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LAGRANGE SPACES WITH CHANGED Z. SHEN SQUARE METRIC

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Abstract. The purpose of present paper to study Lagrange space due to changed Z. Shen square metric $L^* = \frac{(L+\beta)^2}{L}$ and obtained fundamental tensor fields for these space. Further, we studied about the variational problem with fixed endpoints for the Lagrange spaces due to above change.

Keywords: Lagrange space, Z. Shen square metric, Euler-Lagrange equation.

1. Introduction

The notion of (α, β)-metric was introduced by Matsumoto [7] as generalization of Rander's metric $L = \alpha+\beta$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ was a regular Remannian metric and $\beta = b_i(x)y^i$ one-form metric. Rather than Randers metric there are several important (α, β)-metric such as Kropina metric $L = \frac{\alpha^2}{\beta}$, Matsumoto metric $L = \frac{\alpha^2}{\alpha-\beta}$, generalized Kropina metric $L = \frac{\alpha^{n+1}}{\beta^n}$, Z. Shen square metric $L = \frac{(\alpha+\beta)^2}{\alpha}$ etc. Z. Shen square metric [13, 14] was also very interesting because it was constructed from the Berwald metric by using suitable α and β, and it was projectively flat on unit ball with constant flag curvature.

Matsumoto [6] also introduced the transformations of Finsler metric which was given by,

$$L^{'}(x,y) = L(x,y) + \beta(x,y)$$
$$L^{''2}(x,y) = L^2(x,y) + \beta^2(x,y)$$

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where, $\beta = b_i(x)y^i$, $b_i(x)$ are components of covariant vector which was a function of position alone. Further he had obtained the relationship between the imbedding class numbers of $(M^n, L^{'})$, $(M^n, L^{''})$ and (M^n, L). Generalizing above transformations Shibata [3] had studied the properties of Finsler space (M^n, L^*) whose fundamental metric function $L^*(x, y)$ was obtained from L by

$$L^*(x, y) = f(L, \beta)$$

where, f was positively homogeneous function of degree one in L and β.

Now the geometry of a Lagrange space over a real, finite-dimensional manifold M had been introduced and studied as a sub-geometry of the geometry of the tangent bundle TM by R Miron [10]. This geometry was developed together with his collaborators in [8], [9], [10]. Compared to Finsler geometry, when the assumption of homogeneity was relaxed then a new geometry arieses which was known as Lagrange geometry, i.e. the Finsler geometry is a particular case of Lagrange geometry where fundamental function is homogeneous.

Now we state some examples of **Lagrange spaces which are reducible to Finsler spaces**:

Example 1. Every Riemannian space $(M, g_{ij}(x))$ determines a Finsler space $F^n = (M, F(x, y))$ and consequently a Lagrange space $L^n = (M, F^2(x, y))$, where

$$F(x, y) = \sqrt{g_{ij}(x)y^i y^j}.$$

The fundamental tensor of this Finsler space coincides to the metric tensor $g_{ij}(x)$ of the Riemannian manifold $(M, g_{ij}(x))$.

Example 2. Let us consider the function

$$\sqrt[4]{(y^1)^4 + (y^2)^4 + ... + (y^n)^4},$$

defined in a preferential local system of coordinates on $\widetilde{TM}$. The pair $F^n = (M, F(x, y))$, with F defined in above is a Finsler space. The fundamental tensor field g_{ij} can be easily calculated. This was the first example of Finsler space which was given form the lecturer of Riemann in 1854.

Now we give example of **Lagrange spaces, which are not reducible to Finsler spaces**

Example 3 The following Lagrangian from electrodynamics

$$L(x, y) = mc\gamma_{ij}(x)y^i y^j + \frac{2e}{m}A_i(x)y^i + U(x),$$

where $\gamma_{ij}(x)$ is a pseudo-Riemannian metric, $A_i(x)$ a covector field and U(x) a smooth function, *m, c, e* are the well-known constants from physics, determine a Lagrange space L^n.

Example 4 Consider the Lagrangian function

$$L(x, y) = F^2(x, y) + A_i(x)y^i + U(x),$$

where $F(x, y)$ is the fundamental function of a Finsler space, $A_i(x)$ are the components of a covector field and $U(x)$ a smooth function gives rise to a remarkable Lagrange space, called the Almost Finsler-Lagrange space (shortly AFL-space).

Geometric problems derived from the variational problem of a Lagrangian which was studied by J. Kern [4] in details. He said that the variational problem can be formulated for differentiable Lagrangians and can be solved in case when we consider the parameterized curves, even if the integral of action depends on the parameterization of considered the curve.

In the year 2001, B. Nicolaescu [2] studied the lagranges spaces with (α, β)-metric and variational problem with fixed endpoints in the year 2004 [1]. Further in 2011, Pandey and Chaubey [11] considered these problem for the (γ, β)-metric, where $\gamma^3 = a_{ijk}(x)y^i y^j y^k$ was a cubic metric and $\beta = b_i(x)y^i$ a one form metric on TM. In the present paper we transform the Z. Shen square metric as $L = \frac{(L+\beta)^2}{L}$ and studied Lagrange space due to this transformation. The above generalization is very interesting because it enhance our understanding and geometric meaning of non-Riemanian quantities. Further we obtained fundamental tensor fields for these space and also studied about the variational problem with fixed endpoints of Lagrange spaces due to this change.

2. **Lagrange metrics**

In this section we give the definitions of a regular, differentiable Lagrangian over the tangent manifolds TM and $\widehat{TM}$, where M is a differentiable, real manifold of dimension n. Let (TM, τ, M) be the tangent bundle of a C^∞-differentiable real n-dimensional manifold M. If (U, ϕ) is a local chart on M, then the coordinates of a point $u = (x, y) \in \tau^{-1}(U) \subset TM$ will be denoted by (x, y). R. Miron [10] given following defintions:

Definition 1. *A differentiable Lagrangian on TM is a mapping* $L : (x, y) \in TM \longrightarrow L(x, y) \in R, \forall u = (x, y) \in TM$, *which is of class* C^∞ *on* $\widehat{TM} = TM \setminus (0)$ *and is continuous on the null section of the projection* $\tau : TM \longrightarrow M$, *such that*

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L(x, y)}{\partial y^i \partial y^j}, \tag{1}$$

is a (0, 2)-type symmetric d-tensor field on TM.

Definition 2. *A differential Lagrangian L on TM is said to be regular if*

$$rank\|g_{ij}(x, y)\| = n, \qquad \forall (x, y) \in \widehat{TM}.$$

For the Lagrange space $L^n = (M, L(x, y))$ we say that $L(x, y)$ is the fundamental function and $g_{ij}(x, y)$ is the fundamental (or metric) tensor. We will denote by g^{ij} the inverse matrix of g_{ij} . This means that

$$g^{ik} g_{jk} = \delta^i_j.$$

Now the definition of a Lagrange space was given by

Definition 3. *A Lagrange space is a pair* $L^n = (M, L)$ *formed by a smooth, real n-dimensional manifold M and a regular differentiable Lagrangian L on M, for which the d-tensor field* g_{ij} *from (1) has constant signature on* $\widehat{TM}$.

Now, let $L : TM \longrightarrow R$ be a differentiable Lagrangian on the manifold M, which was not necessarily regular. A curve $c : t \in [0, 1] \longrightarrow (x^i(t)) \in U \subset M$

having the image in a domain of a chart U of M, has the extension to $\widehat{TM}$ given by $c^* : t \in [0,1] \longrightarrow (x^i(t), \frac{dx^i(t)}{dt}) \in \tau^{-1}(U)$. The integral of action of the Lagrangian L on the curve c is given by the functional

$$I(c) = \int_0^1 L(x(t), \frac{dx}{dt})dt. \tag{2}$$

Consider the curve $c_\epsilon : t \in [0,1] \longrightarrow (x^i(t) + \epsilon v^i(t)) \in M$, which have the same endpoints $x^i(0), x^i(1)$ as the curve c, $v^i(0) = v^i(1) = 0$ and ϵ is a real number, sufficiently small in absolute value, such that $Imc_\epsilon \in U$. The extension of the curve c_ϵ to TM is

$$c^*_\epsilon : t \in [0,1] \longrightarrow (x^i(t) + \epsilon v^i(t), \frac{dx^i}{dt} + \epsilon \frac{dv^i}{dt}) \in \tau^{-1}(U)$$

The integral of action of the Lagrangian L on the curve c_ϵ is,

$$I(c_\epsilon) = \int_0^1 L(x + \epsilon v, \frac{dx}{dt} + \epsilon \frac{dv}{dt})dt.$$

A necessary condition for I(c) to be an extremal value $I(c_\epsilon)$ is

$$\frac{dI(c_\epsilon)}{d\epsilon}|_{\epsilon=0} = 0.$$

In order that the functional $I(c)$ be an extremal value of $I(c_\epsilon)$ it is necessary that c be the solution of the Euler-Lagrange equations,

$$E_i(L) = \frac{\partial L}{\partial x^i} - \frac{d}{dt}(\frac{\partial L}{\partial y^i}) = 0, \qquad y^i = \frac{dx^i}{dt}.$$

3. The fundamental tensor of a Lagrange space with changed Z. Shen square metric

In General we know that the component b_i is the electromagnetic potential of L^n and the tensor $F_{ij} = \partial_j b_i - \partial_i b_j$ is the electromagnetic tensor field in Lagrange spaces. Now we define the changed Z. Shen square metric as follows

Definition 4. *A Lagrange space $L^{*n} = (M, L^*(x,y))$ is known as changed Z. Shen square metric if the fundamental function $L^*(x,y)$ is a function $\overline{L}$, which depends only on $L(x,y)$ and $\beta(x,y)$,*

$$L^* = \overline{L}\{L(x,y), \beta(x,y)\} = \frac{(L+\beta)^2}{L}.$$

Here, we shall use the following notations throughout the whole paper,

$$\dot\partial_i L = \frac{\partial L}{\partial y^i}, \quad \dot\partial_i \beta = \frac{\partial \beta}{\partial y^i}, \quad \dot\partial_i \dot\partial_j L = \frac{\partial^2 L}{\partial y^i \partial y^j}, \quad \overline{L}_L = \frac{(L^2 - \beta^2)}{L^2}, \quad \overline{L}_\beta = \frac{(L+\beta)}{L},$$
$$\overline{L}_{LL} = \frac{2\beta^2}{L^3}, \quad \overline{L}_{L\beta} = \frac{-2\beta}{L^2}, \quad \overline{L}_{\beta\beta} = \frac{2}{L}.$$

Now we have

Proposition 1. *For the Lagrange space L^n, following relations hold good:*

$$\dot\partial_i L = L^{-1} y_i, \quad \dot\partial_i \dot\partial_j L = 2\dot\partial_j y_i - L^{-3} y_i y_j, \quad \dot\partial_i \beta = b_i(x), \quad \dot\partial_i \dot\partial_j \beta = 0, \tag{3}$$

where $y_i = g_{ij} y^j$.

Now, we introduce the moments of the Lagrangian $L^*(x,y) = \overline{L}(L, \beta(x,y)) = \frac{(L+\beta)^2}{L}$,

$$p_i = \frac{1}{2}\frac{\partial L^*}{\partial y^i} = \frac{(L+\beta)}{2L^2}\{(L-\beta)\dot\partial_i L + 2L\dot\partial_i \beta)\}.$$

Thus we have

Proposition 2. *The moments of the Lagrangian $L^*(x,y)$ with changed Z. Shen square metric is given by*

$$p_i = \rho y_i + \rho_1 b_i, \tag{4}$$

where $\rho = \frac{1}{2}\frac{(L^2-\beta^2)}{L^3}$, *and* $\rho_1 = \frac{(L+\beta)}{L}$.

The two scalar functions defined in (4) are called the principal invariants of the Lagrange space L^{*n}.

Proposition 3. *The derivatives of principal invariants of the Lagrange space L^{*n} due to changed Z. Shen square metric are given by*

$$\dot{\partial}_i \rho = \rho_{-2} y_i + \rho_{-1} b_i, \qquad \dot{\partial}_i \rho_1 = \rho_{-1} y_i + \rho_0 b_i, \tag{5}$$

where, $\rho_{-2} = \frac{1}{2}L^{-5}(3\beta^2 - L^2)$, $\rho_{-1} = -\frac{\beta}{L^3}$, $\rho_0 = \frac{1}{L}$.

Now, the Energy of a Lagrangian is given by

$$E_{L^*} = y^i \frac{\partial L^*}{\partial y^i} - L^*.$$

Thus we have

Proposition 4. *The Energy of a Lagrangian L^* with changed Z. Shen square metric is given by*

$$E_{L^*} = \frac{(L^2-\beta^2)(1-L^2)}{L^2}. \tag{6}$$

Now we can determine the fundamental tensor g_{ij} of the Lagrange space with changed Z. Shen square metric, as follows:

Proposition 5. *The fundamental tensor g^*_{ij} of the Lagrange space L^{*n} with changed Z. Shen square metric is given as*

$$g^*_{ij} = \frac{(L^2-\beta^2)}{L^3} g_{ij} + L^{-1} b_i b_j + \frac{\beta}{L^3}(b_i y_j + b_j y_i) + \frac{(3\beta^2 - L^2)}{2L^5} y_i y_j. \tag{7}$$

The above equation can be rewritten as

$$g^*_{ij} = \frac{(L^2-\beta^2)}{L^3} g_{ij} + c_i c_j,$$

where $c_i = \sqrt{\frac{(3\beta^2-L^2)}{2L^5}}\, y_i + \frac{1}{\sqrt{L}} b_i$ and $g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j}$.

Proposition 6. *The reciprocal tensor g^{*ij} of the fundamental tensor g^*_{ij} in L^{*n} is given by*

$$g^{*ij} = \frac{L^3}{(L^2-\beta^2)} g^{ij} - \frac{1}{(1+c^2)} c^i c^j, \tag{8}$$

where $c^i = \frac{L^3}{(L^2-\beta^2)} a^{ij} c_j$ *and* $c^i c_i = c^2$ *and g^{ij} is reciprocal of the g_{ij}.*

4. Eular-Lagrange equations in Lagrange spaces with changed Z. Shen square metric

The Eular-Lagrange equations of the Lagrange spaces with changed Z. Shen square metric are,

$$E_i(\overline{L}) = \frac{\partial \overline{L}}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial \overline{L}}{\partial y^i}\right) = 0, \qquad y^i = \frac{dx^i}{dt}$$

considering the relations

$$\frac{\partial \overline{L}}{\partial x^i} = \frac{(L+\beta)}{L}\left\{\frac{(L-\beta)}{L}\frac{\partial L}{\partial x^i} + 2\frac{\partial \beta}{\partial x^i}\right\}, \qquad \frac{\partial \overline{L}}{\partial y^i} = \frac{(L+\beta)}{L}\left\{\frac{(L-\beta)}{L}\frac{\partial L}{\partial y^i} + 2\frac{\partial \beta}{\partial y^i}\right\},$$

$$\frac{d}{dt}(\frac{\partial\overline{L}}{\partial y^i}) = \frac{d}{dt}\{\frac{(L^2-\beta^2)}{L^2}\}\frac{\partial L}{\partial y^i} + \frac{d}{dt}\{\frac{2(L+\beta)}{L}\}\frac{\partial\beta}{\partial y^i} + \frac{(L^2-\beta^2)}{L^2}\frac{d}{dt}(\frac{\partial L}{\partial y^i}) + \frac{2(L+\beta)}{L}\frac{d}{dt}(\frac{\partial\beta}{\partial y^i}).$$

By direct calculation, we have

$$E_i(\overline{L}) = \frac{(L^2-\beta^2)}{L^2}E_i(L) + \frac{2(L+\beta)}{L}E_i(\beta) - \frac{\partial L}{\partial y^i}\frac{d}{dt}\{\frac{(L^2-\beta^2)}{L^2}\} - \frac{\partial\beta}{\partial y^i}\frac{d}{dt}\{\frac{2(L+\beta^2)}{L}\},$$
$$y^i = \frac{dx^i}{dt},$$

its give

$$E_i(\overline{L}) = \frac{(L^2-\beta^2)}{L^2}E_i(L) + \frac{2(L+\beta)}{L}E_i(\beta) - \frac{\partial L}{\partial y^i}\frac{d}{dt}(\frac{2\beta^2}{L^3}\frac{dL}{dt} - \frac{2\beta}{L^2}\frac{d\beta}{dt}) + \frac{\partial\beta}{\partial y^i}(\frac{2\beta}{L^2}\frac{dL}{dt} - \frac{2}{\beta}\frac{d\beta}{dt}).$$

As well have

$$E_i(\beta) = F_{ir}\frac{dx^r}{dt},$$

where

$$F_{ir} = \frac{\partial A_r}{\partial x^i} - \frac{\partial A_i}{\partial x^r},$$

is the electromagnetic tensor field. Finally we have the following relation

$$E_i(\overline{L}) = \frac{(L^2-\beta^2)}{L^2}E_i(L) + 2\frac{(L+\beta)}{L}F_{ir}\frac{dx^r}{dt} - \frac{\partial L}{\partial y^i}\frac{d}{dt}(\frac{2\beta^2}{L^3}\frac{dL}{dt} - \frac{2\beta}{L^2}\frac{d\beta}{dt}) + \frac{\partial\beta}{\partial y^i}(\frac{2\beta}{L^2}\frac{dL}{dt} - \frac{2}{\beta}\frac{d\beta}{dt}).$$

Proposition 7. *The Euler-Lagrange equation in the Lagrange space L^{*n} with changed Z. Shen square metric L^* are,*

$$E_i(\overline{L}) = E_i\{\frac{(L+\beta)^2}{L}\} = 0, \qquad y^i = \frac{dx^i}{dt}. \tag{9}$$

For every smooth curve c on the base manifold M, the energy function of the Lagrangian $L^*(x,y)$ can be written as

$$\frac{dE_{L^*}}{dt} = -[\frac{\partial\overline{L}}{\partial x^i} - \frac{d}{dt}(\frac{\partial\overline{L}}{\partial y^i})]y^i = 0, \qquad where \quad y^i = \frac{dx^i}{dt},$$
$$\text{or} \qquad \frac{dE_{L^*}}{dt} = -E_i(\overline{L})\frac{dx^i}{dt}.$$

Thus using proposition (4.1) we have

Theorem 1. *In a differentiable Lagrangian $L^*(x,y)$, the energy function E_{L^*} is conserved along the solution curves c of the Euler-Lagrange equations for changed Z. Shen square metric are $E_i(\overline{L}) = E_i\{\frac{(L+\beta)^2}{L}\} = 0$, $y^i = \frac{dx^i}{dt}$.*

If we have the natural parametrization of the curve $\in [0,1] \longrightarrow (x^i(t) \in M)$, then $L(x, \frac{dx}{dt}) = 1$. Then we get

Proposition 8. *In the canonical parametrization the Euler-Lagrange equations for changed Z. Shen square metric in Lagrange space L^{*n} are*

$$E_i(\overline{L}) = \frac{(L^2-\beta^2)}{L^2}E_i(L) + 2\frac{(L+\beta)}{L}F_{ir}\frac{dx^r}{dt} + \frac{\partial\beta}{\partial y^i}(\frac{2\beta}{L^2}\frac{dL}{dt} - \frac{2}{\beta}\frac{d\beta}{dt}). \tag{10}$$

Proposition 9. *If the 1-form β is constant on the integral curve c of the Euler-Lagrange equations for changed Z. Shen square metric, then (10) rewrite as the Lorentz equations of the space L^{*n}*

$$E_i(\overline{L}) = \frac{(L^2-\beta^2)}{L^2}E_i(L) + 2\frac{(L+\beta)}{L}F_{ir}\frac{dx^r}{dt}. \tag{11}$$

5. Conclution

In this paper, we have continued the investigations on the new introduced changed Z. Shen square metric which is defined as $L^* = \frac{(L+\beta)^2}{L}$ and we succeed to investigate the dually locally flatness and the Cartan tensor for this type of metrics. The above generalization is very interesting because it enhance our understanding and geometric meaning of non-Riemanian quantities. Further, we obtained fundamental tensor fields for these spaces and the variational problem with fixed endpoints for the Lagrange spaces.

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