic* order relation is described by a ternary relation K_3 satisfying the following conditions:
(co1) $\forall x\forall y\forall z(K_3(x,y,z) \to K_3(y,z,x))$;
(co2) $\forall x\forall y\forall z(K_3(x,y,z) \wedge K_3(y,x,z) \leftrightarrow x = y \vee y = z \vee z = x)$;
(co3) $\forall x\forall y\forall z(K_3(x,y,z) \to \forall t[K_3(x,y,t) \vee K_3(t,y,z)])$;
(co4) $\forall x\forall y\forall z(K_3(x,y,z) \vee K_3(y,x,z))$.

Clearly, $\mathrm{ar}(K_3(x,y,z)) = 3$ if the relation has at least three element domain. Hence, theories with infinite circular order relations are at least 3-ary.

The following generalization of circular order produces a *n-ball*, or *n-spherical*, or *n-circular* order relation, for $n \geq 4$, which is described by a n-ary relation K_n satisfying the following conditions:
(nbo1) $\forall x_1,\ldots,x_n(K_n(x_1,x_2,\ldots,x_n) \to K_n(x_2,\ldots,x_n,x_1))$;
(nbo2) $$\forall x_1,\ldots,x_n\Biggl(K_n(x_1,\ldots,x_i,x_{i+1},\ldots,x_n)\wedge \wedge K_n(x_1,\ldots,x_{i+1},x_i,\ldots,x_n) \leftrightarrow \bigvee_{i=1}^{n-1} x_i = x_{i+1}\Biggr);$$
(nbo3) $\forall x_1,\ldots,x_n(K_n(x_1,\ldots,x_n) \to \forall t[K_n(x_1,\ldots,x_{n-1},t) \vee K_n(t,x_2,\ldots,x_n)])$;
(nbo4) $\forall x_1,\ldots,x_n(K_n(x_1,\ldots,x_i,x_{i+1},\ldots,x_n) \vee K_n(x_1,\ldots,x_{i+1},x_i,\ldots,x_n))$, $i < n$.

Clearly, $\mathrm{ar}(K_n(x_1,\ldots,x_n)) = n$ if the relation has at least n-element domain. Thus, theories with infinite n-ball order relations are at least n-ary.

Definition [1]. A T-formula $\varphi(\overline{x})$ is called *n-expansible*, or *n-arizable*, or *n-aritizable*, if T has an expansion T' such that $\varphi(\overline{x})$ is T'-equivalent to a Boolean combination of T'-formulas with n free variables.

A theory T is called *n-expansible*, or *n-arizable*, or *n-aritizable*, if there is an n-ary expansion T' of T.

A theory T is called *arizable* or *aritizable*, if T is n-aritizable for some n.

A 1-aritizable theory is called *unary-able*, or *unary-tizable*. A 2-aritizable theory is called *binary-tizable* or *binarizable*, a 3-aritizable theory is called *ternary-tizable* or *ternarizable*, etc.

Definition. [8] The *disjoint union* $\bigsqcup\limits_{n\in\omega}\mathcal{M}_n$ of pairwise disjoint structures $\mathcal{M}_n$ for pairwise disjoint predicate languages Σ_n, $n\in\omega$, is the structure of language $\bigcup\limits_{n\in\omega}\Sigma_n\cup\{P^{(1)}_n\mid n\in\omega\}$ with the universe $\bigsqcup\limits_{n\in\omega}M_n$, $P_n=M_n$, and interpretations of predicate symbols in Σ_n coinciding with their interpretations in $\mathcal{M}_n$, $n\in\omega$. The *disjoint union of theories* T_n for pairwise disjoint languages Σ_n accordingly, $n\in\omega$, is the theory

$$\bigsqcup_{n\in\omega}T_n\rightleftharpoons\mathrm{Th}\left(\bigsqcup_{n\in\omega}\mathcal{M}_n\right),$$

where $\mathcal{M}_n\models T_n$, $n\in\omega$. Taking empty sets instead of some structures $\mathcal{M}_k$ we obtain disjoint unions of finitely many structures and theories. In particular, we have the disjoint unions $\mathcal{M}_0\sqcup\ldots\sqcup\mathcal{M}_n$ and their theories $T_0\sqcup\ldots\sqcup T_n$.

Theorem 1. [1]. 1. *For any theories T_m, $m\in\omega$, and their disjoint union $\bigsqcup\limits_{m\in\omega}T_m$, all T_m are n-theories iff $\bigsqcup\limits_{m\in\omega}T_m$ is an n-theory, moreover,*

$$\mathrm{ar}\left(\bigsqcup_{m\in\omega}T_m\right)=\max\{\mathrm{ar}(T_m)\mid m\in\omega\}.$$

2. *For any theories T_m, $m\in\omega$, and their disjoint union $\bigsqcup\limits_{m\in\omega}T_m$, all T_m are n-aritizable iff $\bigsqcup\limits_{m\in\omega}T_m$ is n-aritizable.*

Definition [7]. Let $\mathcal{M}$ and $\mathcal{N}$ be structures of relational languages $\Sigma_{\mathcal{M}}$ and $\Sigma_{\mathcal{N}}$ respectively. We define the *composition* $\mathcal{M}[\mathcal{N}]$ of $\mathcal{M}$ and $\mathcal{N}$ satisfying the following conditions:

1) $\Sigma_{\mathcal{M}[\mathcal{N}]}=\Sigma_{\mathcal{M}}\cup\Sigma_{\mathcal{N}}$;

2) $M[N]=M\times N$, where $M[N]$, M, N are universes of $\mathcal{M}[\mathcal{N}]$, $\mathcal{M}$, and $\mathcal{N}$ respectively;

3) if $R\in\Sigma_{\mathcal{M}}\setminus\Sigma_{\mathcal{N}}$, $\mu(R)=n$, then $((a_1,b_1),\ldots,(a_n,b_n))\in R_{\mathcal{M}[\mathcal{N}]}$ if and only if $(a_1,\ldots,a_n)\in R_{\mathcal{M}}$;

4) if $R\in\Sigma_{\mathcal{N}}\setminus\Sigma_{\mathcal{M}}$, $\mu(R)=n$, then $((a_1,b_1),\ldots,(a_n,b_n))\in R_{\mathcal{M}[\mathcal{N}]}$ if and only if $a_1=\ldots=a_n$ and $(b_1,\ldots,b_n)\in R_{\mathcal{N}}$;

5) if $R\in\Sigma_{\mathcal{M}}\cap\Sigma_{\mathcal{N}}$, $\mu(R)=n$, then $((a_1,b_1),\ldots,(a_n,b_n))\in R_{\mathcal{M}[\mathcal{N}]}$ if and only if $(a_1,\ldots,a_n)\in R_{\mathcal{M}}$, or $a_1=\ldots=a_n$ and $(b_1,\ldots,b_n)\in R_{\mathcal{N}}$.

The composition $\mathcal{M}[\mathcal{N}]$ is called *e-definable*, or equ-*definable*, if $\mathcal{M}[\mathcal{N}]$ has an $\emptyset$-definable equivalence relation E whose E-classes are universes of the copies of $\mathcal{N}$ forming $\mathcal{M}[\mathcal{N}]$. If the equivalence relation E is fixed, the e-definable composition is called *E-definable*.

Using a nice basedness of E-definable compositions $T_1[T_2]$ (see [7]) till the formulas of form $E(x,y)$ and generating formulas for T_1 and T_2 we have the following:

Theorem 2. [1]. 1. *For any theories T_1 and T_2 and their E-definable composition $T_1[T_2]$, T_1 and T_2 are n-theories, for $n \geq 2$, iff $T_1[T_2]$ is an n-theory, moreover,* $\mathrm{ar}(T_1[T_2]) = \max\{\mathrm{ar}(T_1), \mathrm{ar}(T_2)\}$*, if models of T_1 and of T_2 have at least two elements, and* $\mathrm{ar}(T_1[T_2]) = \max\{\mathrm{ar}(T_1), \mathrm{ar}(T_2), 2\}$*, if a model of T_1 or T_2 is a singleton.*

2. *For any theories T_1 and T_2 and their E-definable composition $T_1[T_2]$, T_1 and T_2 are n-aritizable iff $T_1[T_2]$ is n-aritizable.*

2. Almost n-ary and n-aritizable theories, their dynamics

Definition. (Cf. [5, 6]) A theory T is called *almost n-ary* if there are finitely many formulae $\varphi_1(\overline{x}), \ldots, \varphi_m(\overline{x})$ such that each T-formula is T-equivalent to a Boolean combination of n-formulae and formulae obtained by substitutions of free variables in $\varphi_1(\overline{x}), \ldots, \varphi_m(\overline{x})$.

In such a case we say that the formulae $\varphi_1(\overline{x}), \ldots, \varphi_m(\overline{x})$ witness that T is almost n-ary.

Almost 1-ary theories are called *almost unary*, almost 2-ary theories are called *almost binary*, almost 3-ary theories are called *almost ternary*, etc.

A theory T is called *almost n-aritizable* if some expansion T' of T is almost n-ary.

Almost 1-aritizable theories are called *almost unary-tizable*, almost 2-aritizable theories are called *almost binarizable*, almost 3-aritizable theories are called *almost ternarizable*, etc.

The following properties are obvious.

1. Any n-ary (respectively, n-aritizable) theory is almost n-ary (almost n-aritizable).

2. Any almost n-ary (respectively, n-aritizable) theory is almost k-ary (almost k-aritizable) for any $k \geq n$.

3. Any theory of a finite structure is almost unary.

Families of weakly circularly minimal structures produce examples of almost binary theories which are not binary [5, 6]. Similarly natural generalizations of weakly circularly minimal structures till n-circular orders give examples of almost $(n-1)$-ary theories T_n with $\mathrm{ar}(T_n) = n$, $n \geq 4$.

Assuming that the witnessing set $\{\varphi_1(\overline{x}), \ldots, \varphi_m(\overline{x})\}$ is minimal for the almost n-ary theory T we have either $m = 0$ of $l(\overline{x}) > n$.

Thus we have two minimal characteristics witnessing the almost n-arity of T: m and $l(\overline{x})$. The pair $(m, l(\overline{x}))$ is called the *degree* of the almost n-arity of T, or the aar-*degree* of T, denoted by $\deg_{\mathrm{aar}}(T)$. Here we assume that n is minimal with almost n-arity of T, this n is denoted by $\mathrm{aar}(T)$. Clearly, $\mathrm{aar}(T) \leq \mathrm{ar}(T)$, and if $m = 0$, i.e., $n = \mathrm{ar}(T) = \mathrm{aar}(T)$ then it is supposed that $l(\overline{x}) = 0$, too.

We have $\mathrm{aar}(T) \in \omega$ if and only if $\mathrm{ar}(T) \in \omega$. So if $\mathrm{ar}(T) = \infty$ then it is natural to put $\mathrm{aar}(T) = \infty$.

Besides, $n = \mathrm{ar}(T) = \mathrm{aar}(T) \in \omega$ means that the set $\Delta_n(T)$ of T-formulae with n free variables allows to express all $\emptyset$-definable sets for T by Boolean combinations and taking any set $\Delta_k(T)$ of T-formulae with $k < n$ free variables all $\emptyset$-definable sets for T can be expressed by Boolean combinations of formulae in Δ_k and substitutions of infinitely many formulae $\varphi(\overline{x})$ only, where $l(\overline{x}) = n$.

By the definition if $\mathrm{aar}(T) = n$ then

$$\deg_{\mathrm{ar}}(T) \in \{(0,0)\} \cup \{(m,r) \mid m \in \omega \setminus \{0\}, r \in \omega, r > n\}. \tag{1}$$

The described pairs in the relation (1) are called *admissible*.

Theorem 3. *For any $m, n \in \omega \setminus \{0\}$ with $m \leq n$ there is a theory T_{mn} with $\mathrm{aar}(Tmn) = m$ and $\mathrm{ar}(T_{mn}) = n$.*

Proof. We use disjoint unions of *dense* n-spherically ordered theories T_n with infinite orders, i.e., theories, generated by n-spherical orders $K_n(x_1, x_2, x_3, \ldots, x_n)$ satisfying the axioms

$$\forall x_1, x_2, \ldots x_n \left(K_n(x_1, x_2, x_3, \ldots, x_n) \wedge \bigwedge_{i \neq j} \neg x_i \approx x_j \rightarrow \right.$$
$$\left. \rightarrow \exists y \left(\bigwedge_{i \leq n} \neg x_i \approx y \wedge K_n(x_1, y, x_3, \ldots, x_n) \right) \right).$$

For $n = 2$ we take the theory T_2 of dense linear order $K_2(x_1, x_2)$ without endpoints, having $\mathrm{ar}(T_2) = 2$, and for $n = 1$ — the theory T_1 of the empty languages, having $\mathrm{ar}(T_1) = 1$.

Similarly to dense linear orders and dense circular orders, dense n-spherical orders produce quantifier eliminations with $\mathrm{ar}(T_n) = n$, $n \in \omega \setminus \{0\}$.

Using Theorem 1 we obtain $m = \mathrm{ar}(T) = \mathrm{aar}(T)$ taking $T_{mm} = \bigsqcup_{r \in \omega} T_m^r$, where T_m^r are copies of T_m in disjoint languages $\{K_m^r\}$, $r \in \omega$. Indeed, disjoint predicates K_m^r producing $\mathrm{ar}(T_m) = m$ witness that $\mathrm{ar}(T_{mm}) = m$. Finitely many these predicates can not define all definable sets for T_{mm} since there are infinitely many of them. Thus, $\mathrm{aar}(T_{mm}) = m$, too.

Now for any $m < n$ we form $T_{mn} = T_n \sqcup \bigsqcup_{r \in \omega} T_m^r$. By T_n in T_{mn} and $m < n$ we have $\mathrm{ar}(T_{mn}) = n$. And $\mathrm{aar}(T_{mn}) = m$ since there are infinitely many disjoint predicates of arity m.

The following theorem shows that all admissible pairs are realized.

Theorem 4. *For any admissible pair (m, r) and $n \in \omega \setminus \{0, 1\}$ there is a theory T with $\mathrm{aar}(T) = n$ and $\deg_{\mathrm{ar}}(T) = (m, r)$.*

Proof. The admissible pair $(0, 0)$ with $\mathrm{ar}(T) = n$ is realized by Theorem 3. Now for an admissible pair $(m, r) \neq (0, 0)$ we can take a disjoint union T of countably many dense n-spherically ordered theories and of m dense r-spherically ordered theories. Using arguments for Theorem 3 we obtain $\mathrm{aar}(T) = n$ and $\deg_{\mathrm{ar}}(T) = (m, r)$. □

Proposition 1. *Any almost n-ary theory T is k-ary for some $k \geq n$.*

Proof. Let T be an almost n-ary theory witnessed by the formulae $\varphi_1(\overline{x}), \ldots, \varphi_m(\overline{x})$. Then taking the set Δ_k of all formulae with $k = \max\{n, l(\overline{x})\}$ we observe, using $\varphi_1(\overline{x}), \ldots, \varphi_m(\overline{x})$, that T is Δ_k-based, i.e., T is k-ary.

Corollary 1. *Any theory T is n-ary for some n iff T is almost m-ary for some m.*

Corollary 2. *Any theory T is n-aritizable for some n iff T is almost m-aritizable for some m.*

3. ω-categorical almost n-ary and n-aritizable theories

Proposition 2. *If T is an almost n-ary ω-categorical theory, for some n, then T is almost k-ary for any $k \in \omega \setminus \{0\}$, i.e.,* $\mathrm{aar}(T) = 1$.

Proof. If $k \geq n$ then T is almost m-ary, as noticed above. If $k < n$ then we collect in a set Z all formulae $\varphi_1(\overline{x}), \dots, \varphi_m(\overline{x})$ witnessing the almost n-arity of T and, by Ryll-Nardzewski Theorem, all non-equivalent formulae $\psi_1(\overline{y}), \dots, \psi_r(\overline{y})$ with $l(\overline{y}) = n$. Clearly, the set Z witnesses that T is almost k-ary. Taking $k = 1$ we obtain $\mathrm{aar}(T) = 1$.

Corollary 3. *For any ω-categorical theory T either* $\mathrm{aar}(T) = 1$ *with* $\mathrm{ar}(T) \in \omega$, *or* $\mathrm{aar}(T) = \infty$ *with* $\mathrm{ar}(T) = \infty$.

The following example illustrates Corollary 3.

Example 2. Taking a dense linear order K_2 without endpoint we can step-by-step extend it to a chain of dense n-spherical orders K_n, $n \geq 2$, in the following way.

We put $(a, b, c) \in K_3$ if $(a, b) \in K_2$ and $(b, c) \in K_2$, or $(b, c) \in K_2$ and $(c, a) \in K_2$, or $(c, a) \in K_2$ and $(a, b) \in K_2$. If K_n, $n \geq 3$, is defined then we put $(a_1, \dots, a_{n+1}) \in K_{n+1}$ if $(a_1, \dots, a_n) \in K_n$ and $(a_2, \dots, a_{n+1}) \in K_n$, or $a_2, \dots, a_{n+1} \in K_n$ and $(a_3, \dots, a_{n+1}, a_1) \in K_n$, or $(a_3, \dots, a_{n+1}, a_1) \in K_n$ and $(a_4, \dots, a_{n+1}, a_1, a_2) \in K_n$. The obtained structure $\mathcal{M}_\infty$ in the language $\Sigma_\infty = \{K_n \mid n \in \omega \setminus \{0, 1\}\}$ has an ω-categorical theory T_∞ with $\mathrm{aar}(T_\infty) = \mathrm{ar}(T_\infty) = \infty$, since each new K_n increases the arity. The same characteristics have restrictions of T_∞ to any infinite sublanguages.

At the same time each restriction $\mathcal{M}$ of $\mathcal{M}_\infty$ to a finite nonempty sublanguage $\{K_{n_1}, \dots, K_{n_m}\}$ produces a theory T with $\mathrm{aar}(T) = 1$ with

$$\mathrm{ar}(T) = \max\{n_1, \dots, n_m\}.$$

By Corollary 3 and the definition of almost aritizability we immediately have:

Corollary 4. *Any restriction T of n-ary ω-categorical theory T' is almost unary-tizable.*

4. Operations for almost n-ary and n-aritizable theories

In this section we consider links for arities of theories with respect to disjoint unions of theories and E-definable compositions of theories.

Theorem 5. 1. *For any theories T_1, T_2 and their disjoint union $T_1 \sqcup T_2$, both T_1 and T_2 are almost n-ary iff $T_1 \sqcup T_2$ is almost n-ary, moreover,* $\mathrm{aar}(T_1 \sqcup T_2) = \max\{\mathrm{aar}(T_1), \mathrm{aar}(T_2)\}$.

2. *For any theories T_1, T_2 and their disjoint union $T_1 \sqcup T_2$, both T_1 and T_2 are almost n-aritizable iff $T_1 \sqcup T_2$ is almost is n-aritizable.*

Proof. 1. Let Φ_1 and Φ_2 be finite sets of formulas witnessing that T_1 and T_2 are almost n-ary, respectively. Using the definition of disjoint union and Theorem 1 we obtain that the finite set $\Phi_1 \cup \Phi_2$ witnesses that $T_1 \sqcup T_2$ is almost n-ary. Conversely, if a finite set Φ of formulas witnesses that $T_1 \sqcup T_2$ is almost n-ary then Phi witnesses that both T_1 and T_2 are almost n-ary.

The equality $\mathrm{aar}(T_1 \sqcup T_2) = \max\{\mathrm{aar}(T_1), \mathrm{aar}(T_2)\}$ follows from the definition of disjoint union since if $\mathrm{aar}(T_1 \sqcup T_2) = k$ then the maximal value of $\mathrm{aar}(T_1)$ and $\mathrm{aar}(T_2)$ is responsible for this equality.

Item 2 follows from Item 1 since expansions of T_1 and T_2 correspond expansions of $T_1 \sqcup T_2$: some expansions of T'_1 and T'_2 of T_1 and T_2, respectively, are almost n-ary iff $T'_1 \sqcup T'_2$ produces an almost n-ary expansion of $T_1 \sqcup T_2$.

Using induction we obtain the following:

Corollary 5. 1. *For any theories $T_1, T_2, \ldots, T_m$ and their disjoint union $\bigsqcup\limits_{i=1}^{m} T_i$, all $T_1, T_2, \ldots, T_m$ are almost n-ary iff $\bigsqcup\limits_{i=1}^{m} T_i$ is almost n-ary, moreover,*

$$\mathrm{aar}\left(\bigsqcup_{i=1}^{m} T_i\right) = \max\{\mathrm{aar}(T_i) \mid i \leq m\}.$$

2. *For any theories $T_1, T_2, \ldots, T_m$ and their disjoint union $\bigsqcup\limits_{i=1}^{m} T_i$, all $T_1, T_2, \ldots, T_m$ are almost n-aritizable iff $\bigsqcup\limits_{i=1}^{m} T_i$ is almost n-aritizable.*

Remark 1. Both almost n-arity and almost n-aritizability can fail taking disjoint unions of infinitely many theories T_i, $i \in I$. Indeed, each theory T_i can have its own finite set Φ_i of formulas witnessing the almost n-arity/n-aritizability, say in disjoint languages, whereas finite unions $\bigcup \Phi_i$ can not witness the almost n-arity/n-aritizability for $\bigsqcup\limits_{i \in I} T_i$.

Generalizing Theorem 2 we obtain:

Theorem 6. 1. *For any theories T_1 and T_2 and their E-definable composition $T_1[T_2]$, T_1 and T_2 are almost n-ary iff $T_1[T_2]$ is almost n-ary, moreover,* $\mathrm{aar}(T_1[T_2]) = \max\{\mathrm{aar}(T_1), \mathrm{aar}(T_2), 2\}$, *if models of T_1 and of T_2 have at least two elements, and* $\mathrm{aar}(T_1[T_2]) = \max\{\mathrm{aar}(T_1), \mathrm{aar}(T_2)\}$, *if a model of T_1 or T_2 is a singleton.*

2. *For any theories T_1 and T_2 and their E-definable composition $T_1[T_2]$, T_1 and T_2 are almost n-aritizable iff $T_1[T_2]$ is almost n-aritizable.*

Proof. 1. Let T_i be Δ_i-based for $i = 1, 2$. Since $T_1[T_2]$ is E-definable it is Δ-based, where Δ consists of formulae in $\Delta_1 \cup \Delta_2$ and $E(x, y)$ [7]. Now assuming that T_1 and T_2 are almost n-ary we can choose Δ_i consisting of n-formulae and finitely many formulae forming Φ_i, $i = 1, 2$. Hence $T_1[T_2]$ is almost n-ary.

Conversely, if $T_1[T_2]$ is almost n-ary and it is witnessed by a set Φ of formulae then Φ witnesses that T_1 and T_2 are almost n-ary.

If models of T_1 and of T_2 have at least two elements then $T_1[T_2]$ is at least binary that witnessed by the formula $E(x, y)$. Thus since $T_1[T_2]$ is Δ-based we have $\mathrm{aar}(T_1[T_2]) = \max\{\mathrm{aar}(T_1), \mathrm{aar}(T_2), 2\}$. If T_1 or T_2 is a theory of singleton then $T_1[T_2]$ is $(\Delta_1 \cup \Delta_2)$-based implying $\mathrm{aar}(T_1[T_2]) = \max\{\mathrm{aar}(T_1), \mathrm{aar}(T_2)\}$.

Item 2 follows from Item 1 repeating the arguments for Item 2 of Theorem 5.

Theorem 6 immediately implies

Corollary 6. *Any composition of finitely many almost n-ary (almost n-aritizable) theories, for $n \geq 2$, is again many almost n-ary (almost n-aritizable).*

5. Conclusion

We considered possibilities for almost arities and almost aritizabilities of theories and their dynamics both in general case, for ω-categorical theories, and with respect to operations for theories. It would be interesting to describe values of almost arities and almost aritizabilities for natural classes of theories.

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Sergey Vladimirovich Sudoplatov
Sobolev Institute of Mathematics
Academician Koptyug avenue, 4
630090, Novosibirsk, Russia.
Email address: sudoplat@math.nsc.ru

Novosibirsk State Technical University
K. Marx avenue, 20
630073, Novosibirsk, Russia.
Email address: sudoplatov@corp.nstu.ru