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THE VOLUME OF A TRIRECTANGULAR HYPERBOLIC TETRAHEDRON

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ABSTRACT. We consider a three-parameter family of tetrahedra in the hyperbolic space, which three edges at one vertex are pairwise orthogonal. It is convenient to determine such tetrahedra by the lengths of these edges. We obtain relatively simple formulas for them expressing the volume and the surface area. This allows us to find normalized volume and investigate its asymptotics.

Keywords: hyperbolic volume, normalized volume, Poincaré upper half-space model, hyperbolic tetrahedron, trirectangular tetrahedron, infinite cone.

1. Introduction

Finding the volume of a hyperbolic polyhedron is a very old and difficult problem. The volume of a biorthogonal hyperbolic tetrahedron (so called *orthoscheme*) was found by N. Lobachevsky [12] and J. Bolyai [17] indepentently. In 1907, G. Sforza [16] proposed a formula expressing the volume of an arbitrary hyperbolic tetrahedron in terms of dihedral angles. Before presenting it, we introduce some definitions.

A *hyperbolic tetrahedron* T is a convex hull of four points in the hyperbolic space $\mathbb{H}^3$. These points are called *vertices* of T. Let us denote them by numbers $1, 2, 3$ and 4 (see Fig.1). Then denote by ℓ_{ij} the length of the edge connecting i-th and j-th vertices. We put θ_{ij} for the dihedral angle along the corresponding edge.

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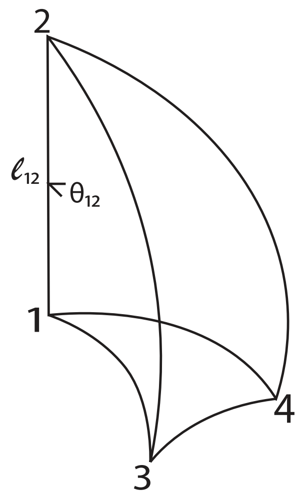

FIG. 1. Hyperbolic tetrahedron T

A *Gram matrix* $G(T)$ of tetrahedron T is defined as

$$G(T) = \begin{pmatrix} 1 & -\cos\theta_{12} & -\cos\theta_{13} & -\cos\theta_{14} \\ -\cos\theta_{12} & 1 & -\cos\theta_{23} & -\cos\theta_{24} \\ -\cos\theta_{13} & -\cos\theta_{23} & 1 & -\cos\theta_{34} \\ -\cos\theta_{14} & -\cos\theta_{24} & -\cos\theta_{34} & 1 \end{pmatrix}.$$

An *Edge matrix* $E(T)$ is formed by hyperbolic cosines of the edge lengths and defined as follows

$$E(T) = \begin{pmatrix} 1 & \cosh\ell_{12} & \cosh\ell_{13} & \cosh\ell_{14} \\ \cosh\ell_{12} & 1 & \cosh\ell_{23} & \cosh\ell_{24} \\ \cosh\ell_{13} & \cosh\ell_{23} & 1 & \cosh\ell_{34} \\ \cosh\ell_{14} & \cosh\ell_{24} & \cosh\ell_{34} & 1 \end{pmatrix}.$$

It is known that a hyperbolic tetrahedron T can be uniquely determined up to isometry either by the Gram matrix $G(T)$ or the edge matrix $E(T)$ (see, e.g., [5]).

Theorem 1 (G. Sforza, 1907)**.** *Let T be a compact hyperbolic tetrahedron given by its Gram matrix G. We assume that all the dihedral angles are fixed exept θ_{34} which is formal variable. Then the volume $V = V(T)$ is given by the formula*

$$V = \frac{1}{4}\int\limits_{t_0}^{\theta_{34}} \log\frac{c_{34}(t) - \sqrt{-\det G(t)}\sin t}{c_{34}(t) + \sqrt{-\det G(t)}\sin t}dt,$$

where t_0 is a suitable root of the equation $\det G(t) = 0$, c_{34} is $(3,4)$-cofactor of the matrix G, and $c_{34}(t), \det G(t)$ are functions in one variable θ_{34} denoted by t.

A new proof of the classical Sforza's formula and its version for the spherical tetrahedron can be found in the paper by the first author and A. Mednykh [3]. An analog of the Sforza's formula for the volume of an arbitrary compact hyperbolic tetrahedron but in terms of edge lengths instead of dihedral angles was given in the recent paper by the first author and B. Vuong [4].

There are also three more known formulas expressing the volume of an arbitrary hyperbolic tetrahedron in terms of dihedral angles. In 1999 solution of the problem was introduced by Yu. Cho and H. Kim [8]. Then another formula was given by J. Murakami and M. Yano [14]. A. Ushijima [18] found a geometrical proof for this result which also covers the case of truncated tetrahedron. In 2005 D. Derevnin and A. Mednykh [9] proposed a closed integral formula.

Known formulas for the volume of a general hyperbolic tetrahedron are rather complicated and not so convenient for applications including volume calculations for more complex polyhedra in $\mathbb{H}^3$. The goal of the present paper is to provide with a new comparatively simple formula for sufficiently large subfamily of hyperbolic tetrahedra.

A *trirectangular tetrahedron* is a tetrahedron where all three face angles at one vertex are right angles. That vertex is called the *right angle vertex* of the trirectangular tetrahedron and the face opposite it is called the *base*. The three edges that meet at the right angle are called the *legs* of a trirectangular tetrahedron.

Corresponding 3-parameter family of hyperbolic tetrahedra was considered in the PhD thesis by G. Baigonakova [6], where trigonometrical relations between the dihedral angles and edge lengths were established.

If we consider reflections in three pairwice orthogonal faces of a trirectangular tetrahedron T then we obtain a hyperbolic octahedron with so called mmm-symmetry. Such an octahedron was investigated in the paper by the first author and G. Baigonakova [1], where the volume was find in terms of its dihedral angles.

A *Coxeter polyhedron* is a polyhedron with all dihedral angles of the form π/n. F. Lannér [11] proved that there are exactly 9 compact Coxeter tetrahedra in $\mathbb{H}^3$. One of them $T(\theta_{12}, \theta_{13}, \theta_{23}, \theta_{34}, \theta_{24}, \theta_{14}) = T(\pi/2, \pi/2, \pi/3, \pi/3, \pi/5, \pi/2)$ is trirectangular tetrahedron. The remaining 8 Coxeter tetrahedra are ortoschemes. Both ortoschemes and trirectangular tetrahedra, not necessarily Coxeter ones, can be used for triangulations of more complex polyhedra in $\mathbb{H}^3$.

In the present paper we consider a trirectangular tetrahedron in the hyperbolic spase given by the lengths of its legs l_1, l_2, l_3. We obtain relatively simple formulas for it expressing the volume and the surface area. This allows us to find normalized volume and investigate its asymptotics.

Most of known results for the volume of a hyperbolic tetrahedron are based on the classical Schläfli differential equation (see, e.g., [5], Ch. 7, Sect. 2.2).

$$-dV = \frac{1}{2} \sum_{ij} \ell_{ij} \, d\theta_{ij},$$

where the sum is taken over all edges.

In the present work instead of using Schläfli equation, we integrate the volume element in $\mathbb{H}^3$ and use Fubini's theorem.

2. Volume formula

Consider a Poincaré model of $\mathbb{H}^3$. That is the upper half-space $\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3 \,|\, z > 0\}$ endowed with the metric

$$ds^2 = \frac{dx^2 + dy^2 + dz^2}{z^2}.$$

The volume element in this model has a form (see [5], Ch. 7, Sect. 2.3)

$$dV = \frac{dx\,dy\,dz}{z^3}. \tag{1}$$

Let $f(x,y)$ be a real function defined on domain $D \subset \mathbb{R}^2$. An *infinite cone* C over the graph of function f is the set formed by vertical lines starting on this graph

$$C = \left\{(x,y,z) \in \mathbb{R}^3_+ \mid z \geq f(x,y),\, (x,y) \in D\right\}.$$

Using (1) and Fubini's theorem we get the volume of infinite cone C over the graph of function $f(x,y)$

$$V(C) = \iiint\limits_C \frac{dx\,dy\,dz}{z^3} = \iint\limits_D \int\limits_{f(x,y)}^{\infty} \frac{dx\,dy\,dz}{z^3} = \frac{1}{2} \iint\limits_D \frac{dx\,dy}{(f(x,y))^2}. \tag{2}$$

Consider a trirectangular tetrahedron $T = T(\ell_1, \ell_2, \ell_3)$ given by the lengths of its legs (i.e. pairwise ortogonal edges). Let us enumerate the vertices $1, \ldots, 4$ in such a way that $\ell_{13} = \ell_1$, $\ell_{14} = \ell_2$, $\ell_{12} = \ell_3$. There exist an isometry of $\mathbb{H}^3$ which maps vertex 1 to point $(0,\,0,\,1) \in \mathbb{R}^3_+$, guides ℓ_3 along the axis $0z$, and put vertices 3 and 4 in coordinate planes $0xz, 0yz$ correspondingly (Fig. 2). We refer this configuration of tetrahedron T in $\mathbb{R}^3_+$ as *standard position.* Further we assume without loss of generality that T is in this position.

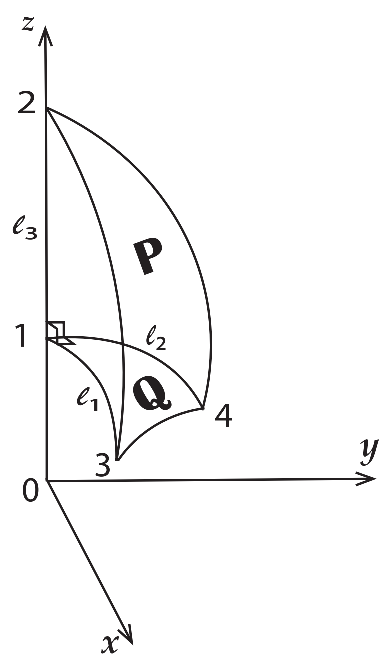

FIG. 2. Hyperbolic tetrahedron T in standart position

Theorem 2. *Let $T = T(\ell_1, \ell_2, \ell_3)$ be a compact hyperbolic tetrahedron defined by lengths of pairwise ortogonal edges. Then the volume $V = V(T)$ is given by the* *formula*

$$V = \frac{1}{2} \int\limits_0^{\tanh \ell_1} \int\limits_0^{\frac{\tanh \ell_2 (\tanh \ell_1 - x)}{\tanh \ell_1}} \left[\frac{1}{1 - x^2 - y^2} + \frac{1}{x(x - 2x_0) + y(y - 2y_0) - e^{2\ell_3}} \right] dx\, dy,$$

where $x_0 = \dfrac{1 - e^{2\,\ell_3}}{2 \tanh \ell_1}$ *and* $y_0 = \dfrac{1 - e^{2\,\ell_3}}{2 \tanh \ell_2}$.

Proof. Let Q and P be the lower and upper bases of T. Denote by V_Q and V_P the volumes of infinite cones over them. Then the required volume $V = V_Q - V_P$. We use formula (2) in order to calculate V_Q and V_P. To do this we define the functions $f_Q(x, y)$ and $f_P(x, y)$ which determine hyperplanes Q, P then find the limits of integration.

Face Q is the part of Euclidean unit sphere $x^2 + y^2 + z^2 = 1$, since it is spanned by the edges ℓ_1 and ℓ_2 orthogonal to Oz. Then

$$f_Q^2(x, y) = 1 - x^2 - y^2.$$

To find $f_P(x, y)$ we define the coordinates of vertices $2, 3$ and 4.

Let the vertex 2 has coordinates $(x_2,\, y_2,\, z_2)$. Since it lies on axis Oz we have $x_2 = y_2 = 0$. In addition, we know that the distance between 1 and 2 is $\rho(1, 2) = \ell_3$. On the other hand, (see [10], Ch. III, Sect. III.4)

$$\rho(1, 2) = \int\limits_1^{z_2} \frac{dz}{z} = \ln z_2.$$

Therefore, $z_2 = e^{\ell_3}$ and 2 has coordinates $(0,\, 0,\, e^{\ell_3})$.

Let φ_1 be the angle between axis $0z$ and radius-vector of vertex 3 (Fig. 3). This vertex lies on the edge ℓ_1 which is the part of unit semi-circle. So 3 has coordinates $(\sin \varphi_1,\, 0,\, \cos \varphi_1)$. Hyperbolic length of the arc connecting vertices 1 and 3 is ℓ_1.

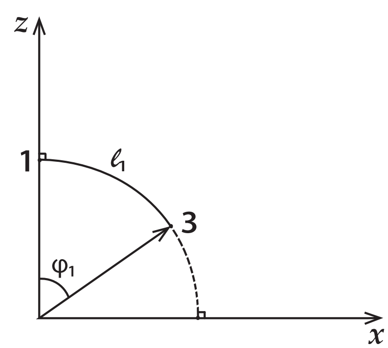

FIG. 3. Angle φ_1 between axis $0z$ and radius-vector of vertex 4

At the same time, ℓ_1 and φ_1 are related as follows ([10], Ch. III, Sect. III.4)

$$\cosh \ell_1 = \frac{1}{\cos \varphi_1},$$

where we can also get $\tanh \ell_1 = \sin \varphi_1$. Thereby, vertex 3 has coordinates $\left(\tanh \ell_1,\, 0,\, \dfrac{1}{\cosh \ell_1} \right)$.

For the 4-th vertex we can get coordinates $\left(0,\, \tanh \ell_2,\, \dfrac{1}{\cosh \ell_2}\right)$ in the same way.

Let hyperbolic plane (i.e. Euclidean semi-sphere) that contains P has a center $(x_0,\, y_0,\, z_0)$. Take into consideration that $z_0 = 0$ as a center of any hyperbolic plane lies on the absolute. In general we have

$$(x - x_0)^2 + (y - y_0)^2 + z^2 = R^2.$$

Substituting the coordinates of vertices 2, 3, 4 in this equation we get the system of three equations with variables $x_0,\, y_0,\, R$

$$\begin{cases} (\tanh \ell_1 - x_0)^2 + y_0^2 + \left(\dfrac{1}{\cosh \ell_1}\right)^2 = R^2, \\ x_0^2 + (\tanh \ell_2 - y_0)^2 + \left(\dfrac{1}{\cosh \ell_2}\right)^2 = R^2, \\ x_0^2 + y_0^2 + e^{2\ell_3} = R^2. \end{cases}$$

Solving that we finally obtain

$$f_P^2(x, y) = -x(x - 2x_0) - y(y - 2y_0) + e^{2l_3},$$

where x_0 and y_0 are given by relations $x_0 = \dfrac{1 - e^{2\,\ell_3}}{2 \tanh \ell_1}$ and $y_0 = \dfrac{1 - e^{2\,\ell_3}}{2 \tanh \ell_2}$.

Integration in formula (2) is taken over the domain of function $f(x, y)$. In our case Q and P are projected on the absolute into triangle D, that is the domains of integration for V_Q and V_P coincides. Thus, we can rewrite difference of integrals as integral of difference.

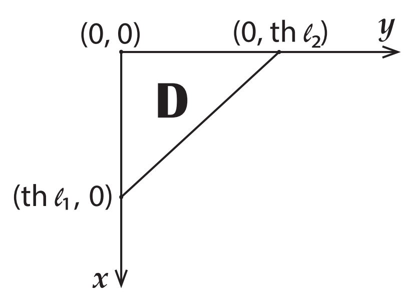

FIG. 4. Projection of faces P and Q on the absolute

Consider triangle D (Fig. 4). Its vertices are $(0, 0)$, $(\tanh \ell_1, 0)$ and $(0, \tanh \ell_2)$. We find equation of the straight line passing through the last two points

$$y = \frac{\tanh \ell_2 (\tanh \ell_1 - x)}{\tanh \ell_1}.$$

Therefore, while x varies within 0 and $\tanh \ell_1$ the second coordinate y runs from 0 to $\dfrac{\tanh \ell_2 (\tanh \ell_1 - x)}{\tanh \ell_1}$.

Now we can apply (2) with functions $f_Q(x, y)$, $f_P(x, y)$ and prescribed integration limits to find V_Q and V_P. Thus we get the fromula from statement of the theorem using the fact that $V = V_Q - V_P$. $\square$

3. Formula verification

To check the formula we have obtained, we use Sforza's formula (see Theorem 1). We take $\ell_1 = \ell_2 = \ell_3 = \ell$.

ℓ	Theorem 1	Theorem 2
0.25	0.0025398399...	0.0025398399...
0.5	0.0188499596...	0.0188499596...
1	0.1124224663...	0.1124224667...
2	0.3463865845...	0.3463863000...
4	0.4538079351...	0.4538079400...
8	0.4579801973...	0.4579810839...

Table 1. Volume of a tetrahedron T, calculated by different formulas

We note that for large values of ℓ, the integral in Sforza's formula accumulates a calculation error while the formula in Theorem 2 works well. As $\ell \to \infty$, the volume tends to 1/8 of the volume of regular ideal octahedron, that is

$$\lim_{\ell \to \infty} V(T) = \frac{1}{2} G = 0,457982797\ldots,$$

where $G = \sum\limits_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 0.9159655941\ldots$ is Catalan's constant.

4. Normalized volume

In the work by O. Nemoul and N. Mebarki [15] the normalized volume of a compact regular hyperbolic tetrahedron is given. Let us find it for 3-parameter family of tetrahedra under consideration. Following [15] we define *normalized volume* as $\nu(T) = \frac{V}{S^{3/2}}$, where V is the volume of T and S is its surface area.

Let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ be the areas of rectangular faces 1–2–3, 1–2–4, 1–3–4 correspondingly and $\mathbf{W}$ is the area of 2–3–4 (Fig. 5).

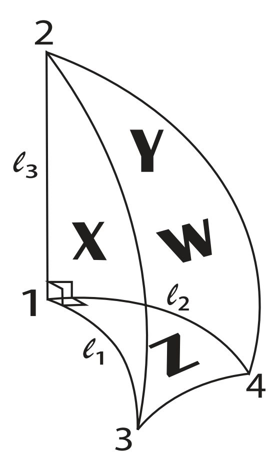

Fig. 5. Areas of the faces of a tetrahedron T

One of the analogs of Heron's formula for the area of a hyperbolic triangle is known as Bilinsky formula [7] (see also [2]).

$$\cos\frac{\mathbf{X}}{2}=\frac{\cosh\ell_1+\cosh\ell_3+\cosh h+1}{4\cosh\frac{\ell_1}{2}\cosh\frac{\ell_3}{2}\cosh\frac{h}{2}}, \tag{3}$$

where h is hypotenuse in the triangle 1–2–3. By Pythagorean theorem

$$\cosh h=\cosh l_1\cosh l_3. \tag{4}$$

We substitute (4) in (3) and use equality $1+\cosh\ell=2\cosh^2\frac{\ell}{2}$ to find

$$\cos\frac{\mathbf{X}}{2}=\frac{\cosh\frac{\ell_1}{2}\cosh\frac{\ell_3}{2}}{\cosh\frac{h}{2}},$$

and equivalently

$$\sin\frac{\mathbf{X}}{2}=\frac{\sinh\frac{\ell_1}{2}\sinh\frac{\ell_3}{2}}{\cosh\frac{h}{2}}.$$

Using these relations for $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ we get the areas of rectangular faces

$$\mathbf{X}=2\arctan\left(\tanh\frac{\ell_1}{2}\tanh\frac{\ell_3}{2}\right),$$
$$\mathbf{Y}=2\arctan\left(\tanh\frac{\ell_2}{2}\tanh\frac{\ell_3}{2}\right),$$
$$\mathbf{Z}=2\arctan\left(\tanh\frac{\ell_1}{2}\tanh\frac{\ell_2}{2}\right).$$

Then we use a hyperbolic version of De Gua's theorem for the faces of a tetrahedron given by B. McConnell [13] to get the face area $\mathbf{W}=$

$$2\arccos\frac{1-\tanh^2\frac{\ell_1}{2}\tanh^2\frac{\ell_2}{2}\tanh^2\frac{\ell_3}{2}}{\sqrt{(1+\tanh^2\frac{\ell_1}{2}\tanh^2\frac{\ell_3}{2})(1+\tanh^2\frac{\ell_2}{2}\tanh^2\frac{\ell_3}{2})(1+\tanh^2\frac{\ell_1}{2}\tanh^2\frac{\ell_2}{2})}}.$$

Thus, the normalized volume $\nu(T)=\dfrac{V}{S^{3/2}}$ is calculated by Theorem 2 and $S=\mathbf{X}+\mathbf{Y}+\mathbf{Z}+\mathbf{W}$.

To find the asymptotics of the normalized volume we take $\ell_1=\ell_2=\ell_3=\ell$. As $\ell\to\infty$ the faces areas $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ attend to $\dfrac{\pi}{2}$ as the areas of right triangles with two vertices at infinity. The face area $\mathbf{W}$ attends to π as the area of a regular ideal hyperbolic triangle. Therefore, $\lim\limits_{\ell\to\infty}S=\dfrac{5\pi}{2}$. Thus,

$$\lim_{\ell\to\infty}\nu=\lim_{\ell\to\infty}\frac{V}{S^{3/2}}=\frac{\sqrt{2}\,G}{(5\pi)^{3/2}}=0.0208071557\ldots,$$

where $G=0.9159655941\ldots$ is Catalan's constant.

Consider the behavior of the normalized volume $\nu(T)=\nu(\ell)$ for the case $\ell_1=\ell_2=\ell_3=\ell$ (Fig. 6). It monotonically decreases for $\ell\in(0,+\infty)$.

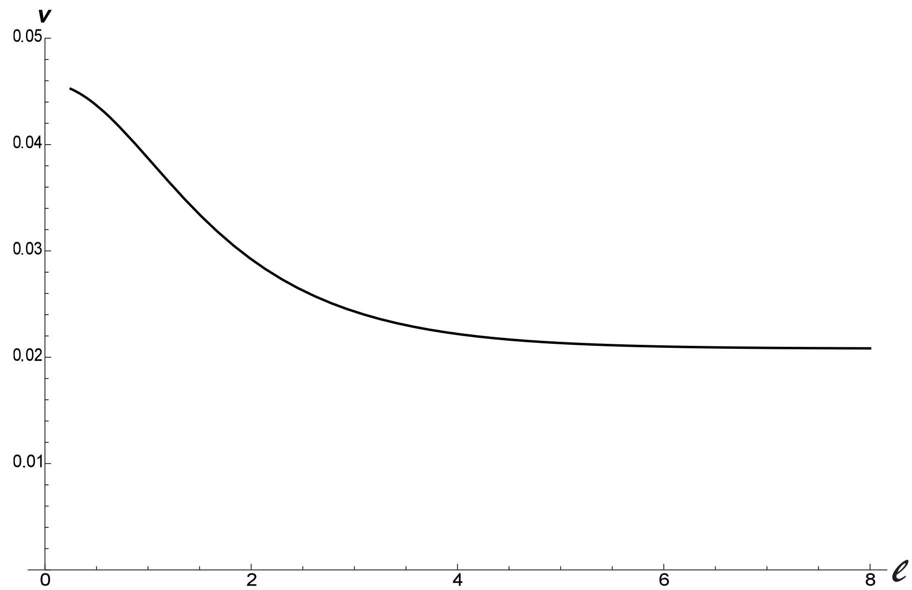

FIG. 6. Graph of the normalized volume of a tetrahedron T

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