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ON THE PRESERVATION OF THE WIENER INDEX OF CUBIC GRAPHS UPON VERTEX REMOVAL

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Abstract. The Wiener index, $W(G)$, is the sum of distances between all vertices of a connected graph G. In 2018, Majstorović, Knor and Škrekovski posed the problem of finding r-regular graphs except cycle C_{11} having at least one vertex v with property $W(G) = W(G - v)$. An infinite family of cubic graphs with four such vertices is constructed.

Keywords: distance invariant, Wiener index, Šoltés problem.

1. Introduction

All graphs G considered in this paper are simple and connected. The cardinality of the vertex set $V(G)$ is called the order of G. Denote by $G-v$ the graph obtained by removing a vertex v from G. The distance $d(u,v)$ between vertices $u, v \in V(G)$ is the number of edges on a shortest path connecting these vertices in G. The vertex distance for $v \in V(G)$ is defined as the sum of distances from v to all the other vertices of G, $d_G(v) = \sum_{u \in V(G)} d(v,u)$. A half of the sum of vertex distances is the Wiener index of G that has found numerous applications [4, 5, 6, 10, 14],

$$W(G) = \frac{1}{2} \sum_{v \in V(G)} d_G(v).$$

One of directions in the study of the Wiener index is that how it changes under graph transformations. In 1991, Šoltés posed the following problem: find all graphs G having the property $W(G) = W(G-v)$ for all vertices v of G [12]. Such graphs will be called Šoltés graphs. The simple cycle C_{11} is the unique known example of Šoltés graphs. Unsuccessful attempts to find new Šoltés graphs led to the formulations of

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various relaxed problems [1, 2, 3, 8, 9, 11, 7]. The following problem was proposed in [9]:

Problem 1. *Are there r-regular connected graphs G other than C_{11} for which the equality $W(G) = W(G - v)$ holds for at least one vertex $v \in V(G)$?*

In this paper, an infinite family of cubic graphs having four such vertices v is constructed.

2. Main result

Consider a cubic graph G of order $2(2k + s + 2)$ shown in Fig. 1. It consists of the left and right symmetrical ladders with k rungs and the middle ladder between them with s rungs. Let $\mathcal{G}_{k,s}$ be a family of such graphs where $k \geq 4$ and $s \geq 2$. The following statement is the basis for solving Problem 1 for the case of cubic graphs ($r = 3$).

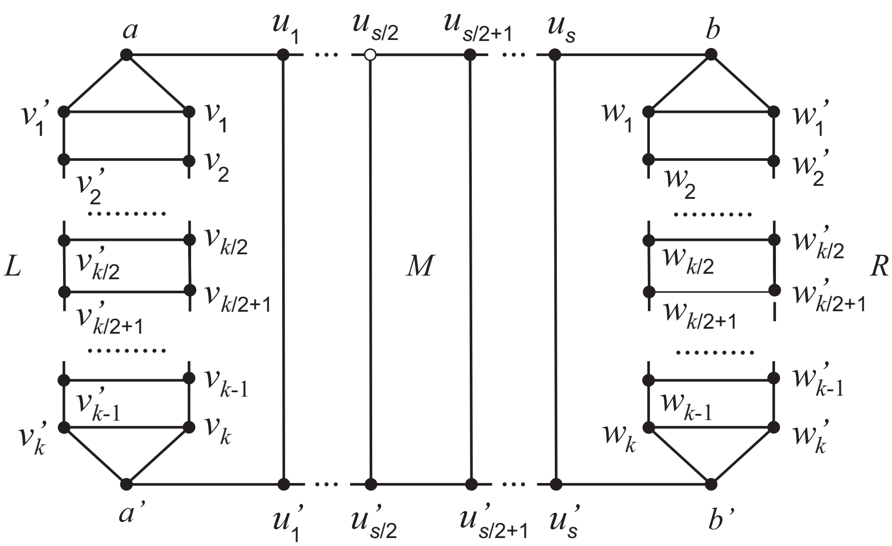

Fig. 1. Cubic graph $G \in \mathcal{G}_{k,s}$.

Proposition 1. *Let $G \in \mathcal{G}_{k,s}$ and integers k and s are even. Then*

$$W(G) = 3k^3 + \frac{1}{2}k^2(12s + 39) + k(4s^2 + 20s + 19) + \frac{1}{3}s(2s^2 + 15s + 34) + 8,$$

$$W(G - u_{s/2}) = 3k^3 + \frac{1}{2}k^2(12s + 41) + k(4s^2 + 20s + 15) + \frac{1}{3}s(2s^2 + 15s + 28) + 4.$$

The proof of this proposition will be given in the next section.

Since $W(G) - W(G - u_{s/2}) = -k^2 + 4k + 2s + 4$ and vertices $u_{s/2}, u_{s/2+1}, u'_{s/2}$, and $u'_{s/2+1}$ belong to the same orbit of the automorphism group of G, we obtain the following result.

Corollary 1. *If $k \geq 6$ is even and $s = (k^2 - 4k - 4)/2$, then a cubic graph $G \in \mathcal{G}_{k,s}$ has four vertices v for which $W(G) = W(G - v)$.*

Table 1 contains order and the Wiener index of graphs $G \in \mathcal{G}_{k,s}$ of Corollary 1 for the initial values of k and s.

TABLE 1. Graphs $G \in \mathcal{G}_{k,s}$ for even $k \geq 6$.

k	s	$\lvert V(G)\rvert$	$W(G)$
6	4	36	3368
8	14	64	19800
10	28	100	77780
12	46	144	236572
14	68	196	603792
16	94	256	1356528
18	124	324	2766300
20	158	400	5227860

3. Proof of Proposition 1

If distances $d_G(u, v)$ are considered in graph G, then we will drop the subscript G. Let $d(X, Y)$ be the sum of distances between vertices of subsets X and Y in a graph, $d(X, Y) = \sum_{x \in X, y \in Y} d(x, y)$. Denote by p_n and c_n the distances of a pendent vertex of the simple path P_n and a vertex of the simple cycle C_n of order n, respectively. It is known that $p_n = n(n-1)/2$ and $c_n = n^2/4$ for even n, and $W(P_n) = n(n^2 - 1)/6$.

Let $G \in \mathcal{G}_{k,s}$ where k and s are even. Divide the vertex set of G into eight disjoint subsets: $V = \{v_1, v_2, \ldots, v_k\}$, $V' = \{v'_1, v'_2, \ldots, v'_k\}$, $W = \{w_1, w_2, \ldots, w_k\}$, $W' = \{w'_1, w'_2, \ldots, w'_k\}$, $U = \{u_1, u_2, \ldots, u_s\}$, $U' = \{u'_1, u'_2, \ldots, u'_s\}$, $A = \{a, a'\}$, and $B = \{b, b'\}$ (see Fig. 1). Denote by L, R, and M the induced subgraphs of G with vertex sets $V \cup V' \cup A$, $W \cup W' \cup B$, and $U \cup U'$, respectively. Then the Wiener index of G can be represented as the sum of several parts:

$$W(G) = \frac{1}{2}[\, d(L, L) + d(R, R) + d(M, M)\,] + d(L, R) + d(L, M) + d(R, M)$$

$$= \frac{1}{2} d(M, M) + d(L, L) + d(L, R) + 2d(L, M). \tag{1}$$

The last equality is valid due to the symmetric structure of G. Next we will calculate the summands of equation (1) .

1. Consider distances between vertices of subgraph M. For vertices $u_i \in V(M)$, $i = 1, 2, \ldots, s$, we have

$$d(u_i) = \sum_{j=1}^{s} d(u_i, u_j) + \sum_{j=1}^{s} d(u_i, u'_j) = 2d_{P_s}(u_i) + s.$$

By symmetry of G, $d(u_i) = d(u'_i)$, $i = 1, 2, \ldots, s$. Then

$$d(M, M) = 2\sum_{i=1}^{s} d(u_i) = 2\sum_{i=1}^{s} (2d_{P_s}(u_i) + s) = 2(4W(P_s) + s^2)$$

$$= \frac{2}{3} s(s + 2)(2s - 1).$$

2. Consider distances between vertices of subgraphs L and M. Since vertices v_i and v_i', $i = 1, 2, \ldots, k$, belong to the same orbit of the automorphism group of G, we can write

$$\begin{aligned} d(u_i, L) &= 2\sum_{j=1}^{k/2}[d(u_i, a) + d(a, v_j)] + 2\sum_{j=k/2+1}^{k}[d(u_i, a') + d(a', v_j)] \\ &\quad + d(u_i, a) + d(u_i, a') \\ &= 2\left(\frac{k}{2}i + \sum_{j=1}^{k/2} d(a, v_j)\right) + 2\left(\frac{k}{2}(i+1) + \sum_{j=k/2+1}^{k} d(a', v_j)\right) + 2i + 1 \\ &= (k+1)(2i+1) + 4p_{k/2+1} = \frac{1}{2}k^2 + 2k(i+1) + 2i + 1. \end{aligned}$$

By symmetry of vertices of U and U',

$$d(L, M) = \sum_{i=1}^{s} d(u_i, L) + \sum_{i=1}^{s} d(u_i', L) = s(k^2 + 2k(s+3) + 2(s+2)).$$

3. Consider distances between vertices of subgraphs L and R. Note that $d(v_i, a) + d(a, u_1) < d(v_i, a') + d(a', u_1')$, $i = 1, 2, \ldots, k/2$ (see Fig. 1). This implies that all shortest paths from vertices of L to vertices of R lie in the subgraph shown in Fig. 2.

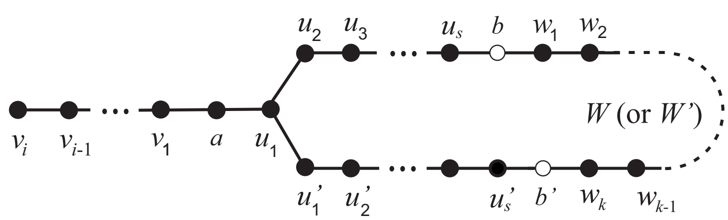

FIG. 2. Subgraphs with the shortest paths for $d(L, R)$.

By symmetry of L and R, it is sufficient to calculate distances from vertices v_i, $i = 1, 2, \ldots, k/2$. Then

$$d(L, R) = 4\sum_{i=1}^{k/2}[2d(v_i, W) + d(v_i, B)] + 2[2d(a, W) + d(a, B)].$$

We have $d(v_i, W) = \sum_{j=1}^{k}[d(v_i, u_1) + d(u_1, w_j)] = k(i+1) + c_{k+2s+2} - p_{s+1} - p_{s+2}$, $d(v_i, B) = 2(i+1) + 2s + 1$, $d(a, W) = k + c_{k+2s+2} - p_{s+1} - p_{s+2}$, and $d(a, B) = 2s + 3$. As a result, we obtain

$$d(L, R) = 2(k+1)(k^2 + k(2s+5) + 2s + 3).$$

4. Consider distances between vertices of subgraph L. By symmetry, it is sufficient to calculate distances for vertex v_i, $i = 1, 2, \ldots, k/2$.

4.1. Distances form vertex v_i to vertices of $V \cup A$. The shortest paths from vertex v_i to vertices of $V \cup A$ are located in subgraphs shown in Fig. 3a. Denote by C the cycle $(v_i, \ldots, v_1, a, u_1, u_1', a', v_k, \ldots, v_{i+1}, v_i)$ of length $k+4$. Then

$$d(v_i, V \cup A) = c_{k+4} - d(v_i, u_1) - d(v_i, u') = c_{k+4} - (i+1) - (i+2) = \frac{1}{4}k^2 + 2k - 2i + 1.$$

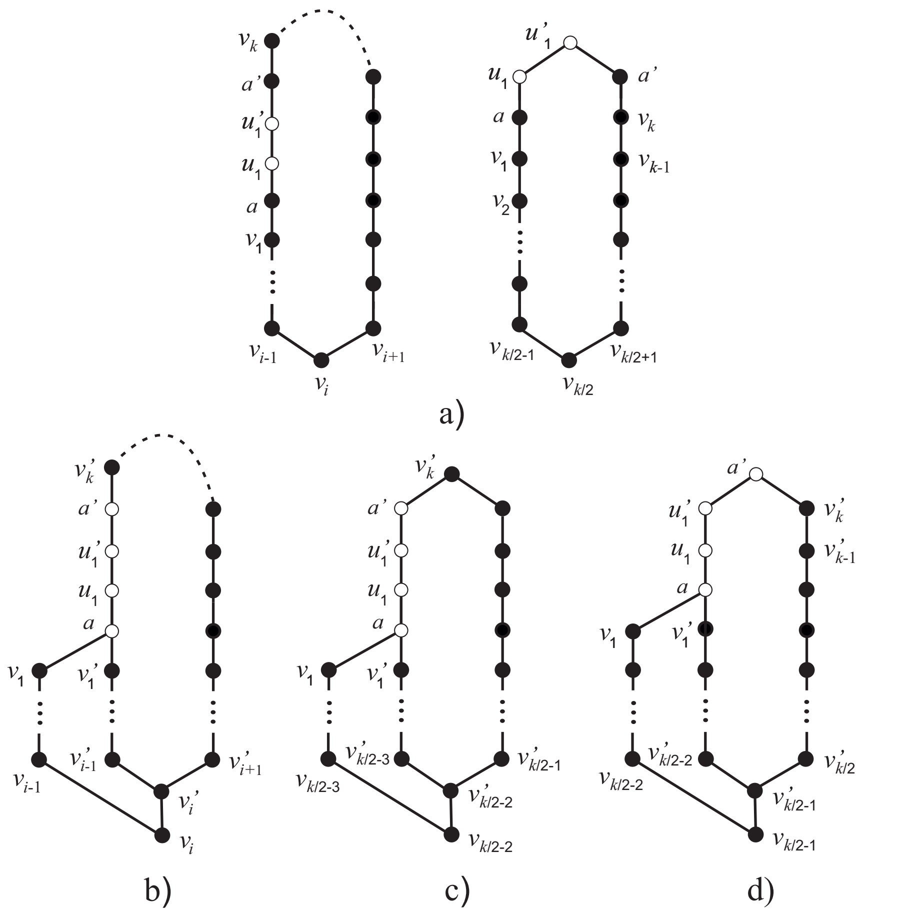

FIG. 3. Subgraphs with the shortest paths for $d(L, L)$.

4.2. Distances form vertex v_i to vertices of V'. The shortest paths from v_i to vertices of V' are located in subgraphs shown in Fig. 3bc. Denote by C the cycle $(v'_i, \dots, v'_1, a, u_1, u'_1, a', v'_k, \dots, v'_{i+1}, v'_i)$. Subpath $(v_i, v_{i-1}, \dots, v_1, a)$ is a part of the shortest paths form v_i to vertices $v'_k, v'_{k-1}, \dots, v'_j$, $i = 1, 2, \dots, k/2-2$. It is convenient to find distances from v_i to all vertices of C subtracting excess values:

$$\begin{aligned} d(v_i, V') &= \sum_{x \in C} [d(v_i, v'_i) + d(v'_i, x))] - d(v'_i, a) - d(v'_i, u_1) - d(v'_i, u'_1) - d(v'_i, a') \\ &\quad - (k/2 - 1 - i) \\ &= (k+4) + c_{k+4} - (i+1) - (i+2) - (i+3) - (i+4) - (k/2 - 1 - i) \\ &= \frac{1}{4}k^2 + \frac{5}{2}k - 3i - 1. \end{aligned}$$

For $i = k/2-1, k/2$, path $(v_i, v_{i-1}, \dots, v_1, a)$ does not affect the shortest distances v_i to vertices of V' (see Fig. 3cd for $i = k/2-2$ and $i = k/2-1$). Then $d(v_i, V') = k + p_i + p_{k+i-1} = k^2/2 - k(2i-3)/2 + i^2 - i$ and, therefore, $d(v_{k/2-1}) = k^2/4 + k + 2$

and $d(v_{k/2-1}) = k^2/2 + k$. Then

$$d(v_i, L) = \begin{cases} \frac{1}{4}k^2 + \frac{5}{2}k - 1 - 3i, & i = 1, 2, \dots, k/2 - 2, \\ k^2/4 + k + 2, & i = k/2 - 1, \\ k^2/2 + k, & i = k/2. \end{cases}$$

To calculate distances $d(a, L)$ and $d(a', L)$, it is sufficient to consider the shortest paths in cycles $(a, u_1, u_1', a', v_k, v_{k-1} \dots, v_1, a)$ and $(a, u_1, u_1', a', v_k', v_{k-1}' \dots, v_1', a)$ (see Fig. 1). Then we can write $d(a, L) = d(a', L) = \sum_{x \in V \cup V'} d(a, x) + d(a, a') = 2c_{k+4} - 2d(a, u_1) - 2d(a, u_1') - d(a, a') = 2c_{k+4} - 9 = k^2/2 + 4k - 1$. Finally, we have

$$d(L, L) = 4 \left(\sum_{i=1}^{k/2-2} d(v_i, L) + \sum_{i=k/2-1}^{k/2} d(v_i, L) \right) + 2d(a, L) = \frac{1}{2}(2k^3 + 15k^2 + 6k + 4).$$

Substituting expressions for $d(L, L)$, $d(M, M)$, $d(L, R)$, and $d(L, M)$ back into equality (1), we obtain

$$W(G) = 3k^3 + k^2(6s + 39/2) + k(4s^2 + 20s + 19) + 2s^3/3 + 5s^2 + 34s/3 + 8.$$

5. To quickly calculate Wiener index of graph $G - u_{s/2}$, we use the following graph operation. Let G_1 and G_2 be graphs of order n_1 and n_2, respectively. If vertices $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ are connected by path P_3, then the Wiener index of the resulting graph H can be written as follows [4]

$$(2)\quad W(H) = W(G_1) + W(G_2) + (n_1 + 1) d_{G_2}(v_2) + (n_2 + 1) d_{G_1}(v_1) + 2n_1 n_2 + n_1 + n_2.$$

Let G_1 and G_2 be graphs of order $n_1 = 2k + s$ and $n_2 = n_1 + 2$ with vertex sets $V \cup V' \cup A \cup \{u_1, \dots, u_{s/2-1}\} \cup \{u_1', \dots, u_{s/2-1}'\}$ and $W \cup W' \cup B \cup \{u_{s/2+1}, \dots, u_s\} \cup \{u_{s/2+1}', \dots, u_s'\}$, respectively. Then we can represent Wiener indices of graphs G_1 and G_2 as $W(G_i) = (d(L, L) + d(M, M))/2 + d(L, M)$, where ladder M has order $2(s/2 - 1)$ in G_1 and $2s$ in G_2. Path P_3 connects vertices $v_1 = u_{s/2-1}$ in G_1 and $v_2 = u_{s/2+1}$ in G_2 .

Now calculate distance of vertex $u_{s/2-1}$ in G_1. Let $U^* = \{u_1, u_2, \dots, u_{s/2-1}, u_1', u_2', \dots, u_{s/2-1}'\}$. It is easy to see that $d(u_{s/2-1}, U^*) = p_{s/2-1} + p_{s/2}$. The shortest paths from vertex $u_{s/2-1}$ to vertices of $V \cup A$ pass on the subgraph of G_1 shown in Fig. 4.

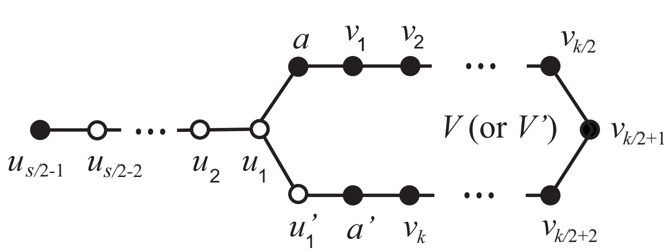

FIG. 4. Subgraph with the shortest paths for $d(u_{s/2-1}, V \cup A)$.

Denote by C the cycle $(u_1, a, v_1, v_2, \dots, v_k, a', u'_1, u_1)$ of length $k+4$. Then

$$\begin{aligned} d_{G_1}(u_{s/2-1}) &= d(u_{s/2-1}, V \cup V' \cup A \cup U') \\ &= 2 \sum_{x \in V(C)} [d(u_{s/2-1}, u_1) + d(u_1, x)] + d(u_{s/2-1}, U') \\ &\quad - 2d(u_{s/2-1}, u_1) - 2d(u_{s/2-1}, u'_1) - d(u_{s/2-1}, a) - d(u_{s/2-1}, a') \\ &= 2(k+4)(s/2-2) + 2c_{k+4} + p_{s/2-1} + p_{s/2} - 2(s/2-2) - 2(s/2-1) \\ &\quad - (s/2-1) - s/2 \\ &= k^2/2 + ks + s^2/4. \end{aligned}$$

By similar consideration, we have $d_{G_2}(u_{s/2+1}) = k^2/2 + k(s+2) + s^2/4 + s + 1$. Substituting the obtained expressions back into equality (2), we get

$$W(G - u_{s/2}) = 3k^3 + k^2(6s + 41/2) + k(4s^2 + 20s + 15) + 2s^3/3 + 5s^2 + 28s/3 + 4.$$

4. Other graphs of $\mathcal{G}_{k,s}$

Computer calculations show that families of cubic graphs G with the property $W(G) = W(G - v)$ for some vertices v exist also for odd $k \geq 7$. Here we present two examples.

The left part of Table 2 contains data for a family of cubic graphs $G \in \mathcal{G}_{k,s}$ of order $2(2k + s + 2)$ with odd $k \geq 7$ and $s = (k+1)(k-5)/2$. As for graphs with even k, vertex $u_{s/2}$ and three symmetrical vertices can be removed from G without changing the Wiener index.

Cubic graphs G of the right part of Table 2 consist of ladders of distinct odd lengths. Namely, the left, right, and central ladders have k, $k+6$, and $s = k+1$ rungs, respectively. In this case, vertex u_4 and symmetrical vertex u'_4 can be removed from graphs G of order $2(2k + s + 8)$ without changing the Wiener index.

TABLE 2. Graphs G with property $W(G) = W(G - v)$ for odd $k \geq 7$.

k	s	$\lvert V(G)\rvert$	$W(G)$	k	$k+6$	s	$\lvert V(G)\rvert$	$W(G)$
7	8	48	8110	7	13	8	60	14764
9	20	80	39166	9	15	10	72	25226
11	36	120	135546	11	17	12	84	39724
13	56	168	377810	13	19	14	96	58914
15	80	224	904854	15	21	16	108	83452
17	108	288	1936950	17	23	18	120	113994
19	140	360	3802626	19	25	20	132	151196
21	176	440	6969386	21	27	22	144	195714

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