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THE JORDAN BLOCK STRUCTURE OF THE IMAGES OF UNIPOTENT ELEMENTS IN IRREDUCIBLE MODULAR REPRESENTATIONS OF CLASSICAL ALGEBRAIC GROUPS OF SMALL DIMENSIONS

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Abstract. For unipotent elements of prime order, the Jordan block structure of their images in infinitesimally irreducible representations of the classical algebraic groups in odd characteristic whose dimensions are at most 100, is determined. The approach proposed can be applied for solving a similar problem for representations of bigger dimensions. A detailed information on small cases is important for stating reasonable conjectures on the behavior of unipotent elements in irreducible representations of the classical algebraic groups.

Keywords: unipotent elements, Jordan block sizes, representations of small dimensions.

1. Introduction

In this paper the canonical Jordan form of the images of unipotent elements of prime order in irreducible p-restricted representations of the classical algebraic groups in odd characteristic p whose dimensions are at most 100, is determined. Observe that in many cases the picture differs heavily from the situation in characteristic 0 even if the dimension of the irreducible representation with a certain highest weight is the same. The approach proposed to find this form can be applied to solve a similar problem for representations of bigger dimensions. The information obtained can be used for stating reasonable conjectures

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on the behaviour of unipotent elements in representations of algebraic groups. The study of such behaviour is important for solving recognition problems on representations of linear groups. At present very little is known on the block structure of images of arbitrary unipotent elements in representations of the classical algebraic groups, hence a detailed study of such images for representations of small dimensions is useful.

Indeed, in the majority of cases only the maximal size of a Jordan block (the degree of the minimal polynomial) of a given element in a fixed representation is known. Such polynomials were found in [27] for unipotent elements of a prime order and all simple algebraic groups and in [29] for arbitrary unipotent elements and the classical algebraic groups in odd characteristic. M. Barry [1, Theorem 2] found recursive formulae for the canonical Jordan form of regular unipotent elements on the wedge and symmetric squares of the standard modules of the special linear groups in odd characteristic. One can apply these results for solving the problem for arbitrary unipotent elements of this groups embedding nonregular elements into proper subsystem subgroups and using the description of the restrictions of relevant modules to this subgroups (see Lemma 19). Recall that in odd characteristic the wedge and symmetric squares of the standard modules are irreducible for the special linear groups, the wedge square is irreducible for the spinor groups and the symmetric square for the symplectic ones (see Proposition 3 and 4 and Theorem 5 and 6).

It is well known that in odd characteristic the following modules have a unique nontrivial composition factor (see, for instance, [18, Theorem 5.1]):
a) the tensor product of the standard module and its dual for a special linear group;
b) the wedge square of the standard module for a symplectic group;
c) the symmetric square of such module for a spinor group.
In all these cases T. Korhonen [16, Theorem 6.1 and Corollaries 6.2 and 6.3] described the canonical Jordan form of unipotent elements acting on these factors, for a special linear group the question is solved without restrictions on the ground field characteristic. These results were applied by him in [15] for classifying irreducible representations of the classical groups where all Jordan blocks in the image of some unipotent element have different sizes.

R. Lawther [17] determined the canonical Jordan form of all unipotent elements of the exceptional algebraic groups in the action on the nontrivial modules of the minimal dimensions and the adjoint modules.

2. Preliminary results

In what follows $\mathbb{N}$ is the set of natural numbers, $\mathbb{C}$ is the complex field, K is an algebraically closed field of odd characteristic p, G is a simply connected algebraic group of a classical type, n is a rank of G, ω_i and α_i with $1 \leq i \leq n$ are the fundamental and the simple roots of G, ε_i with $1 \leq i \leq n$ for $G \neq A_n(K)$ and $1 \leq i \leq n+1$ for $G = A_n(K)$ are weights of the standard G-module defined in [4, §13], $\langle \mu, \alpha \rangle$ is the value of a weight μ on a root α (in the sense of [25, §1]), ρ is the halfsum of the positive roots of G. The symbols ω_i, ε_i and α_i are used not only for the group G, but for other simple algebraic groups, it is clear from the context what group is considered. Recall that an irreducible representation φ in characteristic p with highest weight $\sum_{i=1}^{n} a_i\omega_i$ is called p-restricted if all the coefficients $a_i < p$. If Γ is a simple algebraic group over $\mathbb{C}$ or K, then $\mathfrak{X}_\beta$ is the root subgroup associated with a root β, $\Gamma(\beta_1, ..., \beta_k)$ is the subgroup in Γ generated by the root subgroups $\mathfrak{X}_{\pm\beta_1}, ..., \mathfrak{X}_{\pm\beta_k}$, $W(\Gamma)$ is the Weyl group of Γ, $\Lambda(\Gamma)$ and $\Lambda^+(\Gamma)$ are the set of weights and the set of dominant weights of Γ. Set $\mathfrak{X}_{\pm i} = \mathfrak{X}_{\pm\alpha_i}$, $\Gamma(i_1, ..., i_k) = \Gamma(\alpha_{i_1}, ..., \alpha_{i_k})$. We also use the notation $\Gamma(i_1, ..., i_k, \beta)$ for the group $\Gamma(\alpha_{i_1}, ..., \alpha_{i_k}, \beta)$ and other similar notation. Below $x_{\pm i}(t)$ is the root element of Γ associated with the root $\pm\alpha_i$ and an element t of the field, $X_{\pm i,t}$ is the element in the hyperalgebra of Γ associated with the root $\pm\alpha_i$ and an integer t. For $\Gamma = G$ set $W = W(G)$. The labelling of simple roots is such as in [3]. A subsystem subgroup is the subgroup generated by all root subgroups associated with the roots from some subsystem of the root system of G.

Below $\dim M$ ($\dim\varphi$) is the dimension of a Γ-module M (a representation φ), $\Lambda(M)$ and $\Lambda^+(M)$ ($\Lambda(\varphi)$ and $\Lambda^+(\varphi)$) are the sets of weights and dominant weights of a module M (a representation φ), and M^* is the dual module to the module M. If $\omega\in\Lambda^+(\Gamma)$, then $M(\omega)$, $V(\omega)$, $T(\omega)$, and $\varphi(\omega)$ are the irreducible module, the Weyl module, the indecomposable tilting-module, and the irreducible representation of Γ with highest weight ω; $\omega(\varphi)$ ($\omega(M)$) is the highest weight of a representation φ (a module M); $\omega(m)$ is the weight of a weight vector m from some module; $\operatorname{Irr} M$ is the set of composition factors of a module M (up to isomorphism). If Γ is a simple simply connected algebraic group over K, then $\Gamma_{\mathbb{C}}$ is the simple simply connected algebraic group over $\mathbb{C}$ with the same root system as Γ; for an irreducible representation φ of Γ we denote by $\varphi_{\mathbb{C}}$ the irreducible representation of $\Gamma_{\mathbb{C}}$ with highest weight $\omega(\varphi)$; similarly we define the $\Gamma_{\mathbb{C}}$-module $M_{\mathbb{C}}$ for an irreducible Γ-module M.

If H is a subgroup of Γ, then $M|H$ is the restriction of a Γ-module M to H. The weights and roots of Γ are considered with respect to a fixed maximal torus T. If $T\cap H$ is a maximal torus in H, then $\omega|H$ is the restriction of a weight ω to $T\cap H$. In this case for a weight vector m from some Γ-module put $\omega_H(m)=\omega(m)|H$. Note that $T\cap H$ is a maximal torus in H for a subsystem subgroup H. If M is an irreducible Γ-module, then $v\in M$ is a nonzero highest weight vector. For an element $x\in G$ and a representation φ, we use the symbol $d_\varphi(x)$ to denote the degree of the minimal polynomial of $\varphi(x)$. The set of the weights of the group $A_1(K)$ is identified with the set $\mathbb{Z}$ of integers in the standard way: $a\omega_1\mapsto a$.

For a unipotent element $g\in GL(n,K)$ that has k_1 Jordan blocks of size d_1, k_2 blocks of size d_2, ..., k_t blocks of size d_t with $d_1>d_2>...>d_t$ and $k_1d_1+k_2d_2+...+k_td_t=n$, we shall write $J(g)=(d_1^{k_1},...,d_t^{k_t})$. For $x\in G$ the symbol $J(x)$ denotes the sequence $J(\varphi(x))$ for the standard representation φ. It is well known that the canonical Jordan form of the image of a unipotent element in a representation of G is the same as for its image in the dual representation.

In what follows V is the standard (natural) module of G, $e_1,e_2,...,e_n$ is a basis of V, $\langle v_1,v_2,...,v_r\rangle$ ($\langle V_1,V_2,...,V_r\rangle$) is the subspace generated by vectors v_1, v_2,..., v_r (by subspaces V_1, V_2,..., V_r). We assume that $n>1$ for $G=A_n(K)$ or $C_n(K)$, $n>2$ for $G=B_n(K)$, and $n>3$ for $G=D_n(K)$.

The following facts will be used in this paper.

By [2, Proposition 5.13], the following formula holds for an element m of an arbitrary G-module M, a root γ of G, and an element $t\in K$:

$$x_\gamma(t)m=\sum_{i=0}^{\infty}t^iX_{\gamma,i}m.$$

This formula will be applied without special references.

Lemma 1. [2, Lemma 5.14] *Let γ and δ be roots of a semisimple algebraic group. Then the following formulae hold:*

$$X_{-\gamma}X_{\gamma,d}=X_{\gamma,d}X_{-\gamma}-H_\gamma X_{\gamma,d-1}+(d-1)X_{\gamma,d-1},$$

$$X_{\gamma,d}X_\delta=X_\delta X_{\gamma,d}+\sum_{t=1}^{d}c_tX_{\gamma+\delta,t}X_{\gamma,d-t},\ c_t\in\mathbb{Z}$$

(where $H_\gamma=[X_\gamma,X_{-\gamma}]$). We have $X_{i,k}X_{-j,d}=X_{-j,d}X_{i,k}$ for $i\neq j$.

Probably, Proposition 1, Lemma 3, and Corollary 1 are well known, but we cannot give explicit references.

Proposition 1. *Let Γ be a group and U be a Γ-module. Assume that $\operatorname{Irr} U=S_1\cup S_2$, the subsets $S_i\neq\varnothing$, and that there are no indecomposable Γ-modules with two composition* *factors M_1 and M_2 such that $M_i \in S_i$, $i = 1, 2$. Then $U = N_1 \oplus N_2$ where N_i are Γ-modules and* $\mathrm{Irr}\, N_i = S_i$.

Proof. We use the induction on the number of composition factors of U. Suppose that U has l composition factors and that the proposition holds for modules with a smaller number of composition factors. Obviously, we can assume that $l \geq 3$. Let $M \subset U$ be an irreducible submodule. Set $\overline{U} = U/M$. Without loss of generality we can assume that $M \in S_1$. It is clear that $S_2 \subset \mathrm{Irr}\, \overline{U}$. First suppose that $\mathrm{Irr}\, \overline{U} = S_2$. Obviously, there exists a submodule $U_1 \subset U$ such that $M \subset U_1$ and U_1 has $l-1$ composition factors. It is clear that $\mathrm{Irr}(U_1/M) \subset S_2$. Therefore by the induction hypothesis, $U_1 = M \oplus U_2$ where U_2 is a Γ-module and $\mathrm{Irr}\, U_2 \subset S_2$. Set $P = U/U_2$. It is clear that P has two composition factors, one of them is isomorphic to M and another lies in S_2. Then by the assumptions of the proposition, $P = P_1 \oplus P_2$ where $P_1 \cong M$ and $P_2 \in S_2$. Set $N_1 = M$, and let N_2 denote the full preimage of P_2 in U. It is clear that $\mathrm{Irr}\, N_2 = S_2$, $N_1 \cap N_2 = 0$, and the module N_2 has $l-1$ composition factors. This implies that $N = N_1 \oplus N_2$.

Let $\mathrm{Irr}\, \overline{U} \neq S_2$. By the induction hypothesis, $\overline{U} = \overline{N_1} \oplus \overline{N_2}$ where $\overline{N_i}$ are Γ-modules, $\mathrm{Irr}\, \overline{N_1} \subset S_1$, and $\mathrm{Irr}\, \overline{N_2} = S_2$. Let N_1 be the full preimage of $\overline{N_1}$ in U. It is clear that $\mathrm{Irr}\, N_1 = S_1$ and that for all modules $Q \in S_1$, the multiplicities of the composition factor isomorphic to Q in the modules N_1 and U coincide. Let Z be the full preimage of $\overline{N_2}$ in U. By the induction hypothesis, $Z = M \oplus N_2$ where N_2 is a Γ-module and $\mathrm{Irr}\, N_2 = S_2$. It is clear that for all modules $F \in S_2$, the multiplicities of the composition factor isomorphic to F in the modules U and N_2 coincide. Since $N_1 \cap N_2 = 0$, this yields that $U = N_1 \oplus N_2$.
□

In what follows Γ is a semisimple algebraic group over K unless otherwise stated. We shall use the symbol $Ext^1_\Gamma(M_2, M_1)$ to denote the extension group of a Γ-module M_1 by M_2, i.e. the set of equivalence classes of short exact sequences of Γ-modules of the form

$$0 \to M_1 \xrightarrow{i} M \xrightarrow{j} M_2 \to 0$$

with the standard operation. Recall that for irreducible modules M_1 and M_2 a module M in such sequence is completely reducible if $Ext^1_\Gamma(M_2, M_1) = 0$.

Lemma 2. [13, Part 2, Item 2.12, Formulae (1) and (4)] *Let M_1 and M_2 be irreducible Γ-modules. Then*
1) $Ext^1_\Gamma(M_1, M_1) = 0$;
2) $Ext^1_\Gamma(M_1, M_2) = Ext^1_\Gamma(M_2, M_1)$.

Proposition 2. [13, Part 2, Proposition 2.14] *Let λ and $\mu \in \Lambda^+(\Gamma)$ and $\mu \not> \lambda$. Then*

$$Ext^1_\Gamma(M(\lambda), M(\mu)) \cong Hom_\Gamma(rad_\Gamma V(\lambda), M(\mu)),$$

where $rad_\Gamma V(\lambda)$ is the maximal submodule in $V(\lambda)$.

Lemma 3. *Let U be a Γ-module, $M = M(\omega)$, and $\mathrm{Irr}\, U = M \cup I$. Suppose that the module $V(\omega)$ is irreducible and $M \notin \mathrm{Irr}\, V(\lambda)$ if $M(\lambda) \in I$. Then $U = N_1 \oplus N_2$ where N_1 is the direct sum of several copies of M, N_2 is a Γ-module, and $\mathrm{Irr}\, N_2 = I$.*

Proof. By Proposition 2 and Lemma 2, $Ext^1_\Gamma(M, Q) = Ext^1_\Gamma(Q, M) = 0$ for any $Q \in I$. Hence there are no indecomposable Γ-modules with two composition factors one of which is isomorphic to M and another is contained in I. By Proposition 1, $U = N_1 \oplus N_2$ where N_1 and N_2 are Γ-modules, $\mathrm{Irr}\, N_1 = M$, and $\mathrm{Irr}\, N_2 = I$. It remains to prove that N_1 is completely reducible.

Let l be the number of composition factors of N_1. Apply induction on l. For $l = 1$, our assertion is trivial. For $l = 2$, it follows from Lemma 2. Assume that $l > 2$. Suppose that the assertion holds for modules that have less than l composition factors. Obviously, N_1 contains a submodule S with $l-2$ composition factors. Lemma 2 implies that $N_1/S = \overline{U_1} \oplus \overline{U_2}$ where $\overline{U_i} \cong M$. Let U_i be the full preimage of $\overline{U_i}$ in N_1. It is clear that $N_1 = U_1 + U_2$.

As the modules U_i have $l-1$ composition factors, each of them is completely reducible by the induction hypothesis. This implies that N_1 is completely reducible. □

Corollary 1. *Let U be a Γ-module. Assume that $V(\lambda)$ is irreducible if $M(\lambda) \in \operatorname{Irr} U$. Then U is completely reducible.*

Proof. Apply induction on the order t of $\operatorname{Irr} U$. For $t=1$, argue as for the module N_1 in the proof of Lemma 3. Let $t>1$. Assume that the assertion of the corollary is true for modules F with $|\operatorname{Irr} F| < t$. Fix a module $M \in \operatorname{Irr} U$ and set $I = \operatorname{Irr} U \setminus M$. Since the modules $V(\lambda)$ are irreducible if $M(\lambda) \in \operatorname{Irr} U$, then by Proposition 2 there are no indecomposable modules with two composition factors one of which is isomorphic to M and another lies in I. Hence M and I satisfy the assumptions of Lemma 3. By this lemma, U is the direct sum of several copies of M and a module U_1 with $\operatorname{Irr} U_1 = I$. By the inductive hypothesis, the module U_1 is completely reducible as $|I| = t-1$. This yields the assertion of the corollary. □

Corollary 2 is well known and appears in different forms in numerous publications on representation theory. We include it to show a connection with more general facts.

Corollary 2. *Let $\Gamma = A_1(K)$ and U be a Γ-module all whose weights are less than p. Then U is completely reducible.*

Proof. It is well known that the Weyl modules of Γ with highest weights less than p are irreducible. Therefore our assertion follows from Corollary 1. □

Lemma 4. [13, Part 2, Lemma 2.13b and Formula (2.14.1)] *A Γ-module generated by a nonzero vector of weight λ fixed by all root subgroups associated with positive roots, is a homomorphic image of the Weyl module $V(\lambda)$ of Γ. The quotient module of this module by its maximal submodule is isomorphic to $M(\lambda)$.*

Corollary 3 is well known, but we formulate it for the convenience of the reader.

Corollary 3. *Let $H \subset \Gamma$ be a subsystem subgroup and M be a Γ-module. Suppose that a weight vector $m \in M$ is fixed by all root subgroups associated with positive roots of H. Then the module KHm and hence the restriction $M|H$ has the composition factor with highest weight $\omega(m)|H$ (here $\omega(m)|H$ is defined such as for G).*

Proof. It follows from [25, Theorem 39b] that the module KHm is indecomposable. Then we use Lemma 4. □

Lemma 5. [31, Lemma 1] *Let $\lambda \in \Lambda^+(\Gamma)$ and module $V(\lambda)$ be irreducible. Suppose that λ is the maximal weight of a Γ-module U and the weight subspace of this weight in U is one-dimensional. Then $U = V \oplus N$ where $N \cong M(\lambda)$.*

Lemma 6. [21, 1.5] *Let M be an irreducible Γ-module, $m \in M$ be a nonzero weight vector, β be a root of Γ, $\mathfrak{X}_\beta$ fixes m, and $0 < \langle \omega(M), \beta \rangle < p$. Then $X_{-\beta,c} m \neq 0$ if $0 < c \leq \langle \omega(M), \beta \rangle$.*

Recall that a Γ-module is called a tilting module if it has both a filtration by Weyl modules and a filtration by dual Weyl modules.

Lemma 7. [22, Lemma 1.1]
(a) For each dominant weight λ there exists an indecomposable tilting module $T(\lambda)$, unique up to isomorphism, with highest weight λ.
(b) Any tilting module is the direct sum of tilting modules of form $T(\lambda)$.
(c) A direct summand of a tilting module is a tilting module.
(d) The tensor product of tilting modules is a tilting module.

In Definition 1 and Lemmas 8– 9 Γ is a simple algebraic group of a classical type over K.

Definition 1. *A closed connected subgroup A of type A_1 in Γ is called good if the images of all roots under the homomorphism $\sigma : \Lambda(\Gamma) \to \Lambda(A)$ induced by the restriction of weights from a maximal torus T of Γ to the maximal torus $T_A = T \cap A$ in A are at most $2p-2$.*

Lemma 8. [22, Proposition 2.2] *For any element $x \in \Gamma$ of order p there exists a good subgroup A containing x and all such subgroups are conjugate; one can chose such system of simple roots that $\sigma(\alpha_i) \in \{0,1,2\}$ and coincides with the ith label on the labelled Dynkin diagram of x (here σ is the homomorphism from Definition 1).*

It is well known [8, Chapter 5] that for a regular element x the integer $\sigma(\alpha_i) = 2$ for all i.

Lemma 9. *Let α be the maximal short root and ρ be the halfsum of positive roots of Γ. If $\langle \omega+\rho, \alpha\rangle \le p$, then the Weyl module $V(\omega)$ is irreducible.*

Proof. This lemma follows from [13, Part II, Proposition 8.19] since one easily observes that $\langle \omega+\rho, \beta\rangle \le \langle \omega+\rho, \alpha\rangle$ for any positive root β of Γ. □

Lemma 10. [3, Table I, II, IV] *Let α be the maximal root of G and $\omega = \sum_{i=1}^{n} a_i\omega_i$. Then*

$$\langle \omega, \alpha\rangle = \begin{cases} \sum_{i=1}^{n} a_i & \text{for } G = A_n(K) \text{ or } C_n(K); \\ a_1 + a_n + 2\sum_{i=2}^{n-1} a_i & \text{for } G = B_n(K); \\ a_1 + a_{n-1} + a_n + 2\sum_{i=2}^{n-2} a_i & \text{for } G = D_n(K). \end{cases}$$

If $G = B_n(K)$ or $C_n(K)$ and β is the maximal short root of G, then

$$\langle \omega, \beta\rangle = a_n + 2\sum_{i=1}^{n-1} a_i \text{ in the first case}$$

and

$$\langle \omega, \beta\rangle = a_1 + 2\sum_{i=2}^{n} a_i \text{ in the second one.}$$

Theorem 1. [24] *Let $S = G(i_1, ..., i_k) \subseteq G$, M be an irreducible G-module with highest weight ω, and $v \in M$ be a nonzero highest weight vector. Then a subset $KSv \subseteq M$ is an irreducible S-module with highest weight $\omega|S$ and a direct summand of the S-module M.*

Corollary 4. *Let $S = G(1, 2, ..., i-1, i+1, ..., n)$, M be an irreducible S-module with highest weight ω, $\Omega_k = \{\lambda \in \Lambda(M) | \lambda = \omega - k\alpha_i - \sum_{j\neq i} b_j\alpha_j\}$, and $U_k = \langle M_\mu | \mu \in \Omega_k\rangle$. Then $M|S = U_0 \oplus ... \oplus U_h$ where $h = h(M) = \max\{k | \Omega_k \neq \varnothing\}$, U_k are S-modules, the modules U_0 and U_h are irreducible, $\omega(U_0) = \omega|S$, and the lowest weight of U_h is the restriction to S of the lowest weight of M.*

Proof. It is obvious that U_k are S-modules and direct summands of M. Let $Y \subset G$ be the subgroup generated by all root subgroups $\mathfrak{X}_\alpha$ with negative α. Since $M = KYv$, one easily observes that $KSv = U_0$. Hence the S-module U_0 is irreducible by Theorem 1. Applying Theorem 1 to a lowest weight vector and the negative roots, we conclude that the module U_h is irreducible, too. Now the assertion on the highest and lowest weights is obvious. □

Lemma 11. *Let $1 \le i \le n$, $G_1 = G(1, ..., i-1, i+1, ..., n)$, and M be an irreducible G-module. Suppose that $M \cong M^*$. For $k \in \mathbb{Z}^+$ set $\Omega_k = \{\lambda \in \Lambda(M) | \lambda = \omega - k\alpha_i - \sum_{j\neq i} x_j\alpha_j\}$, $U_k = \langle M_\lambda | \lambda \in \Omega_k\rangle$. Let $h = h(M)$ be the maximal integer with $\Omega_h \neq \varnothing$. Then the G_1-modules U_k and U_{h-k} are dual.*

Proof. Let T be a fixed maximal torus with respect to which the weights and the roots of G are determined. Then T normalizes the subgroup G_1. It is clear that U_k is a direct summand of the restriction $M|TG_1$. Since $M \cong M^*$, then its lowest weight is equal to $-\omega(M)$, $\Omega_{h-k} = \{\lambda \in \Lambda(M) | -\lambda \in \Omega_k\}$, and $M|TG_1$ has a direct summand U isomorphic to U_k^*. Taking into account that $\dim(U_k)_\lambda = \dim U_{-\lambda}$ for any weight $\lambda \in \Lambda(M)$ and analyzing the action of torus T on the subspaces U_j, we get that $U = U_{h-k}$. □

Remark 1. *If $G = B_n(K)$ or $C_n(K)$, Lemma 11 holds for all irreducible modules.*

Indeed, it is well known that for such groups all irreducible modules are self-dual.

Lemma 12. [29, Lemma 2.46] *Let M be an irreducible G-module with highest weight $\sum_{i=1}^{n} a_i\omega_i$ and $v \in M$ be a nonzero highest weight vector. Let $1 \le s,t \le n$ and $s,t < n$ for $G = D_n(K)$. Assume that $0 < a_t < p$. Set $b_k = -\langle \alpha_{k+1}, \alpha_k \rangle$ and $c_k = -\langle \alpha_{k-1}, \alpha_k \rangle$. For an integer d with $0 < d \le a_t$ define the vector $v(s,t,d)$ as follows. Put $d_t = d$. If $s < t$, set $d_k = a_k + d_{k+1}b_k$ for $s \le k < t$. If $s > t$, put $d_k = a_k + d_{k-1}c_k$ for $s \ge k > t$. Now take*

$$v(s,t,d) = X_{-s,d_s} ... X_{-k,d_k} ... X_{-t,d} v.$$

For $s = t$ put $v(s,t,d) = X_{-s,d}v$. Then $v(s,t,d) \ne 0$ and $X_{m,b}v(s,t,d) = 0$ for positive $m \ne s$ and $b > 0$. Hence $\mathfrak{X}_m$ fixes $v(s,t,d)$.

The notation $v(s,t,d)$ is intensively used below.

Corollary 5. *Let $H = G(i, i+1, ..., n)$, $1 < i < n$; and let φ be a p-restricted irreducible representation of G with highest weight*

$$a_1\omega_1 + ... + a_{n-1}\omega_{n-1} + a_n\omega_n, a_i \ne 0.$$

Suppose that t is an integer and $0 < t \le a_i$. Then $\varphi|H$ has a composition factor with highest weight

$$(a_{i-1} + a_i - t)\omega_1 + (a_{i+1} + t)\omega_2 + a_{i+2}\omega_3 + ... + a_n\omega_{n-i+1}.$$

Proof. Put $m = v(i-1, i, t)$. By Lemma 12, $m \ne 0$ and the subgroups $\mathfrak{X}_j$ with $j > i-1$ fix m. Hence m generates an irreducible H-module with highest weight $\omega_H(m)$. One easily concludes that $\omega_H(m)$ is equal to the weight above. This yields the corollary. □

Theorem 2. [20] *Let φ be a p-restricted irreducible representation of G. Then $\Lambda(\varphi) = \Lambda(\varphi_{\mathbb{C}})$.*

For groups of types A_n and D_n this theorem is true for $p = 2$ as well. For groups of type A_n it has been proved in [26].

Lemma 13. *For $d < p$ an exterior power $\wedge^d V$ is a direct summand in $\otimes^d V$ (a special case of the result* [11, Corollary 2.6e]*).*

Corollary 6. *Let $G = A_n(K)$, A be the image of a completely reducible representation of $A_1(K)$ with p-restricted irreducible components, and $i < p$. Then $M(\omega_i)|A$ is a tilting module.*

Proof. This follows immediately from Lemmas 13 and 7(c,d). □

Lemma 14. *Let $A \subset G$ be a connected semisimple closed subgroup and $\omega \in \Lambda^+(G)$. Suppose that the module $V(\omega)$ is irreducible and that there are weights λ_1 and $\lambda_2 \in \Lambda^+(G)$ such that $\omega = \lambda_1 + \lambda_2$ and the restrictions $M(\lambda_1)|A$ and $M(\lambda_2)|A$ are tilting modules. Then $M(\omega)|A$ is a tilting module.*

Proof. It is clear that the dimension of the weight subspace of weight ω in the module $M(\lambda_1) \otimes M(\lambda_2)$ is equal to 1. Then by Lemma 5, $M(\omega)$ is a direct summand of this tensor product. It remains to use Lemma 7(c,d). □

Corollary 7. *Let $A \subset G$ be a connected semisimple closed subgroup and M be an irreducible G-module with highest weight ω. Suppose that the restrictions to A of irreducible G-modules with fundamental highest weights are tilting modules and that $\langle \omega+\rho, \alpha\rangle \leq p$ for the maximal short root α of G. Then $M|A$ is a tilting module.*

Proof. Let $\omega = \sum_{i=1}^{n} a_i\omega_i$. Set $s = \sum_{i=1}^{n} a_i$. Apply induction on s. For $s = 1$ the assertion of the corollary is true by hypothesis. Let $s > 1$ and $a_j \neq 0$ for some j. Put $\lambda_1 = \omega - \omega_j$, $\lambda_2 = \omega_j$. By the induction hypothesis and the assumptions of the corollary, $M(\lambda_i)|A$ is a tilting module for $i = 1$ and 2. The module $V(\omega)$ is irreducible by Lemma 9. Now our assertion follows from Lemma 14. □

Corollary 8. *Let $G = A_n(K)$, A be the image of a completely reducible representation of the group $A_1(K)$ with p-restricted irreducible components, M be the irreducible G-module with highest weight $a_1\omega_1$, and $a_1 < p$. Then $M|A$ is a tilting module.*

Proof. It is clear that $M(\omega_1)|A$ is a tilting module. It is well known that $V(a_1\omega_1)$ is irreducible. It suffices to apply Lemma 14 several times. □

Lemma 15. [5] *Let $G = A_2(K)$ and M be an irreducible p-restricted G-module with highest weight $\omega = a_1\omega_1 + a_2\omega_2$. Then the module $V(\omega)$ is irreducible for $a_1 + a_2 < p - 1$ and has two composition factors $M(\omega)$ and $M(\omega - d(\alpha_1 + \alpha_2))$ for $a_1 + a_2 = p + d - 2$ with $d > 0$.*

Lemma 16. *Let $G = A_2(K)$, M be an irreducible p-restricted G-module with highest weight $\omega = a_1\omega_1 + a_2\omega_2$, $a_1 + a_2 \neq p - 1$, $0 < k \leq a_1$, and $a_2 \neq 0$. Then the multiplicity of the weight $\omega - k\alpha_1 - \alpha_2$ in M is equal to 2.*

Proof. Put $\lambda = \omega - k\alpha_1 - \alpha_2$. Obviously, $\lambda \notin \Lambda(M(\omega - d(\alpha_1 + \alpha_2)))$ for $d > 1$. Hence Lemma 15 yields that $\dim M_\lambda = \dim V(\omega)_\lambda$. By [21, 1.5], this dimension is equal to 2. □

Theorem 3. [10, Chapter $VIII$, Theorem 2.7] *Let $1 \leq s \leq t \leq p$. Then*
$J(J_s \otimes J_t) = (t-s+1,\ t-s+3,\ \dots,\ t+s-1)$ for $s+t \leq p$ and
$J(J_s \otimes J_t) = (p^k,\ t-s+1,\ \dots,\ 2p-t-s-1)$ for $s+t > p$ and $k = s+t-p$. In particular, $J(J_s \otimes J_p) = (p^s)$.

Lemma 17. [23, Lemma 2.3 and the remark after Proposition 2.8] *If $G \neq D_n(K)$ or a unipotent element $x \in G$ has at least one Jordan block of odd size in the standard realization of G, then the conjugacy class containing x is uniquely determined by the Jordan form of its elements in the standard realization. If $G = D_n(K)$ and x has Jordan blocks of even sizes only in the standard realization, then the set of elements with such Jordan form is divided into two conjugacy classes. Furthermore, a unipotent element from $SL_n(K)$ is conjugate to an element from $Sp_n(K)$ if and only if the multiplicities of all Jordan blocks of odd sizes are even, and it is conjugate to an element from $SO_n(K)$ if and only if the multiplicities of all Jordan blocks of even sizes are even (here we mean blocks of this element in the standard realization).*

Lemma 18. [27, Proposition 2.12] *Let $x \in G$ be an element of order p, $k_1 \geq k_2 \geq \dots \geq k_t$ be all sizes of Jordan blocks of x on V, and $m = \dim V$. Suppose that a subgroup A and a homomorphism σ are connected with x as in Lemma 8. Set*

$$N(x) = (k_1 - 1, k_1 - 3, \dots, 1 - k_1, k_2 - 1, \dots, 1 - k_t).$$

Then $N(x)$ coincides with the set $\{\sigma(\omega(e_j)) | 1 \leq j \leq m\}$.

Remark 2. *If $G \neq D_n(K)$ or an element x of order p has at least one block of odd size in the standard realization, then Lemma 8 allows one to determine $\sigma(\varepsilon_i)$.*

Indeed, since all $\sigma(\alpha_j) \geq 0$, one easily observes that $\sigma(\varepsilon_1) \geq \sigma(\varepsilon_2) \geq ... \geq \sigma(\varepsilon_{n+1})$ for $G = A_n(K)$ and $\sigma(\varepsilon_1) \geq \sigma(\varepsilon_2) \geq ... \geq \sigma(\varepsilon_n) \geq 0$ for $G = B_n(K)$ or $C_n(K)$. Let $G = D_n(K)$. We can conclude that

$$\sigma(\varepsilon_1) \geq \sigma(\varepsilon_2) \geq ... \geq \sigma(\varepsilon_{n-1}) \geq |\sigma(\varepsilon_n)|. \tag{1}$$

Since x has a block of odd size, then at least one of the integers $\sigma(\varepsilon_i) = 0$. This yields that $\sigma(\varepsilon_n) = 0$ and $\sigma(\varepsilon_1) \geq \sigma(\varepsilon_2) \geq ... \geq \sigma(\varepsilon_{n-1}) \geq 0$. As the sequence $N(x)$ is known, we can determine all the values $\sigma(\varepsilon_i)$.

If $G = D_n(K)$ and all blocks of x on V have even sizes, the picture is slightly more difficult. This is discussed in detail in Section 6.

In Propositions 3–5 and Lemmas 19– 20 $G = A_n(K)$. It is well known that $A_n(K) \cong SL_{n+1}(K)$.

Denote by $\wedge^r V$, $S^r(V)$, and $S^{r,p}(V)$ the rth exterior, rth symmetric, and the rth truncated symmetric powers of V, respectively.

Proposition 3. [13, Part 2, Item 2.15] *For $1 \leq r \leq n$, the space $\wedge^r V$ is an irreducible G-module with highest weight ω_r. All weight subspaces of this module are one-dimensional.*

Proposition 4. [33, Proposition 1.2] *Let k and j be nonnegative integers and $j < p-1$. For $r = k(p-1) + j \leq (n+1)(p-1)$, the rth truncated symmetric power $S^{r,p}(V)$ is an irreducible module with highest weight*

$$(p-1-j)\omega_k + j\omega_{k+1}.$$

All weight subspaces of this module are one-dimensional. In particular, for $r < p$, the space $S^r(V)$ is an irreducible G-module with highest weight $r\omega_1$.

The following lemma is well known, we state it for the reader's convenience.

Lemma 19. *Let $k + t = n - 1$, $H_1 = A_k(K)$, $H_2 = A_t(K)$, V_i be the standard module for the group H_i, $i = 1, 2$, and $H \cong H_1H_2$ be a subsystem subgroup in G. Then the following formulae hold:*

$$\begin{aligned} \wedge^i V &\cong \bigoplus_{\substack{i_1+i_2=i,\\ i_1\leq k+1,\\ i_2\leq t+1}} \wedge^{i_1} V_1 \otimes \wedge^{i_2} V_2;\\ S^i V &\cong \bigoplus_{0\leq j\leq i} S^j V_1 \otimes S^{i-j} V_2;\\ S^{i,p} V &\cong \bigoplus_{\substack{i_1+i_2=i,\\ 0\leq i_1\leq (k+1)(p-1),\\ 0\leq i_2\leq (t+1)(p-1)}} S^{i_1,p} V_1 \otimes S^{i_2,p} V_2. \end{aligned} \tag{2}$$

Proof. To proof this, consider the action of H in a basis of a relevant module associated with some fixed bases of V_1 and V_2 in the standard way. □

Lemma 20 is well known and follows from the irreducibility of the Weyl modules with highest weights ω_2 and $2\omega_1$ ([13, Part 2, Proposition 2.14] and [21, Item 1.15i]).

Lemma 20. *Let $p > 2$. Then $V \otimes V \cong \wedge^2 V \oplus S^2 V$.*

Proposition 5. *(a part of* [16, Theorem 6.1]*) Let $M_1 = V \otimes V^*$ and $M_2 = M(\omega_1 + \omega_n)$. For an element $x \in G$ of order p and an integer k, denote by $r_k^i(x)$ the number of Jordan blocks of size k for the action of x on M_i, $i = 1, 2$.*

1. *Let x have at least one Jordan block of size less than p in the standard realization. Then $r_k^1(x) = r_k^2(x)$ for $k \neq 1$, $r_1^2(x) = r_1^1(x) - 1$ if $p \nmid n+1$, and $r_1^2(x) = r_1^1(x) - 2$ if $p | n+1$.*

2. *Let x have only blocks of size p in the standard realization. If $p^2 \mid n+1$, then $r_p^2(x) = r_p^1(x) - 2$, $r_{p-1}^2(x) = 2$, and $r_k^2(x) = 0$ for $k \neq p$ or $p-1$. If $p^2 \nmid n+1$, then $r_p^2(x) = r_p^1(x) - 1$, $r_{p-2}^2(x) = 1$, and $r_k^2(x) = 0$ for $k \neq p$ or $p-2$.*

Lemma 21. [19, Lemma 10] *Let $\Gamma = A_1(K)$, φ be an irreducible representation of Γ with highest weight a, and $0 \le a < p$. Then $\dim\varphi = a+1$ and for a nontrivial unipotent element $z \in \Gamma$, an element $\varphi(z)$ has a unique Jordan block of dimension $a+1$.*

In what follows this lemma is sometimes used without an explicit reference. The canonical Jordan form of the image of a unipotent element in an arbitrary irreducible representation of a group of type A_1 can be found with the use of Lemma 21, the Steinberg tensor product theorem [25, Theorem 1.1], and Theorem 3.

Proposition 6. *Let $n < p$, $G_n = A_n(K)$, z_n be a regular unipotent element of G_n, and φ_n^1 and φ_n^2 be the irreducible representations of G_n with highest weights ω_2 and $2\omega_1$, respectively (we assume that φ_1^1 is trivial). For an integer l set $\bar{l} = 1$ or 3 if $l \equiv 1$ or $3 \pmod 4$, respectively. If $n < \frac{p-2}{2}$, then*

$$J(\varphi_n^1(z_n)) = (2n-1,\ 2n-5,\ ...,\ \overline{2n-1}),\ J(\varphi_n^2(z_n)) = (2n+1,\ 2n-3,\ ...,\ \overline{2n+1}).$$

If $\frac{p-2}{2} < n < p-1$, then

$$J(\varphi_n^1(z_n)) = (p^{2n+1-p},\ 2p-2n-3,\ 2p-2n-7,\ ...,\ \overline{2p-2n-3}),$$
$$J(\varphi_n^2(z_n)) = (p^{2n+3-p},\ 2p-2n-5,\ 2p-2n-9,\ ...,\ \overline{2p-2n-5}).$$

We have

$$J(\varphi_{p-1}^1(z_{p-1})) = (p^{\frac{p-1}{2}}) \text{ and } J(\varphi_{p-1}^2(z_{p-1})) = (p^{\frac{p+1}{2}}).$$

Proof. In this proof for $m < n$, a representation φ of G_n, and a representation ψ of G_m, we write $J(\varphi(z_n)) = (a^k, J(\psi(z_m)))$ if $d_\varphi(z_n) = a$, $\varphi(z_n)$ has just k Jordan blocks of size a, and the sequence $J(\varphi(z_n))$ can be obtained from $J(\psi(z_m))$ by adding a^k at the beginning.

For $n = p-1$, the result follows immediately from [1, Theorem 2 (4)].

Let $n < \frac{p-2}{2}$. It is clear that $J(\varphi_1^1(z_1)) = (1)$. By Lemma 21, $J(\varphi_1^2(z_1)) = (3)$. By [1, Theorem 2 (4)],

$$J(\varphi_n^1(z_n)) = J(\varphi_{n-1}^2(z_{n-1})),\ J(\varphi_n^2(z_n)) = (2n+1,\ J(\varphi_{n-1}^1(z_{n-1})))$$

for $n > 1$. This forces that $J(\varphi_2^1(z_2)) = (3)$ and $J(\varphi_2^2(z_2)) = (5,\ 1)$ and that

$$J(\varphi_n^1(z_n)) = (2n-1, J(\varphi_{n-2}^1(z_{n-2}))),\ J(\varphi_n^2(z_n)) = (2n+1, J(\varphi_{n-2}^2(z_{n-2})))$$

for $n > 2$. Now apply induction on n to complete the proof for $n < \frac{p-2}{2}$.

Next, let $\frac{p-2}{2} < n < p-1$. Then by [1, Theorem 2 (1)],

$$J(\varphi_n^1(z_n)) = (p^{\frac{2n-p+1}{2}}, J(\varphi_{p-n-2}^2(z_{p-n-2}))),\ J(\varphi_n^2(z_n)) = (p^{\frac{2n+3-p}{2}},\ J(\varphi_{p-n-2}^1(z_{p-n-2})))$$

(here we assume that $J(\varphi_0^1(z_0)) = \varnothing$ and $J(\varphi_0^2(z_1)) = (1)$). It remains to apply the results for $n < \frac{p-2}{2}$ proven just before. □

Lemma 22. [31, proof of Lemma 4, Formula (1)] *Let $G = A_2(K)$, $\omega = a_1\omega_1 + a_2\omega_2$, and $U = V(\omega)$. Suppose that $\mu = \omega - t\alpha_1 - k\alpha_2 \in \Lambda(U)$. Then the multiplicity n_μ of weight μ in U is determined by the formula*

$$n_\mu = \min\{t,k\} + 1 - \max\{0, \min\{t-a_1-1, k\}+1\} - \max\{0, \min\{t, k-a_2-1\}+1\}.$$

In Lemmas 23–29, Propositions 7–8, Corollary 9, and Remarks 3– 4 $\Gamma = A_1(K)$. Denote by $\Delta(a)$ the Weyl comodule of Γ with highest weight a.

The following lemma describes Weyl modules and indecomposable tilting modules with not large highest weights for a group of type A_1.

Lemma 23. [22, Lemmas 1.2 and 1.3] *For $0 \le c < p$ the module $T(c) \cong V(c) \cong M(c)$. If $p \le c \le 2p-2$, write $c = r+p$. Then the maximal submodule M in $V(c)$ is isomorphic to $M(p-r-2)$ and $V(c)/M \cong M(c)$. The module $T(c)$ has a filtration*

$$T(c) = M_1 \supset M_2 \supset M_3 \supset M_4 = 0$$

with $M_1/M_2 \cong M_3 \cong M(p-r-2)$ *and* $M_2/M_3 \cong M(r+p)$*;*
$\dim(T(c)) = 2p$*. In this case the module* $T(c)$ *is projective for the group* $A_1(p)$*.*

Lemma 24. [22, Lemma 1.4] *Let* $0 \le r \le p-2$*. Set* $c = r+p$ *and* $d = p-r-2$*. Suppose that the composition factors of a* Γ*-module* N *are isomorphic either to* $M(c)$*, or to* $M(d)$*. Then*

$$N = T(c)^r \oplus V(c)^s \oplus \Delta(c)^t \oplus M(c)^u \oplus M(d)^v,$$

where U^a *is the direct sum of* a *copies of a module* U*. Moreover,* N *is self-dual if and only if* $s = t$*.*

Remark 3. *Let* N *be a self-dual* Γ*-module and* N^μ *be the sum of all indecomposable components of* N *with a fixed highest weight* μ*. Then the module* N^μ *and a sum of several such modules are self-dual.*

This assertion is clear since the indecomposable components of a module are determined up to isomorphism and the highest weight of an $A_1(K)$-module is preserved after the transition to the dual one.

The following proposition describes the structure of indecomposable tilting modules with highest weight bigger than p for the group $A_1(K)$.

Proposition 7. [9, Example 2] *Let* $m \ge p$*. Then*

$$T(m) \cong T(p-1+r) \otimes T(s)^{Fr},$$

where Fr *is the Frobenius morphism of the group* $A_1(K)$ *determined by raising elements of the field* K *to the pth power, and* r *and* s *are determined by the equality* $m+1-p = r+ps$*,* $0 \le r \le p-1$*.*

Lemma 25. *Let* X_α *and* $X_{-\alpha}$ *be the root elements of the Lie algebra of* Γ *associated with the positive and the negative roots, respectively, and* $0 \le s < p-2$*. Then the* Γ*-module* $T(p+s)$ *contains a vector* u *of weight* $p-s-2$ *such that* $X_{-\alpha}^{s+1}X_\alpha^{s+1}u \neq 0$*, and a* Γ*-module* U *with* $\operatorname{Irr} U = \{M(p+s),\ M(p-s-2)\}$ *that has no indecomposable components isomorphic to* $T(p+s)$*, does not contain such vectors.*

Proof. It is clear that there is a factor isomorphic to $\Delta(p+s)$ in the filtration of $T(p+s)$ by dual Weyl modules. Then there exists a nonzero vector $u \in T(p+s)$ such that $\omega(u) = p-s-2$ and $X_\alpha^{s+1}u \neq 0$. Obviously, $X_\alpha^{s+1}u$ is a highest weight vector in $T(p+s)$. This forces that $X_{-\alpha}^{s+1}X_\alpha^{s+1}u \neq 0$.

Let U be a Γ-module with $\operatorname{Irr} U = \{M(p+s),\ M(p-s-2)\}$. Assume that U has no indecomposable components isomorphic to $T(p+s)$. Write $U = U^1 \oplus ... \oplus U^k$ where U^1, ..., U^k are the indecomposable components of U. It is clear that

$$U_{p-s-2} = U^1_{p-s-2} \oplus U^2_{p-s-2} \oplus ... \oplus U^k_{p-s-2}.$$

By Lemma 24, $U^i \in \{V(p+s), \Delta(p+s), M(p+s), M(p-s-2)\}$. Let $u \in U_{p-s-2}$. Then $u = u_1 + ... + u_k$ where $u_j \in U^j_{p-s-2}$. Obviously, $u_j = 0$ for $U^j \cong M(p+s)$. One easily observes that $X_\alpha^{s+1}u_j = 0$ for $U^j \cong M(p-s-2)$ or $V(p+s)$. Let $U^j \cong \Delta(p+s)$. Then $h_j = X_\alpha^{s+1}u_j$ is a highest weight vector in U^j. It is well known that $K\Gamma h_j \cong M(p+s)$ for $h_j \neq 0$. Hence $X_{-\alpha}^{s+1}h_j = 0$. This yields that $X_{-\alpha}^{s+1}X_\alpha^{s+1}u_j = 0$. □

Corollary 9. *Let* $0 \le s < p-2$ *and* N *be a* Γ*-module such that* $I(N) = \{2M(p+s), 4M(p-s-2)\}$*. Assume that* $\dim X_{-\alpha}^{s+1}X_\alpha^{s+1}N_{p-s-2} = 2$*. Then* $N \cong T(p+s) \oplus T(p+s)$*.*

Proof. By Lemma 24, any indecomposable component of N is isomorphic to $T(p+s)$, $V(p+s)$, $\Delta(p+s)$, $M(p+s)$, or $M(p-s-2)$. It follows from Lemma 25 that at least one of these components is isomorphic to $T(p+s)$. Write $N = N^1 \oplus N^2$ where $N^1 \cong T(p+s)$. It is clear that $\dim N^1_{p-s-2} = 2$. Since N^1 contains a submodule isomorphic to $V(p+s)$, then $KerX_\alpha \cap N^1_{p-s-2} \neq 0$. This and Lemma 25 imply that $\dim X_{-\alpha}^{s+1}X_\alpha^{s+1}N^1_{p-s-2} = 1$.

Assume that $N^2 \not\cong T(p+s)$. Then it is obvious that N^2 has no such indecomposable components. Hence Lemma 25 yields that $X_{-\alpha}^{s+1}X_{\alpha}^{s+1}N_{p-s-2}^2 = 0$ and $\dim X_{-\alpha}^{s+1}X_{\alpha}^{s+1}N_{p-s-2} = 1$. A contradiction obtained completes the proof. □

Lemma 26. *Let λ and $\mu \in \mathbb{Z}^+$ and one of the following hold:*
a) $\lambda = 2p+b$, $0 \le b < p-1$, $\mu = 2p-b-2$;
b) $p=5$, $\lambda = 15$, $\mu = 13$;
c) $p=5$, $\lambda = 16$, $\mu = 12$.
Then the maximal submodule in $V(\lambda)$ is isomorphic to $M(\mu)$. Hence $V(\lambda)$ has two composition factors: $M(\lambda)$ and $M(\mu)$.

Proof. This facts follow from [7]. It is not difficult to check this directly as we know the weight sets of irreducible modules and Weyl modules of $A_1(K)$. □

Proposition 8. *Let $\Gamma = A_1(K)$. Set $\lambda_1 = 2p+b$ where $0 \le b < p-1$, and $S_1 = \{\lambda_1, \lambda_1 - b - 2, b\}$. For $p=5$ put $\lambda_2 = 15$, $\lambda_3 = 16$ and $S_2 = \{15,\ 13,\ 5,\ 3\}$, $S_3 = \{16,\ 12,\ 6,\ 2\}$. Suppose that M is a Γ-module with the maximal weight λ_i. Then $M = M_1 \oplus M_2$ where M_i is a Γ-module, $\operatorname{Irr} M_1 \subset S_i$, and $\operatorname{Irr} M_2 \cap S_i = \varnothing$ or $M_2 = 0$.*

Proof. Let $a \in S_i$, $b \in \mathbb{Z}^+$, $b \notin S_i$, and $b < \lambda_i$. Show that $Ext^1_\Gamma(M(a), M(b)) = 0$. By Lemmas 23 and 26, $M(c)$ is not a composition factor of $V(d)$ for $(c,d) = (a,b)$. Now Proposition 2 and Lemma 2 imply that $Ext^1_\Gamma(M(b), M(a)) = 0$. To complete the proof, we use Proposition 1. □

Lemma 27. *Let $\Gamma = A_1(K)$. Set $\lambda = p+i$, $0 \le i \le p-2$, and $S = \{p+i, p-i-2\}$. Suppose that M is a Γ-module with the maximal weight λ. Then $M = M_1 \oplus M_2$ where M is a Γ-module, $\operatorname{Irr} M_1 \subset S$, and $\operatorname{Irr} M_2 \cap S = \varnothing$ or $M_2 = 0$.*

Proof. The proof follows from Proposition 1 and Lemma 23. □

Lemma 28. *Let N be an indecomposable Γ-module generated by a highest weight vector v, α be the positive root of Γ, and $\omega(v) = a$. Suppose that $a = p+l$ or $2p+1$ with $l < p-1$ and that $X_{-\alpha}^{l+1}v \ne 0$. Then $N \cong V(a)$.*

Proof. By Lemma 4, N is isomorphic to a quotient module of $V(a)$. Lemmas 23 and 26 yield that in all the cases under consideration the module $V(a)$ has two composition factors. One easily observes that $\omega(N) - (l+1)\alpha \notin \Lambda(M(a))$. Hence $N \not\cong M(a)$. This implies our assertion. □

Lemma 29. *Let N be a self-dual Γ-module with the maximal weight $\omega = p+i$, $0 \le i \le p-2$. Assume that $\dim N_\omega = 1$ and that $X_{-\alpha}^{i+1}m \ne 0$ for a nonzero vector $m \in N_\omega$. Then N has a direct summand isomorphic to $T(p+i)$.*

Proof. Let F be an indecomposable component of N containing the subspace N_ω (such component exists since $\dim N_\omega = 1$). By Lemma 28, $K\Gamma m \cong V(p+i)$. This implies that F is reducible. It is clear that F is self-dual as N is self-dual and ω is not a weight of an indecomposable component of N distinct from F. Now the lemma follows from Lemma 23. □

Remark 4. *If a self-dual Γ-module has a filtration by Weyl modules, then it is a tilting module.*

Indeed, this module has a filtration by dual Weyl modules due to self-duality.

In Theorems 4 and 5 and Proposition 9 $G = C_n(K)$. It is well known that $G \cong Sp_{2n}(K)$.

Theorem 4. [32] *The irreducible representations*

$$\mu_{1,n} = \varphi\left(\omega_{n-1} + \frac{p-3}{2}\omega_n\right) \quad \text{and} \quad \mu_{2,n} = \varphi\left(\frac{p-1}{2}\omega_n\right)$$

of G have the dimensions $\frac{p^n-1}{2}$ and $\frac{p^n+1}{2}$, respectively, and all their weights have multiplicity 1. Let $k < n$, H_1 and $H_2 \subset G$ be commuting subsystem subgroups of types C_k and C_{n-k}, respectively, and $H = H_1H_2$. Then

$$\mu_{1,n}|H \cong \mu_{1,k} \otimes \mu_{2,n-k} \oplus \mu_{2,k} \otimes \mu_{1,n-k},$$
$$\mu_{2,n}|H \cong \mu_{1,k} \otimes \mu_{1,n-k} \oplus \mu_{2,k} \otimes \mu_{2,n-k}$$

(here the first tensor multiplier is a representation of H_1 and the second one is a representation of H_2, $\omega_0 = 0$).

Theorem 5. [21, 8.1] *The irreducible G-modules with highest weights $a\omega_1$ for $a < p$ or $a\omega_i + (p-1-a)\omega_{i+1}$ where $1 \le i < n$ and $a \ne 0$ for $i = n-1$, are equivalent to restrictions to G of the irreducible modules of the group $A_{2n-1}(K)$ with the same highest weights.*

Proposition 9. *(a part of* [16, Corollary 6.2]*) Let $M_1 = \Lambda^2 V$ and $M_2 = M(\omega_2)$. For an element $x \in G$ of order p and an integer k, denote by $r_k^i(x)$ the number of Jordan blocks of size k for the action of x on M_i, $i = 1, 2$.*

1. *Let x have at least one Jordan block of size less than p in the standard realization. Then $r_k^1(x) = r_k^2(x)$ for $k \ne 1$, $r_1^2(x) = r_1^1(x) - 1$ if $p \nmid n$, and $r_1^2(x) = r_1^1(x) - 2$ if $p|n$.*
2. *Let x have only blocks of size p in the standard realization. If $p^2 \mid n+1$, then $r_p^2(x) = r_p^1(x) - 2$, $r_{p-1}^2(x) = 2$, and $r_k^2(x) = r_k^1(x)$ for $k \ne p$ or $p-1$. If $p^2 \nmid n+1$, then $r_p^2(x) = r_p^1(x) - 1$, $r_{p-2}^2(x) = 1$, and $r_k^2(x) = r_k^1(x)$ for $k \ne p$ or $p-2$.*

In Theorems 6 and 7, Lemma 30, and Proposition 10 $G = B_n(K)$ or $D_n(K)$. It is well known that

$$B_n(K) \cong Spin_{2n+1}(K) \quad \text{and} \quad D_n(K) \cong Spin_{2n}(K);$$

the group $B_2(K) \cong C_2(K)$.

Theorem 6. *(see, for example,* [29, Proposition 2.34]*) Let*

$$t = 2n-1, i < n-1 \text{ for } G = D_n(K) \text{ and}$$
$$t = 2n, i < n \text{ for } G = B_n(K).$$

Then the G-module $M(\omega_i)$ is isomorphic to the restriction to G of the irreducible $A_t(K)$-module with the same highest weight.

The following lemma is well known, but we failed to find an explicit reference.

Lemma 30. *Let $G = B_n(K)$, $H_1 = G(1, 2, ..., i-1, \varepsilon_{i-1} + \varepsilon_i)$, $H_2 = G(i+1, ..., n)$, $1 < i < n$, $H = H_1H_2$, and $\Gamma = G(2, 3, ..., n)$. Then*

$$M(\omega_n)|H \cong M(\omega_{i-1}) \otimes M(\omega_{n-i}) \oplus M(\omega_i) \otimes M(\omega_{n-i})$$

(here in each tensor product the first multiplier is an H_1-module and the second one is an H_2-module),

$$M(\omega_n)|\Gamma \cong M(\omega_{n-1}) \oplus M(\omega_{n-1}).$$

Proof. It is clear that $H_1 \cong D_i(K)$ and $H_2 \cong B_{n-i}(K)$. Let $M = M(\omega_n)$ be a G-module. Denote by Ω_1 (respectively, Ω_2) the subspace in $\Lambda(M)$ consisting of all the weights of the form $\{\pm\varepsilon_1 \pm \varepsilon_2 \pm ... \pm \varepsilon_i \pm ... \pm \varepsilon_n\}/2$ with an odd (respectively, an even) number of symbols "minus" for ε_j, $1 \le j \le i$; and let $N_i = \langle M_\lambda | \lambda \in \Omega_i \rangle$. Obviously, $M = N_1 \oplus N_2$. One easily observes that N_1 and N_2 are H-modules. Taking into account the weight structure of M, we conclude that $N_1 \cong M(\omega_{i-1}) \otimes M(\omega_{n-i})$ and $N_2 \cong M(\omega_i) \otimes M(\omega_{n-i})$.

The proof of the second assertion is similar. Here $\Omega_1 = \{(-\varepsilon_1 \pm \varepsilon_2 \pm ... \pm \varepsilon_n)/2\}$ and $\Omega_2 = \{(\varepsilon_1 \pm \varepsilon_2 \pm ... \pm \varepsilon_n)/2\}$. □

Theorem 7. [29] *Let $G = D_n(K)$, $f + g = n - 1$, and let $B \cong B_f(K) \times B_g(K)$ be the subgroup in G determined by the natural embedding $SO_{2f+1}(K) \times SO_{2g+1}(K) \subset SO_{2n}(K)$ (here we put $B_0(K) = 1$). Then the restriction to B of the representations $\varphi(\omega_{n-1})$ and $\varphi(\omega_n)$ are irreducible and equivalent to the tensor product $\varphi(\omega_f) \otimes \varphi(\omega_g)$. For $B \cong B_{n-1}(K)$ these restrictions are equivalent to the spin representation $\varphi(\omega_{n-1})$.*

Proof. The proof follows from the description of the restrictions of weights of G to the relevant subgroups, see [29, Lemma 2.23].

□

Proposition 10. *(a part of* [16, Corollary 6.3]*) Let $M_1 = S^2V$ and $M_2 = M(2\omega_1)$. For an element $x \in G$ of order p and an integer k, denote by $r_k^i(x)$ the number of Jordan blocks of size k for the action of x on M_i, $i = 1, 2$. Set $t = 2n + 1$ for $G = B_n(K)$ and $t = 2n$ for $G = D_n(K)$.*

1. *Let x have at least one Jordan block of size less than p in the standard realization. Then $r_k^1(x) = r_k^2(x)$ for $k \neq 1$, $r_1^2(x) = r_1^1(x) - 1$ if $p \nmid t$, and $r_1^2(x) = r_1^1(x) - 2$ if $p|t$.*
2. *Let x have only blocks of size p in the standard realization. If $p^2 \mid t$, then $r_p^2(x) = r_p^1(x)-2$, $r_{p-1}^2(x) = 2$, and $r_k^2(x) = r_k^1(x)$ for $k \neq p$ or $p-1$. If $p^2 \nmid t$, then $r_p^2(x) = r_p^1(x)-1$, $r_{p-2}^2(x) = 1$, and $r_k^2(x) = r_k^1(x)$ for $k \neq p$ or $p-2$.*

3. The general scheme of proof

The list of p-restricted irreducible representations of the classical algebraic groups whose dimensions are at most 100 can be found in Lubeck's article [18, Theorem 5.1 and Tables in §6]. Obviously, we can omit the standard and trivial modules as well as modules obtained from the standard one with the help of a graph automorphism of a group. Taking this into account, we conclude that the following groups have representations we are interested in: $A_n(K)$, $n \leq 13$, $B_n(K)$, $n \leq 6$, $C_n(K)$ and $D_n(K)$, $n \leq 7$.

In what follows φ is a p-restricted irreducible representation of G, M is a module affording φ, $\omega = \omega(M)$, $M_\lambda \subset M$ is the weight subspace of weight λ, $v \in M$ is a nonzero highest weight vector. If $H_1, H_2 \subset G$ are simple subsystem subgroups and $H = H_1H_2$, then a weight μ of H is written in the form (μ_1, μ_2) where $\mu_1 = \mu|H_1$ and $\mu_2 = \mu|H_2$. An analogical notation is used for subgroups with three simple components. If $A \subset G$ is a fixed subgroup of type A_1, then α is the positive root of A, X_α and $X_{-\alpha}$ are the root operators of its Lie algebra. If N is an A-module, $\mathrm{Irr}(N) = \{M(a_1), ..., M(a_k)\}$, and the multiplicity of the composition factor $M(a_i)$ is equal to b_i, $1 \leq i \leq k$, then set $I(N) = \{b_1M(a_1), b_2M(a_2), ..., b_kM(a_k)\}$. Throughout the text $\mathrm{Inv}(M_\mu)$ is the subspace in M_μ consisting of vectors invariant with respect to $\mathfrak{X}_\alpha$. It is always clear from the context what group is considered. For an element $x \in G$ of order p, let A_x and $\sigma_x : \Lambda(G) \to \mathbb{Z}$ be a good A_1-subgroup containing x and a homomorphism from Lemma 8. It is obvious that $\sigma_x(\omega) = \max_{\mu \in \Lambda(M)} \sigma_x(\mu)$. It occurs that for representations being considered, $\sigma_x(\omega) \leq 3p+3$ and for majority of them, $\sigma_x(\omega) \leq 2p - 2$. To find the dimensions of Jordan blocks of x on M, we determine the indecomposable components of the module $M_x = M|A_x$. First we find the composition factors of M_x. For this purpose, it is necessary to know the dimensions of the weight subspaces of M. By Theorem 2, the set $\Lambda(M) = \Lambda(M_\mathbb{C})$. The latter set can be determined using [4, Chapter 8, §7] if we take into account that $\Lambda(M_\mathbb{C})$ is invariant under the Weyl group W. In some cases the Weyl module $V(\omega)$ is irreducible, i.e. $M \cong V(\omega)$. Then these dimensions are calculated by known formulas [12, §22, Item 3]. The module M often turns out to be a module with one-dimensional weight subspaces from Propositions 3, 4, and Theorem 4. In other cases the composition factors of $V(\omega)$ are determined with the use of the Jantzen filtration [13, Part II, §8, Proposition 8.19] or bases of weight subspaces for dominant weights of M are constructed explicitly.

Lemma 31. *Let $G \neq D_n(K)$ or $\omega = \sum_{i=1}^{n-2} a_i\omega_i$. Then the module M_x is self-dual.*

Proof. For $G = B_n(K)$ or $C_n(K)$ and for $G = D_n(K)$, $\omega = \sum_{i=1}^{n-2} a_i\omega_i$, the module M is self-dual, therefore M_x is self-dual. Let $G = A_n(K)$, φ^* and M^* be the representation and the module dual to the representation φ and the module M. It is clear that $M^*|A_x$ is dual to M_x. Let τ be the graph morphism of G. It is well known that $\varphi^* \cong \varphi \circ \tau$. As the graph automorphism does not change the canonical Jordan form of x on V and therefore fixes its conjugacy class, then the subgroups A_x and $\tau(A_x)$ are conjugate in G. Hence $\varphi^*|A_x \cong \varphi|A_x$. This implies that the restrictions $\varphi|A_x$ and $M|A_x$ are self-dual. □

Corollary 10. *In the assumptions of Lemma* 31 *let* $M_x = N_1 \oplus N_2 \oplus ... \oplus N_j$ *and* $\mathrm{Irr}(N_i) \cap \mathrm{Irr}(N_j) = \varnothing$ *for* $i \neq j$. *Then the modules* N_i *are self-dual. If* $N_j = F_1 \oplus F_2$ *where* F_1 *is an indecomposable module and the maximal weight of* F_1 *is bigger than any weight from* $\Lambda(F_2)$, *then* F_1 *and* F_2 *are self-dual.*

Proof. The corollary follows easily from the self-duality of M_x. □

Lemma 32. *Let* $G = A_2(K)$ *and a weight* $\omega = a_1\omega_1 + a_2\omega_2 \in \Lambda^+(G)$ *be* p*-restricted. Assume that* $a_1 + a_2 = p$ *or* $p + 1$. *Then the restriction* $M(\omega)|G(1)$ *is a tilting module.*

Proof. Put $H = G(1)$, $M = M(\omega)$, and $\Omega_i = \{\mu \in \Lambda(M) | \mu = \omega - i\alpha_2 - k\alpha_1\}$. Since $\omega - (a_1 + a_2)(\alpha_1 + \alpha_2)$ is the lowest weight of M, then $\Omega_i = \varnothing$ for $i > a_1 + a_2$. Set $F_i = \langle M_\mu | \mu \in \Omega_i \rangle$, $0 \leq i \leq a_1 + a_2$. It is clear that F_i is an H-module and $M|H = \bigoplus_{i=0}^{a_1+a_2} F_i$. Put $\mu_i = \omega - i\alpha_2$ for $0 \leq i \leq a_2$ and $\mu_i = \omega - b\alpha_1 - i\alpha_2$ for $i = a_2 + b > a_2$. Obviously, $\dim M_{\mu_i} = 1$ as for $i > a_2$ the weight μ_i lies in the same W-orbit with the weight $\omega - b\alpha_1$. Lemma 12 implies that in all cases $\mu_i \in \Lambda(M)$.

One easily observes that a weight $\nu = \omega - c\alpha_1 - (a_2 + b)\alpha_2 \notin \Lambda(M)$ for $0 \leq c < b$ since $\langle \nu, \alpha_2 \rangle < -(a_2 + b)$. This yields that μ_i is the maximal weight of the H-module F_i. If $k_i = \langle \mu_i, \alpha_1 \rangle < p$, then by Corollary 2, F_i is a direct sum of irreducible p-restricted H-modules. Observe that $k_i = a_1 + i$ for $i \leq a_2$ and $k_i = a_1 + a_2 - b$ for $i = a_2 + b > a_2$. Hence for $a_1 + a_2 = p$, the integer $k_{a_2} = p$ and $k_i < p$ for $i \neq a_2$. If $a_1 + a_2 = p + 1$, then $k_{a_2} = p + 1$, $k_{a_2-1} = k_{a_2+1} = p$, and $k_i < p$ for $i \notin \{a_2 - 1, a_2, a_2 + 1\}$. For $a_1 + a_2 = p$, set $m = X_{-2,a_2}v$. If $a_1 + a_2 = p + 1$, then $a_1, a_2 > 1$ as M is p-restricted. In this case we put $m_1 = X_{-2,a_2-1}v$, $m_2 = X_{-2,a_2}v$, and $m_3 = X_{-2,a_2+1}X_{-1}v$. By Lemma 12, m, m_1, m_2, and m_3 are nonzero. We shall show that

$$(3) \qquad KHm \cong KHm_1 \cong KHm_3 \cong V(p), \quad KHm_2 \cong V(p+1).$$

By Lemma 28, for this it suffices to show that $X_{-1}m$, $X_{-1}m_1$, $X_{-1}m_3$, and $X_{-1}^2m_2 \neq 0$. Put $u = X_{2,a_2}X_{-1}m$, $u_1 = X_{2,a_2-1}X_{-1}m_1$, $u_2 = X_{2,a_2}X_{-1}^2m_2$, and $u_3 = X_{2,a_2+1}X_{-1}m_3$. Taking into account the commutation relations in the Lie algebra of G, we obtain that $u = cX_{-1}v$, $u_1 = c_1X_{-1}v$, $u_2 = c_2X_{-1}^2v$, and $u_3 = c_3X_{-1}v$ where c and $c_i \in K^*$, $1 \leq i \leq 3$. Since $a_1 \neq 0$ and $a_1 > 1$ for $a_1 + a_2 = p + 1$, Lemma 6 implies that all the vectors u, u_1, u_2, and $u_3 \neq 0$. Therefore m, m_1, m_2, and $m_3 \neq 0$ and Formula (3) holds.

For $a_1 + a_2 = p$, set $Q = F_{a_2}$ and denote by N the indecomposable component of module Q containing m. For $a_1 + a_2 = p + 1$, set $Q_1 = F_{a_2-1}$, $Q_2 = F_{a_2}$, and $Q_3 = F_{a_2+1}$ and denote by N_1, N_2, and N_3 the indecomposable components of the modules Q_1, Q_2, and Q_3 containing the vectors m_1, m_2, and m_3, respectively. Such components exist since the weight subspaces of M containing these vectors are one-dimensional.

Since N, N_1, N_2, and N_3 contain the Weyl modules $V(p)$ or $V(p+1)$, then Lemmas 23 and 27 yield that N, N_1, and N_3 are isomorphic to $T(p)$ or $V(p)$ and $N_2 \cong T(p + 1)$ or $V(p + 1)$. By Lemma 31, the module $M|H$ is self-dual. Hence the modules N, N_2, and $N_1 \oplus N_3$ are self-dual since other indecomposable components of $M|H$ have no weights p, $p + 1$, and p in the first, the second, and the third cases, respectively. This implies that $N \cong N_1 \cong N_3 \cong T(p)$ and $N_2 \cong T(p + 1)$. Other indecomposable components of $M|H$ have all weights less than p. Therefore they are p-restricted irreducible modules. □

Next, often it is necessary to prove that for some weight vector $m \in M$ invariant with respect to the group $\mathfrak{X}_\alpha$, the module KA_xm is isomorphic to $V(p+b)$ with $0 \leq b < p-1$ or $2p+b$ with $b \leq 4$. For this, we always use Lemma 28.

In a number of cases Corollary 7 allows one to prove that M_x is a tilting module. Then it is a direct sum of modules $T(\lambda)$. In particular, by Lemma 14, M_x is a tilting module for $\omega = a\omega_1$, $a < p$, if $G = A_n(K)$ or $C_n(K)$. If M_x is a tilting module, then its indecomposable components $T(\lambda)$ are determined with the use of Lemma 23 and Proposition 7. In situations where a priori it is not clear whether M_x is a tilting module, explicit calculations are used to determine indecomposable components of M_x. The action of the root elements of the Lie algebra of A_x and elements from the hyperalgebra of this group on certain weight elements of M_x is considered here. An essential role is played by Lemma 23. Often it is necessary to find out whether a module M_x has a direct summand of the form $T(\lambda) \oplus M(\lambda)$ or $V(\lambda) \oplus \Delta(\lambda)$ for $\lambda = p + a$, $a < p - 1$. For this, we use Lemma 25. Obviously, $\sigma_x(\omega) = \max_{\mu \in \Lambda(M)} \sigma_x(\mu)$ as $\sigma_x(\alpha_i) \geq 0$. So by Corollary 2, M_x is a direct sum of p-restricted modules if $\sigma_x(\omega) < p$. If the module M_x is a direct sum of irreducible submodules and tilting modules, then the block structure of $\varphi(x)$ is determined with the use of Lemmas 23 and 7 and the formula from Theorem 3.

Let y be a regular unipotent element and $M_i = M_i(y)$ be the sum of all weight subspaces M_λ with $\sigma_y(\lambda) = i$. It is well known that for $G \neq D_n(K)$, the restriction $V|A_y$ is an irreducible module $M(a)$ with $a < p$. Recall that in this situation the module $M(a) \cong V(a)$ is infinitesimally irreducible too. Hence V has a basis where the action of the operators $X_{\pm\alpha}$ is determined by the same formulae as for the irreducible module with highest weight a in characteristic 0. These formulae can be found in [4, Chapter 8, §13.1]. For $G = B_n(K)$ or $C_n(K)$, we can determine explicitly an A_y-invariant symmetric or skew-symmetric nonsingular bilinear form on V (unique up to scalars). Then we choose a basis in which this form has a canonic form (as in [4, Chapter 8, §13.2 and §13.3]). Now using the formulae from [4, Chapter 8, §13.1–§13.3] describing the action of the root operators $X_{\pm i}$ in the standard modules for the classical Lie algebras, for all three series one can deduce explicit formulae for expressing the operators X_α and $X_{-\alpha}$ as linear combinations of the operators X_i and X_{-i}, respectively. Analyzing the action of A_y on V, it is not difficult to show that all coefficients in these linear combinations are nonzero.

To investigate the block structure of other unipotent elements, we use an analysis of direct summands of restrictions of φ to subsystem subgroups. For wedge, symmetric and reduced symmetric powers of the standard module for a group of type A_n, the explicit formulae (2) from Lemma 19 are used. Below if x is conjugate to an element from a proper subgroup $\Gamma = G(i_1, i_2, ..., i_k)$ and M is a direct sum of irreducible Γ-modules, then the block structure of $\varphi(x)$ is determined on the base of results obtained earlier and the formulae from Theorem 3. In what follows this approach is applied without special comments. Let x be a regular unipotent element from such subgroup Γ. Then there exist a subgroup $A \subset \Gamma$ and maximal tori $T_A \subset A$ and $T_\Gamma \subset \Gamma$ such that $A \cong A_1(K)$, $x \in A$, and the homomorphism $\sigma : \Lambda(\Gamma) \to \mathbb{Z}$ determined by restricting weights from T_Γ to T_A takes the root α_{i_1}, ..., α_{i_k} (more exactly, their restrictions to Γ) to 2. For such elements in some cases we consider not the subgroup A_x and the homomorphism σ_x, but the restriction $M|\Gamma$, the subgroup A, and the homomorphism σ; here α is the positive root of A. In all these situations it is explicitly indicated in what subsystem subgroup x is contained, the notation A and σ is used.

4. Special linear groups

In this section the problem is solved for representations of special linear groups. We apply Theorem 3 and Proposition 5 to solve the problem for $\omega = \omega_1 + \omega_n$ and Proposition 6 to do this for a regular unipotent element of order p and $\omega = \omega_2$ or $2\omega_1$. If a unipotent element is not regular, then it is conjugate to a regular element from a proper subsystem subgroup. To determine the block structure of such element of order p on a wedge, symmetric, or reduced symmetric power of the standard module, we use Theorem 3, Lemma 19, and an information on such structure for regular unipotent elements in analogical modules for groups of type A_l and smaller rank. This permits us to fill the corresponding positions in Tables 2–12. Hence we do not need to discuss the block structure of nonregular unipotent elements in such modules in more detail. Now we can and shall assume that $\omega \notin \{\omega_2,\ 2\omega_1,\ \omega_1+\omega_n\}$.

It is well known that if $\omega(\varphi) = \sum_{i=1}^{n} a_i\omega_i$, then the highest weight of φ^* equals $\sum_{i=1}^{n} a_{n+1-i}\omega_i$. As we have mentioned in Section 3, it suffices to consider only one of the representations φ and φ^*.

Let x be a transvection. Obviously, then $\sigma_x(\varepsilon_1) = 1$, $\sigma_x(\varepsilon_{n+1}) = -1$, $\sigma_x(\varepsilon_i) = 0$ for $2 \le i \le n$, and $\sigma_x(\omega) = \sum_{i=1}^{n} a_i$. Corollary 2 yields that M_x is a direct sum of p-restricted modules if $\sum_{i=1}^{n} a_i < p$.

Now we shall indicate when M_x is a tilting module for any element x of order p by the results of Section 2. By Lemma 13, M_x is a tilting module for $\omega = \omega_i$, $i < p$. Lemma 14 implies that M_x is a tilting module if $\omega = a\omega_1$ with $2 \le a < p$. Here we choose $(a-1)\omega_1$ and ω_1 for the weights λ_1 and λ_2 from Lemma 14. It is clear that M_x is a tilting module for $\omega = a\omega_n$ with $a < p$ as well since in this case $\omega(\varphi^*) = a\omega_1$.

1. Let $G = A_2(K)$. By Lemma 14, M_x is a tilting module for any element x of order p in the following cases:
for $\omega = 2\omega_1 + \omega_2$ and $p \ge 3$;
for $\omega = 2\omega_1 + 2\omega_2$ and $p \ne 5$;
for $\omega \in \{\omega_1+\omega_2,\ 4\omega_1+\omega_2\}$ and $p \ge 5$;
for $\omega \in \{4\omega_1+2\omega_2,\ 4\omega_1+3\omega_2\}$ and $p \ne 7$;
for $\omega \in \{3\omega_1+\omega_2, 3\omega_1+2\omega_2\}$ and $p \ge 7$;
for $\omega \in \{5\omega_1 + \omega_2, 6\omega_1 + \omega_2, 3\omega_1+3\omega_2, 7\omega_1+\omega_2, 5\omega_1+2\omega_2, 8\omega_1 + \omega_2\}$ and $p \ge 11$.
Here if $\omega = a\omega_1 + b\omega_2$, we take $\lambda_1 = a\omega_1$ and $\lambda_2 = b\omega_2$ in the assumptions of Lemma 14.

2. Let $G = A_3(K)$. By Corollary 7, M_x is a tilting module for any element x of order p
for $\omega \in \{\omega_1+\omega_2, 2\omega_2\}$ and $p \ge 5$;
for $\omega \in \{\omega_1+\omega_2+\omega_3, 3\omega_1+\omega_2, 2\omega_1+2\omega_3\}$ and $p \ge 7$.

Lemma 14 yields that M_x is a tilting module for the following ω and p:
$\omega \in \{\omega_1+\omega_3,\ 2\omega_1+\omega_2,\ \omega_1+2\omega_2\}$ and $p \ge 3$;
$\omega = 2\omega_1+\omega_3$ and $p \ne 5$;
$\omega \in \{3\omega_2,\ 3\omega_1+\omega_3\}$ and $p \ge 5$.

3. Let $G = A_4(K)$. By Lemma 14, M_x is a tilting module for the following ω and p:
$\omega = \omega_1+\omega_2$ and $p \ne 3$;
$\omega = \omega_1+\omega_3$ and $p \ge 3$;
$\omega \in \{2\omega_2,\ 2\omega_1+\omega_4,\ \omega_2+\omega_3\}$ and $p \ge 5$.

4. Let $G = A_5(K)$. Lemma 14 implies that M_x is a tilting module for the following ω and p:
$\omega = \omega_1+\omega_2$ and $p \ne 3$;
$\omega = \omega_1+\omega_4$ and $p \ne 5$.

To apply Lemma 14 to the representations indicated in Items 2–4, we take $\lambda_1 = a\omega_i$ and $\lambda_2 = g\omega_j$ if $\omega = a\omega_i + b\omega_j$, $i < j$, and $\lambda_1 = (a-1)\omega_2$, $\lambda_2 = \omega_2$ if $\omega = a\omega_2$.

In all these cases $V(\omega)$ is irreducible. So for a fixed element x of order p, we can find the composition factors and the indecomposable components of M_x using the approaches described in Section 3.

Now we consider the remaining representations. It what follows Remark 3 and Lemma 28 are used without special comments.

Using the arguments at the end of the previous section, we can assume that for a regular unipotent element y of order p the following holds

$$X_\alpha = X_1 + 2X_2 + ... + iX_i + ... + nX_n,$$
$$X_{-\alpha} = nX_{-1} + (n-1)X_{-2} + ... + X_{-n}. \tag{4}$$

I. Let $G = A_2(K)$. Then $X_\alpha = X_1 + 2X_2$, $X_{-\alpha} = 2X_{-1} + X_{-2}$ for the root elements of the Lie algebra of A_y. First we deal with $\varphi(y)$ and M_y.

I.I. Let $p = 5$ and $\omega = 3\omega_1 + \omega_2$. Then $\dim\varphi = 18$ and $M \cong S^{5,5}(V)$. One can directly verify that $\sigma_y(\omega) = 8$, $\dim M_8 = 1$, $\dim M_6 = 2$, $\dim M_4 = 2$, $\dim M_2 = 3$, and $\dim M_0 = 2$. Hence $I(M_y) = \{M(8), M(6), 2M(2)\}$. By Lemma 23, $M_y = M(8) \oplus N$ where $I(N) = \{M(6), 2M(2)\}$. Prove that $N \cong T(6)$. It is not difficult to see that the subspace M_6 is generated by the vectors $X_{-1}v$ and $X_{-2}v$ and contains a nonzero vector m invariant with respect to $\mathfrak{X}_\alpha$. The vector m is determined up to scalars. We have $m = b_1X_{-1}v + b_2X_{-2}v$ with $b_1 \neq 0$. Since $X_{-1}^2 m \neq 0$, then $X_{-\alpha}^2 m \neq 0$. Hence by Lemma 29, $N \cong T(6)$.

I.II. Let $p = 5$ and $\omega = 2\omega_1 + 2\omega_2$. Then $\dim\varphi = 19$ and $M \cong S^{6,5}(V)$. It is easy to show that $\dim M_8 = 1$, $\dim M_6 = 2$, $\dim M_4 = 3$, $\dim M_2 = 2$, and $\dim M_0 = 3$. Therefore $I(M_y) = \{M(8), M(6), M(4), 2M(0)\}$. By Lemma 23, $M_y = N \oplus M(6) \oplus M(4)$ where $I(N) = \{M(8), 2M(0)\}$. Prove that $N \cong T(8)$. Obviously, $v \in N$. One can directly verify that $3X_{-1}^3X_{-2}v + 3X_{-1}X_{-2}X_{-1}^2v + 3X_{-1}^2X_{-2}X_{-1}v$ is a weight component of $X_{-\alpha}^4 v$. Taking into account that all weight subspaces of M are one-dimensional, we can show that $3X_{-1}^3X_{-2}v + 3X_{-1}X_{-2}X_{-1}^2v + 3X_{-1}^2X_{-2}X_{-1}v = 4X_{-1}^3X_{-2}v \neq 0$ (by Lemma 12) and hence $X_{-\alpha}^4 v \neq 0$. Then $N \cong T(8)$ by Lemma 29.

I.III. Let $p = 7$ and $\omega = 5\omega_1 + \omega_2$. Then $\dim\varphi = 33$ and $M \cong S^{7,7}(V)$. One can directly verify that $\dim M_{12} = 1$, $\dim M_{10} = 2$, $\dim M_8 = 2$, $\dim M_6 = 3$, $\dim M_4 = 3$, $\dim M_2 = 4$, and $\dim M_0 = 3$. We have $I(M_y) = \{M(12), M(10), M(6), 2M(2)\}$. By Lemma 23, $M_y = M(12) \oplus N \oplus M(6)$ where $I(N) = \{M(10), 2M(2)\}$. Prove that $N \cong T(10)$. It is not difficult to check that the vector $m_{10} = X_{-1}v + X_{-2}v$ is invariant under $\mathfrak{X}_\alpha$. Then $X_{-\alpha}^4 m_{10} \neq 0$ since $X_{-1}^5 v \neq 0$. It is clear that $m_{10} \in N$, therefore by Lemma 29, $N \cong T(10)$.

I.IV. Let $p = 7$ and $\omega = 4\omega_1 + 2\omega_2$. Then $\dim\varphi = 36$ and $M \cong S^{8,7}(V)$. One can directly verify that $\dim M_{12} = 1$, $\dim M_{10} = 2$, $\dim M_8 = 3$, $\dim M_6 = 3$, $\dim M_4 = 4$, $\dim M_2 = 3$, and $\dim M_0 = 4$. Hence $I(M_y) = \{M(12), M(10), M(8), 2M(4), 2M(0)\}$. By Lemma 23, $M_y = N_1 \oplus M(10) \oplus N_2$ where $I(N_1) = \{M(12), 2M(0)\}$ and $I(N_2) = \{M(8), 2M(4)\}$. Prove that $N_1 \cong T(12)$ and $N_2 \cong T(8)$. Obviously, $v \in N_1$. One can directly check that $3X_{-2}^4X_{-1}^2v$ is a nonzero weight component of $X_{-\alpha}^6 v$. So $X_{-\alpha}^6 v \neq 0$. By Lemma 29, $N_1 \cong T(12)$.

It is clear that $\dim \operatorname{Inv} M_8 \leq 1$ and it is easy to show that the vector $m_8 = 4X_{-1}^2v + X_{-1}X_{-2}v + 4X_{-2}^2v$ is invariant under $\mathfrak{X}_\alpha$. So $X_{-1}^4 v \neq 0$, then $X_{-\alpha}^2 m_8 \neq 0$. Now it follows from Lemma 29 that $N_2 \cong T(8)$.

I.V. Let $p = 7$ and $\omega = 3\omega_1 + 3\omega_2$. Then $\dim\varphi = 37$ and $M \cong S^{9,7}(V)$. One can directly verify that $\dim M_{12} = 1$, $\dim M_{10} = 2$, $\dim M_8 = 3$, $\dim M_6 = 4$, $\dim M_4 = 3$, $\dim M_2 = 4$, and $\dim M_0 = 3$. Therefore

$$I(M_y) = \{M(12), M(10), M(8), M(6), 2M(2)\}.$$

By Lemma 23, $M_y = M(12) \oplus N \oplus M(8) \oplus M(6)$ where $I(N) = \{M(10), 2M(2)\}$. Prove that $N \cong T(10)$.

One easily observes that $\dim \operatorname{Inv} M_{10} \leq 1$. Set $m_{10} = X_{-1}v + 3X_{-2}v$. One can directly verify that $m_{10} \in \operatorname{Inv} M_{10}$ and that $4X_{-1}^4X_{-2}v$ is a nonzero weight component of $X_{-\alpha}^4 m_{10}$. Hence $\operatorname{Inv} M_{10} = \langle m_{10}\rangle$. Since $X_{-\alpha}^4 m_{10} \neq 0$, then by Lemma 29, $N \cong T(10)$.

I.VI. Let $p=5$ and $\omega = 3\omega_1 + 2\omega_2$. Then $\dim\varphi = 39$. Set

$$\begin{aligned}\Lambda_6 =&\{\omega-2\alpha_1, \omega-2\alpha_2, \omega-\alpha_1-\alpha_2\};\\ \Lambda_4 =&\{\omega-3\alpha_1, \omega-2\alpha_1-\alpha_2, \omega-\alpha_1-2\alpha_2\};\\ \Lambda_2 =&\{\omega-3\alpha_1-\alpha_2, \omega-2\alpha_1-2\alpha_2, \omega-\alpha_1-3\alpha_2\};\\ \Lambda_0 =&\{\omega-3\alpha_1-2\alpha_2, \omega-4\alpha_1-\alpha_2, \omega-2\alpha_1-3\alpha_2\}.\end{aligned}$$

One easily concludes that $M_i = \langle M_\mu | \mu \in \Lambda_i\rangle$. By [21, 6.9], $\dim M_\lambda = 2$ for $\lambda = \omega-\alpha_1-\alpha_2$. By Lemma 15, the maximal submodule of $V(\omega)$ is isomorphic to $M(\omega_1)$. Now using the Freudenthal formula [12, §22, Item 3], we can show that $\dim M_\mu = 2$ for $\mu \in \{\omega-2\alpha_1-\alpha_2,\ \omega-2\alpha_1-2\alpha_2\}$. Put $\Delta = \{\omega-\alpha_1-2\alpha_2, \omega-3\alpha_1-\alpha_2, \omega-2\alpha_1-3\alpha_2, \omega-3\alpha_1-2\alpha_2\}$ and $\Omega = \{\omega-3\alpha_1, \omega-2\alpha_2, \omega-2\alpha_1, \omega-\alpha_1-3\alpha_2, \omega\ -4\alpha_1-\alpha_2\}$. One easily observes that $\omega-\alpha_1-2\alpha_2$ and $\omega-3\alpha_1\ -\ \alpha_2$ lie in the same W-orbit with the weight $\omega-\alpha_1-\alpha_2$; the weight $\omega-2\alpha_1-3\alpha_2$ in the same W-orbit with $\omega-2\alpha_1-\alpha_2$; the weight $\omega-3\alpha_1-2\alpha_2$ in the same W-orbit with $\omega-2\alpha_1-2\alpha_2$; the weight $\omega-\alpha_1-3\alpha_2$ in the same W-orbit with $\omega-\alpha_1$; and the weight $\omega-4\alpha_1-\alpha_2$ in the same W-orbit with $\omega-\alpha_2$. Hence $\dim M_\mu = 2$ for $\mu\in\Delta$ and $\dim M_\mu = 1$ for $\mu\in\Omega$. This implies that $\dim M_{10} = 1$, $\dim M_8 = 2$, $\dim M_6 = 4$, $\dim M_4 = 5$, $\dim M_2 = 5$, and $\dim M_0 = 5$. We have

$$I(M_y) = \{M(10), 2M(8), 2M(6), M(4), 2M(2), M(0)\}.$$

By Proposition 8 and Lemma 27, $M_y = N_1 \oplus N_2 \oplus M(4)$ where $I(N_1) = \{M(10), 2M(8), M(0)\}$ and $I(N_2) = \{2M(6), 2M(2)\}$.

Show that $N_1 \cong T(10)$ and $N_2 \cong T(6)\oplus M(6)$.

Obviously, $v \in N_1$. Since $X_{-1}v \neq 0$, the vector $X_{-\alpha}v \neq 0$. Hence $KA_yv \cong V(10)$ by Lemma 28. Let F be the indecomposable component of N_1 containing v and $\overline{F} = F/KA_yv$. As $M_8 = \langle X_{-1}v, X_{-2}v\rangle$, the subspace $\operatorname{Inv} M_8$ is one-dimensional. Since $F\cap \operatorname{Inv} M_8 \neq 0$, then M_y has no indecomposable components with highest weight 8 and therefore $M(8) \in \operatorname{Irr}\overline{F}$. It is not difficult to show that the pairs $(X_{-1}^3X_{-2}^2v,\ X_{-2}^2X_{-1}^3v)$ and $(X_{-2}X_{-1}^2X_{-2}^2v,\ X_{-2}^3X_{-1}^2v)$ consist of linearly independent vectors. This implies that

$$X_{-1}^4X_{-2}v, X_{-1}^3X_{-2}^2v, X_{-2}^2X_{-1}^3v, X_{-2}X_{-1}^2X_{-2}^2v, X_{-2}^3X_{-1}^2v$$

is a basis in M_0. Using this basis, we can check that there are no nonzero $\mathfrak{X}_\alpha$-invariant vectors in M_0. Hence there are no submodules isomorphic to $M(0)$ in M_y. This forces that $N_1 = F$. The module N_1 is self-dual because M_y is self-dual. Hence we conclude that N_1 has no factor modules isomorphic to $M(0)$. Therefore N_1 has a filtration by Weyl modules. As N_1 is self-dual, it has a filtration by dual Weyl modules as well. Hence N_1 is a tilting module. Therefore $N_1 \cong T(10)$.

It is clear that $\dim \operatorname{Inv} M_6 \leq 2$. Set $u_1 = X_{-1}^2v + X_{-2}^2v + X_{-2}X_{-1}v$ and $u_2 = 2X_{-1}^2v + 4X_{-2}^2v + X_{-1}X_2v$. One can directly verify that u_1 and $u_2 \in \operatorname{Inv} M_6$. Hence $\operatorname{Inv} M_6 = \langle u_1, u_2\rangle$. It is not difficult to check that each of the pairs $(X_{-2}X_{-1}^3v,\ X_{-1}^3X_{-2}v)$ and $(X_{-1}^2X_{-2}^2v,\ X_{-2}^2X_{-1}^2v)$ consists of linearly independent vectors. Therefore the vectors $X_{-2}X_{-1}^3v$, $X_{-1}^3X_{-2}v$, $X_{-2}^3X_{-1}v$, $X_{-1}^2X_{-2}^2v$, and $X_{-2}^2X_{-1}^2v$ form basis in M_2. Set $U = \langle u_1, u_2\rangle \cap X_\alpha^2 M_2$ and $M_2' = N_2 \cap M_2$. It is clear that $U = \langle u_1, u_2\rangle\cap X_\alpha^2 M_2'$. Using the basis of M_2 indicated above, we easily show that $\dim U = 1$ and $X_{-\alpha}^2 U \neq 0$. This yields that $N_2 \neq M(6)\oplus M(6)\oplus M(2)\oplus M(2)$ or $V(6)\oplus\Delta(6)$ (the latter by Lemma 25). Using the self-duality of N_2 and Lemma 23, we get that $N_2 \cong T(6)\oplus M(6)$.

I.VII. Let $p=5$ and $\omega = 3\omega_1 + 3\omega_2$. Then $\dim\varphi = 63$. One easily observes that $\sigma_y(\omega) = 12$. It follows from [18, Table 6.6] that $\dim V(\omega) = \dim M(\omega)+1$. Hence $\dim M_\mu =$

$\dim V(\omega)_\mu$ for $\mu \neq 0$ and $\dim M_\mu = \dim V(\omega)_\mu - 1$ for $\mu = 0$. Set

$$\begin{aligned}\Lambda_8 =&\{\omega - 2\alpha_1, \omega - \alpha_1 - \alpha_2, \omega - 2\alpha_2\};\\ \Lambda_6 =&\{\omega - 3\alpha_1, \omega - 2\alpha_1 - \alpha_2, \omega - \alpha_1 - 2\alpha_2, \omega - 3\alpha_2\};\\ \Lambda_4 =&\{\omega - 3\alpha_1 - \alpha_2, \omega - 2\alpha_1 - 2\alpha_2, \omega - \alpha_1 - 3\alpha_2\};\\ \Lambda_2 =&\{\omega - 4\alpha_1 - \alpha_2, \omega - 3\alpha_1 - 2\alpha_2, \omega - 2\alpha_1 - 3\alpha_2, \omega - \alpha_1 - 4\alpha_2\};\\ \Lambda_0 =&\{\omega - 4\alpha_1 - 2\alpha_2, \omega - 3\alpha_1 - 3\alpha_2, \omega - 2\alpha_1 - 4\alpha_2\}.\end{aligned}$$

One easily concludes that $M_i = \langle M_\mu | \mu \in \Lambda_i\rangle$. Taking into account Lemma 22, we can show that $\dim M_\mu = 2$ for $\mu \in \{\omega - \alpha_1 - \alpha_2, \omega - 2\alpha_1 - \alpha_2, \omega - \alpha_1 - 2\alpha_2\}$ and $\dim M_\mu = 3$ for $\mu = \omega - 2\alpha_1 - 2\alpha_2$. Using the Freudenthal formula, we obtain that $\dim M_\mu = 3$ for $\mu = 0$. It is not difficult to observe that the weights $\omega - 3\alpha_1 - \alpha_2$ and $\omega - \alpha_1 - 3\alpha_2$ lie in the same W-orbit with $\omega - \alpha_1 - \alpha_2$, the weight $\omega - 4\alpha_1 - 2\alpha_2$ lies in the same W-orbit with $\omega - \alpha_1 - 2\alpha_2$, the weight $\omega - 2\alpha_1 - 4\alpha_2$ lies in the same W-orbit with $\omega - 2\alpha_1 - \alpha_2$, the weights $\omega - 3\alpha_1 - 2\alpha_2$ and $\omega - 2\alpha_1 - 3\alpha_2$ lie in the same W-orbit with $\omega - 2\alpha_1 - 2\alpha_2$, the weight $\omega - \alpha_1 - 4\alpha_2$ lies in the same W-orbit with $\omega - \alpha_1$, and the weight $\omega - 4\alpha_1 - \alpha_2$ lies in the same W-orbit with $\omega - \alpha_2$. Hence $\dim M_{12} = 1$, $\dim M_{10} = 2$, $\dim M_8 = 4$, $\dim M_6 = 6$, $\dim M_4 = 7$, $\dim M_2 = 8$, and $\dim M_0 = 7$. Therefore $I(M_y) = \{M(12), M(10), 3M(8), 3M(6), M(4), 3M(2), M(0)\}$.

By Proposition 8 and Lemma 27, $M_y = N_1 \oplus N_2 \oplus M(4)$ where $I(N_1) = \{M(12), 3M(6), 3M(2)\}$ and $I(N_2) = \{M(10), 3M(8), M(0)\}$.

Show that $N_1 \cong T(12) \oplus T(6)$. By Lemma 12, $X^4_{-2}X_{-1}v$ and $X^4_{-1}X_{-2}v \neq 0$. One can verify that the pairs $(X^2_{-1}X_{-2}v,\ X_{-2}X^2_{-1}v)$ and $(X_{-1}X^2_{-2}v,\ X^2_{-2}X_{-1}v)$ and the triples $(X^3_{-1}X^2_{-2}v,\ X_{-2}X^3_{-1}X_{-2}v,\ X^2_{-2}X^3_{-1}v)$ and $(X^3_{-2}X^2_{-1}v,\ X_{-1}X^3_{-2}X_{-1}v,\ X^2_{-1}X^3_{-3}v)$ consist of linearly independent vectors. Hence

$$\begin{aligned}M_6 =&\langle X^3_{-1}v,\ X^2_{-1}X_{-2}v,\ X_{-2}X^2_{-1}v,\ X_{-1}X^2_{-2}v,\ X^2_{-2}X_{-1}v, X^3_{-2}v\rangle,\\ M_2 =&\langle X^4_{-1}X_{-2}v,\ X^3_{-1}X^2_{-2}v,\ X_{-2}X^3_{-1}X_{-2}v,\ X^2_{-2}X^3_{-1}v,\ X^3_{-2}X^2_{-1}v,\\ &X_{-1}X^3_{-2}X_{-1}v,\ X^2_{-1}X^3_{-3}v,\ X^4_{-2}X_{-1}v\rangle.\end{aligned}$$

Set $m^1_6 = 3X^3_{-1}v + X^2_{-1}X_{-2}v + X_{-1}X^2_{-2}v$ and $m^2_6 = 2X_{-2}X^2_{-1}v + X^2_{-2}X_{-1}v + 3X^3_{-2}v$. Using the bases of M_6 and M_2 indicated above, it is not difficult to show that $\operatorname{Inv} M_6 = \langle m^1_6, m^2_6\rangle$ and $\dim \operatorname{Inv} M_2 = 1$. Let $N_1 = F_1 \oplus F_2$ where F_1 is the indecomposable component of N_1 containing v. As $X^3_{-1}v \neq 0$, the vector $X^3_{-\alpha}v$ has a nonzero component of weight $\omega - 3\alpha_1$ and hence $X^3_{-\alpha}v \neq 0$. This and Lemma 26 yield that $KA_yv \cong V(12)$. Put $\overline{F_1} = F_1/KA_yv$. By Corollary 10, F_1 and F_2 are self-dual.

Show that $F_1 \cong T(12)$. Lemma 23 implies that the indecomposable components of $\overline{F_1}$ are isomorphic to $V(6)$, $\Delta(6)$, $T(6)$, $M(6)$, or $M(2)$. Now we prove that there are no components isomorphic to $\Delta(6)$ or $M(6)$ in $\overline{F_1}$.

Suppose that $\overline{F_1}$ has an indecomposable component isomorphic to $\Delta(6)$ or $M(6)$. Now Lemma 23 yields that one of the following holds:

1) $\overline{F_1} \cong \Delta(6) \oplus \Delta(6) \oplus U_1$, $U_1 \oplus F_2 \cong M(2)$;
2) $\overline{F_1} \cong \Delta(6) \oplus V(6) \oplus U_1$, $U_1 \oplus F_2 \cong M(2)$;
3) $\overline{F_1} \cong \Delta(6) \oplus M(6) \oplus U_1$, $U_1 \oplus F_2 \cong M(2) \oplus M(2)$;
4) $\overline{F_1} \cong \Delta(6)$, $F_2 \cong T(6)$;
5) $\overline{F_1} \cong \Delta(6) \oplus U_1$, $F_2 \cong M(6) \oplus U_2$, $U_1 \oplus U_2 \cong M(2) \oplus M(2)$;
6) $\overline{F_1} \cong M(6) \oplus M(6) \oplus U_1$, $U_1 \oplus F_2 \cong M(2) \oplus M(2) \oplus M(2)$;
7) $\overline{F_1} \cong M(6) \oplus V(6) \oplus U_1$, $U_1 \oplus F_2 \cong M(2) \oplus M(2)$;
8) $\overline{F_1} \cong M(6) \oplus U_1$, $F_2 \cong T(6) \oplus U_2$, $U_1 \oplus U_2 \cong M(2) \oplus M(2)$;
9) $\overline{F_1} \cong M(6) \oplus U_1$, $F_2 \cong M(6) \oplus U_2$, $U_1 \oplus U_2 \cong M(2) \oplus M(2) \oplus M(2)$.

We shall show that N_1 has a submodule isomorphic to $M(2) \oplus M(2)$ in all these cases. Put $L = M(2) \oplus M(2) \oplus U_1$ in Case 1), $M(2) \oplus U_1$ in Cases 2), 3), and 5), $M(2)$ in Case 4), and U_1 in Cases 6)–9). As $\Delta(6)$ has a factor module isomorphic to $M(2)$, one easily observes that F_1 has a factor module isomorphic to L in all cases. Then there is a submodule isomorphic to L in F_1 since F_1 is self-dual. Set $L' = L \oplus M(2)$ in Case 4), $L \oplus U_2$ in Cases 5) and 9), $L \oplus M(2) \oplus U_2$ in Case 8), and $L \oplus F_2$ in other cases. Since $T(6)$ has a submodule isomorphic to $M(2)$, it is obvious that N_1 has a submodule isomorphic to L' in all cases. It is not difficult to see that $L' \cong M(2) \oplus M(2)$ or $M(2) \oplus M(2) \oplus M(2)$. This implies that N_1 has a submodule isomorphic to $M(2) \oplus M(2)$. But $\dim \mathrm{Inv}(M_2) = 1$. A contradiction obtained shows that $\overline{F_1}$ has no indecomposable components isomorphic to $\Delta(6)$ or $M(6)$.

Now it is easy to see that F_1 has a filtration by Weyl modules. Then F_1 has a filtration by dual Weyl modules as it is self-dual. Hence F_1 is a tilting module. This forces that $F_1 \cong T(12)$.

Proposition 7 yields that $I(F_2) = \{M(6), 2M(2)\}$. As $\dim \mathrm{Inv}(M_2) = 1$, then $F_2 \not\cong M(6) \oplus M(2) \oplus M(2)$. Since F_2 is self-dual, then by Lemma 23, $F_2 \cong T(6)$.

Show that $N_2 \cong T(10) \oplus M(8)$. One easily observes that the vectors $X_{-1}^2 v$, $X_{-1}X_{-2}v$, $X_{-1}X_{-2}v$, and $X_{-2}^2 v$ form a basis in M_8. Using this basis, we can show that $\dim \mathrm{Inv}\, M_8 = 2$,

$$\mathrm{Inv}\, M_8 = \langle 2X_{-1}^2 v + 4X_{-1}X_{-2}v, 3X_{-1}^2 v + 3X_{-1}X_{-2}v + X_{-2}^2 v\rangle,$$

and that $X_{-\alpha}^4 m = 0$ for $m \in \mathrm{Inv}\, M_8$. Therefore Lemma 4 implies that each nonzero vector from $\mathrm{Inv}\, M_8$ generates a submodule isomorphic to $M(8)$. Put $m_{10} = 3X_{-1}v + X_{-2}v$. One can directly verify that $X_\alpha m_{10} = 0$. Hence $\mathrm{Inv}\, M_{10} = \langle m_{10}\rangle$. Since $X_{-1}^2 v \neq 0$, then $X_{-\alpha} m_{10} \neq 0$ and so $KA_y m_{10} \cong V(10)$. Let F_1 be the indecomposable component of N_2 containing m_{10} (such component exists as $\dim(N_2 \cap M_{10}) = 1$). Set $\overline{F_1} = F_1/KA_y m_{10}$ and $u = X_{-1}^2 v + 4X_{-1}X_{-2}v$. It is obvious that the weights of $\overline{F_1}$ are less than 10. One can directly check that $X_\alpha u = m_{10}$. Let $N_2 = F_1 \oplus F_2$ where F_2 is an A_y-module. Then all weights of F_2 are at most 8. We write $u = u_1 + u_2$ where $u_i \in F_i$. It is clear that $u_i \in M_8$ and that $X_\alpha u_2 = 0$. Thus $X_\alpha u_1 = m_{10}$. We can directly check that $X_{-\alpha,5} m_{10} \neq 0$ and $X_{-\alpha}^4 u \notin \langle X_{-\alpha,5} m_{10}\rangle$. As $u_2 \in M_8 \cap F_2$, then $u_2 \in \mathrm{Inv}\, M_8$ and the facts proved above yield that $X_{-\alpha}^4 u_2 = 0$. Therefore $X_{-\alpha}^4 u_1 \notin \langle X_{-\alpha,5} m_{10}\rangle$.

Let $\overline{u_1}$ be the image of u_1 under the canonical homomorphism $F_1 \to \overline{F_1}$. As the weight subspaces in Weyl modules for A_y are one-dimensional, then it follows from the facts proved above that $X_\alpha \overline{u_1} = 0$ and $X_{-\alpha}^4 \overline{u_1} \neq 0$. Then Lemma 28 implies that the vector $\overline{u_1}$ generates a submodule isomorphic to $V(8)$ in $\overline{F_1}$. Let S be the preimage of this submodule in F_1, $q \in \mathrm{Inv}\, M_8 \setminus \langle X_{-\alpha} m_{10}\rangle$, and $Q = KA_y q$. It follows from above that $Q \cong M(8)$. It is clear that $S \cap \mathrm{Inv}\, M_8 = \langle X_{-\alpha} m_{10}\rangle$. Then $Q \not\subset S$ by the choice of q. Since Q is irreducible, we have $S \cap Q = \{0\}$. Now it is clear that $N_2 = S \oplus Q$ and $F_1 = S$. Since F_1 has a filtration by Weyl modules and is self-dual, it has a filtration by dual Weyl modules. Therefore F_1 is a tilting module and $F_1 \cong T(10)$.

I.VIII. Let $p = 7$ and $\omega = 5\omega_1 + 2\omega_2$. Then $\dim \varphi = 71$. It is not difficult to show that $\sigma_y(\omega) = 14$. Set

$$\begin{aligned}
\Lambda_{10} &= \{\omega - 2\alpha_1, \omega - 2\alpha_2, \omega - \alpha_1 - \alpha_2\},\\
\Lambda_8 &= \{\omega - 3\alpha_1, \omega - 2\alpha_1 - \alpha_2, \omega - \alpha_1 - 2\alpha_2\},\\
\Lambda_6 &= \{\omega - 4\alpha_1, \omega - 3\alpha_1 - \alpha_2, \omega - 2\alpha_1 - 2\alpha_2, \omega - \alpha_1 - 3\alpha_2\},\\
\Lambda_4 &= \{\omega - 5\alpha_1, \omega - 4\alpha_1 - \alpha_2, \omega - 3\alpha_1 - 2\alpha_2, \omega - 2\alpha_1 - 3\alpha_2\},\\
\Lambda_2 &= \{\omega - 5\alpha_1 - \alpha_2, \omega - 4\alpha_1 - 2\alpha_2, \omega - 3\alpha_1 - 3\alpha_2, \omega - 2\alpha_1 - 4\alpha_2\},\\
\Lambda_0 &= \{\omega - 6\alpha_1 - \alpha_2, \omega - 5\alpha_1 - 2\alpha_2, \omega - 4\alpha_1 - 3\alpha_2, \omega - 3\alpha_1 - 4\alpha_2\}.
\end{aligned}$$

One easily concludes that $M_i = \langle M_\mu | \mu \in \Lambda_i\rangle$. It follows from Lemma 15 that $\dim M_\mu = \dim V(\omega)_\mu$ for $\mu = \omega - k\alpha_1 - \alpha_2$ or $\mu = \omega - \alpha_1 - l\alpha_2$, and $\dim M_\mu = \dim V(\omega)_\mu - 1$ for $\mu = \omega - 2\alpha_1 - k\alpha_2$ with $2 \leq k \leq 4$, $\mu = \omega - l\alpha_1 - 2\alpha_2$ with $2 \leq l \leq 5$, and $\mu = 0$. We observe that the weight $\omega - 3\alpha_1 - 3\alpha_2$ lies in the same W-orbit with $\omega - 3\alpha_1 - 2\alpha_2$ and the weight $\omega - 3\alpha_1 - 4\alpha_2$ lies in the same W-orbit with $\omega - 3\alpha_1 - \alpha_2$. Now using Lemma 22, we can show that $\dim M_\mu = 2$ for

$$\mu \in \{\omega-\alpha_1-\alpha_2, \omega-2\alpha_1-\alpha_2, \omega-\alpha_1-2\alpha_2, \omega-3\alpha_1-\alpha_2, \omega-2\alpha_1-2\alpha_2,\\ \omega-4\alpha_1-\alpha_2, \omega-3\alpha_1-2\alpha_2, \omega-2\alpha_1-3\alpha_2, \omega-5\alpha_1-\alpha_2, \omega-4\alpha_1-2\alpha_2,\\ \omega-3\alpha_1-3\alpha_2, \omega-2\alpha_1-4\alpha_2, \omega-5\alpha_1-2\alpha_2, \omega-4\alpha_1-3\alpha_2, \omega-3\alpha_1-4\alpha_2\}$$

and all other weights from the union $\Lambda_{10}\cup\Lambda_8\cup\Lambda_6\cup\Lambda_4\cup\Lambda_2\cup\Lambda_0$ have multiplicity 1. Hence $\dim M_{14}=1$, $\dim M_{12}=2$, $\dim M_{10}=4$, $\dim M_8=5$, $\dim M_6=6$, $\dim M_4=7$, $\dim M_2=7$, and $\dim M_0=7$. This implies that

$$I(M_y)=\{M(14), 2M(12), 2M(10), M(8), M(6), 3M(4), 2M(2), M(0)\}.$$

By Proposition 8 and Lemma 27, $M_y = N_1\oplus N_2\oplus N_3\oplus M(6)$ where

$$I(N_1)=\{M(14), 2M(12), M(0)\},\ I(N_2)=\{2M(10), 2M(2)\},\ \ I(N_3)=\{M(8), 2M(4)\}.$$

Show that $N_1\cong T(14)$, $N_2\cong T(10)\oplus M(10)$, and $N_3\cong T(8)$.

It is clear that $v\in N_1$. Observe that $X_{-1}v\neq 0$. Hence $X_{-\alpha}v\neq 0$ and $KA_yv\cong V(14)$. Let F be the indecomposable component of N_1 containing v, and $\overline{F}=F/KA_yv$. Since $M_{12}=\langle X_{-1}v, X_{-2}v\rangle$, we have $\dim \operatorname{Inv} M_{12}=1$. Then $\operatorname{Inv} M_{12}\subset KA_yv$ and therefore M has no indecomposable components with highest weight 12. It is not difficult to show that each of the pairs $(X_{-1}^5X_{-2}^2v,\ X_{-2}^2X_{-1}^5v)$, $(X_{-2}^3X_{-1}^4v,\ X_{-2}X_{-1}^4X_{-2}^2v)$, and $(X_{-2}^4X_{-1}^3v,\ X_{-2}^2X_{-1}^3X_{-2}^2v)$ consists of linearly independent vectors. Hence the vectors

$$X_{-1}^6X_{-2}v,\ X_{-1}^5X_{-2}^2v,\ X_{-2}^2X_{-1}^5v,\ X_{-2}^3X_{-1}^4v,\ X_{-2}X_{-1}^4X_{-2}^2v,\ X_{-2}^4X_{-1}^3v,\ X_{-2}^2X_{-1}^3X_{-2}^2v$$

form a basis of M_0. Using this basis, we check directly that $\operatorname{Inv} M_0=0$. Therefore M has no submodules isomorphic to $M(0)$. This yields that $N_1=F$ and the module N_1 is self-dual. Now we conclude that N_1 has no factor modules isomorphic to $M(0)$. Then Lemma 23 implies that $\overline{F}\cong V(12)$. Therefore N_1 has a filtration by Weyl modules and so a filtration by dual Weyl modules as N_1 is self-dual. This forces that $N_1\cong T(14)$.

It is clear that $\dim \operatorname{Inv}(M_{10})\le 2$ and the vectors

$$m_{10}^1 = 3X_{-1}^2v + X_{-1}X_{-2}v + 2X_{-2}^2v \text{ and } m_{10}^2 = X_{-1}^2v + X_{-2}X_{-1}v + 4X_{-2}^2v$$

are linearly independent. One can directly verify that they belong to $\operatorname{Inv} M_{10}$ and hence $\operatorname{Inv} M_{10}=\langle m_{10}^1, m_{10}^2\rangle$.

It is not difficult to check that each of the pairs $(X_{-2}^2X_{-1}^4v,\ X_{-1}^4X_{-2}^2v)$, $(X_{-2}^3X_{-1}^3v,\ X_{-2}X_{-1}^3X_{-2}^2v)$, and $(X_{-2}^4X_{-1}^2v,\ X_{-2}^2X_{-1}^2X_{-2}^2v)$ consists of linearly independent vectors. Hence the vectors

$$X_{-1}^5X_{-2}v,\ X_{-2}^2X_{-1}^4v,\ X_{-1}^4X_{-2}^2v,\ X_{-2}^3X_{-1}^3v,\ X_{-2}X_{-1}^3X_{-2}^2v,\ X_{-2}^4X_{-1}^2v,\ X_{-2}^2X_{-1}^2X_{-2}^2v$$

form a basis of M_2. Set $U=\langle m_{10}^1, m_{10}^2\rangle\cap X_\alpha^4M_2$ and $M_2'=N_2\cap M_2$. It is clear that $U=\langle m_{10}^1, m_{10}^2\rangle\cap X_\alpha^4M_2'$. Using the basis of M_2 indicated above, we easily deduce that $\dim U=1$ and $X_{-\alpha}^4U\neq 0$. By Lemma 25, we conclude that $N_2\cong T(10)\oplus M(10)$.

One easily observes that

$$M_8=\langle X_{-1}^3v, X_{-2}X_{-1}^2v, X_{-1}X_{-2}X_{-1}v, X_{-2}^2X_{-1}v, X_{-2}X_{-1}X_{-2}v\rangle.$$

Set $m_8 = 6X_{-1}^3v + 3X_{-2}X_{-1}^2v + X_{-1}X_{-2}X_{-1}v + 3X_{-2}^2X_{-1}v + X_{-2}X_{-1}X_{-2}v$. It is not difficult to verify that $\dim \operatorname{Inv}(M_8)=1$ and $\operatorname{Inv} M_8=\langle m_8\rangle$. It is clear that $m_8\in N_3$. Since $X_{-1}^5v\neq 0$, then $X_{-\alpha}^2m_8\neq 0$. Now Lemma 29 implies that $N_3\cong T(8)$.

I.IX. Let $p = 7$ and $\omega = 4\omega_1 + 3\omega_2$. Then $\dim\varphi = 75$. One easily observes that $\sigma_y(\omega) = 14$. Set

$$\begin{aligned}
\Lambda_{10} &= \{\omega - 2\alpha_1, \omega - 2\alpha_2, \omega - \alpha_1 - \alpha_2\},\\
\Lambda_8 &= \{\omega - 3\alpha_1, \omega - 2\alpha_1 - \alpha_2, \omega - \alpha_1 - 2\alpha_2, 3\alpha_2\},\\
\Lambda_6 &= \{\omega - 4\alpha_1, \omega - 3\alpha_1 - \alpha_2, \omega - 2\alpha_1 - 2\alpha_2, \omega - \alpha_1 - 3\alpha_2\},\\
\Lambda_4 &= \{\omega - 4\alpha_1 - \alpha_2, \omega - 3\alpha_1 - 2\alpha_2, \omega - 2\alpha_1 - 3\alpha_2, \omega - \alpha_1 - 4\alpha_2\},\\
\Lambda_2 &= \{\omega - 5\alpha_1 - \alpha_2, \omega - 4\alpha_1 - 2\alpha_2, \omega - 3\alpha_1 - 3\alpha_2, \omega - 2\alpha_1 - 4\alpha_2\},\\
\Lambda_0 &= \{\omega - 5\alpha_1 - 2\alpha_2, \omega - 4\alpha_1 - 3\alpha_2, \omega - 3\alpha_1 - 4\alpha_2, \omega - 2\alpha_1 - 5\alpha_2\}.
\end{aligned}$$

We can conclude that $M_i = \langle M_\mu | \mu \in \Lambda_i\rangle$. By Lemma 22, it is not difficult to show that $\dim M_\mu = 2$ for

$$\begin{aligned}
\mu \in \{&\omega - 2\alpha_1 - \alpha_2, \omega - \alpha_1 - 2\alpha_2, \omega - 3\alpha_1 - \alpha_2, \omega - 2\alpha_1 - 2\alpha_2, \omega - \alpha_1 - 3\alpha_2,\\
&\omega - 3\alpha_1 - 2\alpha_2, \omega - 2\alpha_1 - 3\alpha_2, \omega - 4\alpha_1 - \alpha_2, \omega - 4\alpha_1 - 2\alpha_2, \omega - 3\alpha_1 - 3\alpha_2,\\
&\omega - 2\alpha_1 - 4\alpha_2, \omega - 5\alpha_1 - 2\alpha_2, \omega - 4\alpha_1 - 3\alpha_2, \omega - 3\alpha_1 - 4\alpha_2\}
\end{aligned}$$

and all other weights from the union $\Lambda_{10} \cup \Lambda_8 \cup \Lambda_6 \cup \Lambda_4 \cup \Lambda_2 \cup \Lambda_0$ have multiplicity 1. Hence $\dim M_{14} = 1$, $\dim M_{12} = 2$, $\dim M_{10} = 4$, $\dim M_8 = 6$, $\dim M_6 = 7$, $\dim M_4 = 7$, $\dim M_2 = 7$, and $\dim M_0 = 7$. This yields that

$$I(M_y) = \{M(14), 2M(12), 2M(10), 2M(8), M(6), 2M(4), 2M(2), M(0)\}.$$

By Proposition 8 and Lemma 27, $M_y = N_1 \oplus N_2 \oplus N_3 \oplus M(6)$ where

$$I(N_1) = \{M(14), 2M(12), M(0)\},\ I(N_2) = \{2M(10), 2M(2)\},\ I(N_3) = \{2M(8), 2M(4)\}.$$

Show that $N_1 \cong T(14)$, $N_2 \cong T(10) \oplus M(10)$, and $N_3 \cong T(8) \oplus M(8)$.

The proofs for N_1 and N_2 are similar to Case I.VIII. For computations related with the subspaces M_2 and M_0, we use their bases

$$X_{-1}^5X_{-2}v,\ X_{-2}^2X_{-1}^4v,\ X_{-1}^4X_{-2}^2v,\ X_{-2}^3X_{-1}^3v,\ X_{-1}^3X_{-2}^3v,\ X_{-2}^4X_{-1}^2v,\ X_{-2}X_{-1}^2X_{-2}^3v \text{ and}$$

$$X_{-2}^5X_{-1}^2v,\ X_{-1}^5X_{-2}^2v,\ X_{-1}X_{-2}^2X_{-1}^4v,\ X_{-2}^3X_{-1}^4v,\ X_{-1}^4X_{-2}^3v,\ X_{-2}^4X_{-1}^3v,\ X_{-2}X_{-1}^3X_{-2}^3v,$$

respectively. Set

$$m_{10}^1 = X_{-1}^2v + X_{-2}X_{-1}v + 3X_{-2}^2v \text{ and } m_{10}^2 = 6X_{-1}^2v + X_{-1}X_{-2}v + 2X_{-2}^2v.$$

One can directly check that $\operatorname{Inv} M_{10} = \langle m_{10}^1, m_{10}^2\rangle$ and $\dim(X_{-\alpha}^4 \operatorname{Inv} M_{10}) = 1$.

Now show that $N_3 \cong T(8) \oplus M(8)$. It is clear that $\dim \operatorname{Inv}(M_8) \le 2$ and the vectors

$$\begin{aligned}
m_8^1 =& X_{-1}^3v + 5X_{-2}X_{-1}^2v + X_{-2}^2X_{-1}v + 3X_{-2}^3v \text{ and}\\
m_8^2 =& X_{-1}^3v + 6X_{-1}^2X_{-2}v + X_{-1}X_{-2}^2v + 6X_{-2}^3v
\end{aligned}$$

are linearly independent. One can directly verify that they belong to $\operatorname{Inv} M_8$. Hence $\operatorname{Inv} M_8 = \langle m_8^1, m_8^2\rangle$. It is not difficult to check that $\dim(X_{-\alpha}^2 \operatorname{Inv} M_8) = 1$ and that each of the pairs $(X_{-2}X_{-1}^4v, X_{-1}^4X_{-2}v)$, $(X_{-2}^2X_{-1}^3v, X_{-1}^3X_{-2}^2v)$, and $(X_{-2}^3X_{-1}^2v, X_{-1}^2X_{-2}^3v)$ consists of linearly independent vectors. Therefore the vectors

$$X_{-2}^4X_{-1}v,\ X_{-2}X_{-1}^4v,\ X_{-1}^4X_{-2}v,\ X_{-2}^2X_{-1}^3v,\ X_{-1}^3X_{-2}^2v,\ X_{-2}^3X_{-1}^2v,\ X_{-1}^2X_{-2}^3v$$

form a basis in M_4. Set $U = \langle m_8^1, m_8^2\rangle \cap X_\alpha^2 M_4$ and $M_4' = N_3 \cap M_4$. It is clear that $U = \langle m_8^1, m_8^2\rangle \cap X_\alpha^2 M_4'$. Using the basis in M_4 indicated above, one easily deduces that $\dim U = 1$ and $X_{-\alpha}^2 U \ne 0$. It follows from Lemma 25 that $N_3 \cong T(8) \oplus M(8)$.

Assume that $p = 11$. Below in Items I.X–I.XIV we analyze representations afforded by reduced symmetric powers of the standard module.

I.X. Let $\omega = 9\omega_1 + \omega_2$. Then $\dim\varphi = 75$ and $M \cong S^{11,11}(V)$. It is not difficult to show that $\sigma_y(\omega) = 20$, $\dim M_{20} = 1$, $\dim M_{18} = 2$, $\dim M_{16} = 2$, $\dim M_{14} = 3$, $\dim M_{12} = 3$,

$\dim M_{10} = 4$, $\dim M_8 = 4$, $\dim M_6 = 5$, $\dim M_4 = 5$, $\dim M_2 = 6$, and $\dim M_0 = 5$. This yields that

$$I(M_y) = \{M(20), M(18), M(14), M(10), 2M(6), 2M(2)\}.$$

By Lemma 27, $M_y = M(20) \oplus N_1 \oplus N_2 \oplus M(10)$ where $I(N_1) = \{M(18), 2M(2)\}$ and $I(N_2) = \{M(14), 2M(6)\}$. Show that $N_1 \cong T(18)$ and $N_2 \cong T(14)$.

It is clear that $M_{18} = \langle X_{-1}v, X_{-2}v\rangle$ and $M_{14} = \langle X_{-1}^3 v, X_{-1}^2 X_{-2}v, X_{-2}^2 X_{-1}v\rangle$ and that $\dim \operatorname{Inv} M_i \leq 1$ for $i = 18$ and 14. Put $m_{18} = X_{-1}v + X_{-2}v$ and $m_{14} = X_{-1}^3 v + 6X_{-1}^2 X_{-2}v + X_{-2}^2 X_{-1}v$. One can directly verify that $m_i \in \operatorname{Inv} M_i$ and hence $\operatorname{Inv} M_i = \langle m_i\rangle$ for $i = 18$ and 14. Since $X_{-1}^9 v \neq 0$, the vectors $X_{-1}^8 m_{18}$ and $X_{-1}^6 m_{14} \neq 0$. Then Lemma 29 forces that $N_1 \cong T(18)$ and $N_2 \cong T(14)$.

I.XI. Let $\omega = 8\omega_1 + 2\omega_2$. Then $\dim \varphi = 82$ and $M \cong S^{12,11}(V)$. We can check that $\sigma_y(\omega) = 20$, $\dim M_{20} = 1$, $\dim M_{18} = 2$, $\dim M_{16} = 3$, $\dim M_{14} = 3$, $\dim M_{12} = 4$, $\dim M_{10} = 4$, $\dim M_8 = 5$, $\dim M_6 = 5$, $\dim M_4 = 6$, $\dim M_2 = 5$, and $\dim M_0 = 6$. Then

$$I(M_y) = \{M(20), M(18), M(16), M(12), 2M(8), 2M(4), 2M(0)\}.$$

By Lemma 27, $M_y = N_1 \oplus M(18) \oplus N_2 \oplus N_3$ where

$$I(N_1) = \{M(20), 2M(0)\},\ I(N_2) = \{M(16), 2M(4)\},\ I(N_3) = \{M(12), 2M(8))\}.$$

We show that $N_1 \cong T(20)$, $N_2 \cong T(16)$, and $N_3 \cong T(12)$.

One can directly check that $9X_{-1}^9 X_{-2}v$ is a nonzero weight component of the vector $X_{-\alpha}^{10}v$. Then Lemma 29 implies that $N_1 \cong T(20)$.

It is clear that $M_{16} = \langle X_{-1}^2 v,\ X_{-1}X_{-2}v,\ X_{-2}^2 v\rangle$, $M_{12} = \langle X_{-1}^4 v,\ X_{-2}X_{-1}^3 v,\ X_{-2}^2 X_{-1}^2 v,\ X_{-2}^3 X_{-1}v\rangle$ and that $\dim \operatorname{Inv} M_i \leq 1$ for $i = 16$ and 12. Set

$$m_{16} = X_{-1}^2 v + 5X_{-2}X_{-1}v + X_{-2}^2 v,\quad m_{12} = 3X_{-1}^4 v + 5X_{-2}X_{-1}^3 v + 9X_{-2}^2 X_{-1}^2 v + X_{-2}^3 X_{-1}v.$$

We can directly verify that $m_i \in \operatorname{Inv} M_i$ and hence $\operatorname{Inv} M_i = \langle m_i\rangle$ for $i = 16$ and 12. As $X_{-1}^8 v \neq 0$, the vectors $X_{-\alpha}^6 m_{16}$ and $X_{-\alpha}^2 m_{12} \neq 0$. Then Lemma 29 implies that $N_2 \cong T(16)$ and $N_3 \cong T(12)$.

I.XII. Let $\omega = 7\omega_1 + 3\omega_2$. Then $\dim \varphi = 87$ and $M \cong S^{13,11}(V)$. We can directly check that $\sigma_y(\omega) = 20$, $\dim M_{20} = 1$, $\dim M_{18} = 2$, $\dim M_{16} = 3$, $\dim M_{14} = 4$, $\dim M_{12} = 4$, $\dim M_{10} = 5$, $\dim M_8 = 5$, $\dim M_6 = 6$, $\dim M_4 = 5$, $\dim M_2 = 6$, and $\dim M_0 = 5$. Then

$$I(M_y) = \{M(20), M(18), M(16), M(14), M(10), 2M(6), 2M(2)\}.$$

By Lemma 27, $M_y = M(20) \oplus N_1 \oplus M(16) \oplus N_2 \oplus M(10)$ where

$$I(N_1) = \{M(18),\ 2M(2)\},\ I(N_2) = \{M(14), 2M(6)\}.$$

We show that $N_1 \cong T(18)$ and $N_2 \cong T(14)$.

It is obvious that $M_{18} = \langle X_{-1}v,\ X_{-2}v\rangle$, $M_{14} = \langle X_{-1}^3 v,\ X_{-2}X_{-1}^2 v,\ X_{-2}^2 X_{-1}v,\ X_{-2}^3 v\rangle$ and that $\dim \operatorname{Inv} M_i \leq 1$ for $i = 18$ and 14. Set

$$m_{18} = 7X_{-1}v + X_{-2}v,\ m_{14} = X_{-1}^3 v + 4X_{-2}X_{-1}^2 v + 7X_{-2}^2 X_{-1}v + X_{-2}^3 v.$$

We can directly check that $m_i \in \operatorname{Inv} M_i$. Hence $\operatorname{Inv} M_i = \langle m_i\rangle$ for $i = 18$ and 14. As $X_{-1}^7 v \neq 0$, then $X_{-\alpha}^4 m_{14} \neq 0$. One can easily verify that $5X_{-1}^8 X_{-2}v$ is a nonzero weight component of the vector $X_{-\alpha}^8 m_{18}$. Then by Lemma 29, $N_1 \cong T(18)$ and $N_2 \cong T(14)$.

I.XIII. Let $\omega = 6\omega_1 + 4\omega_2$. Then $\dim \varphi = 90$ and $M \cong S^{14,11}(V)$. One can check that $\sigma_y(\omega) = 20$, $\dim M_{20} = 1$, $\dim M_{18} = 2$, $\dim M_{16} = 3$, $\dim M_{14} = 4$, $\dim M_{12} = 5$, $\dim M_{10} = 5$, $\dim M_8 = 6$, $\dim M_6 = 5$, $\dim M_4 = 6$, $\dim M_2 = 5$, and $\dim M_0 = 6$. Then

$$I(M_y) = \{M(20), M(18), M(16), M(14), M(12), 2M(8), 2M(4), 2M(0)\}.$$

Lemma 27 implies that $M_y = N_1 \oplus M(18) \oplus N_2 \oplus M(14) \oplus N_3$ where

$$I(N_1) = \{M(20), 2M(0)\},\ I(N_2) = \{M(16), 2M(4)\},\ I(N_3) = \{M(12), 2M(8)\}.$$

We show that $N_1 \cong T(20)$, $N_2 \cong T(16)$, and $N_3 \cong T(12)$.

Obviously, $v \in N_1$. One can directly verify that $9X_{-1}^8 X_{-2}^2 v$ is a nonzero weight component of the vector $X_{-\alpha}^{10} v$. So Lemma 29 yields that $N_1 \cong T(20)$.

It is clear that $M_{16} = \langle X_{-1}^2 v,\ X_{-2}X_{-1}v,\ X_{-2}^2 v\rangle$,

$$M_{12} = \langle X_{-1}^4 v,\ X_{-2}X_{-1}^3 v,\ X_{-2}^2 X_{-1}^2 v,\ X_{-2}^3 X_{-1} v,\ X_{-2}^4 v\rangle$$

and that $\dim \operatorname{Inv} M_i \leq 1$ for $i = 16$ and 12. Set $m_{16} = 8X_{-1}^2 v + X_{-1}X_{-2}v + 4X_{-2}^2 v$,

$$m_{12} = X_{-1}^4 v + 7X_{-2}X_{-1}^3 v + 9X_{-2}^2 X_{-1}^2 v + 6X_{-2}^3 X_{-1} v + X_{-2}^4 v.$$

We can directly check that $\operatorname{Inv} M_i = \langle m_i\rangle$ for $i = 16$ and 12 and that $X_{-1}^7 X_{-2} v$ is a nonzero weight component of the vector $X_{-\alpha}^6 m_{16}$. As $X_{-1}^6 v \neq 0$, then $X_{-1}^2 m_{12} \neq 0$. Then by Lemma 29, $N_2 \cong T(16)$ and $N_3 \cong T(12)$.

I.XIV. Let $\omega = 5\omega_1 + 5\omega_2$. Then $\dim\varphi = 91$ and $M \cong S^{15,11}(V)$. We can directly verify that $\sigma_y(\omega) = 20$, $\dim M_{20} = 1$, $\dim M_{18} = 2$, $\dim M_{16} = 3$, $\dim M_{14} = 4$, $\dim M_{12} = 5$, $\dim M_{10} = 6$, $\dim M_8 = 5$, $\dim M_6 = 6$, $\dim M_4 = 5$, $\dim M_2 = 6$, and $\dim M_0 = 5$. Then

$$I(M_y) = \{M(20), M(18), M(16), M(14), M(12), M(10), 2M(6), 2M(2)\}.$$

Lemma 27 forces that $M_y = M(20) \oplus N_1 \oplus M(16) \oplus N_2 \oplus M(12) \oplus M(10)$ where

$$I(N_1) = \{M(18),\ 2M(2)\},\ I(N_2) = \{M(14), 2M(6)\}.$$

Show that $N_1 \cong T(18)$ and $N_2 \cong T(14)$.

It is clear that $M_{18} = \langle X_{-1}v,\ X_{-2}v\rangle$, $\quad M_{14} = \langle X_{-1}^3 v,\ X_{-2}X_{-1}^2 v,\ X_{-2}^2 X_{-1} v,\ X_{-2}^3 v\rangle$ and that $\dim \operatorname{Inv} M_i \leq 1$ for $i = 18$ and 14. Set

$$m_{18} = X_{-1}v + 5X_{-2}v,\ m_{14} = 8X_{-1}^3 v + 9X_{-2}X_{-1}^2 v + 3X_{-2}^2 X_{-1} v + X_{-2}^3 v.$$

One can check that $\operatorname{Inv} M_i = \langle m_i\rangle$ for $i = 18$ and 14 and that $2X_{-1}^4 X_{-2}^2 X_{-1}^3 v$ and $5X_{-1}^6 X_{-2} v$ are nonzero weight components of the vectors $X_{-\alpha}^8 m_{18}$ and $X_{-\alpha}^4 m_{14}$, respectively. Then by Lemma 29, $N_1 \cong T(18)$ and $N_2 \cong T(14)$.

The problem is solved for the regular unipotent elements in the group $A_2(K)$.

If x is a transvection, then the arguments at the beginning of the section imply that it remains to consider the cases where $p = 5$ and $\omega \in \{3\omega_1 + 2\omega_2, 3\omega_1 + 3\omega_2\}$ or $p = 7$ and $\omega \in \{5\omega_1 + 2\omega_2, 4\omega_1 + 3\omega_2\}$. Notice that in all these situations the dimensions of the weight subspaces of M have been already determined when we found such dimensions for M_y. Hence we can find the dimensions of the subspaces M_i. Set $H = G(1)$. By Lemma 32, $M_x|H$ is a tilting module.

I.XV. Let $\omega = 3\omega_1 + 2\omega_2$ and $p = 5$. Then $\dim\varphi = 39$, $\sigma_x(\omega) = 5$, $\dim M_5 = 1$, $\dim M_4 = 2$, $\dim M_3 = 4$, $\dim M_2 = 5$, $\dim M_1 = 5$, and $\dim M_0 = 5$. Hence

$$M_x|H \cong T(5) \oplus 2M(4) \oplus 2M(3) \oplus 3M(2) \oplus M(1).$$

I.XVI. Let $\omega = 3\omega_1 + 3\omega_2$ and $p = 5$. Then $\dim\varphi = 63$, $\sigma_x(\omega) = 6$, $\dim M_6 = 1$, $\dim M_5 = 2$, $\dim M_4 = 4$, $\dim M_3 = 6$, $\dim M_2 = 7$, $\dim M_1 = 8$, and $\dim M_0 = 7$. Therefore

$$M_x|H \cong T(6) \oplus 2T(5) \oplus 3M(4) \oplus 2M(3) \oplus 2M(2) \oplus 2M(1).$$

I.XVII. Let $\omega = 5\omega_1 + 2\omega_2$ and $p = 7$. Then $\dim\varphi = 71$, $\sigma_x(\omega) = 7$, $\dim M_7 = 1$, $\dim M_6 = 2$, $\dim M_5 = 4$, $\dim M_4 = 5$, $\dim M_3 = 6$, $\dim M_2 = 7$, $\dim M_1 = 7$, and $\dim M_0 = 7$. Hence

$$M_x|H \cong T(7) \oplus 2M(6) \oplus 2M(5) \oplus 3M(4) \oplus 2M(3) \oplus 2M(2) \oplus M(1).$$

I.XVIII. Let $\omega = 4\omega_1 + 3\omega_2$ and $p = 7$. Then $\dim\varphi = 75$, $\sigma_x(\omega) = 7$, $\dim M_7 = 1$, $\dim M_6 = 2$, $\dim M_5 = 4$, $\dim M_4 = 6$, $\dim M_3 = 7$, $\dim M_2 = 7$, $\dim M_1 = 7$, and $\dim M_0 = 7$. Therefore

$$M_x|H \cong T(7) \oplus 2M(6) \oplus 2M(5) \oplus 2M(4) \oplus 3M(3) \oplus M(2).$$

For $G = A_2(K)$ the problem is solved.

II. Let $G = A_3(K)$.

First consider the behaviour of a regular unipotent element. As before, let y be such element. Since $|y| = p$, we have $p \geq 5$. Using the arguments at the end of Section 3, we can assume that $X_\alpha = X_1 + 2X_2 + 3X_3$ and $X_{-\alpha} = 3X_{-1} + 2X_{-2} + X_{-3}$. As $V_y \cong M(3)$, one easily observes that $\sigma_y(\varepsilon_1) = 3$, $\sigma_y(\varepsilon_2) = 1$, $\sigma_y(\varepsilon_3) = -1$, and $\sigma_y(\varepsilon_4) = -3$. This forces that $\sigma_y(\omega) = 3a_1 + 4a_2 + 3a_3$.

II.I. Let $p = 5$ and $\omega = 2\omega_1 + \omega_3$. Then $\dim\varphi = 32$ and $\sigma_y(\omega) = 9$. It is clear that $\dim M_9 = 1$ and $\dim M_7 = 2$. Set

$$\begin{aligned}\Lambda_5 =&\{\omega - 2\alpha_1, \omega - \alpha_1 - \alpha_2, \omega - \alpha_3 - \alpha_2, \omega - \alpha_1 - \alpha_3\},\\ \Lambda_3 =&\{\omega - 2\alpha_1 - \alpha_2, \omega - \alpha_1 - \alpha_2 - \alpha_3, \omega - 2\alpha_1 - \alpha_3\},\\ \Lambda_1 =&\{\omega - 2\alpha_1 - 2\alpha_2, \omega - 2\alpha_1 - \alpha_2 - \alpha_3, \omega - \alpha_1 - \alpha_2 - 2\alpha_3, \omega - \alpha_1 - 2\alpha_2 - \alpha_3\},\end{aligned}$$

and $\lambda = \omega - \alpha_1 - \alpha_2 - \alpha_3$. Then $\lambda = \omega_1$. We can show that $M_i = \langle M_\mu | \mu \in \Lambda_i\rangle$. Using the Jantzen filtration [13, Part 2, §8, Proposition 8.19], we get that the maximal submodule in $V(\omega)$ is isomorphic to $M(\lambda)$. As $\dim V(\omega)_\lambda = 3$ by Freudenthal's formula, we have $\dim M_\lambda = 2$. Since the weight $\nu = \omega - 2\alpha_1 - \alpha_2 - \alpha_3$ lies in the same W-orbit with λ, then $\dim M_\nu = 2$. Obviously, $\dim M_\mu = 1$ for $\mu \in \{\omega - \alpha_1 - \alpha_2, \omega - \alpha_2 - \alpha_3, \omega - \alpha_1 - \alpha_3, \omega - 2\alpha_1 - \alpha_2, \omega - 2\alpha_1 - \alpha_3, \omega - 2\alpha_1 - 2\alpha_2\}$ as all weight subspaces of the $A_2(K)$-modules $M(\omega_1)$ and $M(2\omega_1)$ are one-dimensional and the operators X_{-1} and X_{-3} commute. Let $\beta = \omega - \alpha_1 - 2\alpha_2 - \alpha_3$ or $\omega - \alpha_1 - \alpha_2 - 2\alpha_3$. Then β lies in the same W-orbit with $\omega - \alpha_1 - \alpha_3$ or $\omega - \alpha_1 - \alpha_2$, respectively, therefore $\dim M_\beta = 1$. This yields that $\dim M_5 = \dim M_3 = 4$ and $\dim M_1 = 5$. Then $I(M_y) = \{M(9), M(7), 2M(5), 2M(3), 2M(1)\}$. By Lemma 27, $M_y = M(9) \oplus N_1 \oplus N_2$ where $I(N_1) = \{M(7),\ 2M(1)\}$ and $I(N_2) = \{2M(5), 2M(3)\}$. Show that $N_1 \cong T(7)$ and $N_2 \cong T(5) \oplus M(5)$.

Let $m = X_{-1}v + X_{-3}v$. One can directly verify that $m \in \operatorname{Inv} M_7$ and that $X_{-2}^2 X_{-1}^2 v$ is a nonzero weight component of $X_{-\alpha}^3 m$. Lemma 29 implies that $N_1 \cong T(7)$.

Put $m_5^1 = X_{-1}^2 v + X_{-1}X_{-3}v - X_{-2}X_{-3}v$ and $m_5^2 = X_{-2}X_{-1}v + X_{-1}X_{-3}v - X_{-2}X_{-3}v$. We can check that the vectors m_5^1 and $m_5^2 \in \operatorname{Inv} M_5$ and that they are linearly independent. As $\dim \operatorname{Inv} M_5 \leq 2$, then $\operatorname{Inv} M_5 = \langle m_5^1, m_5^2\rangle$. It is not difficult to show that the vectors $X_{-2}X_{-3}X_{-1}v$ and $X_{-1}X_{-2}X_{-3}v$ are linearly independent. Hence $X_{-2}X_{-3}X_{-1}v$, $X_{-1}X_{-2}X_{-3}v$, $X_{-2}X_{-1}^2 v$, and $X_{-3}X_{-1}^2 v$ form a basis in M_3. Set $U = \langle m_5^1, m_5^2\rangle \cap X_\alpha M_3$. It is clear that $U = \langle m_5^1, m_5^2\rangle \cap X_\alpha(M_3 \cap N_2)$. Using the basis of M_3 indicated above, it is not difficult to check that $\dim U = 1$ and $X_{-\alpha}U \neq 0$. Lemma 25 yields that $N_2 \cong T(5) \oplus M(5)$.

II.II. Let $p = 5$ and $\omega = 3\omega_1 + \omega_2$. Then $\dim\varphi = 52$ and $M \cong S^{5,5}(V)$. One can easily verify that $\sigma_y(\omega) = 13$, $\dim M_{13} = 1$, $\dim M_{11} = 2$, $\dim M_9 = 3$, $\dim M_7 = 4$, $\dim M_5 = 4$, $\dim M_3 = 6$, and $\dim M_1 = 6$. Then

$$I(M_y) = \{M(13), M(11), M(9), 2M(7), M(5), 2M(3), M(0)\}.$$

By Proposition 8 and Lemma 27, $M_y = N_1 \oplus N_2 \oplus M(9)$ where

$$I(N_1) = \{M(13), M(5), 2M(3)\},\ I(N_2) = \{M(11), 2M(7), M(1)\}.$$

Show that $N_1 \cong M(13) \oplus T(5)$ and $N_2 \cong T(11)$.

One can check that $X_{-\alpha}^4 v = 0$. Hence $KA_y v \cong M(13)$. Set $\overline{N_1} = N_1/KA_y v$. Obviously, $I(\overline{N_1}) = \{M(5), 2M(3)\}$. As N_1 is self-dual, we conclude that it contains a submodule U with $I(U) = I(\overline{N_1})$. It is clear that $U \cap KA_y v = 0$. Therefore $N_1 = KA_y v \oplus U$. We show that $U \cong T(5)$. Put

$$m_5 = X_{-2}^2 X_{-1}^2 v + 4X_{-3}X_{-2}X_{-1}^2 v + X_{-3}X_{-2}^2 X_{-1}v.$$

We can verify that $\mathfrak{X}_\alpha$ fixes m_5 and that $X_{-\alpha}m_5$ has a nonzero weight component $2X_{-2}^3 X_{-1}^2 v$. One easily observes that $m_5 \in N_1$ and so $m_5 \in U$. By Lemma 29, $U \cong T(5)$.

Put $m_{11} = X_{-1}v + X_{-2}v$. One can check that $\mathfrak{X}_\alpha$ fixes m_{11}. Since $X_{-1}^3 v \neq 0$, then $X_{-\alpha}^2 m_{11} \neq 0$. Therefore $KA_y m_{11} \cong V(11)$. Taking into account Lemmas 12 and 6, it is not difficult to conclude that the vectors

$X_{-3}^2 X_{-2}^2 X_{-1}^2 v,\ X_{-3}X_{-2}^3 X_{-1}^2 v,\ X_{-3}X_{-2}^2 X_{-1}^3 v,\ X_{-2}^3 X_{-1}^3 v,\ X_{-1}^4 X_{-3}X_{-2}v,\ X_{-2}X_{-1}^4 X_{-2}v$

form a basis of M_1. We use this basis to show that $\operatorname{Inv} M_1 = 0$. Since N_2 has no submodules isomorphic to $M(1)$, then N_2 has no such factor modules as it is self-dual. Now Lemma 23 yields that $N_2/KA_y m_{11} \cong V(7)$ and N_2 has a filtration by Weyl modules. Then N_2 has a filtration by dual Weyl modules since it is self-dual. Hence N_2 is a tilting module and so $N_2 \cong T(11)$.

II.III. Let $p = 5$ and $\omega = \omega_1 + \omega_2 + \omega_3$. Then $\dim \varphi = 58$ and $\sigma_y(\omega) = 10$. Set

$$\Lambda_6 = \{\omega - \alpha_1 - \alpha_2, \omega - \alpha_1 - \alpha_3, \omega - \alpha_2 - \alpha_3\};$$

$$\Lambda_4 = \{\omega - 2\alpha_1 - \alpha_2, \omega - \alpha_1 - 2\alpha_2, \omega - \alpha_1 - \alpha_2 - \alpha_3,\ \omega - 2\alpha_2 - \alpha_3, \omega - \alpha_2 - 2\alpha_3\};$$

$$\Lambda_2 = \{\omega - 2\alpha_1 - 2\alpha_2, \omega - 2\alpha_1 - \alpha_2 - \alpha_3, \omega - \alpha_1 - 2\alpha_2 - \alpha_3, \omega - \alpha_1 - \alpha_2 - 2\alpha_3, \omega - 2\alpha_2 - 2\alpha_3\};$$

$$\Lambda_0 = \{\omega - 2\alpha_1 - 2\alpha_2 - \alpha_3, \omega - 2\alpha_1 - \alpha_2 - 2\alpha_3, \omega - \alpha_1 - 3\alpha_2 - \alpha_3, \omega - \alpha_1 - 2\alpha_2 - 2\alpha_3\}.$$

It is not difficult to show that $M_i = \langle M_\mu | \mu \in \Lambda_i\rangle$. By Lemma 22, $\dim M_\lambda = 2$ for $\lambda \in \{\omega - \alpha_1 - \alpha_2,\ \omega - \alpha_2 - \alpha_3\}$. One easily observes that $\dim M_\mu = 2$ for $\mu \in \{\omega - \alpha_1 - \alpha_2 - 2\alpha_3,\ \omega - 2\alpha_1 - \alpha_2 - \alpha_3\}$ since each of these weights lies in the same W-orbit with $\omega - \alpha_1 - \alpha_2$ or $\omega - \alpha_2 - \alpha_3$. Put

$$\Sigma = \{\omega - \alpha_1 - \alpha_3,\ \omega - 2\alpha_1 - \alpha_2,\ \omega - \alpha_1 - 2\alpha_2,\ \omega - 2\alpha_2 - \alpha_3,\ \omega - \alpha_2 - 2\alpha_3,\ \omega - 2\alpha_1 - 2\alpha_2,\ \omega - 2\alpha_2 - 2\alpha_3,\ \omega - 2\alpha_1 - \alpha_2 - 2\alpha_3,\ \omega - \alpha_1 - 3\alpha_2 - \alpha_3\}.$$

It is clear that $\dim M_\sigma = 1$ for $\sigma \in \Sigma$ as all weights from Σ lie in the same W-orbit with ω. Set $\mu = \omega - \alpha_1 - \alpha_2 - \alpha_3$. Then $\mu = \omega_2$. Using the Jantzen filtration [13, Part 2, §8, Proposition 8.19], we can find out that $V(\omega)$ has two composition factors: $M(\omega)$ and $M(\mu)$. Hence $\dim M_\mu = \dim V(\omega)_\mu - 1$. By Freudenthal's formula, we get $\dim V(\omega)_\mu = 4$ and $\dim M_\mu = 3$. The weights $\tau \in \{\omega - \alpha_1 - 2\alpha_2 - \alpha_3, \omega - 2\alpha_1 - 2\alpha_2 - \alpha_3, \omega - \alpha_1 - 2\alpha_2 - 2\alpha_3\}$ lie in the same W-orbit with μ, therefore $\dim M_\tau = 3$. This implies that $\dim M_{10} = 1$, $\dim M_8 = 3$, $\dim M_6 = 5$, $\dim M_4 = 7$, $\dim M_2 = 9$, and $\dim M_0 = 8$. Then

$$I(M_y) = \{M(10), 3M(8), 2M(6), 2M(4), 4M(2), M(0)\}.$$

By Proposition 8 and Lemma 27, $M_y = N_1 \oplus N_2 \oplus 2M(4)$ where $I(N_1) = \{M(10), 3M(8), M(0)\}$ and $I(N_2) = \{2M(6), 4M(2)\}$. We show that $N_1 \cong T(10) \oplus M(8)$ and $N_2 \cong 2T(6)$.

Obviously, $X_{-\alpha} v \neq 0$. Hence $KA_y v \cong V(10)$. Set $m_8 = X_{-1}v$ and $m_8' = 2X_{-1}v + X_{-3}v$. One can directly verify that $\mathfrak{X}_\alpha$ fixes m_8', $X_{-\alpha}^4 m_8' = 0$, $X_\alpha m_8 = v$, that the vector $X_{-\alpha,5}v$ has a weight component $u_1 = 3X_{-2}^2 X_{-3} X_{-2} X_{-1} v$, and the vector $X_{-\alpha}^4 m_8$ has a weight component $u_2 = 3X_{-1}X_{-3}X_{-2}X_{-3}X_{-1}v$ and a trivial component of weight $\omega(u_1)$. We claim that u_1 and $u_2 \neq 0$. Indeed, we have $X_2 u_1 = X_{-2}^2 X_{-3} X_{-1} v \neq 0$ and $X_3 X_1 u_2 = 3X_{-1}X_{-3}X_{-2}v \neq 0$ by Lemma 6. It is clear that $m_8 \in N_1$ and $X_{-\alpha}^4 m_8 \notin KA_y v$. Put $\overline{N_1} = N_1/KA_y v$ and denote by the symbol $\overline{m_8}$ the image of m_8 under the canonical homomorphism $N_1 \to \overline{N_1}$. The arguments above yield that $KA_y \overline{m_8} \cong V(8)$. Let N_1' be the full preimage of $KA_y\overline{m_8}$ in N_1. It is clear that $\operatorname{Irr} N_1' = \{M(10), M(8), M(0)\}$ and that $N_1' \cap \operatorname{Inv} M_8 = \langle X_{-\alpha} v\rangle$. Let $N_2' = KA_y m_8'$. As $X_{-\alpha}^4 m_8' = 0$, the A_y-module $N_2' \cong M(8)$. Since $m_8' \notin X_{-\alpha} v$, one easily observes that $N_1' \cap N_2' = 0$. This yields that $N_1 = N_1' \oplus N_2'$. Show that N_1' is indecomposable. Obviously, N_1' has only one indecomposable component with highest weight 10. Denote it by N^+. As $X_{-\alpha} v \in N^+$, then $N_1' \cap \operatorname{Inv} M_8 \subset N^+$. Therefore N_1' has no indecomposable components with highest weight 8.

Show that $\operatorname{Inv} M_0 = 0$. Suppose this it false. Then N_1 has a submodule isomorphic to $M(0)$. As N_1 is self-dual, it also has a factor module isomorphic to $M(0)$. Then $N_1 \cong M(0) \oplus U$ where $I(U) = \{M(10), 3M(8)\}$. It is clear that M_{10} and $M_8 \subset U$. Hence $KA_y v \subset U$ and $m_8 \in U$. We come to a contradiction as $KA_y \overline{m_8} \cong V(8)$. Hence $\operatorname{Inv} M_0 = 0$ and N_1' has no indecomposable components with highest weight 0. This implies that N_1' is indecomposable. By Corollary 10, N_1' is self-dual. Since N_1' has a filtration by Weyl modules, it has a filtration by dual Weyl modules as well. Therefore N_1' is a tilting module. Now it is obvious that $N_1' \cong T(10)$.

Set $m_6^1 = 4X_{-2}X_{-1}v + 3X_{-3}X_{-1}v + X_{-3}X_{-2}v$ and $m_6^2 = X_{-1}X_{-2}v + X_{-3}X_{-1}v + X_{-2}X_{-3}v$. One can directly verify that $\mathfrak{X}_\alpha$ fixes m_6^1 and m_6^2, $X_{-\alpha}^2 m_6^1 = X_{-1}X_{-2}^2X_{-1}v + 3X_{-2}X_{-3}^2X_{-2}v + u_1$, and $X_{-\alpha}^2 m_6^2 = 4X_{-1}X_{-2}^2X_{-1}v + 3X_{-2}X_{-3}^2X_{-2}v + u_2$ where the vectors u_i have no nontrivial components of weights $\omega - 2\alpha_1 - 2\alpha_2$ and $\omega - 2\alpha_2 - 2\alpha_3$. Lemma 12 implies that $X_{-1}X_{-2}^2X_{-1}v$ and $X_{-2}X_{-3}^2X_{-2}v \neq 0$. As $\dim \operatorname{Inv} M_6 \leq 2$, this yields that $\operatorname{Inv} M_6 = \langle m_6^1, m_6^2 \rangle$ and $\dim X_{-\alpha}^2 \operatorname{Inv} M_6 = 2$. Now Corollary 9 forces that $N_2 \cong T(6) \oplus T(6)$.

II.IV. Let $p = 5$ and $\omega = 2\omega_1 + 2\omega_2$. Then $\dim \varphi = 68$ and $M \cong S^{6,5}(V)$. It is not difficult to show that $\sigma_y(\omega) = 14$, $\dim M_{14} = 1$, $\dim M_{12} = 2$, $\dim M_{10} = 4$, $\dim M_8 = 4$, $\dim M_6 = 6$, $\dim M_4 = 6$, $\dim M_2 = 7$, and $\dim M_0 = 8$. Hence

$$I(M_y) = \{M(14), M(12), 2M(10), 2M(8), 3M(6), 3M(2), M(0)\}.$$

By Proposition 8 and Lemma 27, $M_y = M(14) \oplus N_1 \oplus N_2$ where

$$I(N_1) = \{M(12), 3M(6), 3M(2)\},\ I(N_2) = \{2M(10), 2M(8), M(0)\}.$$

Show that $N_1 \cong T(12) \oplus T(6)$ and $N_2 \cong T(10) \oplus M(10)$.

Put $m_{12} = X_{-1}v + 2X_{-2}v$. One can directly verify that $\mathfrak{X}_\alpha$ fixes m_{12} and the vector $X_{-\alpha}^3 m_{12}$ has a weight component $X_{-3}^2X_{-2}^2v$ and hence $X_{-\alpha}^3 m_{12} \neq 0$. By Lemma 28, $KA_y m_{12} \cong V(12)$. Set $\overline{N_1} = N_1/KA_y m_{12}$. Applying Lemma 12, it is not difficult to see that the vectors

$$X_{-1}^4X_{-2}^2v,\ X_{-2}^2X_{-1}^3X_{-2}v,\ X_{-3}X_{-1}^3X_{-2}^2v,\ X_{-3}^2X_{-1}^2X_{-2}^2v,$$
$$X_{-3}X_{-2}^3X_{-1}^2v,\ X_{-2}^4X_{-1}^2v,\ X_{-3}^2X_{-2}^3X_{-1}v$$

form a basis in M_2. Using this basis, we can check that $\dim \operatorname{Inv} M_2 = 1$. Show that N_1 is a tilting module. One easily observes that N_1 has a filtration by Weyl modules if it has no direct summands isomorphic to $\Delta(6)$ or $M(6)$. Since N_1 has no submodules isomorphic to $M(2) \oplus M(2)$ and is self-dual, it has no such factor modules. This yields that $\overline{N_1}$ has no direct summands of the form $\Delta(6) \oplus U$ where $U \in \{\Delta(6), M(2), T(6)\}$, and $M(2) \oplus F$ where $F \cong M(2)$ or $T(6)$. Since $I(\overline{N_1}) = \{2M(6), 3M(2)\}$, now Lemma 23 yields that $\overline{N_1}$ has no direct summands isomorphic to $\Delta(6)$ or $M(6)$. This implies that N_1 has a filtration by Weyl modules and therefore it has a filtration by dual Weyl modules as well since N_1 is self-dual. Hence N_1 is a tilting module. As we know $I(N_1)$, we conclude that $N_1 \cong T(12) \oplus T(6)$.

Set $m_{10}^1 = 2X_{-1}^2v + 4X_{-1}X_{-2}v + X_{-3}X_{-2}v$, $m_{10}^2 = X_{-2}^2v + 2X_{-1}X_{-2}v + X_{-1}^2v$, and $m_8 = 4X_{-1}^2X_{-2}v + 4X_{-1}X_{-2}^2v$. One can directly verify that $\mathfrak{X}_\alpha$ fixes m_{10}^i, $X_\alpha m_8 = m_{10}^2$, $X_{-\alpha}m_{10}^2 \neq 0$, $X_{-\alpha}m_{10}^1 = 0$, and $X_{-\alpha}^4 m_8 \notin \langle X_{-\alpha,5}m_{10}^2 \rangle$. Then $KA_y m_{10}^2 \cong V(10)$ and $N_2/KA_y m_{10}^2$ has a submodule isomorphic to $V(8)$. Let U be the full preimage of this submodule in N_2. Show that U is indecomposable. We can check that $\dim \operatorname{Inv} M_8 = 1$. Since $X_{-\alpha}m_{10}^2 \in \operatorname{Inv} M_8$, then U has no indecomposable components with highest weight 8. As $U \cap M_0 = \langle X_{-\alpha,5}m_{10}^1, X_{-\alpha}^4 m_8 \rangle$, then U has no indecomposable components with highest weight 0. Since $\dim(U \cap M_{10}) = 1$, this implies that U is indecomposable.

It is clear that $KA_y m_{10}^1 \cong M(10)$. As $m_{10}^1 \notin U \cap M_{10} = \langle m_{10}^2 \rangle$ and $KA_y m_{10}^1$ is irreducible, we conclude that $KA_y m_{10}^1 \cap U = 0$. Hence $N_2 = U \oplus KA_y m_{10}^1$. The module U is self-dual as N_2 is self-dual. Since U has a filtration by Weyl modules, it has a filtration by dual Weyl modules as well. Therefore U is a tilting module. This forces that $U \cong T(10)$ and completes the analysis of the case under consideration.

II.V. Let $p = 5$ and $\omega = \omega_1 + 3\omega_2$. Then $\dim \varphi = 80$ and $M \cong S^{7,5}(V)$. One can directly verify that $\sigma_y(\omega) = 15$, $\dim M_{15} = 1$, $\dim M_{13} = 2$, $\dim M_{11} = 3$, $\dim M_9 = 5$, $\dim M_7 = 6$, $\dim M_5 = 7$, $\dim M_3 = 8$, and $\dim M_1 = 8$. This implies that

$$I(M_y) = \{M(15), 2M(13), M(11), 2M(9), 2M(7), 2M(5), 2M(3), M(1)\}.$$

By Proposition 8 and Lemma 27, $M_y \cong N_1 \oplus N_2 \oplus M(9) \oplus M(9)$ where

$$I(N_1) = \{M(15), 2M(13), 2M(5), 2M(3)\}, I(N_2) = \{M(11), 2M(7), M(1)\}.$$

Show that $N_1 \cong T(15) \oplus T(5)$ and $N_2 \cong T(11)$.

Set $m_{13} = X_{-2}v$. Since $X_{-1}v \neq 0$, one easily observes that $X_{-\alpha}v \neq 0$. It is not difficult to check that $X_\alpha m_{13} = v$ and $X^4_{-\alpha}m_{13} \notin \langle X_{-\alpha,5}v\rangle$. Set $\overline{N_1} = N_1/KA_yv$ and denote by the symbol $\overline{m_{13}}$ the image of m_{13} under the canonical homomorphism $N_1 \to \overline{N_1}$. The arguments above and Lemma 28 yield that $KA_yv \cong V(15)$ and $KA_y\overline{m_{13}} \cong V(13)$. Let U be the full preimage of $KA_y\overline{m_{13}}$ in N_1. Then $I(N_1/U) = \{M(5), 2M(3)\}$. Since N_1 is self-dual, it has a submodule F with $I(F) = \{M(5), 2M(3)\}$.

Obviously, the vectors

$$X^3_{-1}X^2_{-2}v,\ X^2_{-1}X_{-3}X^2_{-2}v,\ X^2_{-1}X^3_{-2}v,\ X^4_{-2}X_{-1}v,$$
$$X_{-3}X^3_{-2}X_{-1}v,\ X^2_{-3}X^3_{-2}v,\ X^2_{-3}X^2_{-2}X_{-1}v$$

form a basis of M_5. Set $Q = KerX_\alpha \cap M_5$, $m_5 = 3X^3_{-1}X^2_{-2}v + X^2_{-1}X^3_{-2}v$, and $m^*_5 = X^4_{-2}X_{-1}v + 3X_{-3}X^3_{-2}X_{-1}v + 4X^2_{-3}X^3_{-2}v + X^2_{-3}X^2_{-2}X_{-1}v$. Show that $\operatorname{Inv} M_5 = \langle m_5\rangle$. Using the basis indicated above, we can check that $Q = \langle m_5, m^*_5\rangle$, $X_{\alpha,5}m_5 = 0$, and $X_{\alpha,5}m^*_5 \neq 0$. As $\operatorname{Inv} M_5 \subset Q$, this implies that $\operatorname{Inv} M_5 = \langle m_5\rangle$. It is not difficult to verify that $X_{-\alpha}m_5 \neq 0$. Obviously, $m_5 \in F$. One easily observes that $U \cap F = 0$ as $M(3) \notin \operatorname{Irr} U$. Hence $N_1 = U \oplus F$ and the modules U and F are self-dual since N_1 is such. As U has a filtration by Weyl modules, this yields that it has a filtration by dual Weyl modules as well. Therefore U is a tilting module. This forces that $U \cong T(15)$. As $m_5 \in F$, $X_{-\alpha}m_5 \neq 0$, and F is self-dual, then by Lemma 29, $F \cong T(5)$.

Set $m_{11} = X^2_{-2}v + 4X_{-3}X_{-2}v$. One can directly verify that $\mathfrak{X}_\alpha$ fixes m_{11} and that the vector $X^2_{-\alpha}m_{11}$ has a weight component $4X^2_{-1}X^2_{-2}v \neq 0$. Lemma 28 implies that $KA_ym_{11} \cong V(11)$.

Put $\overline{N_2} = N_2/KA_ym_{11}$. It is clear that $I(\overline{N_2}) = \{M(7), M(1)\}$. We can check that $\operatorname{Inv} M_1 = 0$. Therefore N_2 has no submodules isomorphic to $M(1)$. As N_2 is self-dual, we conclude that N_2 and $\overline{N_2}$ have no factor modules isomorphic to $M(1)$. Hence by Lemma 23, $\overline{N_2} \cong V(7)$ and N_2 has a filtration by Weyl modules. Since N_2 is self-dual, it has a filtration by dual Weyl modules as well. Therefore N_2 is a tilting module and $N_2 \cong T(11)$.

II.VI. Let $p = 5$ and $\omega = 2\omega_1 + 2\omega_3$. Then $\dim\varphi = 83$ and $\sigma_y(\omega) = 12$. Set

$$\Lambda_8 = \{\omega - 2\alpha_1, \omega - \alpha_1 - \alpha_2, \omega - \alpha_1 - \alpha_3, \omega - \alpha_2 - \alpha_3, \omega - 2\alpha_3\};$$
$$\Lambda_6 = \{\omega - 2\alpha_1 - \alpha_2, \omega - 2\alpha_1 - \alpha_3, \omega - \alpha_1 - \alpha_2 - \alpha_3, \omega - \alpha_2 - 2\alpha_3, \omega - \alpha_1 - 2\alpha_3\};$$
$$\Lambda_4 = \{\omega - 2\alpha_1 - 2\alpha_2,\ \omega - 2\alpha_1 - \alpha_2 - \alpha_3,\ \omega - 2\alpha_1 - 2\alpha_3,\ \omega - \alpha_1 - 2\alpha_2 - \alpha_3,$$
$$\omega - \alpha_1 - \alpha_2 - 2\alpha_3,\ \omega - 2\alpha_2 - 2\alpha_3\};$$
$$\Lambda_2 = \{\omega - 2\alpha_1 - 2\alpha_2 - \alpha_3,\ \omega - 2\alpha_1 - \alpha_2 - 2\alpha_3,\ \omega - 3\alpha_1 - \alpha_2 - \alpha_3,$$
$$\omega - \alpha_1 - 2\alpha_2 - 2\alpha_3,\ \omega - \alpha_1 - \alpha_2 - 3\alpha_3\};$$
$$\Lambda_0 = \{\omega - 2\alpha_1 - 2\alpha_2 - 2\alpha_3, \omega - 3\alpha_1 - 2\alpha_2 - \alpha_3, \omega - 3\alpha_1 - \alpha_2 - 2\alpha_3,$$
$$\omega - 2\alpha_1 - \alpha_2 - 3\alpha_3, \omega - 2\alpha_1 - 3\alpha_2 - \alpha_3, \omega - \alpha_1 - 3\alpha_2 - 2\alpha_3, \omega - \alpha_1 - 2\alpha_2 - 3\alpha_3\}.$$

One easily observes that $M_i = \langle M_\mu | \mu \in \Lambda_i\rangle$. By [18, Table 6.7], the maximal submodule in $V(\omega)$ is isomorphic to $M(0)$. Hence $\dim M_\mu = \dim V(\omega)_\mu$ for $\mu \neq 0$ and $\dim M_\mu = \dim V(\omega)_\mu - 1$ for $\mu = 0$. Put

$$\Sigma = \{\mu \in \Lambda(M) | \mu = \omega - a\alpha_1 - b\alpha_2 - c\alpha_3, abc = 0\}.$$

One easily concludes that $\dim M_\sigma = 1$ for $\sigma \in \Sigma$. Using Freudenthal's formula, we get that $\dim M_\mu = 3$ for $\mu = \omega - \alpha_1 - \alpha_2 - \alpha_3$ and $\dim M_\mu = 5$ for $\mu = \omega - 2\alpha_1 - 2\alpha_2 - 2\alpha_3 = 0$. It is not difficult to show that the weight λ lies in the same W-orbit with $\mu = \omega - \alpha_1 - \alpha_2 - \alpha_3$

for

$$\lambda \in \{\omega - 2\alpha_1 - \alpha_2 - \alpha_3,\ \omega - \alpha_1 - \alpha_2 - 2\alpha_3,\\ \omega - 2\alpha_1 - 2\alpha_2 - \alpha_3,\ \omega - 2\alpha_1 - \alpha_2 - 2\alpha_3,\ \omega - \alpha_1 - 2\alpha_2 - 2\alpha_3\},$$

so $\dim M_\lambda = 3$ for these λ. Finally, the weight μ lies in the same W-orbit with a weight from Σ for

$$\mu \in \{\omega - \alpha_1 - 2\alpha_2 - \alpha_3,\ \omega - 3\alpha_1 - \alpha_2 - \alpha_3,\ \omega - \alpha_1 - \alpha_2 - 3\alpha_3,\\ \omega - 3\alpha_1 - 2\alpha_2 - \alpha_3,\ \omega - 3\alpha_1 - \alpha_2 - 2\alpha_3,\ \omega - 2\alpha_1 - \alpha_2 - 3\alpha_3,\\ \omega - 2\alpha_1 - 3\alpha_2 - \alpha_3,\ \omega - \alpha_1 - 3\alpha_2 - 2\alpha_3,\ \omega - \alpha_1 - 2\alpha_2 - 3\alpha_3\}.$$

Therefore $\dim M_\mu = 1$ for all these μ. Now we easily deduce that $\dim M_{12} = 1$, $\dim M_{10} = 2$, $\dim M_8 = 5$, $\dim M_6 = 7$, $\dim M_4 = 10$, $\dim M_2 = 11$, and $\dim M_0 = 11$. This yields that $I(M_y) = \{M(12), M(10), 4M(8), 3M(6), 3M(4), 3M(2), 3M(0)\}$. By Proposition 8 and Lemma 27, $M_y = N_1 \oplus N_2 \oplus 3M(4)$ where $I(N_1) = \{M(12), 3M(6), 3M(2)\}$ and $I(N_2) = \{M(10), 4M(8), 3M(0)\}$. Show that $N_1 \cong T(12) \oplus T(6)$ and $N_2 \cong T(10) \oplus T(8) \oplus M(8)$.

It is not difficult to verify that the vector $X_{-\alpha}^3 v$ has a nonzero weight component $X_{-3}X_{-1}^2 v$. Hence $KA_y v \cong V(12)$ by Lemma 28. Set $\overline{N_1} = N_1/KA_y v$. Then $I(\overline{N_1}) = \{2M(6), 3M(2)\}$. Put

$$m_2 = X_{-1}^3 X_{-2} X_{-3} v + 4X_{-3}X_{-2}^2X_{-1}^2 v + 2X_{-2}^2X_{-3}X_{-1}^2 v + 2X_{-3}^2X_{-2}X_{-1}^2 v + \\ + 2X_{-1}X_{-2}^2X_{-3}X_{-1}v + 3X_{-1}^2X_{-2}X_{-3}^2 v + 2X_{-1}X_{-2}^2X_{-3}^2 v + \\ + X_{-2}^2X_{-1}X_{-3}^2 v + X_{-3}X_{-2}^2X_{-3}X_{-1}v + X_{-3}^3X_{-2}X_{-1}v.$$

One can directly verify that each of the triples

$$(X_{-3}X_{-2}^2X_{-1}^2 v,\ X_{-2}^2X_{-3}X_{-1}^2 v,\ X_{-1}X_{-2}^2X_{-3}X_{-1}v),\\ (X_{-1}X_{-2}^2X_{-3}^2 v,\ X_{-2}^2X_{-1}X_{-3}^2 v,\ X_{-3}X_{-2}^2X_{-3}X_{-1}v),\\ (X_{-2}X_{-3}^2X_{-1}^2 v,\ X_{-3}^2X_{-2}X_{-1}^2 v,\ X_{-1}^2X_{-2}X_{-3}^2 v)$$

consists of linearly independent vectors. Therefore the vectors

$$X_{-3}X_{-2}^2X_{-1}^2 v,\ X_{-2}^2X_{-3}X_{-1}^2 v,\ X_{-1}X_{-2}^2X_{-3}X_{-1}v,\ X_{-1}X_{-2}^2X_{-3}^2 v,\\ X_{-2}^2X_{-1}X_{-3}^2 v,\ X_{-3}X_{-2}^2X_{-3}X_{-1}v,\ X_{-2}X_{-3}^2X_{-1}^2 v,\ X_{-3}^2X_{-2}X_{-1}^2 v,\\ X_{-1}^2X_{-2}X_{-3}^2 v,\ X_{-1}^3X_{-2}X_{-3}v,\ X_{-3}^3X_{-2}X_{-1}v$$

form a basis in M_2. Using this basis, we can show that $\dim \operatorname{Inv} M_2 = 1$ and $\operatorname{Inv} M_2 = \langle m_2 \rangle$.

Hence N_1 has no submodules isomorphic to $M(2) \oplus M(2)$. As N_1 is self-dual, this yields that N_1 and $\overline{N_1}$ have no such factor modules. Hence $\overline{N_1}$ has no direct summands of the form $T(6) \oplus \Delta(6)$, $T(6) \oplus M(2)$, $\Delta(6) \oplus M(2)$, or $M(2) \oplus M(2)$. Then by Lemma 27, $\overline{N_1} \cong T(6) \oplus V(6)$ or $V(6) \oplus V(6) \oplus M(2)$. Therefore N_1 has a filtration by Weyl modules and it has a filtration by dual Weyl modules as N_1 is self-dual. Hence N_1 is a tilting module. As we know $I(N_1)$, we conclude that $N_1 \cong T(12) \oplus T(6)$.

Set $m_{10} = X_{-1}v + 3X_{-3}v$. It is clear that $X_{-\alpha}m_{10} \neq 0$ since $X_{-1}^2 v \neq 0$. One can directly verify that $\mathfrak{X}_\alpha$ fixes m_{10}. Therefore $\operatorname{Inv} M_{10} = \langle m_{10} \rangle$. By Lemma 28, $KA_y m_{10} \cong V(10)$. As $\dim(N_2 \cap M_{10}) = 1$, the module N_2 has an indecomposable component $N_{2,1}$ containing the vector m_{10}. Put $\overline{N_{2,1}} = N_{2,1}/KA_y m_{10}$ and write $N_2 = N_{2,1} \oplus N_{2,2}$ where $N_{2,2}$ is a KA_y-module. By Corollary 10, $N_{2,1}$ and $N_{2,2}$ are self-dual. Show that $N_{2,1} \cong T(10)$ and $N_{2,2} \cong T(8) \oplus M(8)$.

It is not difficult to observe that the vectors $X_{-3}^2X_{-2}^2X_{-1}^2 v$, $X_{-2}^2X_{-3}^2X_{-1}^2 v$, $X_{-3}X_{-2}^2X_{-3}X_{-1}^2 v$, $X_{-1}X_{-3}X_{-2}^2X_{-3}X_{-1}v$, and $X_{-1}X_{-2}^2X_{-3}^2X_{-1}v$ are linearly independent.

Taking into account Lemmas 12 and 6, we conclude that the vectors

$$X_{-1}^2X_{-2}^2X_{-3}X_{-1}v,\ X_{-1}^3X_{-2}X_{-3}^2v,\ X_{-2}^3X_{-3}X_{-1}^2v,\ X_{-2}^3X_{-3}^2X_{-1}v,$$
$$X_{-3}^3X_{-2}X_{-1}^2v,\ X_{-2}X_{-3}^3X_{-2}X_{-1}v,\ X_{-3}^2X_{-2}^2X_{-1}^2v,\ X_{-2}^2X_{-3}^2X_{-1}^2v,$$
$$X_{-3}X_{-2}^2X_{-3}X_{-1}^2v,\ X_{-1}X_{-3}X_{-2}^2X_{-3}X_{-1}v,\ X_{-1}X_{-2}^2X_{-3}^2X_{-1}v$$

form a basis of M_0. Using this basis, we check directly that $\dim \operatorname{Inv} M_0 = 1$. Hence N_2 has no submodules isomorphic to $M(0) \oplus M(0)$. As N_2 is self-dual, it has no such factor modules. This implies that there are no direct summands isomorphic to $T(8) \oplus \Delta(8)$, $T(8) \oplus M(0)$, $\Delta(8) \oplus \Delta(8)$, $\Delta(8) \oplus M(0)$, or $M(0) \oplus M(0)$ in $\overline{N_{2,1}}$ and $N_{2,2}$, and that there are no direct summands isomorphic to $T(8)$ or $M(0)$ in $N_{2,2}$ if $\overline{N_{2,1}}$ has an indecomposable component isomorphic to $T(8)$, $\Delta(8)$, or $M(0)$.

Suppose that there is an indecomposable component isomorphic to $\Delta(8)$ in $\overline{N_{2,1}}$. Lemma 23 and the arguments above imply that $\overline{N_{2,1}} \cong \Delta(8)$ or $V(8) \oplus \Delta(8)$. Using Lemma 23 and the self-duality of N_2, we come to a contradiction with the facts proven above on indecomposable components of $\overline{N_{2,1}}$ and $N_{2,2}$. Hence $\overline{N_{2,1}}$ has no direct summands isomorphic to $\Delta(8)$.

Assume now that $\overline{N_{2,1}}$ has an indecomposable component isomorphic to $M(8)$. As $N_{2,2}$ is self-dual, Lemma 23 and the arguments above yield that $\overline{N_{2,1}} \cong T(8) \oplus V(8) \oplus M(8)$, $V(8) \oplus V(8) \oplus M(8)$, $V(8) \oplus V(8) \oplus M(8) \oplus M(0)$, or $V(8) \oplus M(8)$. Check that all these possibilities cannot be realized.

Obviously, the vectors X_{-1}^2v, $X_{-2}X_{-1}v$, $X_{-2}X_{-3}v$, X_{-3}^2v, and $X_{-3}X_{-1}v$ form a basis of M_8. Set $m_8^1 = 3X_{-1}^2v + 2X_{-2}X_{-1}v + X_{-2}X_{-3}v + 3X_{-3}^2v$,
$m_8^2 = 2X_{-1}^2v + X_{-3}X_{-1}v + 4X_{-2}X_{-3}v$, and $m_8^3 = 2X_{-2}X_{-3}v + X_{-3}^2v$.
It is clear that the vectors m_8^i, $1 \leq i \leq 3$, are linearly independent. One can directly verify that $\operatorname{Inv} M_8 = \langle m_8^1, m_8^2, m_8^3\rangle$ and that $\dim X_{-\alpha}^4 \operatorname{Inv} M_8 = 1$. Hence $\dim(KerX_{-\alpha}^4 \cap \operatorname{Inv} M_8) = 2$. Set $S = KerX_{-\alpha}^4 \cap \operatorname{Inv} M_8$.

First we assume that the multiplicity of the composition factor $M(8)$ in the module $\overline{N_{2,1}}$ equals 3. Then $\operatorname{Inv} M_8 \subset N_{2,1}$ as $\operatorname{Inv} M_8 \subset N_2$. Observe that $\overline{N_{2,1}} \cong M(8) \oplus \overline{U}$ where

$$\overline{U} \in \{T(8) \oplus V(8),\ V(8) \oplus V(8),\ V(8) \oplus V(8) \oplus M(0)\}.$$

In all cases we have

$$X_{-\alpha}^4 u \neq 0 \tag{5}$$

for a nonzero vector $u \in \overline{U}$ of weight 8. Let U be the full preimage of $\overline{U}$ in $N_{2,1}$. Let $m \in S$ and $m \notin KA_ym_{10}$. Such vectors exist since $\dim(KA_ym_{10} \cap M_8) = 1$ and $\dim S = 2$. Set $F = KA_ym$. Since $X_{-\alpha}^4 m = 0$, then by Lemma 28, $F \cong M(8)$. As $m \notin KA_ym_{10}$, Formula (5) implies that $m \notin U$. Then $U \cap F = 0$ since F is irreducible. Therefore $N_{2,1} = U \oplus F$ which yields a contradiction as $N_{2,1}$ is indecomposable.

Let $\overline{N_{2,1}} \cong V(8) \oplus M(8)$. Then $I(N_{2,2}) = \{M(8), 2M(0)\}$. Since $N_{2,2}$ has no submodules isomorphic to $M(0) \oplus M(0)$ and is self-dual, Lemma 23 forces that $N_{2,2} \cong T(8)$. Then $KA_yz \cong V(8)$ for any vector $z \in (M_8 \cap N_{2,2}) \setminus \{0\}$. Therefore $X_{-\alpha}^4 z \neq 0$ and so $S \subset N_{2,1}$. As $\dim S = 2$, then $S \not\subset KA_ym_{10}$. Let $m \in S$ and $m \notin KA_ym_{10}$. Arguing as above, it is not difficult to show that the module $N' = KA_ym$ is isomorphic to $M(8)$ and that $N_{2,1} = N' \oplus N''$ where $KA_ym_{10} \subset N''$ and $N''/KA_ym_{10} \cong V(8)$. We again come to a contradiction since $N_{2,1}$ is indecomposable. This implies that $\overline{N_{2,1}}$ has no indecomposable components isomorphic to $\Delta(8)$ or $M(8)$. Then Lemma 23 implies that $N_{2,1}$ has a filtration by Weyl modules and a filtration by dual Weyl modules as $N_{2,1}$ is self-dual. Hence $N_{2,1}$ is a tilting module and this forces that $N_{2,1} \cong T(10)$.

It is clear that $I(N_{2,2}) = \{2M(8), 2M(0)\}$. As $N_{2,2}$ has no direct summands isomorphic to $M(0) \oplus M(0)$ and is self-dual, Lemma 23 yields that $N_{2,2} \cong T(8) \oplus M(8)$ or $V(8) \oplus \Delta(8)$. Show that the first possibility is realized. Set $Q = X_\alpha^4 M_0 \cap \operatorname{Inv} M_8$ and

$$z = 2X_{-1}^2v + 2X_{-2}X_{-3}v + X_{-1}X_{-3}v + 4X_{-3}^2v.$$

Using the basis of M_0 indicated above, we can directly check that $\dim Q = 2$, $Q = \langle X_{-\alpha}m_{10}, z\rangle$, and $X_{-\alpha}^4 z \neq 0$. It is clear that $Q \subset N_2$ since $\operatorname{Inv} M_8 \subset N_2$. Let $z = X_\alpha^4 z_0$ where $z_0 \in M_0$. We can assume that $z_0 \in N_2$. Write $z_0 = z_0^1 + z_0^2$ and $z = z^1 + z^2$ where $z_0^i \in N_{2,i} \cap M_0$ and $z^i \in N_{2,i} \cap M_8$. Then $X_\alpha^4 z_0^i = z^i$. As $N_{2,2} \cap M_8 \subset \operatorname{Inv} M_8$, $\dim(N_{2,2} \cap M_8) = 2$, and $\dim M_8 = 3$, it is clear that $\dim(N_{2,1} \cap \operatorname{Inv} M_8) = 1$. Since $z^1 \in \operatorname{Inv} M_8$, we conclude that $z^1 \in \langle X_{-\alpha}m_{10}\rangle \subset KA_y m_{10}$ and $X_{-\alpha}^4 z^1 = 0$. Therefore $X_{-\alpha}^4 z^2 \neq 0$. This yields that $X_{-\alpha}^4(X_\alpha^4 z_0^2) \neq 0$. Now Lemma 25 forces that $N_2 \cong T(10) \oplus T(8) \oplus M(8)$.

II.VII. Let $p = 5$ and $\omega = 4\omega_2$. Then $\dim \varphi = 85$ and $M \cong S^{8,5}(V)$. One can directly verify that $\sigma_y(\omega) = 16$, $\dim M_{16} = 1$, $\dim M_{14} = 1$, $\dim M_{12} = 3$, $\dim M_{10} = 4$, $\dim M_8 = 6$, $\dim M_6 = 7$, $\dim M_4 = 8$, $\dim M_2 = 8$, and $\dim M_0 = 9$. This yields that $I(M_y) = \{M(16), 3M(12), M(10), 3M(8), 3M(6), M(4), M(2), 3M(0)\}$. By Proposition 8 and Lemma 27, $M_y = N_1 \oplus N_2 \oplus M(4)$ where $I(N_1) = \{M(16), 3M(12), 3M(6), M(2)\}$ and $I(N_2) = \{M(10), 3M(8), 3M(0)\}$. Show that $N_1 \cong T(16) \oplus T(12)$ and $N_2 \cong T(10) \oplus T(8)$. As $X_{-2}^2 v \neq 0$, then $X_{-\alpha}^2 v \neq 0$. Hence $KA_y v \cong V(16)$. Set $m_{12} = 3X_{-1}X_{-2}v + X_{-2}^2 v$ and $u = X_{-2}^2 v$. We can check that $\mathfrak{X}_\alpha$ fixes m_{12}, $X_\alpha^2 u = v$, and

$$X_{-\alpha}^3 u \notin \langle X_{-\alpha,5}v\rangle, \;\; X_{-\alpha}^3 m_{12} \notin \langle X_{-\alpha,5}v, \; X_{-\alpha}^3 u\rangle. \tag{6}$$

Put $\overline{N_1} = N_1/KA_y v$ and denote by $\overline{u}$ the image of u under the natural homomorphism $N_1 \to \overline{N_1}$. Let U be the full preimage of $KA_y\overline{u}$ in N_1 and $F = KA_y m_{12}$. Formula (6) implies that $KA_y\overline{u} \cong F \cong V(12)$ and $U \cap F = 0$. Indeed, any nontrivial submodule of F contains the vector $X_{-\alpha}^3 m_{12}$ which does not lie in U. Set $N_1' = N_1/(U \oplus F)$. Then $I(N_1') = \{M(6), M(2)\}$. It is clear that the vectors

$$X_{-3}X_{-1}^2X_{-2}^4v,\; X_{-1}X_{-3}^2X_{-2}^4v,\; X_{-1}^3X_{-2}^4v,\; X_{-3}^3X_{-2}^4v,$$
$$X_{-1}^2X_{-3}^2X_{-2}^3v,\; X_{-1}X_{-3}^3X_{-2}^3v,\; X_{-1}^3X_{-3}X_{-2}^3v,\; X_{-2}X_{-1}X_{-3}X_{-2}^4v$$

form a basis of M_2. Using this basis, we can show that $\operatorname{Inv} M_2 = 0$. Hence N_1 has no submodules isomorphic to $M(2)$. As N_1 is self-dual, the modules N_1 and N_1' have no such factor modules. Now Lemma 23 yields that $N_1' \cong V(6)$. Hence N_1 has a filtration by Weyl modules and N_1 is a tilting module. Since we know $I(N_1)$, we conclude that $N_1 \cong T(16) \oplus T(12)$.

Set $m_{10} = 3X_{-2}^3 v + X_{-1}X_{-2}^2 v + 2X_{-3}X_{-2}^2 v + X_{-1}X_{-3}X_{-2}v$. One can directly verify that $\mathfrak{X}_\alpha$ fixes m_{10} and that $X_{-\alpha}m_{10}$ has a nonzero weight component $X_{-2}^4 v$. Therefore $KA_y m_{10} \cong V(10)$. Put $\overline{N_2} = N_2/KA_y m_{10}$. Then $I(\overline{N_2}) = \{2M(8), 3M(0)\}$. Show that $\overline{N_2}$ has no indecomposable components isomorphic to $\Delta(8)$ or $M(8)$. Taking into account Lemmas 6 and 12, we can conclude that the vectors

$$X_{-3}X_{-1}^3X_{-2}^4v,\; X_{-1}X_{-3}^3X_{-2}^4v,\; X_{-1}^2X_{-3}^2X_{-2}^4v,\; X_{-1}^4X_{-2}^4v,\; X_{-3}^4X_{-2}^4v,$$
$$X_{-1}^3X_{-3}^2X_{-2}^3v,\; X_{-1}^2X_{-3}^3X_{-2}^3v,\; X_{-2}^3X_{-1}^2X_{-3}X_{-2}^2v,\; X_{-2}^3X_{-1}X_{-3}^2X_{-2}^2v$$

form a basis of M_0. Using this basis, one can directly check that $\dim \operatorname{Inv} M_0 = 1$. Hence N_2 has no submodules isomorphic to $M(0) \oplus M(0)$. Since N_2 is self-dual, the modules N_2 and $\overline{N_2}$ have no such factor modules. Therefore $\overline{N_2}$ has no direct summands isomorphic to $T(8) \oplus \Delta(8)$, $T(8) \oplus M(8)$, $\Delta(8) \oplus \Delta(8)$, $\Delta(8) \oplus M(0)$, or $M(0) \oplus M(0)$. Taking this into account and using Lemma 23, we conclude that $\overline{N_2}$ has no indecomposable components isomorphic to $\Delta(8)$ or $M(8)$. Then Lemma 23 forces that N_2 has a filtration by Weyl modules. Therefore N_2 is a tilting module. We know $I(N_2)$ and can conclude that $N_2 \cong T(10) \oplus T(8)$.

II.VIII. Let $p = 7$ and $\omega = 4\omega_1 + \omega_3$. Then $\dim \varphi = 100$ and $\sigma_y(\omega) = 15$. It is clear that $\dim M_{15} = 1$ and $\dim M_{13} = 2$. Set

$\Lambda_{11} = \{\omega - 2\alpha_1, \omega - \alpha_1 - \alpha_2, \omega - \alpha_2 - \alpha_3, \omega - \alpha_1 - \alpha_3\}$;

$\Lambda_9 = \{\omega - 3\alpha_1, \omega - 2\alpha_1 - \alpha_2, \omega - 2\alpha_1 - \alpha_3, \omega - \alpha_1 - \alpha_2 - \alpha_3\}$;

$\Lambda_7 = \{\omega - 4\alpha_1, \omega - 3\alpha_1 - \alpha_2, \omega - 3\alpha_1 - \alpha_3, \omega - 2\alpha_1 - 2\alpha_2,$
$\omega - 2\alpha_1 - \alpha_2 - \alpha_3, \omega - \alpha_1 - 2\alpha_2 - \alpha_3, \omega - \alpha_1 - \alpha_2 - 2\alpha_3\}$;

$\Lambda_5 = \{\omega - 4\alpha_1 - \alpha_2, \omega - 4\alpha_1 - \alpha_3, \omega - 3\alpha_1 - 2\alpha_2, \omega - 3\alpha_1 - \alpha_2 - \alpha_3,$
$\omega - 2\alpha_1 - 2\alpha_2 - \alpha_3, \omega - 2\alpha_1 - \alpha_2 - 2\alpha_3, \omega - \alpha_1 - 2\alpha_2 - 2\alpha_3\}$;

$\Lambda_3 = \{\omega - 4\alpha_1 - 2\alpha_2,\ \omega - 4\alpha_1 - \alpha_2 - \alpha_3,\ \omega - 3\alpha_1 - 3\alpha_2,\ \omega - 3\alpha_1 - 2\alpha_2 - \alpha_3,$
$\omega - 3\alpha_1 - \alpha_2 - 2\alpha_3,\ \omega - 2\alpha_1 - 3\alpha_2 - \alpha_3,\ \omega - 2\alpha_1 - 2\alpha_2 - 2\alpha_3\}$;

$\Lambda_1 = \{\omega - 5\alpha_1 - \alpha_2 - \alpha_3, \omega - 4\alpha_1 - 3\alpha_2, \omega - 4\alpha_1 - 2\alpha_2 - \alpha_3, \omega - 4\alpha_1 - \alpha_2 - 2\alpha_3,$
$\omega - 3\alpha_1 - 3\alpha_2 - \alpha_3, \omega - 3\alpha_1 - 2\alpha_2 - 2\alpha_3, \omega - 2\alpha_1 - 3\alpha_2 - 2\alpha_3, \omega - 2\alpha - 2\alpha_2 - 3\alpha_3\}$.

One easily observes that $M_i = \langle M_\mu | \mu \in \Lambda_i \rangle$ for $1 \leq i \leq 11$. Now we shall determine the dimensions of the subspaces M_μ for $\mu \in \cup_{i=1}^{11} \Lambda_i$. Put

$\Delta = \{\mu \in \Lambda(M) | \mu = \omega - b_1\alpha_1 - b_2\alpha_2 - b_3\alpha_3,\ b_1b_2b_3 = 0\}$;

$\Delta' = \{\omega - \alpha_1 - 2\alpha_2 - \alpha_3,\ \omega - \alpha_1 - \alpha_2 - 2\alpha_3,\ \omega - 2\alpha_1 - \alpha_2 - 2\alpha_3,\ \omega - \alpha_1 - 2\alpha_2 - 2\alpha_3,$
$\omega - 3\alpha_1 - \alpha_2 - 2\alpha_3,\ \omega - 2\alpha_1 - 3\alpha_2 - \alpha_3,\ \omega - 5\alpha_1 - \alpha_2 - \alpha_3,\ \omega - 4\alpha_1 - \alpha_2 - 2\alpha_3,$
$\omega - 2\alpha_1 - 3\alpha_2 - 2\alpha_3,\ \omega - 2\alpha_1 - 2\alpha_2 - 3\alpha_3\}$;

$\Sigma = \{\omega - \alpha_1 - \alpha_2 - \alpha_3,\ \omega - 2\alpha_1 - \alpha_2 - \alpha_3,\ \omega - 3\alpha_1 - 2\alpha_2 - \alpha_3\}$;

$\Sigma' = \{\omega - 3\alpha_1 - \alpha_2 - \alpha_3,\ \omega - 2\alpha_1 - 2\alpha_2 - \alpha_3,$
$\omega - 2\alpha_1 - 2\alpha_2 - 2\alpha_3,\ \omega - 4\alpha_1 - 2\alpha_2 - \alpha_3,\ \omega - 3\alpha_1 - 3\alpha_2 - \alpha_3,$
$\omega - 4\alpha_1 - \alpha_2 - \alpha_3,\ \omega - 3\alpha_1 - 2\alpha_2 - 2\alpha_3\}$.

Using Theorem 1 and Proposition 4, we can conclude that $\dim M_\delta = 1$ for $\delta \in \Delta$. It is clear that $\dim M_\gamma = 1$ for $\gamma \in \Delta'$ since γ lies in the same W-orbit with a weight from Δ. By [13, Part II, 8.20], the maximal submodule in $V(\omega)$ is isomorphic to $M(3\omega_1) = M(\omega - \alpha_1 - \alpha_2 - \alpha_3)$. So it is clear that $\dim M_\sigma = \dim V(\omega)_\sigma - 1$ for $\sigma \in \Sigma$. Using Freudenthal's formula, we get that $\dim M_\sigma = 2$ for $\sigma \in \Sigma$. If $\nu \in \Sigma'$, then ν lies in the same W-orbit with a weight from Σ, hence $\dim M_\nu = 2$. Now all dimensions of the subspaces M_μ we need are found.

The arguments above imply that $\dim M_{11} = 4$, $\dim M_9 = 5$, $\dim M_7 = 8$, $\dim M_5 = 9$, $\dim M_3 = 10$, and $\dim M_1 = 11$. Now we can see that

$$I(M) = \{M(15), M(13), 3M(11), M(9), 3M(7), 4M(5), 2M(3), 3M(1)\}.$$

By Proposition 8 and Lemma 27, $M_y = N_1 \oplus M(13) \oplus N_2 \oplus N_3$ where

$$I(N_1) = \{M(15), 3M(11), 3M(1)\},\ I(N_2) = \{M(9), 2M(3)\},\ I(N_3) = \{3M(7), 4M(5)\}.$$

Remark 3 implies that the modules N_i are self-dual. Show that $N_1 \cong T(15) \oplus T(11)$, $N_2 \cong T(9)$, and $N_3 \cong 2T(7) \oplus M(7)$.

Since $X_{-1}^2 v \neq 0$, then $X_{-\alpha}^2 v \neq 0$. This forces that $KA_y v \cong V(15)$. One easily check that the pairs $(X_{-3}X_{-2}^2X_{-1}^4 v,\ X_{-2}^2X_{-3}X_{-1}^4 v)$, $(X_{-2}^3X_{-3}X_{-1}^3 v,\ X_{-3}X_{-2}^3X_{-1}^3 v)$, and $(X_{-3}^2X_{-2}^2X_{-1}^3 v,\ X_{-3}X_{-2}^2X_{-3}X_{-1}^3 v)$ consist of linearly independent vectors. Now it is obvious that the vectors

$$X_{-3}X_{-2}^2X_{-1}^4 v,\ X_{-2}^2X_{-3}X_{-1}^4 v,\ X_{-2}^3X_{-3}X_{-1}^3 v,\ X_{-3}X_{-2}^3X_{-1}^3 v,$$
$$X_{-3}^2X_{-2}^2X_{-1}^3 v,\ X_{-3}X_{-2}^2X_{-3}X_{-1}^3 v,\ X_{-1}^5X_{-2}X_{-3} v,\ X_{-2}^3X_{-1}^4 v,$$
$$X_{-3}^2X_{-2}X_{-1}^4 v,\ X_{-2}^2X_{-3}^2X_{-2}X_{-1}^2 v,\ X_{-3}^3X_{-2}^2X_{-1}^2 v$$

form a basis of M_1. Using this basis, we directly verify that $\dim \operatorname{Inv} M_1 = 1$.

Set $\overline{N_1} = N_1/KA_yv$. Then $I(\overline{N_1}) = \{2M(11), 3M(1)\}$. As the module N_1 has no submodules isomorphic to $M(1) \oplus M(1)$ and is self-dual, both N_1 and $\overline{N_1}$ have no such factor modules. Hence $\overline{N_1}$ has no direct summands of the form $M(1) \oplus M(1)$, $\Delta(11) \oplus M(1)$, $T(11) \oplus M(1)$, $\Delta(11) \oplus \Delta(11)$, or $T(11) \oplus \Delta(11)$. Using Lemma 23, we conclude that $\overline{N_1} \cong T(11) \oplus V(11)$ or $V(11) \oplus V(11) \oplus M(1)$. Therefore N_1 has a filtration by Weyl modules. By remark 4, N_1 is a tilting module. As we know $I(N_1)$, we conclude that $N_1 \cong T(15) \oplus T(11)$.

Set $m_9 = 2X_{-1}^3v + 6X_{-2}X_{-1}^2v + 2X_{-3}X_{-1}^2v + X_{-3}X_{-2}X_{-1}v$. One can directly check that $\mathfrak{X}_\alpha$ fixes m_9 and that the vector $X_{-\alpha}^3 m_9$ has a nonzero weight component $6X_{-2}^2X_{-1}^4v$. Hence $KA_ym_9 \cong V(9)$. By Lemma 29, $N_2 \cong T(9)$.

We can show that the pairs $(X_{-3}X_{-2}X_{-1}^3v,\ X_{-1}^3X_{-2}X_{-3}v)$ and $(X_{-3}X_{-2}^2X_{-1}^2v,\ X_{-1}X_{-2}^2X_{-1}X_{-3}v)$ consist of linearly independent vectors. Then the vectors

$$X_{-2}X_{-1}^4v,\ X_{-3}X_{-1}^4v,\ X_{-2}^2X_{-1}^2v,\ X_{-3}^2X_{-2}X_{-1}^2v,\ X_{-3}X_{-2}^2X_{-3}X_{-1}v,$$
$$X_{-3}X_{-2}X_{-1}^3v,\ X_{-1}^3X_{-2}X_{-3}v,\ X_{-3}X_{-2}^2X_{-1}^2v,\ X_{-1}X_{-2}^2X_{-1}X_{-3}v$$

form a basis in M_5. Put

$$\begin{aligned} m_7^1 = {} & 2X_{-1}^4v + 6X_{-2}X_{-1}^3v + 5X_{-3}X_{-1}^3v + 5X_{-2}^2X_{-1}^2v + \\ & +5X_{-3}^2X_{-2}X_{-1}v + 6X_{-2}^2X_{-3}X_{-1}v + 6X_{-1}^2X_{-2}X_{-3}v, \\ m_7^2 = {} & 4X_{-1}^4v + 2X_{-2}X_{-1}^3v + 3X_{-2}^2X_{-1}^2v + 6X_{-3}^2X_{-2}X_{-1}v + \\ & +3X_{-3}X_{-2}X_{-1}^2v + X_{-2}^2X_{-3}X_{-1}v + X_{-1}^2X_{-2}X_{-3}v. \end{aligned}$$

Using the basis indicated above, we check that $X_\alpha M_5 \cap \operatorname{Inv} M_7 = \langle m_7^1, m_7^2 \rangle$. Set $u_1, u_2 \in M_5$ and $X_\alpha u_i = m_7^i$. As we know the structure of $N_1 \oplus N_2$, we conclude that $M_y = N \oplus N_3$ and the module N has no composition factors isomorphic to $M(7)$. Write $u_i = u_i^1 + u_i^2$ where $u_i^1 \in N$ and $u_i^2 \in N_3$. The arguments above yield that $X_\alpha u_i^2 = m_7^i$. One can directly verify that $X_{-\alpha} m_7^1$ has nonzero weight components $3X_{-3}X_{-1}^4v$ and $4X_{-2}X_{-1}^4v$ and the vector $X_{-\alpha}m_7^2$ has a nonzero weight component $4X_{-3}X_{-1}^4v$ and the zero component of weight $\omega - 4\alpha_1 - \alpha_2$. Hence the vectors $X_{-\alpha}m_7^1$ and $X_{-\alpha}m_7^2$ are linearly independent and $\dim X_{-\alpha}(X_\alpha(M_5 \cap N_3) \cap \operatorname{Inv} M_7) = 2$. Arguing as in the proof of Corollary 9, we conclude that $N_3 \cong 2T(7) \oplus M(7)$.

All possibilities are considered for regular unipotent elements in $A_3(K)$.

II.IX. Let x be not regular. The arguments at the beginning of the section imply that it suffices to consider the following cases:

a) $p = 5$, $\omega = 2\omega_1 + \omega_3$;
b) $p = 3$ or 5, $\omega = \omega_1 + \omega_2 + \omega_3$;
c) $p = 3$ or 5, $\omega = 2\omega_1 + 2\omega_3$;
d) $p = 7$, $\omega = 4\omega_1 + \omega_3$.

First we assume that $J(x) = (3,1)$ or $(2,1^2)$. Then x is conjugate to an element from the subgroup $H = G(1,2)$. Suppose that x is such element.

II.IX.I. Let $p = 5$, $\omega = 2\omega_1 + 2\omega_3$, and $J(x) = (3,1)$. In this case x is a regular unipotent element of H. Assume that $A \subset H$ and $\sigma : \Lambda(H) \to \mathbb{Z}$ are a subgroup of type A_1 containing x, and the homomorphism described at the end of Section 3; X_α and $X_{-\alpha}$ are the root operators of the Lie algebra of A. Taking into account Formula (4), we can suppose that $X_\alpha = X_1 + 2X_2$ and $X_{-\alpha} = 2X_{-1} + X_{-2}$. Set

$$\Omega_i = \{\lambda \in \Lambda(M) | \lambda = \omega - a\alpha_1 - b\alpha_2 - i\alpha_3\},\ 0 \le i \le 4,\ U_i = \langle M_\lambda | \lambda \in \Omega_i \rangle,$$

$\Omega_2^+ = \{\lambda \in \Omega_2 | \sigma(\lambda) \ge 0\}$. It is clear that U_i are H-modules and $M|H = U_0 \oplus U_1 \oplus U_2 \oplus U_3 \oplus U_4$. By Lemma 11, $U_0 \cong U_4^*$ and $U_1 \cong U_3^*$. Corollary 4 implies that $U_0 \cong M(2\omega_1)$ and $U_4 \cong M(2\omega_2)$.

It is not difficult to show that

$$\Omega_1 = \{\omega-\alpha_3, \omega-\alpha_2-\alpha_3, \omega-\alpha_1-\alpha_3, \omega-2\alpha_1-\alpha_3, \omega-\alpha_1-\alpha_2-\alpha_3,$$
$$\omega-2\alpha_1-\alpha_2-\alpha_3, \omega-\alpha_1-2\alpha_2-\alpha_3, \omega-2\alpha_1-2\alpha_2-\alpha_3, \omega-3\alpha_1-\alpha_2-\alpha_3,$$
$$\omega-3\alpha_1-2\alpha_2-\alpha_3, \omega-2\alpha_1-3\alpha_2-\alpha_3, \omega-3\alpha_1-3\alpha_2-\alpha_3\}.$$

Recall that $\dim M_\mu = 3$ for $\mu \in \{\omega-\alpha_1-\alpha_2-\alpha_3, \omega-2\alpha_1-\alpha_2-\alpha_3, \omega-2\alpha_1-2\alpha_2-\alpha_3\}$ and $\dim M_\mu = 1$ for other weights $\mu \in \Omega_1$. This yields that $\dim U_1 = 18$. Put $w = X_{-3}v$. It is obvious that the vector $w \in U_1$ and is invariant under $\mathfrak{X}_1$ and $\mathfrak{X}_2$. Hence U_1 has a composition factor with highest weight $\lambda_1 = \omega_H(w) = 2\omega_1 + \omega_2$.

Set $\nu = \omega-\alpha_1-\alpha_2-\alpha_3$. As $\dim M_{\nu+\alpha_1} = \dim M_{\nu+\alpha_2} = 1$ and $\dim M_\nu = 3$, one easily observes that

$$\dim(KerX_1 \cap M_\nu) = \dim(KerX_2 \cap M_\nu) = 2.$$

Now it is clear that there exists a nonzero vector $u \in M_\nu$ such that $X_1u = X_2u = 0$. This yields that $\mathfrak{X}_\beta$ fixes u for all positive roots β of H. Therefore U_1 has a composition factor with highest weight $\lambda_2 = \omega_H(u) = \omega_1$. Since $\dim M(\lambda_1) = 15$ and $\dim M(\lambda_2) = 3$, the H-module U_1 has no other composition factors. Observe that the modules $V(\lambda_1)$ and $V(\lambda_2)$ are irreducible (see [18, Table 6.6]). Hence $U_1 \cong M(\lambda_1) \oplus M(\lambda_2)$ by Corollary 1.

One easily checks that

$$\Omega_2 =\{\omega-2\alpha_3,\ \omega-\alpha_2-2\alpha_3,\ \omega-\alpha_1-2\alpha_3,\ \omega-2\alpha_1-2\alpha_3,\ \omega-2\alpha_2-2\alpha_3,$$
$$\omega-\alpha_1-\alpha_2-2\alpha_3,\ \omega-2\alpha_1-\alpha_2-2\alpha_3,\ \omega-\alpha_1-2\alpha_2-2\alpha_3,$$
$$\omega-2\alpha_1-2\alpha_2-2\alpha_3,\ \omega-3\alpha_1-\alpha_2-2\alpha_3,\ \omega-\alpha_1-3\alpha_2-2\alpha_3\}.$$

Recall that $\dim M_\mu = 3$ for

$$\mu \in \{\omega-\alpha_1-\alpha_2-2\alpha_3,\ \omega-2\alpha_1-\alpha_2-2\alpha_3,\ \omega-\alpha_1-2\alpha_2-2\alpha_3\},$$

$\dim M_\nu = 5$ for $\nu = \omega-2\alpha_1-2\alpha_2-2\alpha_3$, and $\dim M_\lambda = 1$ for all weights $\lambda \in \Omega_3$. This yields that $\dim U_2 = 35$. Now one easily concludes that

$$I(U_2|A) = \{M(8), M(6), 3M(4), 2M(2), 2M(0)\}.$$

By Lemma 27, we get that $U_2|A = N_1 \oplus N_2 \oplus 3M(4)$ where $I(N_1) = \{M(8), 2M(0)\}$ and $I(N_2) = \{M(6), 2M(2)\}$. Show that $N_1 \cong T(8)$ and $N_2 \cong T(6)$.

Set $m_8 = X_{-3}^2v$, $m_6 = X_{-2}X_{-3}^2v + 2X_{-1}X_{-3}^2v$, $f_1 = X_{-1}^2X_{-2}X_{-3}^2v$, and $f_2 = X_{-2}X_{-1}^2X_{-3}^2v$. It is clear that $m_8 \in N_1$. One can directly verify that the vector $X_{-\alpha}^3m_8$ has a nonzero weight component $2X_{-2}^3X_{-1}X_{-3}^2v$, $m_6 \in \mathrm{Inv}\, M_6$, the vector $X_{-\alpha}^2m_6$ has a weight component f_1+f_2, and that $X_3^2f_1 = 0$, $X_3^2f_2 \neq 0$, and $X_1^2f_1 \neq 0$.

Hence the vectors f_1 and f_2 are linearly independent and $X_{-\alpha}^2m_6 \neq 0$. Since U_3 is self-dual, now Lemma 23 implies that $N_1 \cong T(8)$ and $N_2 \cong T(6)$.

Using the results from Item I and Lemma 23, we can determine $J(\varphi(x))$.

Now let $J(x) = (2, 1^2)$. In this case one easily concludes that $\sigma_x(\omega) = 4 < p$. Hence by Corollary 2, M_x is a direct sum of p-restricted irreducible modules. As the dimensions of all weight subspaces of M are known (see Item II.IV), it is not difficult to find irreducible components of the module M_x and to determine $J(\varphi(x))$.

II.IX.II. In other cases using the results of [6], we show that the modules $M|H$ are completely reducible and find their irreducible components. After that we apply the results of Item I.

Now we need some additional notation. Let $\Gamma = A_t(K)$ and $\lambda \in \Lambda(\Gamma)$ be a dominant weight. For $\lambda \neq 0$ we write $\lambda = b_1\omega_{i_1} + ... + b_k\omega_{i_k}$ where $i_1 < ... < i_k$ and b_1, ..., $b_k > 0$. For $\lambda = 0$ we assume that $k = 0$. For $1 \le l < m \le k$ set

$$B_{l,m}^\lambda = i_m - i_l + \sum_{j=l}^{m} b_j \in K.$$

We say that λ satisfies Condition (JS1) if for any l, m, and b with $1 \leq l < m \leq k$ and $0 < b < b_l + b_m$ the difference $B^{\lambda}_{l,m} - b \neq 0$ (in K). We say that λ satisfies Condition (JS2) if $B^{\lambda}_{l,l+1} = 0$ for all $l < k$.

For $1 \leq i \leq t$ set $\alpha(i,t) = \alpha_i + ... + \alpha_t$. We say that the weight $\mu \in \Lambda(\Gamma)$ is λ-admissible if $\mu = \lambda - n_1\alpha(i_1,t) - ... - n_k\alpha(i_k,t)$, $n_j \in \mathbb{Z}^+$ and the following conditions hold:
(1) $0 \leq n_j \leq b_j$, $1 \leq j \leq k$;
(2) if $n_j \neq 0$ and $B^{\lambda}_{j,m} = 0$ for some m with $j < m \leq k$, then $n_m = b_m$.

Let λ be p-restricted and $\Gamma_1 = \Gamma(1,2,...,t-1)$. By [6, Theorem 1.4], the restriction $M(\lambda)|\Gamma_1$ is completely reducible if and only if $\lambda = 0$ or satisfies one of Conditions (JS1) or (JS2). In these cases

$$M(\lambda)|\Gamma_1 \cong \bigoplus M(\mu|\Gamma_1) \tag{7}$$

where μ runs over the set of all admissible weights. Now we use Formula (7) in all remaining cases.

Let $p = 5$ and $\omega = 2\omega_1 + \omega_3$. Then $k = 2$, $B^{\omega}_{1,2} = 5 = 0$, ω satisfies Condition (JS1), and

$$M|H = M(2\omega_1) \oplus M(2\omega_1 + \omega_2) \oplus M(\omega_1 + \omega_2) \oplus M(\omega_2).$$

Let $p = 3$ and $\omega = \omega_1 + \omega_2 + \omega_3$. Then $k = 3$, $B^{\omega}_{1,2} = B^{\omega}_{2,3} = 3 = 0$, $B^{\omega}_{1,3} = 5 = 2$, ω satisfies Condition (JS2), and

$$M|H = M(\omega_1 + \omega_2) \oplus M(\omega_1 + 2\omega_2) \oplus M(2\omega_1 + \omega_2) \oplus M(\omega_1 + \omega_2).$$

Let $p = 5$ and $\omega = \omega_1 + \omega_2 + \omega_3$. Then $k = 3$, $B^{\omega}_{1,2} = B^{\omega}_{2,3} = 3$, $B^{\omega}_{1,3} = 5 = 0$, ω satisfies Condition (JS1), and

$$M|H = M(\omega_1 + \omega_2) \oplus M(\omega_1 + 2\omega_2) \oplus M(2\omega_1 + \omega_2) \oplus M(2\omega_1) \oplus M(\omega_1 + \omega_2) \oplus M(2\omega_2).$$

Let $p = 3$ and $\omega = 2\omega_1 + 2\omega_3$. Then $k = 2$, $B^{\omega}_{1,2} = 6 = 0$, ω satisfies Condition (JS2), and

$$M|H = M(2\omega_1) \oplus M(2\omega_1 + \omega_2) \oplus M(2\omega_1 + 2\omega_2) \oplus M(\omega_1 + 2\omega_2) \oplus M(2\omega_2).$$

Let $p = 7$ and $\omega = 4\omega_1 + \omega_3$. Then $k = 2$, $B^{\omega}_{1,2} = 7 = 0$, ω satisfies Condition (JS2), and

$$M|H = M(4\omega_1) \oplus M(4\omega_1 + \omega_2) \oplus M(3\omega_1 + \omega_2) \oplus M(2\omega_1 + \omega_2) \oplus M(\omega_1 + \omega_2) \oplus M(\omega_2).$$

Hence for $J(x) = (3,1)$ or $(2,1^2)$, all possibilities are considered.

II.IX.III. Let $J(x) = (2^2)$. Then x is conjugate to a regular unipotent element of the subsystem subgroup $H = G(1,3)$ and we can assume that $x \in H$. Obviously, $H = G(1)G(3)$. Recall that the set $\Lambda(A_1(K))$ is identified with $\mathbb{Z}$, hence we can suppose that $\mu \in \Lambda(H)$ is equal to (a,b) where $a,b \in \mathbb{Z}$. It is not difficult to check that $\sigma_x(\omega) < p$ for $p \geq 5$. Therefore it remains to consider the representations with highest weights $\omega = \omega_1 + \omega_2 + \omega_3$ and $2\omega_1 + 2\omega_3$ for $p = 3$. While analyzing these representations, we will consider the restriction of M to a subgroup A constructed as at the end of Section 3 and the homomorphism $\sigma : \Lambda(H) \to A$ described there. One easily observes that for $\mu = (a,b) \in \Lambda(H)$ the image $\sigma(\mu) = a + b$.

Let $p = 3$ and $\omega = \omega_1 + \omega_2 + \omega_3$. Then $\sigma_x(\omega) = 4$. Set
$\Omega_i = \{\lambda \in \Lambda(H) | \lambda = \omega - a\alpha_1 - i\alpha_2 - b\alpha_3\}$, $0 \leq i \leq 4$, $U_i = \langle M_\lambda | \lambda \in \Omega_i \rangle$,
$\Omega_i^+ = \{\lambda \in \Omega_i | \sigma(\lambda) \geq 0\}$. It is clear that U_i are H-modules and
$M|H = U_0 \oplus U_1 \oplus U_2 \oplus U_3 \oplus U_4$. Lemma 11 implies that $U_0 \cong U_4^*$ and $U_1 \cong U_3^*$. Corollary 4 yields that $U_0 \cong M(\omega_1) \otimes M(\omega_1)$.

One easily concludes that

$$\Omega_1 = \{\omega-\alpha_2,\ \omega-\alpha_1-\alpha_2,\ \omega-\alpha_2-\alpha_3,\ \omega-2\alpha_1-\alpha_2,\ \omega-\alpha_1-\alpha_2-\alpha_3,$$
$$\omega-\alpha_2-2\alpha_3,\ \omega-2\alpha_1-\alpha_2-\alpha_3,\ \omega-\alpha_1-\alpha_2-2\alpha_3,\ \omega-2\alpha_1-\alpha_2-2\alpha_3\};$$
$$\Omega_2 = \{\omega-\alpha_1-2\alpha_2,\ \omega-2\alpha_2-\alpha_3,\ \omega-2\alpha_1-2\alpha_2,\ \omega-\alpha_1-2\alpha_2-\alpha_3,$$
$$\omega-2\alpha_2-2\alpha_3,\ \omega-2\alpha_1-2\alpha_2-\alpha_3, \omega-\alpha_1-2\alpha_2-2\alpha_3\}.$$

Put $\mu=\omega-\alpha_1-\alpha_2-\alpha_3$, $\Lambda_1=\{\omega,\ \omega-\alpha_1-\alpha_2,\ \omega-\alpha_2-\alpha_3\}$,

$$\Lambda_2 = \{\omega-\alpha_1,\ \omega-\alpha_2,\ \omega-\alpha_3,\ \omega-\alpha_1-\alpha_3,\ \omega-2\alpha_1-\alpha_2,\ \omega-\alpha_2-2\alpha_3,$$
$$\omega-2\alpha_1-\alpha_2-\alpha_3,\ \omega-\alpha_1-\alpha_2-2\alpha_3,\ \omega-2\alpha_1-\alpha_2-2\alpha_3,$$
$$\omega-\alpha_1-2\alpha_2,\ \omega-2\alpha_2-\alpha_3,\ \omega-2\alpha_1-2\alpha_2,\ \omega-2\alpha_2-2\alpha_3\},$$
$$\Lambda_3 = \{\omega-\alpha_1-2\alpha_2-\alpha_3, \omega-2\alpha_1-2\alpha_2-\alpha_3, \omega-\alpha_1-2\alpha_2-2\alpha_3\}.$$

Show that $\dim M_\mu=2$. It is clear that M_μ is generated by the vectors $X_{-1}X_{-2}X_{-3}v$, $X_{-1}X_{-3}X_{-2}v$, $X_{-3}X_{-2}X_{-1}v$, and $X_{-2}X_{-3}X_{-1}v$. By Theorem 1 and Proposition 4, the weight subspaces in M of the weights $\omega-\alpha_1-\alpha_2$ and $\omega-\alpha_2-\alpha_3$ are one-dimensional. This implies that M_μ is generated by the vectors $X_{-1}X_{-3}X_{-2}v$ and $X_{-2}X_{-1}X_{-3}v$. One can directly verify that they are linearly independent.

Obviously, $\dim M_\lambda=1$ for $\lambda\in\Lambda_1$. One easily observes that $\dim M_\tau=1$ for $\tau\in\Lambda_2$ since τ lies in the same W-orbit with a weight from Λ_1. It is clear that $\dim M_\nu=2$ for $\nu\in\Lambda_3$ as ν lies in the same W-orbit with μ.

Now one easily concludes that $\dim U_1=10$ and $\dim U_2=16$. Let $w=X_{-2}v$. It is clear that the vector $w\in U_1$ and is invariant under $\mathfrak{X}_1$ and $\mathfrak{X}_3$. Therefore U_1 has a composition factor with highest weight $\theta=\omega_H(w)=(2,2)$. Since $\dim M_\theta=9$, then U_1 has two composition factors: $M(\theta)$ and the trivial one. As $V(\theta)$ is irreducible, Proposition 2 implies that $U_1\cong M(2)\otimes M(2)\oplus M(0)\otimes M(0)$.

Denote by $U_{2,i}$ the weight subspace of weight i in the module U_2. Using Formula (4), we can assume that $X_\alpha=X_1+X_3$ and $X_{-\alpha}=X_{-1}+X_{-3}$. We know the dimensions of the weight subspaces in M and so can show that $\dim U_{2,4}=2$, $\dim U_{2,2}=4$, and $\dim U_{2,0}=4$. Hence $I(U_2|A)=\{2M(4),2M(2),2M(0)\}$. By Lemma 23, $U_2|A=N\oplus N'$ where $I(N)=\{2M(4),2M(0)\}$ and $N'\cong 2M(2)$. Show that $N\cong T(4)\oplus M(4)$. Set $m_0=X_{-2}X_{-3}X_{-1}^2X_{-2}v$. One can check that $X_\alpha^2 m_0=X_{-2}X_{-3}X_{-2}v+2X_{-2}X_{-1}X_{-3}v$ and $X_{-\alpha}^2X_\alpha^2m_0\neq 0$. Since $U_2=N\oplus N'$, then $m_0=m_0^1+m_0^2$ where $m_0^1\in N$, $m_0^2\in N'$, and $m_0^1,m_0^2\in U_{2,0}$. It is clear that $X_\alpha^2m_0^2=0$, therefore $X_{-\alpha}^2X_\alpha^2m_0^1\neq 0$. Hence by Lemma 25, $N\cong T(4)\oplus M(4)$.

Now we can determine the canonical Jordan form of $\varphi(x)$ using Lemma 21 and Theorem 3.

Let $\omega=2\omega_1+2\omega_3$ and $p=3$. Set $\Omega_i=\{\omega-a\alpha_1-i\alpha_2-b\alpha_3\}$, $0\le i\le 4$, $U_i=\langle M_\lambda|\lambda\in\Omega_i\rangle$, $\Omega_i^+=\{\lambda\in\Omega_i|\sigma(\lambda)\ge 0\}$. It is clear that U_i are H-modules and $M|H=U_0\oplus U_1\oplus U_2\oplus U_3\oplus U_4$. By Lemma 11, $U_0\cong U_4^*$ and $U_1\cong U_3^*$. Corollary 4 implies that $U_0\cong M(2)\oplus M(2)$.

It is not difficult to see that

$$\Omega_1 = \{\omega-\alpha_1-\alpha_2,\ \omega-\alpha_2-\alpha_3,\ \omega-2\alpha_1-\alpha_2,\ \omega-\alpha_1-\alpha_2-\alpha_3,$$
$$\omega-\alpha_2-2\alpha_3,\ \omega-2\alpha_1-\alpha_2-\alpha_3,\ \omega-\alpha_1-\alpha_2-2\alpha_3\};$$
$$\Omega_2 = \{\omega-2\alpha_1-2\alpha_2,\ \omega-2\alpha_2-2\alpha_3,\ \omega-\alpha_1-2\alpha_2-\alpha_3,\ \omega-2\alpha_1-2\alpha_2-\alpha_3,$$
$$\omega-\alpha_1-2\alpha_2-2\alpha_3,\ \omega-2\alpha_1-2\alpha_2-2\alpha_3,\ \omega-3\alpha_1-2\alpha_2-\alpha_3, \omega-\alpha_1-2\alpha_2-3\alpha_3\}.$$

Set $\mu=\omega-\alpha_1-\alpha_2-\alpha_3=\omega_1+\omega_3$. Show that $\dim M_\mu=2$. Using the Jantzen filtration [13, Part 2, §8, Proposition 8.19], we can conclude that the maximal submodule in $V(\omega)$ is isomorphic to $M(\mu)$. Hence $\dim M_\mu=\dim V(\omega)_\mu-1$. Then we use Freudenthal's formula.

Put

$$\Lambda_1 = \{\omega - 2\alpha_1 - \alpha_2 - \alpha_3, \omega - \alpha_1 - \alpha_2 - 2\alpha_3,\ \omega - 2\alpha_1 - 2\alpha_2 - \alpha_3, \omega - \alpha_1 - 2\alpha_2 - 2\alpha_3\},$$
$$\Lambda_2 = \{\omega, \omega - \alpha_1, \omega - \alpha_3, \omega - \alpha_1 - \alpha_3\},$$
$$\Lambda_3 = \{\omega - \alpha_1 - \alpha_2,\ \omega - \alpha_2 - \alpha_3,\ \omega - 2\alpha_1 - \alpha_2,\ \omega - \alpha_2 - 2\alpha_3,\ \omega - 2\alpha_2 - 2\alpha_3,\ \omega - 2\alpha_1 - 2\alpha_2,\ \omega - \alpha_1 - 2\alpha_2 - \alpha_3,\ \omega - 3\alpha_1 - 2\alpha_2 - \alpha_3,\ \omega - \alpha_1 - 2\alpha_2 - 3\alpha_3\}.$$

Since a weight $\nu \in \Lambda_1$ lies in the same W-orbit with μ, then $\dim M_\nu = 2$. It is clear that $\dim M_\lambda = 1$ for $\lambda \in \Lambda_2$. One easily observes that $\dim M_\eta = 1$ for $\eta \in \Lambda_3$ as η lies in the same W-orbit with a weight from Λ_2. Since $\dim M = 69$, then $\dim M_\tau = 3$ for $\tau = \omega - 2\alpha_1 - 2\alpha_2 - 2\alpha_3 = 0$.

Now one easily deduces that $\dim U_1 = 16$ and $\dim U_2 = 19$. Below $U_{1,i}$ and $U_{2,j}$ are the weight subspaces of the weights i and j in the A-modules U_1 and U_2, respectively. It is not difficult to check that $\dim U_{1,4} = 2$, $\dim U_{1,2} = 4$, and $\dim U_{1,0} = 4$. Arguing as for $\omega = \omega_1 + \omega_2 + \omega_3$, we can see that $U_1|A = N \oplus 2M(2)$ where $I(N) = \{2M(4), 2M(0)\}$. Show that $N \cong T(4) \oplus M(4)$. Set $m_0 = X_{-2}X_{-3}X_{-1}^2 v$. We can directly verify that $X_\alpha^2 m_0 = X_{-2}X_{-3}v + 2X_{-2}X_{-1}v$ and $X_{-\alpha}^2 X_\alpha^2 m_0$ has a nonzero weight component $f = X_{-1}^2 X_{-2}X_{-3}v + X_{-3}X_{-1}X_{-2}X_{-1}v$ (one easily observes that $X_2 f \neq 0$). Arguing as for $\omega = \omega_1 + \omega_2 + \omega_3$, we get that there exists a vector $m_0^1 \in N$ such that $X_{-\alpha}^2 X_\alpha^2 m_0^1 \neq 0$. Then by Lemma 25, $N \cong T(4) \oplus M(4)$.

As we know the dimensions of weight subspaces in M, it is not difficult to check that $\dim U_{2,4} = 3$, $\dim U_{2,2} = 4$, and $\dim U_{2,0} = 5$. Then $I(U_2|A) = \{3M(4),\ M(2),\ 4M(0)\}$. By Lemma 23, $U_2|A = N \oplus M(2)$ where $I(N) = \{3M(4),\ 4M(0)\}$. Show that $N \cong 2T(4) \oplus M(4)$.

One can directly verify that $X_{-2}X_{-1}X_{-3}X_{-2}X_{-3}X_{-1}v$, $X_{-3}X_{-2}X_{-1}^2X_{-2}X_{-3}v$, and $X_{-3}^2X_{-2}^2X_{-1}^2v$ are linearly independent. Then the vectors

$$X_{-1}^2X_{-2}^2X_{-3}X_{-1}v, X_{-3}^2X_{-2}^2X_{-3}X_{-1}v, X_{-2}X_{-1}X_{-3}X_{-2}X_{-3}X_{-1}v,$$
$$X_{-3}^2X_{-2}^2X_{-1}^2v, \text{ and } X_{-3}X_{-2}X_{-1}^2X_{-2}X_{-3}v$$

form a basis of $U_{2,0}$. Put $u_1 = X_{-2}^2X_{-3}X_{-1}v$, $u_2 = X_{-2}^2X_{-3}^2v + X_{-2}^2X_{-1}^2v$, $w_1 = X_{-\alpha}^2 u_1$, and $w_2 = X_{-\alpha}^2 u_2$. We can check that $\dim X_\alpha^2 U_{2,0} = 2$, $X_\alpha^2 U_{2,0} = \langle u_1, u_2\rangle$ and that the vector w_1 has a nonzero weight component $X_{-3}^2X_{-2}^2X_{-3}X_{-1}v$, but the vector $w_2 = X_{-1}^2X_{-2}^2X_{-3}^2v + X_{-3}^2X_{-2}^2X_{-1}^2v \neq 0$. This implies that w_1 and w_2 are linearly independent. Arguing as before for the A-module U_1, we get that there exists vectors m_1 and $m_2 \in N \cap U_{2,0}$ such that $X_{-\alpha}^2 X_\alpha^2 m_1 = w_1$ and $X_{-\alpha}^2 X_\alpha^2 m_2 = w_2$. Arguing as in the proof of Corollary 9, we conclude that $N \cong 2T(4) \oplus M(4)$.

For $G = A_3(K)$, the problem is solved. The arguments at the beginning of the section yield that it is solved for $G = A_4(K)$ and $A_6(K)$ as well.

III. Let $G = A_5(K)$. Observe that the order of a regular unipotent element is greater than p for $p = 3$ or 5. The arguments at the beginning of the section imply that it remains to consider only the case where $p = 5$ and $\omega = \omega_1 + \omega_4$. Then $\dim \varphi = 78$. If x has a Jordan block of size 2 in the standard realization of G, then x is conjugate to a unipotent element of the subgroup $\Gamma = G(1,2,3,5)$. Show that the restriction $M|\Gamma$ is completely reducible. Set $U_i = \langle M_\lambda | \lambda = \omega - i\alpha_4 - \sum_{j\neq 4} b_j\alpha_j\rangle$. One easily concludes that $M|\Gamma = U_0 \oplus U_1 \oplus U_2 \oplus U_3$. Corollary 4 implies that $U_0 \cong M((\omega_1, 0))$ and $U_3 \cong M((\omega_2, 1))$.

Set $m_1 = X_{-4}v$, $m_{2,1} = X_{-4}^2X_{-3}X_{-2}X_{-1}v$, and $m_{2,2} = X_{-4}X_{-5}X_{-3}X_{-4}v$. It is obvious that $m_1 \in U_1$ and $m_{2,1}$ and $m_{2,2} \in U_2$. By Lemma 12, the vectors m_1 and $m_{2,1} \neq 0$ and they are invariant under the subgroups $\mathfrak{X}_i$ for $i \neq 4$. Using Lemma 6 several times, one easily observes that $X_4 m_{2,2} = X_{-5}X_{-3}X_{-4}v \neq 0$. This yields that $m_{2,2} \neq 0$. Taking into account the weight structure of M, we conclude that the subgroups $\mathfrak{X}_i$ for $i \neq 4$ fix $m_{2,2}$. Hence U_1 has a composition factor isomorphic to $M(\omega_\Gamma(m_1)) = M((\omega_1 + \omega_3, \omega_1))$

and U_2 has composition factors Z_1 and Z_2 with highest weights $\mu_i = \omega_\Gamma(m_{2,i})$, $i = 1, 2$. One easily deduces that $\mu_1 = (\omega_3, 2)$ and $\mu_2 = (\omega_1 + \omega_2, 0)$.

Now dimensional considerations imply that U_1 is irreducible and U_2 has exactly two composition factors. Since the modules $V(\mu_1)$ and $V(\mu_2)$ are irreducible, then by Lemma 3, $U_2 \cong Z_1 \oplus Z_2$. Therefore $M|\Gamma \cong U_0 \oplus U_1 \oplus Z_1 \oplus Z_2 \oplus U_3$. One can apply Theorem 3, Lemma 21, and the results of Item II to determine the block structure of $\varphi(x)$.

Let $J(x) = (3, 3)$. Set $\Gamma = G(1, 2, 4, 5)$ and $U_i = \langle M_\lambda | \lambda = \omega - i\alpha_3 - \sum_{j\neq 3} b_j\alpha_j\rangle$. One easily observes that $M|\Gamma \cong U_0 \oplus U_1 \oplus U_2 \oplus U_3$. Corollary 4 implies that $U_0 \cong M((\omega_1, \omega_1))$ and $U_3 \cong M((\omega_1, \omega_1))$. Set $m_{1,1} = X_{-3}X_{-2}X_{-1}v$, $m_{1,2} = X_{-3}X_{-4}v$, and $m_{2,1} = X_{-3}^2X_{-4}X_{-2}X_{-1}v$. It is clear that $m_{1,1}$ and $m_{1,2} \in U_1$ and $m_{2,1} \in U_2$. By Lemma 12, $m_{1,i} \neq 0$ and the groups $\mathfrak{X}_i$ for $i \neq 3$ fix these vectors. Hence U_1 has composition factors Z_1 and Z_2 with highest weights $\mu_1 = \omega_\Gamma(m_{1,1}) = (0, 2\omega_1)$ and $\mu_2 = \omega_\Gamma(m_{1,2}) = (\omega_1+\omega_2, \omega_2)$. Applying Lemma 6 several times, we show that $m_{2,1} \neq 0$. One easily observes that $\omega(m_{2,1}) + \alpha_i \notin \Lambda(M)$ for $i \neq 3$. Therefore the subgroups $\mathfrak{X}_i$ fix $m_{2,1}$ for $i \neq 3$. This yields that U_2 has a composition factor with highest weight $\mu_3 = \omega_\Gamma(m_{2,1}) = (\omega_2, \omega_1 + \omega_2)$.

Put $\lambda = \omega - \alpha_2 - 2\alpha_3 - 2\alpha_4 - \alpha_5$. One easily concludes that λ lies in the same W-orbit with ω, hence $\lambda \in \Lambda(M)$. Let $m_{2,2} \in M_\lambda \setminus \{0\}$. Taking into account the weight structure of M, we get that $X_2m_{2,2} = X_5m_{2,2} = X_4m_{2,2} = 0$. This forces that the groups $\mathfrak{X}_i$ fix the vector $m_{2,2}$ for $i \neq 3$. Therefore U_2 has a composition factor with highest weight $\mu_4 = \omega_\Gamma(m_{2,2}) = (2\omega_1, 0)$. Now dimensional considerations imply that the modules U_1 and U_2 have exactly two composition factors. Observe that the modules $V(\mu_i)$ are irreducible for $1 \leq i \leq 4$. Hence by Lemma 3, the modules U_1 and U_2 and $M|\Gamma$ are completely reducible. Using Theorem 3 and the results of Item I, we can determine the block structure of $\varphi(x)$.

All other unipotent elements distinct from a regular unipotent element, have blocks of size 1 in the standard realization of G. Such elements are conjugate to elements from the subsystem subgroup $H = G(1, 2, 3, 4)$. Set $U_i = \langle M_\lambda | \lambda = \omega - i\alpha_5 - \sum_{j\neq 5} b_j\alpha_j\rangle$. One easily observes that $M|H \cong U_0 \oplus U_1 \oplus U_2$. Corollary 4 implies that $U_0 \cong M(\omega_1 + \omega_4)$ and $U_2 \cong M(\omega_3)$. Let $m = X_{-5}X_{-4}v$. By Lemma 12, $m \neq 0$ and the subgroups $\mathfrak{X}_i$ fix m for $i \neq 5$. Hence U_1 has a composition factor with highest weight $\omega_H(m) = \omega_1 + \omega_3$. By dimensional considerations, U_1 is irreducible. Therefore the restriction $M|H$ is completely reducible. Using the results of Item III, we can determine the canonical Jordan form of $\varphi(x)$.

For $G = A_5(K)$ the problem is solved.

IV. Let $7 \leq n \leq 13$. Recall the assumptions at the beginning of the section and conclude that it suffices to consider the cases where $\omega = \omega_3$ or ω_4. Then M_x is a tilting module by Lemmas 13. As we know the weight multiplicities of M, we can determine the indecomposable components of M_x and the canonical Jordan form of $\varphi(x)$. For nonregular elements, Lemma 19 is applied.

All p-restricted representations of $A_n(K)$ of dimension ≤ 100 have been considered.

5. Symplectic groups

In this section $G = C_n(K)$. We apply Propositions 4 and 9, Theorem 5, and the results of Section 4 to solve the problem for $\omega = a\omega_1$ or $a\omega_i + (p - 1 - a)\omega_{i+1}$ where $a \neq 0$ for $i = n - 1$, and for $\omega = \omega_2$. Thereafter these cases are not considered. It follows from [4, §13, Item 3] that the formal character of the G-module $\wedge^3 V$ is equal to the sum of the formal characters of the Weyl modules $V(\omega_3)$ and $V(\omega_1)$ and the formal character of the G-module $\wedge^4 V$ is equal to the sum of the formal characters of the modules $V(\omega_4)$, $V(\omega_2)$, and $V(0)$. The group G can be naturally embedded into the group $G_1 = A_{2n-1}(K)$. Let N_3 and N_4 be the irreducible G_1-modules with highest weights ω_3 and ω_4. Then Proposition 3 and Corollary 1 imply that $N_3|G \cong M(\omega_3) \oplus M(\omega_1)$ if $V(\omega_3)$ is irreducible, and $N_4|G \cong M(\omega_4) \oplus M(\omega_2) \oplus M(0)$ if $V(\omega_4)$ and $V(\omega_2)$ are irreducible. These facts are used for solving the problem for $\omega = \omega_3$ and ω_4.

Observe that the order of a regular unipotent element is greater than p for $p = 3$. As in Section 4, y is a regular unipotent element.

I. Let $G = C_2(K)$. One can check that $\sigma_y(\omega) = 3a_1 + 4a_2$ for $\omega = a_1\omega_1 + a_2\omega_2$ and that $\sigma_x(\omega_2) < p$ for any element x of order p. Hence in this case M_x is a direct sum of p-restricted modules and certainly is a tilting module. Theorem 5 and Lemma 14 imply that M_x is a tilting module for $\omega = a\omega_1$, $a < p$.

Now using Lemma 14, we conclude that M_x is a tilting module in the following cases:
$p \neq 5$ and $\omega = 2\omega_2$ or $\omega_1 + \omega_2$;
$p > 7$ and $\omega \in \{3\omega_2,\ 4\omega_2,\ 3\omega_1 + \omega_2,\ \omega_1 + 3\omega_2\}$;
$p \neq 3$ and $\omega = 2\omega_1 + \omega_2$;
$p \neq 5, 7$ and $\omega = \omega_1 + 2\omega_2$ or $2\omega_1 + 2\omega_2$;
$p > 11$ and $\omega = 5\omega_2$.
Here in the notation of Lemma 14 we take $\lambda_1 = a\omega_1$ and $\lambda_2 = b\omega_2$ if $\omega = a\omega_1 + b\omega_2$ and $\lambda_1 = (a-1)\omega_2$ and $\lambda_2 = \omega_2$ for $\omega = a\omega_2$.

First consider the block structure of $\varphi(y)$. Using arguments given at the end of Section 3, we can assume that $X_\alpha = X_1 - 6X_2$ and $X_{-\alpha} = 3X_{-1} - \frac{2}{3}X_{-2}$.

In Items I.I – I.IV by Theorem 4, $\dim M_\mu = 1$ for all $\mu \in \Lambda(M)$. Taking this into account, it is not difficult to find the dimensions of the subspaces M_i.

I.I. Let $p = 5$ and $\omega = \omega_1 + \omega_2$. Then $\dim\varphi = 12$ and $\sigma_y(\omega) = 7$. One easily observes that $\dim M_7 = 1$, $\dim M_5 = 2$, $\dim M_3 = 1$, and $\dim M_1 = 2$. Hence $I(M_y) = \{M(7), M(5), 2M(1)\}$. By Lemma 27, $M_y = N \oplus M(5)$ where $I(N) = \{M(7), 2M(1)\}$. Prove that $N \cong T(7)$.

One can directly verify that $X^3_{-\alpha}v = X^2_{-1}X_{-2}v + X_{-2}X_{-1}X_{-2}v \neq 0$. As N is self-dual, Lemma 29 implies that $N \cong T(7)$.

I.II. Let $p = 5$ and $\omega = 2\omega_2$. Then $\dim\varphi = 13$ and $\sigma_y(\omega) = 8$. One easily observes that $\dim M_8 = 1$, $\dim M_6 = 1$, $\dim M_4 = 2$, $\dim M_2 = 2$, and $\dim M_0 = 1$. Then $I(M_y) = \{M(8), M(4)\}$. By Lemma 27, $M_y = M(8) \oplus M(4)$.

I.III. Let $p = 7$ and $\omega = \omega_1 + 2\omega_2$. Then $\dim\varphi = 24$ and $\sigma_y(\omega) = 11$. One easily concludes that $\dim M_{11} = 1$, $\dim M_9 = 2$, $\dim M_7 = 2$, $\dim M_5 = 2$, $\dim M_3 = 3$, and $\dim M_1 = 2$. Hence $I(M_y) = \{M(11), M(9), 2M(3)\}$. By Lemma 27, $M_y = M(11) \oplus N$ where $I(N) = \{M(9), 2M(3)\}$. Show that $N \cong T(9)$.

Set $m_9 = 5X_{-1}v + X_{-2}v$. One can directly verify that $m_9 \in \operatorname{Inv} M_9$ and $X^3_{-\alpha}m_9 = 2X^2_{-1}X^2_{-2}v + 3X_{-2}X_{-1}X^2_{-2}v \neq 0$. Hence $m_9 \in N$. As N is self-dual, Lemma 29 implies that $N \cong T(9)$.

I.IV. Let $p = 7$ and $\omega = 3\omega_2$. Then $\dim\varphi = 25$ and $\sigma_y(\omega) = 12$. One can check that $\dim M_{12} = 1$, $\dim M_{10} = 1$, $\dim M_8 = 2$, $\dim M_6 = 3$, $\dim M_4 = 2$, $\dim M_2 = 2$, and $\dim M_0 = 3$. Then $I(M_y) = \{M(12), M(8), M(6), 2M(0)\}$. By Lemma 27, $M_y = N \oplus M(8) \oplus M(6)$ where $I(N) = \{M(12), 2M(0)\}$. Show that $N \cong T(12)$. One can directly verify that the vector $X^6_{-\alpha}v$ has a nonzero weight component $2X^4_{-1}X^2_{-2}v$ and so $X^6_{-\alpha}v \neq 0$. Since N is self-dual, $N \cong T(12)$ by Lemma 29.

I.V. Let $p = 5$ and $\omega = 3\omega_2$. Then $\dim\varphi = 30$ and $\sigma_y(\omega) = 12$. It is clear that $\dim M_{12} = 1$, $\dim M_{10} = 1$, and $\dim M_8 = 2$. Set

$$\begin{aligned}
\Lambda_6 &= \{\omega - 3\alpha_2, \omega - \alpha_1 - 2\alpha_2, \omega - 2\alpha_1 - \alpha_2\};\\
\Lambda_4 &= \{\omega - \alpha_1 - 3\alpha_2, \omega - 2\alpha_1 - 2\alpha_2\};\\
\Lambda_2 &= \{\omega - 2\alpha_1 - 3\alpha_2, \omega - 3\alpha_1 - 2\alpha_2\};\\
\Lambda_0 &= \{\omega - 2\alpha_1 - 4\alpha_2, \omega - 3\alpha_1 - 3\alpha_2, \omega - 4\alpha_1 - 2\alpha_2\}.
\end{aligned}$$

One easily observes that $M_i = \langle M_\lambda | \lambda \in \Lambda_i \rangle$ for $i \in \{0, 2, 4, 6\}$. Since $M(\omega) \cong V(\omega)$ (see, for example, [18, Table 6.22]), the weight multiplicities of M can be determined with the help of Freudenthal's formula [12, §22, Item 3]. Using this formula, we get that $\dim M_\mu = 2$

for $\mu=\omega-2\alpha_1-2\alpha_2$ and $\omega-3\alpha_1-3\alpha_2$ and $\dim M_\tau=1$ for $\tau=\omega-\alpha_1-\alpha_2$. The weight $\nu=\omega-2\alpha_1-3\alpha_2$ lies in the same W-orbit with the weight $\omega-2\alpha_1-2\alpha_2$ and the weight $\eta=\omega-3\alpha_1-\alpha_2$ lies in the same W-orbit with τ. Hence $\dim M_\nu=2$ and $\dim M_\eta=1$. It is clear that $\dim M_\theta=1$ for $\theta=\omega-\alpha_1-\alpha_2$ and $\dim M_\sigma=1$ for $\sigma=\omega-\alpha_1-3\alpha_2$ since σ lies in the same W-orbit with θ. All other weights $\lambda\in\Lambda_0\cup\Lambda_2\cup\Lambda_4\cup\Lambda_6$ lie in the same W-orbit with ω, therefore $\dim M_\lambda=1$. Now we can deduce that $\dim M_6=3$, $\dim M_4=3$, $\dim M_2=3$, and $\dim M_0=4$. Then $I(M_y)=\{M(12),M(8),2M(6),M(2),2M(0)\}$. By Proposition 8 and Lemma 27, $M_y=N_1\oplus N_2$ where $I(N_1)=\{M(12),2M(6),M(2)\}$ and $I(N_2)=\{M(8),2M(0)\}$. Show that $N_1\cong T(12)$ and $N_2\cong T(8)$.

Since $X_{-2}^3v\neq 0$, then $X_{-\alpha}^3v\neq 0$. Then Lemma 28 implies that $KA_yv\cong V(12)$. Set $N=N_1/KA_yv$. One can directly verify that the vectors $X_{-1}^2X_{-2}^3v$ and $X_{-2}^2X_{-1}^2X_{-2}v$ are linearly independent. This implies that the vectors $X_{-1}^2X_{-2}^3v$, $X_{-2}^2X_{-1}^2X_{-2}v$, and $X_{-1}^3X_{-2}^2v$ form a basis of M_2. Using this basis, we can directly check that $\operatorname{Inv}M_2=0$. Since N_1 has no submodules isomorphic to $M(2)$ and is self-dual, it has no such factor modules. Now Lemma 23 yields that $N\cong V(6)$ and N_1 has filtrations by Weyl modules and by dual Weyl modules. Hence N_1 is a tilting module and $N_1\cong T(12)$.

Set $m_8=X_{-2}^2v+2X_{-1}X_{-2}v$. One can directly verify that $m_8\in\operatorname{Inv}M_8$ and that $X_{-1}^4X_{-2}^2v$ is a nonzero weight component of $X_{-\alpha}^4m_8$. As N_2 is self-dual, by Lemma 29, we get $N_2\cong T(8)$.

I.VI. Let $p=5$ and $\omega=\omega_1+2\omega_2$. Then $\dim\varphi=40$ and $\sigma_y(\omega)=11$. It is clear that $\dim M_{11}=1$ and $\dim M_9=2$. Set

$$\begin{aligned}\Lambda_7=&\{\omega-2\alpha_2,\omega-\alpha_1-\alpha_2\};\\ \Lambda_5=&\{\omega-\alpha_1-2\alpha_2,\omega-2\alpha_1-\alpha_2\};\\ \Lambda_3=&\{\omega-\alpha_1-3\alpha_2,\omega-2\alpha_1-2\alpha_2,\omega-3\alpha_1-\alpha_2\};\\ \Lambda_1=&\{\omega-2\alpha_1-3\alpha_2,\omega-3\alpha_1-2\alpha_2\}.\end{aligned}$$

One easily observes that $M_i=\langle M_\lambda|\lambda\in\Lambda_i\rangle$ for $i\in\{1,3,5,7\}$. Since $M(\omega)\cong V(\omega)$ (see, for example, [18, Table 6.22]), the weight multiplicities of M can be determined with the use of Freudenthal's formula. By this formula, $\dim M_\tau=2$ for $\tau=\omega-\alpha_1-\alpha_2$ and $\dim M_\theta=3$ for $\theta=\omega-2\alpha_1-2\alpha_2$. It is clear that $\dim M_\mu=1$ for $\mu\in\{\omega-2\alpha_2,\ \omega-\alpha_1-3\alpha_2,\ \omega-3\alpha_1-\alpha_2\}$ since μ lies in the same W-orbit with ω, $\dim M_\sigma=2$ for $\sigma\in\{\omega-2\alpha_1-\alpha_2,\omega-\alpha_1-2\alpha_2,\omega-2\alpha_1-3\alpha_2\}$ as σ lies in the same W-orbit with τ, and $\dim M_\nu=3$ for $\nu=\omega-3\alpha_1-2\alpha_2$ since ν lies in the same W-orbit with θ. Now one easily deduces that $\dim M_7=3$, $\dim M_5=4$, $\dim M_3=5$, and $\dim M_1=5$. Then $I(M_y)=\{M(11),M(9),2M(7),M(5),2M(3),M(1)\}$. Proposition 8 and Lemma 27 imply that $M_y=N_1\oplus M(9)\oplus N_2$ where $I(N_1)=\{M(11),2M(7),M(1)\}$ and $I(N_2)=\{M(5),2M(3)\}$. Show that $N_1\cong T(11)$ and $N_2\cong T(5)$.

Since $X_{-2}^2v\neq 0$, then $X_{-\alpha}^2v\neq 0$. By Lemma 28, $KA_yv\cong V(11)$. Set $N=N_1/KA_yv$. It is clear that $I(N)=\{M(7),M(1)\}$. One can directly verify that the pair ($X_{-2}X_{-1}^3X_{-2}^2v$, $X_{-1}X_{-2}^3X_{-1}v$) and the triple ($X_{-1}^3X_{-2}^2v$, $X_{-1}^2X_{-2}^2X_{-1}v$, $X_{-2}X_{-1}^3X_{-2}v$) consist of linearly independent vectors. Hence the vectors

$$X_{-2}X_{-1}^3X_{-2}^2v,\ X_{-1}X_{-2}^3X_{-1}v,\ X_{-1}^3X_{-2}^2v,\ X_{-1}^2X_{-2}^2X_{-1}v,\ X_{-2}X_{-1}^3X_{-2}v$$

form a basis of M_1. Using this basis, it is not difficult to show that $\operatorname{Inv}M_1=0$. Therefore N_1 has no submodules isomorphic to $M(1)$. As N_1 is self-dual, it has no such factor modules. Now Lemma 23 yields that $N\cong V(7)$ and N_1 has filtrations by Weyl modules and by dual Weyl modules. Hence N_1 is a tilting module and so $N_1\cong T(11)$.

Set $m_5=2X_{-1}X_{-2}^2v+X_{-1}^2X_{-2}v$. One can directly check that $m_5\in\operatorname{Inv}M_5$ and that the vector $X_{-\alpha}m_5$ has a nonzero weight component $3X_{-1}^3X_{-2}v$. As N_2 is self-dual, Lemma 29 implies that $N_2\cong T(5)$.

I.VII. Let $p = 7$ and $\omega = 3\omega_1 + \omega_2$. Then $\dim\varphi = 44$ and $\sigma_x(\omega) = 13$. One easily observes that $\dim M_{13} = 1$ and $\dim M_{11} = 2$. Set

$$\Lambda_9 = \{\omega - 2\alpha_1, \omega - \alpha_1 - \alpha_2\},$$
$$\Lambda_7 = \{\omega - 3\alpha_1, \omega - 2\alpha_1 - \alpha_2, \omega - \alpha_1 - 2\alpha_2\},$$
$$\Lambda_5 = \{\omega - 3\alpha_1 - \alpha_2, \omega - 2\alpha_1 - 2\alpha_2\},$$
$$\Lambda_3 = \{\omega - 4\alpha_1 - \alpha_2, \omega - 3\alpha_1 - 2\alpha_2, \omega - 2\alpha_1 - 3\alpha_2\},$$
$$\Lambda_1 = \{\omega - 5\alpha_1 - \alpha_2, \omega - 4\alpha_1 - 2\alpha_2, \omega - 3\alpha_1 - 3\alpha_2\}.$$

It is not difficult to conclude that $M_i = \langle M_\lambda | \lambda \in \Lambda_i\rangle$ for $i \in \{1,3,5,7,9\}$. Put $\mu = \omega-\alpha_1-\alpha_2$, $\nu = \omega-2\alpha_1-\alpha_2$, and $\sigma = \omega-3\alpha_1-2\alpha_2$. By [13, Part 2, §8, Proposition 8.19], the maximal submodule of the module $V(\omega)$ is isomorphic to $M(\mu)$. Observe that $\nu = \mu - \alpha_1$ and $\sigma = \mu - 2\alpha_1 - \alpha_2$. It is clear that $\dim(M_\mu)_\nu = 1$. Taking into account Theorem 5 and Lemma 14, we can show that $\dim(M_\mu)_\sigma = 2$. Hence $\dim M_\mu = \dim V(\omega)_\mu - 1$, $\dim M_\nu = \dim V(\omega)_\nu - 1$, and $\dim M_\sigma = \dim V(\omega)_\sigma - 2$. Applying Freudenthal's formula for calculating the weight multiplicities of $V(\omega)$, we get that $\dim M_\mu = 1$ and $\dim M_\nu = \dim M_\sigma = 2$. Set $\Delta = \{\omega - a\alpha_1, 0 \le a \le 3;\ \omega - \alpha_2\}$. It is clear that $\dim M_\delta = 1$ for $\delta \in \Delta$. Obviously, $\dim M_\tau = 1$ for

$$\tau \in \{\omega - \alpha_1 - 2\alpha_2,\ \omega - 2\alpha_1 - 3\alpha_2,\ \omega - 4\alpha_1 - \alpha_2,\ \omega - 5\alpha_1 - 2\alpha_2\}$$

since τ lies in the same W-orbit with a weight from Δ or with μ. One easily observes that $\dim M_\theta = 2$ for

$$\theta \in \{\omega - 3\alpha_1 - \alpha_2, \omega - 2\alpha_1 - 2\alpha_2, \omega - 4\alpha_1 - 2\alpha_2, \omega - 3\alpha_1 - 3\alpha_2\}$$

as θ lies in the same W-orbit with ν or σ. It is not difficult to show that $\dim M_9 = 2$, $\dim M_7 = 4$, $\dim M_5 = 4$, $\dim M_3 = 4$, and $\dim M_1 = 5$. Hence

$$I(M_y) = \{M(13), M(11), 2M(7), 2M(5), 2M(1)\}.$$

By Lemma 27, $M_y = M(13) \oplus N_1 \oplus N_2$ where $I(N_1) = \{M(11), 2M(1)\}$ and $I(N_2) = \{2M(7), 2M(5)\}$. Show that $N_1 \cong T(11)$ and $N_2 \cong T(7) \oplus M(7)$.

Set $m_{11} = X_{-1}v + 4X_{-2}v$. One can directly verify that $m_{11} \in \operatorname{Inv} M_{11}$ and that the vector $X^5_{-\alpha}m_{11}$ has a nonzero weight component $5X^5_{-1}X_{-2}v$. As N_1 is self-dual, Lemma 29 implies that $N_1 \cong T(11)$.

Put $u = 6X^3_{-1}v+3X^2_{-1}X_{-2}v+X^2_{-2}X_{-1}v$. One easily observes that the pairs $(X_{-2}X^3_{-1}v$, $X^3_{-1}X_{-2}v)$ and $(X^2_{-2}X^2_{-1}v$, $X_{-2}X^2_{-1}X_{-2}v)$ consist of linearly independent vectors. Hence the vectors $X_{-2}X^3_{-1}v$, $X^3_{-1}X_{-2}v$, $X^2_{-2}X^2_{-1}v$, and $X_{-2}X^2_{-1}X_{-2}v$ form a basis in M_5. Using this basis, it is not difficult to show that

$$X_\alpha M_5 = \langle 4X^3_{-1}v + 4X_{-2}X^2_{-1}v + X^2_{-1}X_{-2}v;$$
$$4X_{-2}X^2_{-1}v + X^2_{-1}X_{-2}v + 6X^2_{-2}X_{-1}v, 3X^3_{-1}v + X_{-2}X^2_{-1}v + 4X^2_{-2}X_{-1}v\rangle,$$

$u \in X_\alpha M_5$, and the vector $X_{-\alpha}u$ has a nonzero weight component $3X_{-2}X^3_{-1}v+2X^3_{-1}X_{-2}v$. Since $\operatorname{Inv} M_7 \subset N_2$, this yields that there exists a vector $m \in N_2 \cap M_5$ such that $X_{-\alpha}X_\alpha m \neq 0$. As N_2 is self-dual, Lemmas 24 and 25 imply that $N_2 \cong T(7) \oplus M(7)$.

I.VIII. Let $p = 7$ and $\omega = 4\omega_2$. Then $\dim\varphi = 54$ and $\sigma_y(\omega) = 16$. It is clear that $\dim M_{16} = 1$, $\dim M_{14} = 1$, and $\dim M_{12} = 2$. Set

$$\Lambda_{10} = \{\omega - 3\alpha_2, \omega - \alpha_1 - 2\alpha_2, \omega - 2\alpha_1 - \alpha_2\};$$
$$\Lambda_8 = \{\omega - 2\alpha_1 - 2\alpha_2, \omega - \alpha_1 - 3\alpha_2, \omega - 4\alpha_2\};$$
$$\Lambda_6 = \{\omega - 3\alpha_1 - 2\alpha_2, \omega - 2\alpha_1 - 3\alpha_2, \omega - \alpha_1 - 4\alpha_2\};$$
$$\Lambda_4 = \{\omega - 4\alpha_1 - 2\alpha_2, \omega - 3\alpha_1 - 3\alpha_2, \omega - 2\alpha_1 - 4\alpha_2\};$$
$$\Lambda_2 = \{\omega - 4\alpha_1 - 3\alpha_2, \omega - 3\alpha_1 - 4\alpha_2, \omega - 2\alpha_1 - 5\alpha_2\};$$
$$\Lambda_0 = \{\omega - 5\alpha_1 - 3\alpha_2, \omega - 4\alpha_1 - 4\alpha_2, \omega - 3\alpha_1 - 5\alpha_2\}.$$

One easily observes that $M_i = \langle M_\lambda | \lambda \in \Lambda_i \rangle$ for $i \in \{0, 2, 4, 6, 8, 10\}$. It follows from [18, Table 6.22] that the maximal submodule of the module $V(\omega)$ is isomorphic to $M(0)$. Hence $\dim M_\mu = \dim V(\omega)_\mu$ for $\mu \neq 0$ and $\dim M_\mu = \dim V(\omega)_\mu - 1$ for $\mu = 0$. Set $\Sigma = \{\omega - a\alpha_2, 0 \leq a \leq 4; \omega - \alpha_1 - \alpha_2\}$, $\gamma = \omega - \alpha_1 - 2\alpha_2$, $\delta = \omega - 2\alpha_1 - 2\alpha_2$, $\tau = \omega - 2\alpha_1 - 3\alpha_2$, $\eta = \omega - 3\alpha_1 - 3\alpha_2$, and $\theta = \omega - 4\alpha_1 - 4\alpha_2$. It is clear that $\dim M_\sigma = 1$ for $\sigma \in \Sigma$. Using the formula in [4, Chapter 8, §9, Item 3], it is not difficult to deduce that $\dim M_\gamma = 1$ and $\dim M_\delta = \dim M_\tau = \dim M_\eta = \dim M_\theta = 2$. It is clear that $\dim M_\lambda = 1$ for

$$\lambda \in \{\omega - 2\alpha_1 - \alpha_2, \omega - \alpha_1 - 3\alpha_2, \omega - 3\alpha_1 - 2\alpha_2, \omega - \alpha_1 - 4\alpha_2,$$
$$\omega - 4\alpha_1 - 2\alpha_2, \omega - 2\alpha_1 - 5\alpha_2, \omega - 3\alpha_1 - 5\alpha_2, \omega - 5\alpha_1 - 3\alpha_2\}$$

since λ lies in the same W-orbit with γ or with a weight from the set Σ, and $\dim M_\nu = 2$ for $\nu \in \{\omega - 2\alpha_1 - 4\alpha_2, \omega - 4\alpha_1 - 3\alpha_2, \omega - 3\alpha_1 - 4\alpha_2\}$ as ν lies in the same W-orbit with one of the weights δ, τ, or η. Now one can observe that $\dim M_{10} = 3$, $\dim M_8 = 4$, $\dim M_6 = 4$, $\dim M_4 = 5$, $\dim M_2 = 5$, and $\dim M_0 = 4$. Therefore

$$I(M_y) = \{M(16),\ M(12),\ 2M(10),\ M(8),\ 2M(4),\ M(2)\}.$$

By Proposition 8 and Lemma 27, $M_y = N_1 \oplus M(12) \oplus N_2$ where

$$I(N_1) = \{M(16),\ 2M(10),\ M(2)\} \text{ and } I(N_2) = \{M(8), 2M(4)\}.$$

Obviously, $X_{-\alpha}^3 v \neq 0$. Then by Lemma 28, $KA_y v \cong V(16)$. It is not difficult to check that the pairs $(X_{-1}^3 X_{-2}^4 v,\ X_{-2}^2 X_{-1}^3 X_{-2}^2 v)$ and $(X_{-1}^4 X_{-2}^2 v,\ X_{-1}^2 X_{-2}^2 X_{-1}^2 X_{-2} v)$ consist of linearly independent vectors. Now it is clear that the vectors $X_{-1}^3 X_{-2}^4 v$, $X_{-2}^2 X_{-1}^3 X_{-2}^2 v$, $X_{-1}^4 X_{-2}^2 v$, $X_{-1}^2 X_{-2}^2 X_{-1}^2 X_{-2} v$, and $X_{-2}^4 X_{-1}^2 X_{-2} v$ form a basis of M_2. Using this basis, it is easy to check that $\operatorname{Inv} M_2 = 0$. This yields that N_1 has no submodules isomorphic to $M(2)$. Since N_1 is self-dual, it has no such factor modules. Now Lemma 23 implies that $N_1/KA_y v \cong V(10)$ and N_1 has filtrations by Weyl modules and by dual Weyl modules. Hence N_1 is a tilting module and so $N_1 \cong T(16)$.

Set $m_8 = 4X_{-1}^2 X_{-2}^2 v + X_{-2} X_{-1}^2 X_{-2} v$. It is not difficult to check that $\operatorname{Inv} M_8 = \langle m_8 \rangle$ and $X_{-\alpha}^2 m_8$ has a nonzero weight component $6X_{-1}^4 X_{-2}^2 v$. As N_2 is self-dual, then $N_2 \cong T(8)$ by Lemma 29.

In Items I.IX and I.X by Lemma 4, $\dim M_\lambda = 1$ for all $\lambda \in \Lambda(M)$. This fact is used for calculating the dimensions of the subspaces M_i.

I.IX. Let $p = 11$ and $\omega = \omega_1 + 4\omega_2$. Then $\dim \varphi = 60$ and $\sigma_y(\omega) = 19$. It is not difficult to show that $\dim M_{19} = 1$, $\dim M_{17} = 2$, $\dim M_{15} = 2$, $\dim M_{13} = 3$, $\dim M_{11} = 4$, $\dim M_9 = 3$, $\dim M_7 = 4$, $\dim M_5 = 4$, $\dim M_3 = 3$, and $\dim M_1 = 4$. Then

$$I(M_y) = \{M(19), M(17), M(13), M(11), 2M(7), 2M(1)\}.$$

By Lemma 27, $M_y = N_1 \oplus M(17) \oplus N_2 \oplus M(11)$ where $I(N_1) = \{M(19), 2M(1)\}$ and $I(N_2) = \{M(13), 2M(7)\}$.

Set $m_{13} = 9X_{-1}^2 X_{-2} v + X_{-1} X_{-2} v + 9X_{-2}^3 v$. One can directly verify that $m_{13} \in \operatorname{Inv} M_{13}$, the vector $X_{-\alpha}^9 v$ has a nonzero weight component $7X_{-2}^5 X_{-1}^3 X_{-2} v$, and the vector $X_{-\alpha}^3 m_{13}$ has a nonzero weight component $8X_{-2}^5 X_{-1} v$. As N_1 and N_2 are self-dual, now Lemma 29 implies that $N_1 \cong T(19)$ and $N_2 \cong T(13)$.

I.X. Let $p = 11$ and $\omega = 5\omega_2$. Then $\dim \varphi = 61$ and $\sigma_y(\omega) = 20$. One easily deduces that $\dim M_{20} = 1$, $\dim M_{18} = 1$, $\dim M_{16} = 2$, $\dim M_{14} = 3$, $\dim M_{12} = 3$, $\dim M_{10} = 4$, $\dim M_8 = 4$, $\dim M_6 = 3$, $\dim M_4 = 4$, $\dim M_2 = 4$, and $\dim M_0 = 3$. Then

$$I(M_y) = \{M(20), M(16), M(14), M(10), 2M(4)\}.$$

By Lemma 27, $M_y \cong M(20) \oplus N \oplus M(14) \oplus M(10)$ where $I(N) = \{M(16), 2M(4)\}$. Show that $N \cong T(16)$.

Set $m_{16} = 6X_{-2}^2 v + X_{-1} X_{-2} v$. One can directly verify that $m_{16} \in \operatorname{Inv} M_{16}$ and the vector $X_{-\alpha}^6 m_{16}$ has a nonzero weight component $X_{-1}^5 X_{-2}^3 v$. As N is self-dual, $N \cong T(16)$ by Lemma 29.

I.XI. Let $p = 7$ and $\omega = 2\omega_1 + 2\omega_2$. Then $\dim \varphi = 71$ and $\sigma_y(\omega) = 14$. It is clear that $\dim M_{14} = 1$ and $\dim M_{12} = 2$. Set

$$\begin{aligned}\Lambda_{10} =&\{\omega - 2\alpha_1,\ \omega - \alpha_1 - \alpha_2,\ \omega - 2\alpha_2\};\\ \Lambda_8 =&\{\omega - 2\alpha_1 - \alpha_2,\ \omega - \alpha_1 - 2\alpha_2\};\\ \Lambda_6 =&\{\omega - 3\alpha_1 - \alpha_2,\ \omega - 2\alpha_1 - 2\alpha_2,\ \omega - \alpha_1 - 3\alpha_2\};\\ \Lambda_4 =&\{\omega - 4\alpha_1 - \alpha_2,\ \omega - 3\alpha_1 - 2\alpha_2,\ \omega - 2\alpha_1 - 3\alpha_2\};\\ \Lambda_2 =&\{\omega - 4\alpha_1 - 2\alpha_2,\ \omega - 3\alpha_1 - 3\alpha_2,\ \omega - 2\alpha_1 - 4\alpha_2\};\\ \Lambda_0 =&\{\omega - 5\alpha_1 - 2\alpha_2,\ \omega - 4\alpha_1 - 3\alpha_2,\ \omega - 3\alpha_1 - 4\alpha_2\}.\end{aligned}$$

One easily observes that $M_i = \langle M_\lambda | \lambda \in \Lambda_i\rangle$. By [13, Part 2, §8, Proposition 8.19], $M(\omega) \cong V(\omega)/M(2\omega_1)$. We have $2\omega_1 = \omega - 2\alpha_1 - 2\alpha_2$. Set $N = M(2\omega_1)$. Taking into account Theorem 5, one easily concludes that $\dim N_\mu = 1$ for $\mu \in \Lambda(N) \setminus \{0\}$ and $\dim N_0 = 2$. Now the weight multiplicities of M can be found with the help of Freudenthal's formula for the weights of $V(\omega)$. Set $\Sigma = \{\omega,\ \omega - \alpha_1,\ \omega - \alpha_2\}$, $\mu = \omega - \alpha_1 - \alpha_2$, $\nu = \omega - 2\alpha_1 - \alpha_2$, $\tau = \omega - 2\alpha_1 - 2\alpha_2$, $\eta = \omega - 3\alpha_1 - 2\alpha_2$, and $\psi = \omega - 4\alpha_1 - 3\alpha_2 = 0$. One can show that $\dim M_\mu = \dim V(\omega)_\mu = 2$, $\dim M_\nu = \dim V(\omega)_\nu = 3$, $\dim M_\theta = \dim V(\omega)_\theta - 1 = 3$ for $\theta = \tau$ or η, and $\dim M_\psi = \dim V(\omega)_\psi - 2 = 3$. It is clear that $\dim M_\lambda = 1$ for

$$\lambda \in \{\omega - 2\alpha_1, \omega - 2\alpha_2, \omega - \alpha_1 - 3\alpha_2, \omega - 4\alpha_1 - \alpha_2, \omega - 2\alpha_1 - 4\alpha_2\}$$

since λ lies in the same W-orbit with a weight from Σ, $\dim M_\delta = 2$ for

$$\delta \in \{\omega - 2\alpha_1 - 3\alpha_2, \omega - 3\alpha_1 - \alpha_2, \omega - 5\alpha_1 - 2\alpha_2, \omega - 3\alpha_1 - 4\alpha_2\}$$

as δ lies in the same W-orbit with μ, and $\dim M_\gamma = 3$ for

$$\gamma \in \{\omega - 2\alpha_1 - 3\alpha_2, \omega - 4\alpha_1 - 2\alpha_2, \omega - 3\alpha_1 - 3\alpha_2\}$$

since γ lies in the same W-orbit with ν, τ, or η. Now one easily observes that $\dim M_{10} = 4$, $\dim M_8 = 5$, $\dim M_6 = 6$, $\dim M_4 = 7$, $\dim M_2 = 7$, and $\dim M_0 = 7$. This yields that

$$I(M_y) = \{M(14), 2M(12), 2M(10), M(8), M(6), 2M(4), 2M(2), M(0)\}.$$

Proposition 8 and Lemma 27 imply that $M_y \cong N_1 \oplus N_2 \oplus N_3 \oplus M(6)$ where

$$I(N_1) = \{M(14), 2M(12), M(0)\},\ I(N_2) = \{2M(10), 2M(2)\},\ I(N_3) = \{M(8), 2M(4)\}.$$

Show that $N_1 \cong T(14)$, $N_2 \cong T(10) \oplus M(10)$, and $N_3 \cong T(8)$.

Obviously, $X_{-\alpha}v \neq 0$ since $X_{-1}v \neq 0$. Then by Lemma 28, $KA_yv \cong V(14)$. Set $\overline{N_1} = N_1/KA_yv$. It is clear that $I(\overline{N_1}) = \{M(12), M(0)\}$. One can directly verify that the pairs $(X_{-1}^5X_{-2}^2v,\ X_{-1}^3X_{-2}^2X_{-1}^2v)$ and $(X_{-2}^2X_{-1}^3X_{-2}^2v,\ X_{-1}X_{-2}^4X_{-1}^2v)$ and the triple $(X_{-2}X_{-1}^4X_{-2}^2v,\ X_{-2}^2X_{-1}^3X_{-2}X_{-1}v,\ X_{-1}^2X_{-2}^3X_{-1}^2v)$ consist of linearly independent vectors. Then the vectors $X_{-2}^2X_{-1}^3X_{-2}^2v$, $X_{-1}X_{-2}^4X_{-1}^2v$, $X_{-2}X_{-1}^4X_{-2}^2v$, $X_{-2}^2X_{-1}^3X_{-2}X_{-1}v$, $X_{-1}^2X_{-2}^3X_{-1}^2v$, $X_{-1}^5X_{-2}^2v$, and $X_{-1}^3X_{-2}^2X_{-1}^2v$ form a basis of M_0. Using this basis, it is not difficult to check that $\operatorname{Inv} M_0 = 0$. Hence N_1 has no submodules isomorphic to $M(0)$. As N_1 is self-dual, it has no such factor modules. Now Lemma 23 yields that $N_1/KA_yv \cong V(12)$ and N_1 has filtrations by Weyl modules and by dual Weyl modules. Hence N_1 is a tilting module and so $N_1 \cong T(14)$.

Set $m_2 = X_{-1}X_{-2}^3X_{-1}^2v$ and $m_8 = 4X_{-1}^2X_{-2}v + 4X_{-2}X_{-1}^2v + 2X_{-1}X_{-2}^2v + X_{-2}^2X_{-1}v$. One can directly verify that $X_{-\alpha}^4X_\alpha^4m_2 \neq 0$, $X_\alpha^5m_2 = 0$, $m_8 \in \operatorname{Inv} M_8$, and the vector $X_{-\alpha}^2m_8$ has a nonzero weight component $5X_{-1}^4X_{-2}v$. It is clear that $M_y = N_2 \oplus N'$ where $N' = N_1 \oplus N_3 \oplus M(6)$. As $N' \cap \operatorname{Inv} M_8 = 0$, the arguments above imply that there exists a vector $m_2' \in N_2 \cap M_2$ such that $X_{-\alpha}^4X_\alpha^4m_2' \neq 0$. Then by Lemma 25, $N_2 \cong T(10) \oplus M(10)$ and by Lemma 29, $N_3 \cong T(8)$.

I.XII. Let $p = 7$ and $\omega = \omega_1 + 3\omega_2$. Then $\dim\varphi = 76$ and $\sigma_y(\omega) = 15$. It is clear that $\dim M_{15} = 1$ and $\dim M_{13} = 2$. Set

$$\begin{aligned}
\Lambda_{11} =&\{\omega - \alpha_1 - \alpha_2,\ \omega - 2\alpha_2\};\\
\Lambda_9 =&\{\omega - 2\alpha_1 - \alpha_2,\ \omega - \alpha_1 - 2\alpha_2,\ \omega - 3\alpha_2\};\\
\Lambda_7 =&\{\omega - 3\alpha_1 - \alpha_2,\ \omega - 2\alpha_1 - 2\alpha_2,\ \omega - \alpha_1 - 3\alpha_2\};\\
\Lambda_5 =&\{\omega - 3\alpha_1 - 2\alpha_2,\ \omega - 2\alpha_1 - 3\alpha_2,\ \omega - \alpha_1 - 4\alpha_2\};\\
\Lambda_3 =&\{\omega - 4\alpha_1 - 2\alpha_2,\ \omega - 3\alpha_1 - 3\alpha_2,\ \omega - 2\alpha_1 - 4\alpha_2\};\\
\Lambda_1 =&\{\omega - 5\alpha_1 - 2\alpha_2,\ \omega - 4\alpha_1 - 3\alpha_2,\ \omega - 3\alpha_1 - 4\alpha_2\}.
\end{aligned}$$

One easily observes that $M_i = \langle M_\lambda | \lambda \in \Lambda_i\rangle$ for $i \in \{1, 3, 5, 7, 9, 11\}$. By [13, Part 2, §8, Proposition 8.19], $M(\omega) \cong V(\omega)/M(\omega_1)$. We have $\omega_1 = \omega - 3\alpha_1 - 3\alpha_2$. Now the weight multiplicities of M can be found with the use of Freudenthal's formula for the weights of $V(\omega)$. Set $\Sigma = \{\omega, \omega-\alpha_2, \omega-2\alpha_2\}$, $\mu = \omega-\alpha_1-\alpha_2$, $\nu = \omega-\alpha_1-2\alpha_2$, $\tau = \omega-2\alpha_1-2\alpha_2$, and $\eta = \omega - 3\alpha_1 - 3\alpha_2$. We can show that $\dim M_\theta = \dim V(\omega)_\theta = 2$ for $\theta = \mu$ or ν, $\dim M_\tau = \dim V(\omega)_\tau = 3$, and $\dim M_\eta = \dim V(\omega)_\eta - 1 = 3$. Obviously, $\dim M_\sigma = 1$ for $\sigma \in \Sigma$, $\dim M_\lambda = 1$ for $\lambda \in \{\omega - 3\alpha_2, \omega - 3\alpha_1 - \alpha_2, \omega - \alpha_1 - 4\alpha_2, \omega - 5\alpha_1 - 2\alpha_2\}$ since λ lies in the same W-orbit with a weight from Σ, $\dim M_\delta = 2$ for $\delta \in \{\omega - 2\alpha_1 - \alpha_2, \omega - \alpha_1 - 3\alpha_2,\ \omega - 2\alpha_1 - 4\alpha_2, \omega - 4\alpha_1 - 2\alpha_2\}$ as δ lies in the same W-orbit with μ or ν, and $\dim M_\gamma = 3$ for $\gamma \in \{\omega - 3\alpha_1 - 2\alpha_2, \omega - 2\alpha_1 - 3\alpha_2,\ \omega - 3\alpha_1 - 4\alpha_2, \omega - 4\alpha_1 - 3\alpha_2\}$ since γ lies in the same W-orbit with τ or η. Now it is not difficult to show that $\dim M_{11} = 3$, $\dim M_9 = 5$, $\dim M_7 = 6$, $\dim M_5 = 7$, $\dim M_3 = 7$, and $\dim M_1 = 7$. Hence

$$I(M_y) = \{M(15), M(13), 2M(11), 2M(9), M(7), 2M(5), 2M(3), M(1)\}.$$

By Proposition 8 and Lemma 27, $M_y \cong N_1 \oplus M(13) \oplus N_2 \oplus N_3$ where

$$I(N_1) = \{M(15), 2M(11), M(1)\},\ I(N_2) = \{2M(9), 2M(3)\},\ I(N_3) = \{M(7), 2M(5)\}.$$

Show that $N_1 \cong T(15)$, $N_2 \cong T(9) \oplus M(9)$, and $N_3 \cong T(7)$.

Since $X_{-2}^2 v \neq 0$, we have $X_{-\alpha}^2 v \neq 0$. Then Lemma 28 implies that $KA_y v \cong V(15)$. Set $\overline{N_1} = N_1/KA_y v$. Then $I(\overline{N_1}) = \{M(11), M(1)\}$.

One can check that the triples $(X_{-1}^4 X_{-2}^3 v,\ \ X_{-1}^3 X_{-2}^3 X_{-1} v,\ \ X_{-2} X_{-1}^4 X_{-2}^2 v)$ and $(X_{-1}^2 X_{-2}^4 X_{-1} v, X_{-2} X_{-1}^3 X_{-2}^3 v, X_{-2}^2 X_{-1}^2 X_{-2}^2 X_{-1} v)$ consist of linearly independent vectors. Then the vectors $X_{-1}^5 X_{-2}^2 v$, $X_{-1}^4 X_{-2}^3 v$, $X_{-1}^3 X_{-2}^3 X_{-1} v$, $X_{-2} X_{-1}^4 X_{-2}^2 v$, $X_{-1}^2 X_{-2}^4 X_{-1} v$, $X_{-2} X_{-1}^3 X_{-2}^3 v$, and $X_{-2}^2 X_{-1}^2 X_{-2}^2 X_{-1} v$ form a basis of M_1. Using this basis, we can show that $\operatorname{Inv} M_1 = 0$. Therefore N_1 has no submodules isomorphic to $M(1)$. As N_1 is self-dual, it has no such factor modules. Now Lemma 23 yields that $\overline{N_1} \cong V(11)$ and N_1 has filtrations by Weyl modules and by dual Weyl modules. Hence N_1 is a tilting module and so $N_1 \cong T(15)$.

Put

$$\begin{aligned}
m_3 &= 6X_{-1}^4 X_{-2}^2 v + 2X_{-1}^2 X_{-2} X_{-1}^2 X_{-2} v + 6X_{-2} X_{-1} X_{-2}^3 v + X_{-1}^2 X_{-2}^3 X_{-1} v,\\
m_7 &= 4X_{-1}^3 X_{-2} v + 4X_{-1}^2 X_{-2}^2 v + X_{-1} X_{-2}^3 v.
\end{aligned}$$

One can directly verify that $X_{-\alpha}^3 X_\alpha^3 m_3$ has a nonzero weight component $6X_{-1}^4 X_{-2}^2 v + 6X_{-1} X_{-2} X_{-1}^3 X_{-2} v$, $X_{-\alpha}^4 m_3 = 0$, $m_7 \in \operatorname{Inv} M_7$, and the vector $X_{-\alpha} m_7$ has a nonzero weight component $X_{-2}^4 X_{-1} v$. It is clear that $M_y = N_2 \oplus N$ where $N \cong N_1 \oplus M(13) \oplus N_3$. As $N \cap \operatorname{Inv} M_9 = 0$, there exists a vector $m_3' \in N_2 \cap M_3$ such that $X_{-\alpha}^3 X_\alpha^3 m_3' \neq 0$. Then by Lemma 25, $N_2 \cong T(9) \oplus M(9)$. It is clear that $m_7 \in N_3$. Since N_3 is self-dual, then $N_3 \cong T(7)$ by Lemma 29.

In Items I.XIII and I.XIV by Lemma 4, $\dim M_\mu = 1$ for any $\mu \in \Lambda(M)$. This fact is used for calculating the dimensions of the subspaces M_i.

I.XIII. Let $p = 13$ and $\omega = \omega_1 + 5\omega_2$. Then $\dim\varphi = 84$ and $\sigma_y(\omega) = 23$. We can show that $\dim M_{23} = 1$, $\dim M_{21} = 2$, $\dim M_{19} = 2$, $\dim M_{17} = 3$, $\dim M_{15} = 4$, $\dim M_{13} = 4$, $\dim M_{11} = 4$, $\dim M_9 = 5$, $\dim M_7 = 4$, $\dim M_5 = 4$, $\dim M_3 = 5$, and $\dim M_1 = 4$. Hence

$I(M_y) = \{M(23), M(21), M(17), M(15), 2M(9), 2M(3)\}$. Lemma 27 implies that $M_y \cong M(23) \oplus N_1 \oplus M(17) \oplus N_2$ where $I(N_1) = \{M(21), 2M(3)\}$ and $I(N_2) = \{M(15), 2M(9)\}$. Show that $N_1 \cong T(21)$ and $N_2 \cong T(15)$.

Set $m_{21} = 4X_{-1}v + X_{-2}v$ and $m_{15} = 7X_{-1}^3X_{-2}v + 11X_{-1}^2X_{-2}^2v + 5X_{-1}X_{-2}^3v + X_{-2}^4v$. One can directly verify that $m_{21} \in \operatorname{Inv} M_{21}$, $m_{15} \in \operatorname{Inv} M_{15}$, the vector $X_{-\alpha}^9 m_{21}$ has a nonzero weight component $3X_{-1}^7X_{-2}^3v$, and the vector $X_{-\alpha}^3 m_{15}$ has a nonzero weight component $X_{-2}^6X_{-1}v$. It is clear that $m_{21} \in N_1$ and $m_{15} \in N_2$. As N_1 and N_2 are self-dual, $N_1 \cong T(21)$ and $N_2 \cong T(15)$ by Lemma 29.

I.XIV. Let $p = 13$ and $\omega = 6\omega_2$. Then $\dim \varphi = 85$ and $\sigma_y(\omega) = 24$. One can check that $\dim M_{24} = 1$, $\dim M_{22} = 1$, $\dim M_{20} = 2$, $\dim M_{18} = 3$, $\dim M_{16} = 3$, $\dim M_{14} = 4$, $\dim M_{12} = 5$, $\dim M_{10} = 4$, $\dim M_8 = 4$, $\dim M_6 = 5$, $\dim M_4 = 4$, $\dim M_2 = 4$, and $\dim M_0 = 5$. Hence

$$I(M_y) = \{M(24), M(20), M(18), M(14), M(12), 2M(6), 2M(0)\}.$$

By Lemma 27, $M_y \cong N_1 \oplus M(20) \oplus N_2 \oplus M(14) \oplus M(12)$ where $I(N_1) = \{M(24), 2M(0)\}$ and $I(N_2) = \{M(18), 2M(6)\}$. Show that $N_1 \cong T(24)$ and $N_2 \cong T(18)$.

Put $m_{18} = 7X_{-1}^2X_{-2}v + 5X_{-1}X_{-2}^2v + X_{-2}^3v$. One can directly verify that $m_{18} \in \operatorname{Inv} M_{18}$, the vector $X_{-\alpha}^{12}v$ has a nonzero weight component $4X_{-1}^8X_{-2}^4v$, and the vector $X_{-\alpha}^6 m_{18}$ has a nonzero weight component $12X_{-1}^3X_{-2}^6v$. It is clear that $v \in N_1$ and $m_{18} \in N_2$. Since N_1 and N_2 are self-dual, Lemma 29 yields that $N_1 \cong T(24)$ and $N_2 \cong T(18)$.

I.XV. Let $p = 5$ and $\omega = 2\omega_1 + 3\omega_2$. Then $\dim \varphi = 86$ and $\sigma_y(\omega) = 18$. One easily observes that $\dim M_{18} = 1$ and $\dim M_{16} = 2$. Set $\mu = 2\omega_1 + 2\omega_2$, $M' = M(\mu)$,

$$\begin{aligned}
\Lambda_{14} =& \{\omega - 2\alpha_1,\ \omega - \alpha_1 - \alpha_2,\ \omega - 2\alpha_2\};\\
\Lambda_{12} =& \{\omega - 2\alpha_1 - \alpha_2,\ \omega - \alpha_1 - 2\alpha_2,\ \omega - 3\alpha_2\};\\
\Lambda_{10} =& \{\omega - 3\alpha_1 - \alpha_2,\ \omega - 2\alpha_1 - 2\alpha_2,\ \omega - \alpha_1 - 3\alpha_2\};\\
\Lambda_{8} =& \{\omega - 4\alpha_1 - \alpha_2,\ \omega - 3\alpha_1 - 2\alpha_2,\ \omega - 2\alpha_1 - 3\alpha_2,\ \omega - \alpha_1 - 4\alpha_2\};\\
\Lambda_{6} =& \{\omega - 4\alpha_1 - 2\alpha_2,\ \omega - 3\alpha_1 - 3\alpha_2,\ \omega - 2\alpha_1 - 4\alpha_2\};\\
\Lambda_{4} =& \{\omega - 5\alpha_1 - 2\alpha_2,\ \omega - 4\alpha_1 - 3\alpha_2,\ \omega - 3\alpha_1 - 4\alpha_2,\ \omega - 2\alpha_1 - 5\alpha_2\};\\
\Lambda_{2} =& \{\omega - 6\alpha_1 - 2\alpha_2,\ \omega - 5\alpha_1 - 3\alpha_2,\ \omega - 4\alpha_1 - 4\alpha_2,\ \omega - 3\alpha_1 - 5\alpha_2\};\\
\Lambda_{0} =& \{\omega - 6\alpha_1 - 3\alpha_2,\ \omega - 5\alpha_1 - 4\alpha_2,\ \omega - 4\alpha_1 - 5\alpha_2\}.
\end{aligned}$$

One easily deduces that $M_i = \langle M_\lambda | \lambda \in \Lambda_i \rangle$ for $i \in \{0, 2, 4, 6, 8, 10, 12, 14\}$. By [13, Part 2, § 8, Proposition 8.19], $M(\omega) \cong V(\omega)/M'$. We have $\mu = \omega - \alpha_1 - \alpha_2$. By Proposition 4 and Theorem 5, $M' \cong S^{6,5}(V)$. This fact is used to find the weight multiplicities of M'. After that we can find such multiplicities for M taking into account Freudenthal's formula for the weights of $V(\omega)$. Set $\nu = \omega - 2\alpha_1 - \alpha_2$, $\tau = \omega - \alpha_1 - 2\alpha_2$, $\xi = \omega - 2\alpha_1 - 2\alpha_2$, $\gamma = \omega - 3\alpha_1 - 2\alpha_2$, $\delta = \omega - 3\alpha_1 - 3\alpha_2$, $\theta = \omega - 4\alpha_1 - 3\alpha_2$, $\chi = \omega - 5\alpha_1 - 4\alpha_2 = 0$, and $\Sigma = \{\omega,\ \omega - \alpha_1,\ \omega - \alpha_2,\ \mu,\ \tau\}$. Obviously, $\dim M'_\mu = \dim M'_\nu = \dim M'_\tau = 1$. Using the realization of M' mentioned above, it is not difficult to show that $\dim M'_\xi = \dim M'_\gamma = 2$, $\dim M'_\delta = \dim M'_\theta = 3$, and $\dim M'_\chi = 4$. Now we can conclude that $\dim M_\sigma = 1$ for $\sigma \in \Sigma$, $\dim M_\nu = \dim M_\xi = \dim M_\delta = \dim M_\chi = 2$, and $\dim M_\theta = 3$. It is clear that $\dim M_\lambda = 1$ for

$$\begin{aligned}
\lambda \in \{&\omega - 3\alpha_1 - \alpha_2,\ \omega - \alpha_1 - 3\alpha_2,\ \omega - \alpha_1 - 4\alpha_2,\ \omega - 4\alpha_1 - \alpha_2,\\
&\omega - 2\alpha_1 - 5\alpha_2,\ \omega - 5\alpha_1 - 2\alpha_2,\ \omega - 3\alpha_1 - 5\alpha_2,\ \omega - 6\alpha_1 - 2\alpha_2\}
\end{aligned}$$

since λ lies in the same W-orbit with a weight from Σ, $\dim M_\psi = 2$ for

$$\begin{aligned}
\psi \in \{&\omega - 2\alpha_1 - 3\alpha_2,\ \omega - 2\alpha_1 - 4\alpha_2,\ \omega - 4\alpha_1 - 2\alpha_2,\\
&\omega - 3\alpha_1 - 4\alpha_2,\ \omega - 5\alpha_1 - 3\alpha_2,\ \omega - 6\alpha_1 - 3\alpha_2,\ \omega - 4\alpha_1 - 5\alpha_2\}
\end{aligned}$$

as ψ lies in the same W-orbit with ν, ξ, γ, or δ, and $\dim M_\eta = 3$ for $\eta = \omega - 4\alpha_1 - 4\alpha_2$ since η lies in the same W-orbit with θ. One easily observes that $\dim M_{14} = 3$, $\dim M_{12} = 4$,

$\dim M_{10} = 4$, $\dim M_8 = 6$, $\dim M_6 = 6$, $\dim M_4 = 7$, $\dim M_2 = 7$, and $\dim M_0 = 6$. Hence

$$I(M_y) = \{M(18), M(16), M(14),\ 2M(12), M(10), 2M(8), M(6), M(4), M(0)\}.$$

By Proposition 8 and Lemma 27, $M_y \cong N_1 \oplus N_2 \oplus M(14) \oplus M(4)$ where

$$I(N_1) = \{M(18), M(10), 2M(8), M(0)\},\ I(N_2) = \{M(16), 2M(12), M(6)\}.$$

Show that $N_1 \cong M(18) \oplus T(10)$ and $N_2 \cong T(16)$.

Set $m_{10} = X_{-1}^3 X_{-2} v + 4X_{-1}^2 X_{-2}^2 v$. One can directly verify that $X_{-\alpha}^4 v = 0$, $m_{10} \in \operatorname{Inv} M_{10}$, and the vector $X_{-\alpha} m_{10}$ has a nonzero weight component $3X_{-1}^4 X_{-2} v$. Therefore $KA_y v \cong M(18)$ and $KA_y m_{10} \cong V(10)$. Set $\overline{N_1} = N_1/KA_y v$. It is clear that $I(\overline{N_1}) = \{M(10), 2M(8), M(0)\}$. As N_1 is self-dual, it contains a submodule U with $I(U) = I(\overline{N_1})$. We have $U \cap KA_y v = 0$. Hence $N_1 = KA_y v \oplus U$. Show that $U \cong T(10)$. Obviously, $m_{10} \in U$. It is not difficult to check that the pairs $(X_{-1}^4 X_{-2}^3 X_{-1}^2 v,\ X_{-1}^4 X_{-2}^2 X_{-1}^2 X_{-2} v)$, $(X_{-1}^3 X_{-2}^4 X_{-1}^2 v,\ X_{-1}^3 X_{-2}^3 X_{-1}^2 X_{-2} v)$, and $(X_{-2}^4 X_{-1}^2 X_{-2} X_{-1}^2 v,\ X_{-1}^2 X_{-2}^4 X_{-1}^2 X_{-2} v)$ consist of linearly independent vectors. Therefore the vectors $X_{-1}^4 X_{-2}^3 X_{-1}^2 v$, $X_{-1}^4 X_{-2}^2 X_{-1}^2 X_{-2} v$, $X_{-1}^3 X_{-2}^4 X_{-1}^2 v$, $X_{-1}^3 X_{-2}^3 X_{-1}^2 X_{-2} v$, $X_{-2}^4 X_{-1}^2 X_{-2} X_{-1}^2 v$, and $X_{-1}^2 X_{-2}^4 X_{-1}^2 X_{-2} v$ form a basis of M_0. Using this basis, we can deduce that $\operatorname{Inv} M_0 = 0$. Hence U has no submodules isomorphic to $M(0)$. The module U is self-dual as N_1 is such. Therefore U has no factor modules isomorphic to $M(0)$. Hence $U/KA_y m_{10}$ has no direct summands isomorphic to $M(0)$ or $\Delta(8)$. Lemma 23 implies that $U/KA_y m_{10} \cong V(8)$. Therefore U has a filtration by Weyl modules and is a tilting module by Remark 4. As we know $I(U)$, we conclude that $U \cong T(10)$.

Set $m_{16} = 2X_{-1} v + 3X_{-2} v$. We can directly verify that $m_{16} \in \operatorname{Inv} M_{16}$. Since $X_{-2}^3 v \neq 0$, then $X_{-\alpha}^2 m_{16} \neq 0$. Then by Lemma 28, $KA_y m_{16} \cong V(16)$. Put $\overline{N_2} = N_2/KA_y m_{16}$. It is clear that $I(\overline{N_2}) = \{M(12), M(6)\}$. It is not difficult to check that the pairs $(X_{-1}^2 X_{-2} X_{-1}^2 X_{-2} v,\ X_{-1}^2 X_{-2}^2 X_{-1}^2 v)$, $(X_{-1} X_{-2}^2 X_{-1}^2 X_{-2} v,\ X_{-1} X_{-2}^3 X_{-1}^2 v)$, and $(X_{-2}^3 X_{-1}^2 X_{-2} v,\ X_{-2}^4 X_{-1}^2 v)$ consist of linearly independent vectors. Hence the vectors $X_{-1}^2 X_{-2} X_{-1}^2 X_{-2} v$, $X_{-1}^2 X_{-2}^2 X_{-1}^2 v$, $X_{-1} X_{-2}^2 X_{-1}^2 X_{-2} v$, $X_{-1} X_{-2}^3 X_{-1}^2 v$, $X_{-2}^3 X_{-1}^2 X_{-2} v$, and $X_{-2}^4 X_{-1}^2 v$ form a basis of M_6. Using this basis, we can directly verify that $\operatorname{Inv} M_6 = 0$. Therefore N_2 has no submodules isomorphic to $M(6)$. As N_2 is self-dual, the modules N_2 and $\overline{N_2}$ have no such factor modules. Hence $\overline{N_2}$ is indecomposable. Now Lemma 4 yields that $\overline{N_2}$ is generated by a nonzero vector of weight 12 and $\overline{N_2} \cong V(12)$. Therefore N_2 has a filtration by Weyl modules and it is a tilting module by Remark 4. As we know $I(N_2)$, we conclude that $N_2 \cong T(16)$.

For regular unipotent elements of the group $C_2(K)$ all possibilities are considered.

I.XVI. Let $J(x) = (2, 2)$. Then $\sigma_x(a_1\omega_1 + a_2\omega_2) = a_1 + 2a_2$. One easily observes that $\sigma_x(\omega) < p$ for $\omega \in \{\omega_1 + \omega_2,\ 2\omega_1 + \omega_2,\ 2\omega_2\}$ and $p > 3$, for $\omega \in \{\omega_1 + 2\omega_2,\ 3\omega_2,\ 2\omega_1 + 2\omega_2,\ 3\omega_1 + \omega_2\}$ and $p > 5$, and for $\omega \in \{4\omega_2,\ 5\omega_2,\ \omega_1 + 3\omega_2,\ \omega_1 + 4\omega_2\}$ and $p > 7$. In this case M_x is a direct sum of irreducible p-restricted modules. As we know the weight multiplicities of M, it is not difficult to find these modules and determine the block structure of $\varphi(x)$. Let $p = 3$ and $\omega = 2\omega_2$, or $p = 5$ and $\omega \in \{3\omega_2, 4\omega_2, \omega_1 + 3\omega_2\}$, or $p = 7$ and $\omega = 5\omega_2$. Set $\mu_1 = \mu_2 = \omega_2$ for $p = 3$ and $\omega = 2\omega_2$; $\mu_1 = 2\omega_2$ and $\mu_2 = \omega_2$ for $p = 5$ and $\omega = 3\omega_2$; $\mu_1 = 2\omega_2$ and $\mu_2 = \omega_1 + \omega_2$ for $p = 5$ and $\omega = \omega_1 + 3\omega_2$; $\mu_1 = \mu_2 = 2\omega_2$ for $p = 5$ and $\omega = 4\omega_2$; and $\mu_1 = 3\omega_2$ and $\mu_2 = 2\omega_2$ for $p = 7$ and $\omega = 5\omega_2$. In all these cases it is clear that the restrictions $M(\mu_i)|A_x$ are tilting modules since all their weights are less than p. As the module $V(\omega)$ is irreducible for such weights ω, then M_x is a tilting module by Lemma 14. Using Lemma 14 again, we conclude that M_x is also a tilting module for $\omega = \omega_1 + 2\omega_2$ and $p = 3$ (set $\mu_1 = \omega_1$ and $\mu_2 = 2\omega_2$). Since we know the weight multiplicities for all these modules, we can present the module M_x in the form of a direct sum of indecomposable tilting modules and determine the block structure of $\varphi(x)$ using Lemma 23 and Proposition 7.

Set $\Gamma = G(2\varepsilon_1, \alpha_2)$. It is clear that $\Gamma \cong C_1(K) \times C_1(K) \cong A_1(K) \times A_1(K)$ and that x is conjugate to a regular unipotent element of Γ. We use Theorems 4 and 3 and Lemma 21 to determine the canonical Jordan form of $\varphi(x)$ when $p = 11$ and $\omega \in \{\omega_1 + 4\omega_2, 5\omega_2\}$ or $p = 13$ and $\omega \in \{\omega_1 + 5\omega_2, 6\omega_2\}$. Recall that for $p = 3$ the representation $\varphi(2\omega_1 + 2\omega_2)$ is the Steinberg representation and hence $\varphi(x)$ has only blocks of size 3 for all elements x of order 3 (see [30]). Taking into account the arguments at the beginning of this section, we can conclude that it remains to consider the following cases:

1) $p = 3$, $\omega = 2\omega_1 + \omega_2$;
2) $p = 5$, $\omega = 2\omega_1 + 3\omega_2$ or $\omega_1 + 2\omega_2$;
3) $p = 7$, $\omega = 4\omega_2$ or $\omega_1 + 3\omega_2$.

Observe that x is conjugate to a short root element. Set $H = G(1)$. We can assume that $x \in H$. To determine the block structure of $\varphi(x)$, we consider the restriction $M|H$. Put $\Omega_i = \{\mu \in \Lambda(M) | \mu = \omega - k\alpha_1 - i\alpha_2\}$, $U_i = \langle M_\mu | \mu \in \Omega_i\rangle$, and $\Omega_i^+ = \{\lambda \in \Omega_i | \sigma(\lambda) \geq 0\}$. Below $i(M)$ is the maximal index i such that $\Omega_i \neq \varnothing$ and $U_{i,j}$ is the weight subspace of weight j in the H-module U_i. Theorem 1 and Lemma 11 imply that $U_j^* \cong U_{i(M)-j}$, the H-modules U_0 and $U_{i(M)}$ are irreducible, and $U_0 \cong U_{i(M)}$.

I.XVI.1) Let $p = 3$ and $\omega = 2\omega_1 + \omega_2$. Then $i(M) = 4$ and $U_0 \cong U_4 \cong M(2)$. One easily observes that $\Omega_1^+ = \{\omega - i\alpha_1 - \alpha_2 | 0 \leq i \leq 2\}$ and $\Omega_2^+ = \{\omega - i\alpha_1 - 2\alpha_2 | 1 \leq i \leq 3\}$. Show that $U_1 \cong T(4)$ and $U_2 \cong M(4) \oplus M(2)$.

Set $N = M(2\omega_1)$. By [13, Part 2, §8, Proposition 8.19], $M(\omega) \cong V(\omega)/N$. We have $2\omega_1 = \omega - \alpha_1 - \alpha_2$. Using Theorem 5, one easily concludes that $\dim N_\mu = 1$ for $\mu \in \Lambda(N) \setminus \{0\}$ and $\dim N_0 = 2$. Now the weight multiplicities in M can be found with the use of Freudenthal's formula for the weights of $V(\omega)$. Put $\Sigma = \{\omega,\ \omega - \alpha_1,\ \omega - \alpha_1 - \alpha_2\}$ and $\lambda = \omega - 2\alpha_1 - \alpha_2$. One has $\omega - 3\alpha_1 - 2\alpha_2 = 0$. Taking into account the arguments above, one can verify that $\dim M_\delta = 1$ for $\delta = \omega - \alpha_1 - 2\alpha_2$ or 0 and $\dim M_\lambda = 2$. Now it is clear that $\dim M_\sigma = 1$ for $\sigma \in \Sigma$. Obviously, $\dim M_\tau = 1$ for $\tau = \omega - 2\alpha_1 - \alpha_2$ since τ lies in the same W-orbit with a weight from Σ, and $\dim M_\nu = 2$ for $\nu = \omega - 2\alpha_1 - 2\alpha_2$ as ν lies in the same W-orbit with λ. This yields that $\dim U_{1,4} = 1$, $\dim U_{1,2} = 1$, $\dim U_{1,0} = 2$, $\dim U_{2,4} = 1$, $\dim U_{2,2} = 2$, and $\dim U_{2,0} = 1$. Hence $I(U_1) = \{M(4), 2M(0)\}$ and $I(U_2) = \{M(4), M(2)\}$. By Lemma 27, $U_2 \cong M(4) \oplus M(2)$.

Set $m_1 = X_{-2}v$ and $m_2 = X_{-2,3}X_{-1}^2 v$. It is clear that $m_1 \in L_1$ and $m_2 \in L_3$. By Lemma 12, the vectors m_1 and $m_2 \neq 0$ and the group $\mathfrak{X}_1$ fixes these vectors. Show that $KA_x m_i \cong V(4)$. For this, it suffices to show that $X_{-1}^2 m_i \neq 0$. Put $u_1 = X_{-1}^2 m_1 = X_{-1}^2 X_{-2} v$ and $u_2 = X_{-1}^2 m_2 = X_{-1}^2 X_{-2,3} X_{-1}^2 v$. As $X_2 X_{-1}^2 X_{-2} v = X_{-1}^2 v \neq 0$, then $u_1 \neq 0$. Since

$$\begin{aligned} X_2X_1^2X_2^2u_2 =& X_2X_1^2X_{-1}^2X_{-2}X_{-1}^2v = X_2X_1(X_{-1}X_{-2}X_{-1}^2v + 2X_{-1}^2X_{-2}X_{-1}v) = \\ =& X_2(2X_{-1}X_{-2}X_{-1}v + X_{-1}X_{-2}X_{-1}v + X_{-1}^2X_{-2}v) = X_{-1}^2v \neq 0, \end{aligned}$$

then $u_2 \neq 0$. This implies that the H-modules U_1 and U_3 contain submodules isomorphic to $V(4)$. As M is self-dual, now Lemma 24 yields that $U_1 \cong U_3 \cong T(4)$.

I.XVI.2) Let $p = 5$ and $\omega = 2\omega_1 + 3\omega_2$. Then $i(M) = 8$ and $U_0 \cong U_8 \cong M(2)$. One can check that

$$\begin{aligned} \Omega_1^+ =& \{\omega - i\alpha_1 - \alpha_2 | 0 \leq i \leq 2\};\ \Omega_2^+ = \{\omega - i\alpha_1 - 2\alpha_2 | 0 \leq i \leq 3\}; \\ \Omega_3^+ =& \{\omega - i\alpha_1 - 3\alpha_2 | 0 \leq i \leq 4\};\ \Omega_4^+ = \{\omega - i\alpha_1 - 4\alpha_2 | 1 \leq i \leq 5\}. \end{aligned}$$

The dimensions of the weight subspaces with the weights from the sets Ω_i^+ are determined in Item I.XV. Now one easily observes that $\dim U_{1,4} = 1$, $\dim U_{1,2} = 1$, $\dim U_{1,0} = 2$, $\dim U_{2,6} = 1$, $\dim U_{2,4} = 1$, $\dim U_{2,2} = 2$, $\dim U_{2,0} = 2$, $\dim U_{3,8} = 1$, $\dim U_{3,6} = 1$, $\dim U_{3,4} = 2$, $\dim U_{3,2} = 2$, $\dim U_{3,0} = 3$, $\dim U_{4,8} = 1$, $\dim U_{4,6} = 2$, $\dim U_{4,4} = 2$, $\dim U_{4,2} = 3$, and $\dim U_{4,0} = 2$. This yields that

$$\begin{aligned} &I(U_1) = \{M(4), M(0)\}, && I(U_2) = \{M(6), 2M(2)\}, \\ &I(U_3) = \{M(8), M(4), 2M(0)\}, && I(U_4) = \{M(8), M(6), 2M(2)\}. \end{aligned}$$

By Lemma 27, $U_1 \cong M(4) \oplus M(0)$, $U_2 \cong T(6)$, $U_3 \cong N \oplus M(4)$ where $I(N) = \{M(8), 2M(0)\}$, and $U_4 \cong M(8) \oplus N_1$ where $I(N_1) = \{M(6), 2M(2)\}$. As the modules $U_3 \oplus U_5$ and U_4 are self-dual, we conclude that N_1 is self-dual and U_5 has a direct summand N' such that $I(N) = I(N')$ and the module $N \oplus N'$ is self-dual. Show that $U_2 \cong T(6)$, $N \cong N' \cong T(8)$, and $N_1 \cong T(6)$. First we prove that the H-modules U_2, N_1, and U_6 contain submodules isomorphic to $V(6)$, and N and N' contain submodules isomorphic to $V(8)$.

Set $m_1 = X_{-2}^2 v$, $m_2 = X_{-2,5} X_{-1}^4 X_{-2} v$, $m_3 = X_{-2}^3 v$, $m_4 = X_{-2,5} X_{-1}^2 v$, and $m = X_{-2}^4 X_{-1}^2 v + X_{-2}^3 X_{-1}^2 X_{-2} v$. It is clear that $m_1 \in U_2$, $m_2 \in U_6$, $m_3 \in U_3$, $m_4 \in U_5$, and $m \in U_4$. One can directly verify that $m \in \operatorname{Inv} U_{4,6}$. Observe that the group $\mathfrak{X}_1$ fixes the vectors m_1 and m_3. Since $\omega(u) + \alpha_1 \notin \Lambda(M)$ for $u = m_2$ or m_4, then the group $\mathfrak{X}_1$ fixes u. Obviously, $m_3 \in N$, $m_4 \in N'$, and $m \in N_1$. By Lemma 28, it suffices to show that $X_{-1}^2 m_1 \neq 0$, $X_{-1}^2 m_2 \neq 0$, $X_{-1}^4 m_3 \neq 0$, $X_{-1}^4 m_4 \neq 0$, and $X_{-1}^2 m \neq 0$. One can check that

$$
\begin{aligned}
&X_2^2 X_{-1}^2 m_1 = 2X_{-1}^2 v \neq 0,\\
&X_2^3 X_1^4 X_2^3 X_{-1}^2 m_2 = 3X_2^3 X_1^4 X_{-1}^2 X_{-2}^2 X_{-1}^4 X_{-2} v =\\
&\quad = X_2^3 X_1^3 (4X_{-1} X_{-2}^2 X_{-1}^4 X_{-2} v + 2X_{-1}^2 X_{-2}^2 X_{-1}^3 X_{-2} v) = 2X_2^3 X_1^2 X_{-1}^2 X_{-2}^2 X_{-1}^2 X_{-2} v =\\
&\quad = 2X_2^3 X_1 (X_{-1} X_{-2}^2 X_{-1}^2 X_{-2} v + X_{-1}^2 X_{-2}^2 X_{-1} X_{-2} v) =\\
&\quad = X_2^3 (3X_{-2}^2 X_{-1}^2 X_{-2} v + 2X_{-1} X_{-2}^2 X_{-1} X_{-2} v + 3X_{-1}^2 X_{-2}^3 v) =\\
&\quad = X_2^2 (2X_{-2} X_{-1}^2 X_{-2} v + 4X_{-2}^2 X_{-1}^2 v + 4X_{-1} X_{-2} X_{-1} X_{-2} v + X_{-1} X_{-2}^2 X_{-1} v +\\
&\quad + 4X_{-1}^2 X_{-2}^2 v) == X_2 (3X_{-2} X_{-1}^2 v + 3X_{-1} X_{-2} X_{-1} v) = 2X_{-1}^2 v \neq 0,\\
&X_2 X_1^2 X_2^2 X_{-1}^4 m_3 = 2X_2 X_1^2 X_{-1}^4 X_{-2} v = 3X_2 X_{-1}^2 X_{-2} v = 4X_{-1}^2 v \neq 0,
\end{aligned}
$$

$$
\begin{aligned}
&X_2^3 X_1^4 X_2^2 X_{-1}^4 m_4 = 2X_2^3 X_1^4 X_{-1}^4 X_{-2,3} X_{-1}^2 v =\\
&\quad = X_2^3 X_1^3 (3X_{-1}^3 X_{-2,3} X_{-1}^2 v + 4X_{-1}^4 X_{-2,3} X_{-1} v) =\\
&\quad = X_2^3 X_1^2 (3X_{-1}^2 X_{-2,3} X_{-1}^2 v + 4X_{-1}^3 X_{-2,3} X_{-1} v + 3X_{-1}^4 X_{-2,3} v) =\\
&\quad = X_2^3 X_1 (3X_{-1} X_{-2,3} X_{-1}^2 v + 4X_{-1}^2 X_{-2,3} X_{-1} v + 3X_{-1}^3 X_{-2,3} v) =\\
&\quad = X_2^3 (2X_{-2,3} X_{-1}^2 v + X_{-1} X_{-2,3} X_{-1} v + 2X_{-1}^2 X_{-2,3} v) = X_{-1}^2 v \neq 0,\\
&X_2^2 X_1^2 X_2^2 X_{-1}^2 m = X_2^2 X_1^2 X_2 (3X_{-1}^2 X_{-2}^2 X_{-1}^2 X_{-2} v + X_{-1}^2 X_{-2}^3 X_{-1}^2 v) =\\
&\quad = X_2^2 X_1^2 (2X_{-1}^2 X_{-2} X_{-1}^2 X_{-2} v + 3X_{-1}^2 X_{-2}^2 X_{-1}^2 v) =\\
&\quad = X_2^2 X_1 (4X_{-1} X_{-2} X_{-1}^2 X_{-2} v + 2X_{-1}^2 X_{-2} X_{-1} X_{-2} v + X_{-1} X_{-2}^2 X_{-1}^2 v +\\
&\quad + X_{-1}^2 X_{-2}^2 X_{-1} v) = X_2^2 (3X_{-2} X_{-1}^2 X_{-2} v + X_{-1} X_{-2} X_{-1} X_{-2} v + 2X_{-2}^2 X_{-1}^2 v +\\
&\quad + 3X_{-1} X_2^2 X_{-1} v) = X_2 (X_{-1}^2 X_{-2} v + X_{-1} X_{-2} X_{-1} v) = 2X_{-1}^2 v \neq 0.
\end{aligned}
$$

Hence the H-modules U_2, N, N_1, N', and U_6 have the required submodules. As the modules $U_2 \oplus U_6$, $N \oplus N'$, and N_1 are self-dual, we conclude that $U_2 \cong U_6 \cong T(6)$, $N \cong N' \cong T(8)$, and $N_1 \cong T(6)$.

I.XVI.3) Let $p = 5$ and $\omega = \omega_1 + 2\omega_2$. Then $i(M) = 5$ and $U_0 \cong M(1)$. One easily observes that $\Omega_1^+ = \{\omega - \alpha_2, \omega - \alpha_1 - \alpha_2\}$ and $\Omega_2^+ = \{\omega - i\alpha_1 - 2\alpha_2 | 0 \leq i \leq 2\}$. The dimensions of the weight subspaces with the weights from Ω_i^+ are determined in Item I.IV. Now we can conclude that $\dim U_{1,3} = 1$, $\dim U_{1,1} = 2$, $\dim U_{2,5} = 1$, $\dim U_{2,3} = 2$, and $\dim U_{2,1} = 3$. This yields that $I(U_1) = \{M(3), M(1)\}$ and $I(U_2) = \{M(5), 2M(3), M(1)\}$. Lemma 27 implies that $U_1 \cong M(3) \oplus M(1)$ and $U_2 \cong N \oplus M(1)$ where $I(N) = \{M(5), 2M(3)\}$. As $U_3 \cong U_2^*$, then $U_3 \cong N' \oplus M(1)$ where $N' \cong N^*$. Show that $N \cong N' \cong T(5)$. First we prove that the H-modules N and N' contain submodules isomorphic to $V(5)$. Set $m_1 = X_{-2}^2 v$ and $m_2 = X_{-2}^3 X_{-1} v$. It is clear that $m_1 \in N$ and $m_2 \in N'$. By Lemma 12, the vectors m_1 and $m_2 \neq 0$ and the group $\mathfrak{X}_1$ fixes these vectors.

By Lemma 28, it suffices to show that $X_{-1}m_1 \neq 0$ and $X_{-1}m_2 \neq 0$. One can directly verify that $X_2^2 X_{-1} m_1 = 4X_{-1}v \neq 0$,

$$X_1^2 X_2^2 X_{-1} m_2 = 2X_1^2 X_{-1} X_{-2} X_{-1} v = 2X_1(X_{-2}X_{-1}v + X_{-1}X_{-2}v) = 3X_{-2}v \neq 0.$$

Hence N and N' have the required submodules. As $N \oplus N'$ is self-dual, Lemma 24 yields that $N \cong N' \cong T(5)$.

I.XVI.4) Let $p = 7$ and $\omega = 4\omega_2$. Then $i(M) = 8$ and $U_0 \cong M(0)$. One easily observes that

$$\Omega_1^+ = \{\omega - i\alpha_1 - \alpha_2 | 0 \leq i \leq 1\};\ \ \Omega_2^+ = \{\omega - i\alpha_1 - 2\alpha_2 | 0 \leq i \leq 2\};$$
$$\Omega_3^+ = \{\omega - i\alpha_1 - 3\alpha_2 | 0 \leq i \leq 3\};\ \Omega_4^+ = \{\omega - i\alpha_1 - 4\alpha_2 | 0 \leq i \leq 4\}.$$

The dimensions of the weight subspaces with the weights from Ω_i^+ are determined in Item I.VIII. Now we can show that $\dim U_{1,2} = 1$, $\dim U_{1,0} = 1$; $\dim U_{2,4} = 1$, $\dim U_{2,2} = 1$, $\dim U_{2,0} = 2$; $\dim U_{3,6} = 1$, $\dim U_{3,4} = 1$, $\dim U_{3,2} = 2$, $\dim U_{3,0} = 2$; $\dim U_{4,8} = 1$, $\dim U_{4,6} = 1$, $\dim U_{4,4} = 2$, $\dim U_{4,2} = 2$, and $\dim U_{4,0} = 2$. This yields that $I(U_1) = \{M(2)\}$, $I(U_2) = \{M(4), M(0)\}$, $I(U_3) = \{M(6), M(2)\}$, and $I(U_4) = \{M(8), 2M(4)\}$. By Lemma 27, $U_1 \cong M(2)$, $U_2 \cong M(4) \oplus M(0)$, and $U_3 \cong M(6) \oplus M(2)$. Show that $U_4 \cong T(8)$. First we prove that the H-module U_4 contains a submodule isomorphic to $V(8)$. Set $m = X_{-2}^4 v$. It is clear that $m \in U_4$. By Lemma 28, it suffices to show that $X_{-1}^2 m \neq 0$. One can directly verify that $X_2 X_{-1}^2 m = 4X_{-1}^2 X_{-2}^3 v \neq 0$ by Lemma 6. As U_4 is self-dual, Lemma 24 imply that $U_4 \cong T(8)$.

I.XVI.5) Let $p = 7$ and $\omega = \omega_1 + 3\omega_2$. Then $i(M) = 7$ and $U_0 \cong M(1)$. One easily observes that

$$\Omega_1^+ = \{\omega - \alpha_2, \omega - \alpha_1 - \alpha_2\};\ \Omega_2^+ = \{\omega - i\alpha_1 - 2\alpha_2 | 0 \leq i \leq 2\};\ \Omega_3^+ = \{\omega - i\alpha_1 - 3\alpha_2 | 0 \leq i \leq 3\}.$$

The dimensions of the weight subspaces with the weights from the sets Ω_i^+ are determined in Item I.XII. Now we can show that $\dim U_{1,3} = 1$, $\dim U_{1,1} = 2$; $\dim U_{2,5} = 1$, $\dim U_{2,3} = 2$, $\dim U_{2,1} = 3$; $\dim U_{3,7} = 1$, $\dim U_{3,5} = 2$, $\dim U_{3,3} = 3$, and $\dim U_{3,1} = 3$. Then

$$I(U_1) = \{M(3), M(1)\},\ I(U_2) = \{M(5), M(3), M(1)\},\ I(U_3) = \{M(7), 2M(5), M(3)\}.$$

By Lemma 27, $U_1 \cong M(3) \oplus M(1)$, $U_2 \cong M(5) \oplus M(3) \oplus M(1)$, and $U_3 \cong N \oplus M(3)$ where $I(N) = \{M(7), 2M(5)\}$. Since $U_4 \cong U_3^*$, then $U_4 \cong N' \oplus M(3)$ where $N' \cong N^*$. Show that $N \cong N' \cong T(7)$. For this we prove that the H-modules N and N' contain submodules isomorphic to $V(7)$. Set $m_1 = X_{-2}^3 v$ and $m_2 = X_{-2}^4 X_{-1} v$. It is clear that $m_1 \in N$ and $m_2 \in N'$. By Lemma 12, m_1 and $m_2 \neq 0$ and the group $\mathfrak{X}_1$ fixes these vectors. By Lemma 28, it suffices to show that $X_{-1}m_i \neq 0$. One can directly verify that

$$X_2 X_{-1} m_1 = 3X_{-1}X_{-2}^2 v \neq 0,$$
$$X_1^2 X_2 X_{-1} m_2 = 4X_1^2 X_{-1} X_{-2}^3 X_{-1} v = X_1(6X_{-2}^3 X_{-1} v + 4X_{-1}X_{-2}^3 v) = 6X_{-2}^3 v \neq 0.$$

As $N \oplus N'$ is self-dual, Lemma 24 yields that $N \cong N' \cong T(7)$.

For an element x with $J(x) = (2, 2)$ all possibilities are considered.

I.XVII. Let $J(x) = (2, 1, 1)$. Then $\sigma_x(a_1\omega_1 + a_2\omega_2) = a_1 + a_2$. Obviously, $M(\lambda)|A_x$ is a tilting module if $\sigma_x(\lambda) < p$. Using Lemma 14 and arguing as in Item I.XVI, we can show that $M|A_x$ is a tilting module for $p = 3$ and $\omega = \omega_1 + 2\omega_2$ (in the assumptions of Lemma 14 set $\lambda_1 = \omega_1$ and $\lambda_2 = 2\omega_2$). Therefore it remains to consider the following cases:

1) $p = 3$, $\omega = 2\omega_1 + \omega_2$;
2) $p = 5$, $\omega = 2\omega_1 + 3\omega_2$.

Recall that x is conjugate to a long root element. Set $H = G(2)$. We can assume that $x \in H$. To determine the block structure of $\varphi(x)$, we consider the restriction $M|H$. Set $\Omega_i = \{\mu \in \Lambda(M) | \mu = \omega - i\alpha_1 - k\alpha_2\}$, $U_i = \langle M_\mu | \mu \in \Omega_i \rangle$, and $\Omega_i^+ = \{\lambda \in \Omega_i | \sigma(\lambda) \geq 0\}$. Define $i(M)$ and $U_{i,j}$ as in Item I.XVI. By Theorem 1 and Lemma 11, the H-modules U_0 and $U_{i(M)}$ are irreducible, $U_{i(M)-j} \cong U_j^*$, and $\omega(U_0) = \omega|H$.

I.XVII.1) Let $p=3$ and $\omega=2\omega_1+\omega_2$. Then $i(M)=6$ and $U_0\cong M(1)$. One easily observes that $\Omega_1^+=\{\omega-\alpha_1,\omega-\alpha_1-\alpha_2\}$, $\Omega_2^+=\{\omega-2\alpha_1,\omega-2\alpha_1-\alpha_2\}$, and $\Omega_3^+=\{\omega-3\alpha_1-\alpha_2,\omega-3\alpha_1-2\alpha_2\}$. The dimensions of the weight subspaces with weights from the sets Ω_i^+ are determined in Item I.XVI.1). Now we can show that $\dim U_{i,2}=1$ and $\dim U_{i,0}=1$ for $i=1$ or 3; $\dim U_{2,3}=1$, and $\dim U_{2,1}=2$. This yields that $I(U_1)=I(U_3)=\{M(2)\}$ and $I(U_2)=\{M(3),2M(1)\}$. Obviously, $U_1\cong U_3\cong M(2)$. Show that $U_2\cong T(3)$. First we prove that the H-modules U_2 and U_4 contain submodules isomorphic to $V(3)$. Set $m_1=X_{-1}^2v$ and $m_2=X_{-1,4}X_{-2}v$. It is clear that $m_1\in U_2$ and $m_2\in U_4$. By Lemma 12, m_1 and $m_2\neq 0$ and the group $\mathfrak{X}_2$ fixes these vectors. By Lemma 28, it suffices to show that $X_{-2}m_i\neq 0$. One can directly verify that

$$X_1^2X_{-2}m_1=X_{-2}v\neq 0,$$
$$X_2^2X_1^2X_{-2}m_2=X_2^2X_1X_{-2}X_{-1,3}X_{-2}v=X_2^2X_{-2}X_{-1}^2X_{-2}v=$$
$$=X_2(X_{-1}^2X_{-2}v+X_{-2}X_{-1}^2v)=X_{-1}^2v\neq 0.$$

As $U_2\oplus U_4$ is self-dual, Lemma 24 implies that $U_2\cong U_4\cong T(3)$.

I.XVII.2) Let $p=5$ and $\omega=2\omega_1+3\omega_2$. Then $i(M)=10$ and $U_0\cong M(3)$. One easily observes that $\Omega_1^+=\{\omega-\alpha_1-i\alpha_2|0\leq i\leq 2\}$;

$$\Omega_2^+=\{\omega-2\alpha_1-i\alpha_2|0\leq i\leq 2\};\quad \Omega_3^+=\{\omega-3\alpha_1-i\alpha_2|1\leq i\leq 3\};$$
$$\Omega_4^+=\{\omega-4\alpha_1-i\alpha_2|1\leq i\leq 3\};\quad \Omega_5^+=\{\omega-5\alpha_1-i\alpha_2|2\leq i\leq 4\}.$$

The dimensions of the weight subspaces with weights from Ω_i^+ are determined in Item I.XV. Now we can conclude that $\dim U_{1,4}=1$, $\dim U_{1,2}=1$, $\dim U_{1,0}=1$, $\dim U_{2,5}=1$, $\dim U_{2,3}=2$, $\dim U_{2,1}=2$, $\dim U_{i,4}=1$, $\dim U_{i,2}=2$, $\dim U_{i,0}=2$ for $i=3$ and 5, $\dim U_{4,5}=1$, $\dim U_{4,3}=2$, and $\dim U_{4,1}=3$. Then

$$I(U_1)=\{M(4)\},\qquad I(U_2)=\{M(5),2M(3)\},$$
$$I(U_3)=I(U_5)=\{M(4),M(2)\},\quad I(U_4)=\{M(5),2M(3),M(1)\}.$$

By Lemma 27, $U_1\cong M(4)$, $U_3\cong U_5\cong M(4)\oplus M(2)$, and $U_4\cong N\oplus M(1)$ where $I(N)=\{M(5),2M(3)\}$. As $U_6\cong U_4^*$, then $U_6\cong N'\oplus M(1)$ where $N'\cong N^*$. Show that $U_2\cong U_8\cong N\cong N'\cong T(5)$. First we prove that each of these modules contains a submodule isomorphic to $V(5)$. Set $m_1=X_{-1}^2v$, $m_2=X_{-1}^4X_{-2}v$, $m_3=X_{-1,6}X_{-2}^2v$, and $m_4=X_{-1,8}X_{-2}^3v$. It is clear that $m_1\in U_2$, $m_2\in N$, $m_3\in N'$, and $m_4\in U_8$. By Lemma 12, $m_i\neq 0$ and the group $\mathfrak{X}_2$ fixes these vectors. By Lemma 28, it suffices to show that $X_{-2}m_i\neq 0$. One can directly verify that $X_1^2X_{-2}m_1=4X_{-2}v\neq 0$, $X_1^4X_{-2}m_2=X_{-2}^2v\neq 0$,

$$X_1^2X_2X_1^4X_{-2}m_3=2X_1^2X_2X_{-2}X_{-1}^2X_{-2}^2v=$$
$$=X_1^2(2X_{-1}^2X_{-2}^2v+3X_{-2}X_{-1}^2X_{-2}v)=2X_{-2}^2v\neq 0,$$
$$X_1^4X_2^2X_1^4X_{-2}m_4=4X_1^4X_2^2X_{-2}X_{-1,4}X_{-2}^3v=X_1^4X_2(4X_{-1,4}X_{-2}^3v+2X_{-2}X_{-1,4}X_{-2}^2v)=$$
$$=X_1^4(3X_{-1,4}X_{-2}^2v+3X_{-2}X_{-1,4}X_{-2}v)=X_1^3(4X_{-1,3}X_{-2}^2v+3X_{-2}X_{-1,3}X_{-2}v)=$$
$$=X_1^2(3X_{-1}^2X_{-2}^2v+3X_{-2}X_{-1}^2X_{-2}v)=3X_1X_{-2}X_{-1}X_{-2}v=2X_{-2}^2v\neq 0.$$

As $U_2\oplus U_8$ and $N\oplus N'$ are self-dual, Lemma 24 implies that $U_2\cong U_8\cong N\cong N'\cong T(5)$.

For the group $C_2(K)$ all possibilities are considered.

II. Let $G=C_3(K)$. It follows from [18, Table 6.32] that the module $V(\omega_3)$ is irreducible. Proposition 3 and the arguments at the beginning of the section imply that for all elements x of order p the restriction $M(\omega)|A_x$ is a tilting module for $\omega=\omega_2$ and $p>3$ and for $\omega=\omega_3$. Now Lemma 14 implies that M_x is a tilting module for $\omega=\omega_1+\omega_2$ or $2\omega_2$ and $p\neq 3,7$, for $\omega=\omega_1+\omega_3$ and $p>3$, and for $\omega=2\omega_3$ and $p\neq 5$.

Obviously, $p>5$ if $|y|=p$. Taking into account the arguments at the beginning of the section, one easily observes that for a regular unipotent element it suffices to consider the case where $p = 7$ and $\omega = \omega_1 + \omega_2$ or $2\omega_2$. Observe that $\sigma_y(\omega) = 5a_1 + 8a_2 + 9a_3$ for $\omega = a_1\omega_1 + a_2\omega_2 + a_3\omega_3$. Using the arguments at the end of Section 3, we can assume that $X_\alpha = X_1 + 2X_2 + 2X_3$ and $X_{-\alpha} = 5X_{-1} + 4X_{-2} + X_{-3}$.

II.I. Let $p = 7$ and $\omega = \omega_1 + \omega_2$. Then $\dim \varphi = 58$ and $\sigma_y(\omega) = 13$. One easily concludes that $\dim M_{13} = 1$ and $\dim M_{11} = 2$. Set $\mu = \omega - \alpha_1 - \alpha_2$,

$$\begin{aligned}
\Lambda_9 &= \{\omega - \alpha_1 - \alpha_2,\ \omega - \alpha_2 - \alpha_3\};\\
\Lambda_7 &= \{\omega - \alpha_1 - \alpha_2 - \alpha_3,\ \omega - 2\alpha_1 - \alpha_2,\ \omega - \alpha_1 - 2\alpha_2,\ \omega - 2\alpha_2 - \alpha_3\};\\
\Lambda_5 &= \{\omega - 2\alpha_1 - \alpha_2 - \alpha_3,\ \omega - \alpha_1 - 2\alpha_2 - \alpha_3,\ \omega - 2\alpha_1 - 2\alpha_2\};\\
\Lambda_3 &= \{\omega - 2\alpha_1 - 2\alpha_2 - \alpha_3,\ \omega - \alpha_1 - 3\alpha_2 - \alpha_3,\ \omega - \alpha_1 - 2\alpha_2 - 2\alpha_3\};\\
\Lambda_1 &= \{\omega - 3\alpha_1 - 2\alpha_2 - \alpha_3,\ \omega - 2\alpha_1 - 3\alpha_2 - \alpha_3,\ \omega - 2\alpha_1 - 2\alpha_2 - 2\alpha_3,\\
&\qquad \omega - \alpha_1 - 3\alpha_2 - 2\alpha_3\}.
\end{aligned}$$

It is not difficult to show that $M_i = \langle M_\lambda | \lambda \in \Lambda_i\rangle$ for $i \in \{1, 3, 5, 7, 9\}$. It follows from [13, Part 2, § 8, Proposition 8.19] that the maximal submodule in $V(\omega)$ is isomorphic to $M(\omega_1)$. We have $\omega_1 = \omega - \alpha_1 - 2\alpha_2 - \alpha_3$. Now using Freudenthal's formula for the weights of $V(\omega)$, we can deduce that $\dim M_\mu = \dim V(\omega)_\mu = 2$ and $\dim M_{\omega_1} = \dim V(\omega)_{\omega_1} - 1 = 3$. One easily observes that $\dim M_\nu = 2$ for $\nu \in \{\omega - \alpha_1 - \alpha_2 - \alpha_3,\ \omega - \alpha_1 - 3\alpha_2 - \alpha_3,\ \omega - \alpha_1 - 3\alpha_2 - 2\alpha_3\}$ since ν lies in the same W-orbit with μ, $\dim M_\delta = 3$ for $\delta = \omega - 2\alpha_1 - 2\alpha_2 - \alpha_3$ or $\omega - 2\alpha_1 - 3\alpha_2 - \alpha_3$ as δ lies in the same W-orbit with ω_1, and $\dim M_\tau = 1$ for other weights $\tau \in \Lambda_9 \cup \Lambda_7 \cup ... \cup \Lambda_1$ since they lie in the same W-orbit with ω. Now it is not difficult to show that $\dim M_9 = 3$, $\dim M_7 = 5$, $\dim M_5 = 5$, $\dim M_3 = 6$, and $\dim M_1 = 7$. This yields that

$$I(M_y) = \{M(13), M(11), M(9), 2M(7), 2M(5), 2M(3), 2M(1)\}.$$

By Lemma 27, $M_y \cong M(13) \oplus N_1 \oplus N_2 \oplus N_3$ where $I(N_1) = \{M(11), 2M(1)\}$, $I(N_2) = \{M(9), 2M(3)\}$ and $I(N_3) = \{2M(7), 2M(5)\}$. Show that $N_1 \cong T(11)$, $N_2 \cong T(9)$, and $N_3 \cong T(7) \oplus M(7)$.

Set $m_{11} = 5X_{-1}v + X_{-2}v$, $m_9 = X_{-1}X_{-2}v + 3X_{-2}X_{-1}v + X_{-3}X_{-2}v$, $u_1 = X_{-3}X_{-2}v$, $u_2 = X^3_{-1}X_{-2}u_1$, and $w = X^2_{-3}X^2_{-2}X_{-1}v$. Lemma 12 implies that the vector $u_1 \neq 0$ and is fixed by the subgroups $\mathfrak{X}_1$ and $\mathfrak{X}_2$. Using Lemma 6 several times, we get that u_2 and $w \neq 0$ (when we consider the vector u_2, Lemma 6 is applied to the vector u_1 in the $G(1, 2)$-module generated by it). One can directly verify that $m_{11} \in \operatorname{Inv} M_{11}$, $m_9 \in \operatorname{Inv} M_9$, and the vectors $X^5_{-\alpha}m_{11}$ and $X^3_{-\alpha}m_9$ have nonzero weight components $4u_1$ and $5w$, respectively. Hence $X^5_{-\alpha}m_{11}$ and $X^3_{-\alpha}m_9 \neq 0$. As N_1 and N_2 are self-dual, then by Lemma 29, $N_1 \cong T(11)$ and $N_2 \cong T(9)$.

Put $U = X_\alpha M_5$,

$$a = 2X_{-2}X_{-1}X_{-2}v + 5X_{-1}X_{-2}X_{-1}v + 2X_{-1}X_{-3}X_{-2}v + 6X_{-3}X_{-2}X_{-1}v + 3X_{-2}X_{-3}X_{-2}v.$$

It is not difficult to check that the vectors $X_{-3}X_{-2}X_{-1}X_{-2}v$, $X_{-2}X_{-3}X_{-2}X_{-1}v$, and $X_{-1}X_{-2}X_{-3}X_{-2}v$ are linearly independent. Now one easily observes that the vectors $X_{-1}X^2_{-2}X_{-1}v$, $X^2_{-1}X_{-3}X_{-2}v$, $X_{-3}X_{-2}X_{-1}X_{-2}v$, $X_{-2}X_{-3}X_{-2}X_{-1}v$, and $X_{-1}X_{-2}X_{-3}X_{-2}v$ form a basis of M_5. Using this basis, we can verify that

$$\begin{aligned}
U = \langle &X_{-2}X_{-1}X_{-2}v + 2X_{-1}X_{-2}X_{-1}v,\\
&X_{-1}X_{-3}X_{-2}v + 2X_{-1}X_{-2}X_{-1}v, X_{-3}X_{-2}X_{-1}v + 2X_{-2}X_{-1}X_{-2}v,\\
&X_{-2}X_{-3}X_{-2}v + 4X_{-3}X_{-2}X_{-1}v + 4X_{-2}X_{-1}X_{-2}v\rangle,
\end{aligned}$$

$U \cap \operatorname{Inv} M_7 = \langle a\rangle$, and the vector $X_{-\alpha}a$ has a nonzero weight component $6X^2_{-1}X_{-3}X_{-2}v$. As $\operatorname{Inv} M_7 \subset N_3$, there exists a vector $m \in M_5 \cap N_3$ such that $X_{-\alpha}X_\alpha m \neq 0$. Since N_3 is self-dual, now Lemmas 24 and 25 imply that $N_3 \cong T(7) \oplus M(7)$.

II.II. Let $p = 7$ and $\omega = 2\omega_2$. Then $\dim \varphi = 89$ and $\sigma_y(\omega) = 16$. One easily concludes that $\dim M_{16} = 1$, $\dim M_{14} = 1$, and $\dim M_{12} = 3$. Set $\mu = \omega - 2\alpha_2 - \alpha_3$, $\nu = \omega - \alpha_1 -$

$2\alpha_2-\alpha_3$,

$$\Lambda_{10}=\{\omega-\alpha_1-2\alpha_2,\ \omega-\alpha_1-\alpha_2-\alpha_3,\ \omega-2\alpha_2-\alpha_3\};$$
$$\Lambda_8=\{\omega-2\alpha_1-\alpha_2,\ \omega-\alpha_1-2\alpha_2-\alpha_3,\ \omega-2\alpha_2-2\alpha_3,\ \omega-3\alpha_2-\alpha_3\};$$
$$\Lambda_6=\{\omega-2\alpha_1-2\alpha_2-\alpha_3,\ \omega-\alpha_1-2\alpha_2-2\alpha_3,\ \omega-\alpha_1-3\alpha_2-\alpha_3,\ \omega-3\alpha_2-2\alpha_3\};$$
$$\Lambda_4=\{\omega-2\alpha_1-2\alpha_2-2\alpha_3,\ \omega-2\alpha_1-3\alpha_2-\alpha_3,\ \omega-\alpha_1-3\alpha_2-2\alpha_3,\ \omega-\alpha_1-4\alpha_2-\alpha_3,\ \omega-4\alpha_2-2\alpha_3\};$$
$$\Lambda_2=\{\omega-3\alpha_1-3\alpha_2-\alpha_3,\ \omega-2\alpha_1-3\alpha_2-2\alpha_3,\ \omega-2\alpha_1-4\alpha_2-\alpha_3,\ \omega-\alpha_1-4\alpha_2-2\alpha_3\};$$
$$\Lambda_0=\{\omega-3\alpha_1-3\alpha_2-2\alpha_3,\ \omega-3\alpha_1-4\alpha_2-\alpha_3,\ \omega-2\alpha_1-4\alpha_2-2\alpha_3,\ \omega-\alpha_1-4\alpha_2-3\alpha_3,\ \omega-\alpha_1-5\alpha_2-2\alpha_3\}.$$

One easily observes that $M_i=\langle M_\lambda|\lambda\in\Lambda_i\rangle$ for $i\in\{0,2,...,10\}$. By [18, Table 6.32], $\dim M(\omega)=\dim V(\omega)-1$. Therefore $M(\omega)\cong V(\omega)/M(0)$. This implies that $\dim M_\lambda=\dim V(\omega)_\lambda$ for $\lambda\neq 0$ and $\dim M_\lambda=\dim V(\omega)_\lambda-1$ for $\lambda=0$. Using Freudenthal's formula, we can show that $\dim M_\mu=2$, $\dim M_\nu=3$, and $\dim M_\lambda=5$ for $\lambda=0$. One easily deduces that $\dim M_\gamma=2$ for $\gamma=\omega-2\alpha_1-2\alpha_2-\alpha_3$ or $\omega-2\alpha_1-4\alpha_2-\alpha_3$ since because γ lies in the same W-orbit with μ, $\dim M_\tau=3$ for $\tau\in\{\omega-\alpha_1-3\alpha_2-\alpha_3,\ \omega-2\alpha_1-3\alpha_2-\alpha_3,\ \omega-\alpha_1-3\alpha_2-2\alpha_3,\ \omega-2\alpha_1-3\alpha_2-2\alpha_3,\ \omega-\alpha_1-4\alpha_2-2\alpha_3\}$ as τ lies in the same W-orbit with ν, all other weights $\delta\in\Lambda_0\cup\Lambda_2\cup...\cup\Lambda_{10}$ lie in the same W-orbit with ω or $\omega-\alpha_2$, hence $\dim M_\delta=1$. Now one easily observes that $\dim M_{10}=4$, $\dim M_8=6$, $\dim M_6=7$, $\dim M_4=9$, $\dim M_2=9$, and $\dim M_0=9$. This implies that

$$I(M_y)=\{M(16),2M(12),2M(10),2M(8),M(6),4M(4),M(2),2M(0)\}.$$

Proposition 8 and Lemma 27 yield that $M_y\cong N_1\oplus N_2\oplus N_3\oplus M(6)$ where $I(N_1)=\{M(16),2M(10),M(2)\}$, $I(N_2)=\{2M(12),2M(0)\}$, and $I(N_3)=\{2M(8),4M(4)\}$. Show that $N_1\cong T(16)$, $N_2\cong T(12)\oplus M(12)$, and $N_3\cong 2T(8)$.

One easily concludes that $X^3_{-\alpha}v$ has a nonzero weight component $5X_{-3}X_{-1}X_{-2}v$. Then by Lemma 28, $KA_yv\cong V(16)$. Set $\overline{N_1}=N_1/KA_yv$. It is clear that $I(\overline{N_1})=\{M(10),M(2)\}$. One can directly verify that the triples

$$(X_{-2}X_{-3}X_{-1}X^2_{-2}X_{-3}X_{-2}v,\ X_{-3}X^2_{-2}X_{-1}X_{-3}X^2_{-2}v,\ X^2_{-2}X_{-1}X^2_{-3}X^2_{-2}v)$$

and $(X_{-3}X_{-2}X^2_{-1}X_{-2}X_{-3}X_{-2}v,\ X^2_{-1}X_{-2}X^2_{-3}X^2_{-2}v,\ X_{-2}X^2_{-1}X^2_{-3}X^2_{-2}v)$, and the pair $(X^2_{-2}X^2_{-1}X_{-3}X^2_{-2}v,\ X_{-2}X^2_{-1}X^2_{-2}X_{-3}X_{-2}v)$ consist of linearly independent vectors. Set $u=X_{-3}X_{-2}v$. By Lemma 12, the vector $u\neq 0$ and is fixed by the subgroups $\mathfrak{X}_1$ and $\mathfrak{X}_2$. Applying Lemma 6 to the $G(1,2)$-submodule generated by u, we can get that the vector $X^3_{-1}X^2_{-2}X_{-3}X_{-2}v\neq 0$. Therefore the vectors

$$X_{-2}X_{-3}X_{-1}X^2_{-2}X_{-3}X_{-2}v, X_{-3}X^2_{-2}X_{-1}X_{-3}X^2_{-2}v, X^2_{-2}X_{-1}X^2_{-3}X^2_{-2}v,$$
$$X_{-3}X_{-2}X^2_{-1}X_{-2}X_{-3}X_{-2}v, X^2_{-1}X_{-2}X^2_{-3}X^2_{-2}v, X_{-2}X^2_{-1}X^2_{-3}X^2_{-2}v,$$
$$X^2_{-2}X^2_{-1}X_{-3}X^2_{-2}v, X_{-2}X^2_{-1}X^2_{-2}X_{-3}X_{-2}v, X^3_{-1}X^2_{-2}X_{-3}X_{-2}v$$

form a basis of M_2. Using this basis, we can directly check that $\operatorname{Inv} M_2=0$. This yields that N_1 has no submodules isomorphic to $M(2)$. As N_1 is self-dual, N_1 and $\overline{N_1}$ have no such factor modules. Now Lemma 23 forces that $\overline{N_1}\cong V(10)$ and N_1 has a filtration by Weyl modules. By Remark 4, N_1 is a tilting module. As we know $I(N_1)$, we conclude that $N_1\cong T(16)$.

Set $m=X^2_{-2}X^2_{-1}X^2_{-3}X^2_{-2}v$. It is clear that $m\in M_0$. One can directly verify that the vector $X^6_{-\alpha}X^6_\alpha m$ has a nonzero weight component $2X_{-3}X^2_{-2}X_{-1}X^2_{-3}X^2_{-2}v$. Obviously, $X^6_\alpha m\in\operatorname{Inv} M_{12}$ since $X^7_\alpha w=0$ for any $w\in M$. The facts proved earlier yield that $M_y=N_2\oplus N'$ where $N'\cap\operatorname{Inv} M_{12}=0$. This implies that there exists a vector $m_0\in N_2\cap M_0$ such that $X^6_{-\alpha}X^6_\alpha m_0=X^6_{-\alpha}X^6_\alpha m\neq 0$. Hence by Lemmas 24 and 25, $N_2\cong T(12)\oplus M(12)$.

Put $t_1 = X_{-2}^2 X_{-3}^2 X_{-2}^2 v$, $t_2 = X_{-2}^2 X_{-3} X_{-1} X_{-2}^2 v$, $f_1 = X_{-\alpha}^2 X_{\alpha}^2 t_1$, and $f_2 = X_{-\alpha}^2 X_{\alpha}^2 t_2$. One can directly check that $X_{\alpha}^2 t_i \in \operatorname{Inv} M_8$, the vector f_1 has nonzero weight components $X_{-1}^2 X_{-3}^2 X_{-2}^2 v$ and $6X_{-2}^2 X_{-3}^2 X_{-2}^2 v$, and f_2 has such components $3X_{-1}^2 X_{-3}^2 X_{-2}^2 v$ and $X_{-2}^2 X_{-3}^2 X_{-2}^2 v$. This forces that f_1 and f_2 are linearly independent. It is clear that $M_y = N_3 \oplus N''$ where $N'' \cap \operatorname{Inv} M_8 = 0$. Therefore there exist vectors l_1 and $l_2 \in N_3 \cap M_4$ such that $X_{-\alpha}^2 X_{\alpha}^2 l_i = f_i$, $i = 1, 2$. Corollary 9 implies that $N_3 \cong 2T(8)$.

For regular unipotent elements of the group $C_3(K)$ all possibilities are considered.

II.III. Let x have a block of size 4 or 2 in the standard realization of G. Set $H = G(2\varepsilon_1, \alpha_2, \alpha_3)$. It is clear that $H \cong C_1(K) \times C_2(K)$. We can assume that $x \in H$.

To determine the block structure of element $\varphi(x)$, we analyze the restriction $M|H$. For $p = 5$ and $\omega = \omega_2 + \omega_3$ or $2\omega_3$, we use Theorems 4 and 3 and Lemma 21. Taking into account the arguments at the beginning of section, it suffices to consider the following cases:

A) $p = 3$, $\omega = \omega_1 + \omega_3$;
B) $p = 7$, $\omega = \omega_1 + \omega_2$;
C) $p = 7$, $2\omega_2$.

Show that in all these cases the module $M|H$ is completely reducible. Set $u_1 = X_{-1}v$ and $u_2 = X_{-1}^2 X_{-2} X_{-3} v$ in Case A), $u_1 = X_{-1}v$ and $u_2 = X_{-1}^2 X_{-2} v$ in Case B), and $u_1 = X_{-1} X_{-2} v$ and $u_2 = X_{-1}^2 X_{-2}^2 v$ in Case C); $\mu_0 = \omega|H$ and $\mu_i = \omega(u_i)|H$ for $i = 1, 2$. In Case A) one easily observes that $X_2 X_1^2 u_2 = 2X_{-3}v \neq 0$ and that $\omega(u_2) + \alpha_2$ and $\omega(u_2) + \alpha_3 \notin \Lambda(M)$. Hence the groups $\mathfrak{X}_2$ and $\mathfrak{X}_3$ fix u_2. In the other cases Lemma 12 implies that $u_i \neq 0$ and the groups $\mathfrak{X}_2$ and $\mathfrak{X}_3$ fix u_i. Obviously, the groups $\mathfrak{X}_{2\varepsilon_1}$ always fix u_i. This yields that the restriction $M|H$ has composition factors with highest weights μ_i, $0 \leq i \leq 2$.

In Case B) set $\lambda = \omega - \alpha_1 - \alpha_2$. Recall that $\dim M_\lambda = 2$. Since $\dim M_{\omega - \alpha_1} = 1$, there exists a nonzero vector $m \in M_\lambda$ such that $X_2 m = 0$. Therefore $\mathfrak{X}_2$ fixes m. This forces that $M|H$ has a composition factor with highest weight $\mu_3 = \lambda|H$.

In Case C) put $\tau = \omega - \alpha_1 - 2\alpha_2 - \alpha_3$, $\tau_1 = \tau + \alpha_2$, $\tau_2 = \tau + \alpha_3$, and $\mu_3 = \tau|H$. Recall that $\dim M_\tau = 3$ and $\dim M_{\tau_i} = 1$ for $i = 1, 2$. This yields that

$$\dim(Ker X_2 \cap M_\tau) = \dim(Ker X_3 \cap M_\tau) = 2.$$

Hence $Ker X_2 \cap Ker X_3 \cap M_\tau \neq \{0\}$. Let $w \in Ker X_2 \cap Ker X_3 \cap M_\tau$ and $w \neq 0$. Obviously, the group $\mathfrak{X}_{2\varepsilon_1}$ fixes w. Therefore $M|H$ has a composition factor $M(\mu_3)$.

One easily observes that in Case A) $\mu_0 = (2\omega_1, \omega_2)$, $\mu_1 = (\omega_1, \omega_1 + \omega_2)$, and $\mu_2 = (0, 2\omega_1)$; in Case B) $\mu_0 = (2\omega_1, \omega_1)$, $\mu_1 = (\omega_1, 2\omega_1)$, $\mu_2 = (0, \omega_1 + \omega_2)$, and $\mu_3 = (\omega_1, \omega_2)$; in Case C) $\mu_0 = (2\omega_1, 2\omega_1)$, $\mu_1 = (\omega_1, \omega_1 + \omega_2)$, $\mu_2 = (0, 2\omega_2)$, and $\mu_3 = (\omega_1, \omega_1)$. As we know the dimension of M, we can conclude that in Cases A) and B) $M|H$ has only three composition factors $M(\mu_i)$. In Case C) set $\mu_4 = (0, \omega_2)$. Since we know the weight multiplicities of M, we can show that the formal character of the restriction $M|H$ is equal to the sum of the formal characters of the modules $M(\mu_i)$ for $0 \leq i \leq 4$. Therefore in Case C) the restriction $M|H$ has five composition factors $M(\mu_i)$ with $0 \leq i \leq 4$. Observe that in all Cases A), B), and C) the modules $V(\mu_i)$ are irreducible. Now Corollary 1 yields that $M|H$ is completely reducible. To determine the block structure of $\varphi(x)$, we use the results of Item I and Theorem 3.

II.IV. It remains to consider the case where $J(x) = (3, 3)$. Set $H = G(1, 2)$. Then $H \cong A_2(K)$ and we can assume that x is a regular unipotent element from H. In this case $\sigma_x(a_1\omega_1 + a_2\omega_2 + a_3\omega_3) = 2a_1 + 4a_2 + 4a_3$. It is clear that $\sigma_x(\omega) < p$ for $p = 7$, $\omega = \omega_1 + \omega_2$. For $p = 3$ and $\omega = \omega_2$ we use Proposition 9. The arguments at the beginning of the section yields that it suffices to consider the following cases:

1) $p = 5$, $\omega = \omega_2 + \omega_3$ or $2\omega_3$;
2) $p = 3$, $\omega = \omega_1 + \omega_3$;
3) $p = 7$, $\omega = 2\omega_2$.

In all these cases we analyze the restriction $M|H$. Set $\Omega_i = \{\lambda \in \Lambda(M) | \lambda = \omega - x\alpha_1 -$

$y\alpha_2 - i\alpha_3\}$ and $U_i = \langle M_\lambda | \lambda \in \Omega_i \rangle$. Define the parameter $i(M)$ as above. By Theorem 1 and Lemma 11, $U_j^* \cong U_{i(M)-j}$ and $U_0 \cong M(\omega|H)$.

II.IV.1) Assume that one of the following holds:
A) $p = 5$, $\omega = \omega_2 + \omega_3$;
B) $p = 5$, $\omega = 2\omega_3$.
Cases A) and B) are considered together since the approach is the same. Show that the H-modules U_i are irreducible and indicate their highest weights. One easily concludes that $i(M) = 5$ in Case A) and 6 in Case B). Set $\mu_0 = \omega$; $\mu_1 = \omega - \alpha_3$, and $\mu_2 = \omega - \alpha_2 - 2\alpha_3$ in Case A); $\mu_1 = \omega - \alpha_3$, $\mu_2 = \omega - 2\alpha_3$, and $\mu_3 = \omega - 2\alpha_2 - 3\alpha_3$ in Case B); in both cases let $\delta_i = \mu_i|H$ and $N_i = M(\delta)$. One easily observes that in Case B) the weight $\omega - x\alpha_1 - y\alpha_2 - 3\alpha_3 \notin \Lambda(H)$ for $y < 2$ (consider the W-orbit of such weight). Now it is clear that δ_i is a maximal weight in the H-module U_i. Show that $U_i \cong N_i$. It is clear that U_i has such composition factor. The Weyl group of H can be naturally identified with a subgroup $W_H \subset W$. Any weight $\lambda \in \Omega_i$ lies in the same W_H-orbit with a weight $\tau \in \Omega_i$ such that $\tau|H \in \Lambda^+(H)$. It is clear that $\tau \leq \mu_i$. By [4, Chapter VIII, § 7, Proposition 5.iv], $\tau|H$ is a weight of the irreducible $A_2(\mathbb{C})$-module with highest weight δ_i. By [26], the set of weights of such module coincides with $\Lambda(N_i)$. Hence $\tau|H$ and $\lambda|H \in \Lambda(N_i)$. By Theorem 4, all weight subspaces in M are one-dimensional. This yields that the H-module U_i has one composition factor. Therefore $U_i \cong N_i$. One easily checks that in Case A) $\delta_0 = \omega_2$, $\delta_1 = 3\omega_2$, and $\delta_2 = \omega_1 + 3\omega_2$; in Case B) $\delta_0 = 0$, $\delta_1 = 2\omega_2$, $\delta_2 = 4\omega_2$, and $\delta_3 = 2\omega_1 + 2\omega_2$. Now we can use the results of Item I of Section 4 to determine the block structure of $\varphi(x)$.

II.IV.2) Here and in Item II.IV.3) $A \subset H$ is a good A_1-subgroup containing x, $\sigma : \Lambda(H) \to \mathbb{Z}$ is the homomorphism determined by the restriction of weights from a maximal torus $T_H \subset H$ to a maximal torus $T_A \subset A$ such that $\sigma(\alpha_i) = 2$ for $i = 1, 2$; X_α and $X_{-\alpha}$ are the root elements of the Lie algebra of A; $\Omega_i^+ = \{\lambda \in \Omega_i | \sigma(\lambda) \geq 0\}$, and $U_{i,j}$ is the weight subspace of weight j in the A-module U_i. Using the formulae from [4, Chapter 8, §13.1], we can assume that $X_\alpha = X_1 + 2X_2$ and $X_{-\alpha} = 2X_{-1} + X_{-2}$.

Let $p = 3$ and $\omega = \omega_1 + \omega_3$. Then $\dim \varphi = 57$, $i(M) = 4$, and $U_0 \cong M(\omega_1)$. One easily observes that

$$\begin{aligned}\Omega_1 = \{&\omega - \alpha_3,\ \omega - \alpha_2 - \alpha_3,\ \omega - \alpha_1 - \alpha_2,\ \omega - 2\alpha_2 - \alpha_3,\\ &\omega - \alpha_1 - \alpha_2 - \alpha_3,\ \omega - \alpha_1 - 2\alpha_2 - \alpha_3, \omega - 2\alpha_1 - \alpha_2 - \alpha_3,\ \omega - 2\alpha_1 - 2\alpha_2 - \alpha_3,\\ &\omega - \alpha_1 - 3\alpha_2 - \alpha_3,\ \omega - 2\alpha_1 - 3\alpha_2 - \alpha_3, \omega - 3\alpha_1 - 3\alpha_2 - \alpha_3\};\end{aligned}$$

$$\begin{aligned}\Omega_2^+ = \{&\omega - 2\alpha_2 - 2\alpha_3,\ \omega - \alpha_1 - \alpha_2 - 2\alpha_3,\ \omega - \alpha_1 - 2\alpha_2 - 2\alpha_3,\ \omega - \alpha_1 - 3\alpha_2 - 2\alpha_3,\\ \omega - &\alpha_1 - 4\alpha_2 - 2\alpha_3,\ \omega - 2\alpha_1 - 2\alpha_2 - 2\alpha_3,\ \omega - 2\alpha_1 - 3\alpha_2 - 2\alpha_3,\ \omega - 3\alpha_1 - 2\alpha_2 - 2\alpha_3\}.\end{aligned}$$

Set $\eta = \omega - \alpha_1 - \alpha_2 - \alpha_3$ and $\nu = \omega - 2\alpha_1 - 3\alpha_2 - 2\alpha_3$. Then $\eta = \omega_2$ and $\nu = 0$. We can verify that $\dim M_\eta = 2$. Since $\dim V(\omega)_\eta = 3$ by Freudenthal's formula, this yields that $V(\omega)$ has a composition factor $M(\eta)$. Taking into account the dimensions of the modules $M(\omega)$ and $M(\eta)$, we get that $M(\omega) = V(\omega)/M(\eta)$. Using the same Freudenthal's formula and taking into account that all weight subspaces of $M(\eta)$ are one-dimensional, we get that $\dim M_\nu = 3$. One easily concludes that $\dim M_\mu = 2$ for

$$\begin{aligned}\mu \in \{&\omega - \alpha_1 - \alpha_2 - \alpha_3, \omega - \alpha_1 - 2\alpha_2 - \alpha_3, \omega - 2\alpha_1 - 2\alpha_2 - \alpha_3,\\ &\omega - \alpha_1 - 2\alpha_2 - 2\alpha_3, \omega - 2\alpha_1 - 2\alpha_2 - 2\alpha_3, \omega - \alpha_1 - 3\alpha_2 - 2\alpha_3\}\end{aligned}$$

since μ lies in the same W-orbit with η, and that $\dim M_\tau = 1$ for other nonzero weights $\tau \in \Omega_1 \cup \Omega_2^+$ as τ lies in the same W-orbit with ω or $\omega - \alpha_2 - \alpha_3$. This implies that $\dim U_1 = 15$, $\dim U_{2,6} = 2$, $\dim U_{2,4} = 2$, $\dim U_{2,2} = 4$, and $\dim U_{2,0} = 5$. Set $\delta = (\omega - \alpha_3)|H = \omega_1 + 2\omega_2$. Obviously, the H-module U_1 has a composition factor $M(\delta)$. Since $\dim U_1 = \dim M(\delta)$, then $U_1 \cong M(\delta)$. The canonical Jordan form of x in U_1 can be determined with the use of the results of Item I of Section 4.

By the arguments above, we get that $I(U_2|A) = \{2M(6), 2M(4), 2M(2), M(0)\}$. Then by Proposition 8, the restriction $U_2|A \cong N \oplus 2M(2)$ where $I(N) = \{2M(6), 2M(4), M(0)\}$.

Show that $N \cong T(6) \oplus M(6)$. Set $u_1 = X_{-3}^2 X_{-2} X_{-1} v$, $u_2 = X_{-3} X_{-2}^2 X_{-3} v$, $m_6^1 = u_1 + u_2$, $m_6^2 = u_1 + 2u_2$, and $m_4 = X_{-3} X_{-2}^2 X_{-3} X_{-1} v$. By Lemma 12, $u_1 \neq 0$ and the subgroups $\mathfrak{X}_i$ fix u_1 for $i = 1, 2$. Hence $m_6^i \neq 0$. One easily observes that $\mathfrak{X}_1$ and $\mathfrak{X}_2$ fix u_2 as $\omega(u_2) + \alpha_i \notin \Lambda(M)$ for $i = 1, 2$. This yields that $m_6^i \in \operatorname{Inv} U_{2,6}$. We can directly verify that $X_\alpha m_4 = m_6^2$, $X_{-\alpha} m_6^1 = 0$, $X_{-\alpha} m_6^2 \neq 0$, and $X_{-\alpha}^2 m_4 \notin \langle X_{-\alpha,3} m_6^2 \rangle$. Then $KAm_6^2 \cong V(6)$. One easily deduces that N/KAm_6^2 has a submodule isomorphic to $V(4)$, this submodule is generated by the image of m_4 under the canonical homomorphism. Let F be the full preimage of this submodule in N. Show that F is indecomposable. We can directly check that $\dim \operatorname{Inv} U_{2,4} = 1$. Since $X_{-\alpha} m_6^2 \in \operatorname{Inv} U_{2,4}$, then F has no indecomposable components with highest weight 4. As $F \cap U_{2,0} = \langle X_{-\alpha,3} m_6^1, X_{-\alpha}^2 m_4 \rangle$, F has no such components with highest weight 0. Since $\dim(F \cap U_{2,6}) = 1$, this yields that F is indecomposable.

It is clear that $KAm_6^1 \cong M(6)$. As $m_6^1 \notin F \cap U_{2,6} = \langle m_6^2 \rangle$ and KAm_6^1 is irreducible, we conclude that $KAm_6^1 \cap F = 0$. Hence $N = F \oplus KAm_6^1$. The module F is self-dual since N is such. As F has a filtration by Weyl modules, then by Remark 4, F is a tilting module. We know $I(F)$ and can prove that $F \cong T(6)$.

II.IV.3) Let $p = 7$ and $\omega = 2\omega_2$. Then $i(M) = 4$ and $U_0 \cong M(2\omega_2)$. One easily observes that

$$\begin{aligned}\Omega_1^+ =&\{\omega - \alpha_2 - \alpha_3, \omega - \alpha_1 - \alpha_2 - \alpha_3, \omega - 2\alpha_2 - \alpha_3, \omega - 3\alpha_2 - \alpha_3,\\ &\omega - \alpha_1 - 2\alpha_2 - \alpha_3, \omega - 2\alpha_1 - 2\alpha_2 - \alpha_3, \omega - \alpha_1 - 3\alpha_2 - \alpha_3\};\end{aligned}$$

$$\begin{aligned}\Omega_2^+ =&\{\omega - 2\alpha_2 - 2\alpha_3, \omega - \alpha_1 - 2\alpha_2 - 2\alpha_3, \omega - 3\alpha_2 - 2\alpha_3, \omega - 4\alpha_2 - 2\alpha_3,\\ &\omega - 2\alpha_2 - 2\alpha_2 - 2\alpha_3, \omega - \alpha_1 - 3\alpha_2 - 2\alpha_3, \omega - 2\alpha_1 - 3\alpha_2 - 2\alpha_3,\\ &\omega - \alpha_1 - 4\alpha_2 - 2\alpha_3, \omega - 3\alpha_1 - 3\alpha_2 - 2\alpha_3, \omega - \alpha_1 - 5\alpha_2 - 2\alpha_3, \omega - 2\alpha_1 - 4\alpha_2 - 2\alpha_3\}.\end{aligned}$$

The dimensions of the weight subspaces with weights from Ω_1^+ and Ω_2^+ are determined in Item II.II. Now one can show that $\dim U_{1,6} = 1$, $\dim U_{1,4} = 3$, $\dim U_{1,2} = 4$, $\dim U_{1,0} = 5$, $\dim U_{2,8} = 1$, $\dim U_{2,6} = 2$, $\dim U_{2,4} = 5$, $\dim U_{2,2} = 6$, and $\dim U_{2,0} = 7$. Then $I(U_1|A) = \{M(6), 2M(4), M(2), M(0)\}$ and $I(U_2|A) = \{M(8), M(6), 4M(4), M(2), M(0)\}$. By Lemma 27, $U_1|A \cong M(6) \oplus 2M(4) \oplus M(2) \oplus M(0)$ and $U_2|A \cong N \oplus M(6) \oplus M(2) \oplus M(0)$ where $I(N) = \{M(8), 4M(4)\}$. Show that $N \cong T(8) \oplus 2M(4)$.

Set $m = X_{-3}^2 X_{-2}^2 v$. Since $X_{-1}^2 m \neq 0$, then $X_{-\alpha}^2 m \neq 0$. The module N is self-dual as U_2 is self-dual. Then by Lemma 29, $N \cong T(8) \oplus 2M(4)$.

For $n = 3$ all possibilities are considered.

III. Let $n \geq 4$. As the multiplicities of all blocks of odd sizes in the canonical Jordan form of x in the standard realization are even, then for $n = 4$ all unipotent elements distinct from regular are conjugate to elements from subsystem subgroups of types $C_3 \times C_1$ or $C_2 \times C_2$. Therefore we can use Theorems 4 and 3, the results of Items I, II, and Lemma 21 to solve the problem for the elements of order 3 for $n = 4$, $p = 3$, and $\omega = \omega_3$ or ω_4. It follows from [18, Table 6.33] that the module $V(\omega_4)$ is irreducible for $n = 4$ and $p > 3$. Taking into account the arguments at the beginning of the section, it suffices to consider the cases where $\omega = \omega_2$, $n = 5$, 6, or 7, and $p = 5$, 3, or 7, respectively. In these cases each element of order p is conjugate to an element from a subsystem subgroup H and the restriction $M|H$ is completely reducible. Then to determine the block structure of $\varphi(x)$, one can apply the results of Section 4 and the results of this section for the groups of smaller ranks. If all block sizes in the canonical Jordan form of x in the standard realization are odd, we can set $H = G(1, ..., n-1)$ and use Proposition 9. Assume that x has a block even size in the standard realization. Below $\mu_0 = \omega|H$. Obviously, in all cases $M|H$ has a composition factor $M(\mu_0)$.

a) Let $G = C_5(K)$ and $p = 5$. Since $|x| = p$, then x has a block of size 4, or two blocks of size 2, or a block of size 2 and two blocks of size 1 in the standard realization of G. Set $H = G(\alpha_1, 2\varepsilon_2, \alpha_3, \alpha_4, \alpha_5)$. It is clear that $H \cong C_2(K) \times C_3(K)$. We can assume that $x \in H$. Obviously, $\mu_1 = (\omega_2, 0)$. Let $v_1 = X_{-2} v$, $v_2 = X_{-2} X_{-3} X_{-1} X_{-2} v$, and $\mu_i = \omega(v_i)|H$. Then $\mu_1 = (\omega_1, \omega_1)$ and $\mu_2 = (0, \omega_2)$. By Lemma 6, $v_1 \neq 0$. One easily observes that $X_1X_3X_2v_2 = X_{-2}v \neq 0$ and that $\omega(v_2)+\alpha_1$ and $\omega(v_2)+\alpha_3 \notin \Lambda(M)$. Now it is clear that the vectors v_1 and v_2 are fixed by all subgroups $\mathfrak{X}_\beta \in H$ with positive roots β. Hence $M|H$ has composition factors $M(\mu_1)$ and $M(\mu_2)$.

b) Let $G = C_6(K)$ and $p = 3$. As $|x| = 3$, then x has a block of size 2 in the standard realization of G. Set $H = G(\alpha_1, ..., \alpha_4, 2\varepsilon_5, \alpha_6)$. It is clear that $H \cong C_5(K) \times C_1(K)$ and $\mu_0 = (\omega_2, 0)$. We can assume that $x \in H$. Set $v_1 = X_{-5}X_{-4}X_{-3}X_{-2}v$ and $\mu_1 = \omega(v_1)|H$. Then $\mu_1 = (\omega_1, \omega_1)$. By Lemma 12, $v_1 \neq 0$ and the groups $\mathfrak{X}_i$ fix v_1 for $i \neq 5$. Obviously, $\mathfrak{X}_\beta$ fixes v_1 for all positive roots β of the subgroup H. Therefore $M|H$ has a composition factor $M(\mu_1)$.

c) Let $G = C_7(K)$ and $p = 7$. Recall that x has Jordan blocks of size at most 7 in the standard realization of G.

(i) Let x has a block of size 6 or 3 in the standard realization. Set $H = G(\alpha_1, \alpha_2, 2\varepsilon_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7)$. One easily observes that $H \cong C_3(K) \times C_4(K)$ and $\mu_0 = (\omega_2, 0)$. As x has an even number of blocks of size 3 in the standard realization, we can assume that $x \in H$.

Set $v_1 = X_{-3}X_{-2}v$, $v_2 = X_{-3}X_{-2}X_{-4}X_{-3}X_{-1}X_{-2}v$, and $\mu_i = \omega(v_i)|H$. Then $\mu_1 = (\omega_1, \omega_1)$ and $\mu_2 = (0, \omega_2)$. By Lemma 12, the vector $v_1 \neq 0$ and is fixed by the subgroups $\mathfrak{X}_i$ for $i \neq 3$. One can directly verify that $X_1X_2X_3v_2 = X_{-4}X_{-3}X_{-2}v \neq 0$. It is not difficult to check that $\omega(v_2) + \alpha_i \notin \Lambda(M)$ for $i \neq 3$. It is clear that $\mathfrak{X}_{2\varepsilon_3}$ fixes the vectors v_1 and v_2. This yields that $M|H$ has composition factors $M(\mu_1)$ and $M(\mu_2)$.

(ii) Assume that x has no blocks of sizes 6 and 3 in the standard realization of G. Set $H = G(\alpha_1, \alpha_2, \alpha_3, \alpha_4, 2\varepsilon_5, \alpha_6, \alpha_7)$. It is not difficult to conclude that $H \cong C_5(K) \times C_2(K)$ and $\mu_0 = (\omega_2, 0)$. Arguing as in Item a), we can assume that $x \in H$. Set $v_1 = X_{-5}X_{-4}X_{-3}X_{-2}v$, $v_2 = X_{-5}X_{-4}X_{-3}X_{-2}X_{-1}X_{-6}X_{-5}X_{-4}X_{-3}X_{-2}v$, and $\mu_i = \omega(v_i)|H$. Then $\mu_1 = (\omega_1, \omega_1)$ and $\mu_2 = (0, \omega_2)$. Lemma 12 implies that the vector $v_1 \neq 0$ and is fixes by the subgroups $\mathfrak{X}_i$ for $i \neq 5$. Let $m = v(6, 2)$. Lemma 12 yields that the vector $m \neq 0$ and generates an indecomposable $G(1, ..., 5)$-module with highest weight ω_1. The vector v_2 coincides with the vector $m(5, 1)$ constructed in this module. By Lemma 12, $v_2 \neq 0$ and is fixed by $\mathfrak{X}_i$ for $i < 5$. It is not difficult to show that $\omega(v_2) + \alpha_6$ lies in the same W-orbit with $\omega - \alpha_1$ and therefore does not belong to $\Lambda(M)$. Then it is clear that $\mathfrak{X}_\beta$ fixes v_1 and v_2 for all positive roots β of H. This forces that $M|H$ has composition factors $M(\mu_1)$ and $M(\mu_2)$.

Taking into account the dimension of M, we can conclude that in all Cases a), b), and c) the composition factors of the restriction $M|H$ are exhausted by the modules $M(\mu_i)$. Observe that for all weights μ_i the modules $V(\mu_i)$ are irreducible. Hence by Corollary 1, the restriction $M|H$ is completely reducible.

For the groups of type C_n the problem is solved.

6. Spinor groups

In this section $G = B_n(K)$ or $D_n(K)$.

We use Theorem 6, Proposition 10, and results of Section 4 to solve the problem for $\omega = \omega_i$ where $i < n$ for $G = B_n(K)$ and $i < n-1$ for $G = D_n(K)$, and for $\omega = 2\omega_1$. So these weights will not be considered afterwards.

To handle certain representations, we need to describe representatives of the unipotent conjugacy classes in $D_n(K)$ that have only Jordan blocks of even sizes in the standard realization of the group. Let $G = D_n(K)$, $k_1 \geq k_2 \geq ... \geq k_s$, $k_1 + k_2 + ... + k_s = n$, and all integers k_1, k_2, ..., k_s are even. Obviously, n is even. Set

$$\begin{aligned} I &= \{k_1, k_1+k_2, ..., k_1+k_2+...+k_{s-1}, n\}, \\ I_1 &= \{1, ..., n\} \setminus I, I_2 = I_1 \setminus \{n-1\}, \\ z_1 &= z_1(k_1, ..., k_s) = \prod_{i \in I_1} x_i(1) \in G, \end{aligned}$$

$$z_2 = z_2(k_1, ..., k_s) = x_n(1) \prod_{i \in I_2} x_i(1) \in G;$$

$H_1 = G(1, 2, ..., n-1)$, and $H_2 = G(1, ..., n-2, n)$. Then $H_1 \cong H_2 \cong A_{n-1}(K)$, $z_1 \in H_1$, and $z_2 \in H_2$. In [27, Subsection 2.1] it is proved that $J(z_1) = J(z_2) = (k_1^2, ..., k_s^2)$ and that the elements z_1 and z_2 are not conjugate. Let φ_1, $\varphi_2 \in \operatorname{Irr} G$, $\omega(\varphi_1) = a_1\omega_1 + ... + a_{n-2}\omega_{n-2} + b\omega_{n-1}$, and $\omega(\varphi_2) = a_1\omega_1 + ... + a_{n-2}\omega_{n-2} + b\omega_n$. Then the representation φ_2 can be obtained from φ_1 with the help of the graph morphism of G permuting the elements z_1 and z_2. Hence the canonical Jordan form of $\varphi_2(z_j)$ is the same as of $\varphi_1(z_i)$ for $\{i, j\} = \{1, 2\}$. We use the notation $N(z_i)$ introduced in Lemma 18.

Obviously, the collection $N(z_i)$ contains no zeros and at least two ones. Therefore Formula (1) implies that $\sigma_{z_i}(\varepsilon_n) = \pm 1$. It is clear that $\sigma_x(\omega_n) > \sigma_x(\omega_{n-1})$ if $\sigma_x(\varepsilon_n) = 1$, and $\sigma_x(\omega_n) < \sigma_x(\omega_{n-1})$ if $\sigma_x(\varepsilon_n) = -1$. It follows from [27, Lemma 2.5 and Corollary 2.13] that $\sigma_{z_1}(\omega_n) > \sigma_{z_1}(\omega_{n-1})$ and $\sigma_{z_2}(\omega_n) < \sigma_{z_2}(\omega_{n-1})$ for $n \equiv 0 \pmod 4$ and that $\sigma_{z_1}(\omega_n) < \sigma_{z_1}(\omega_{n-1})$ and $\sigma_{z_2}(\omega_n) > \sigma_{z_2}(\omega_{n-1})$ for $n \equiv 2 \pmod 4$. This yields that $\sigma_{z_1}(\varepsilon_n) = 1$ if $n \equiv 0 \pmod 4$, and -1 if $n \equiv 2 \pmod 4$, and that $\sigma_{z_2}(\varepsilon_n) = -\sigma_{z_1}(\varepsilon_n)$ in both cases. Hence the values $\sigma_x(\varepsilon_i)$ can be determined for all elements x of order p. It is well known that the graph morphism of the group $D_n(K)$ mentioned above fixes the conjugacy classes of unipotent elements that have at least one block of odd size in the standard realization of the group. Therefore for $G = D_n(K)$, it suffices to consider only one of the representations $\varphi(\omega_n)$ or $\varphi(\omega_{n-1})$.

I. Spinor and semispinor representations

In this item $\omega = \omega_n$ for $G = B_n(K)$ and $G = D_n(K)$. Recall that $\Lambda(M)$ consists of all linear combinations of the form $\{\frac{\pm\varepsilon_1 \pm ... \pm \varepsilon_n}{2}\}$ for $G = B_n(K)$ and of all such combinations with an even number of the symbols "minus" for $G = D_n(K)$; all weight subspaces of M are one-dimensional.

One can directly verify that $\sigma_x(\omega) < p$ for all elements x of order p, except the following cases:

a)	$G = B_4(K)$,	$p = 3$,	$J(x) = (3^3)$;
b)	$G = D_5(K)$,	$p = 7$,	$J(x) = (7, 3)$;
		$p = 5$,	$J(x) = (5^2)$;
		$p = 3$,	$J(x) \in \{(3^3, 1), (3^2, 2^2)\}$;
c)	$G = B_5(K)$,	$p = 11, 13$,	$J(x) = (11)$;
		$p = 7$,	$J(x) \in \{(7, 3, 1), (7, 2^2)\}$;
		$p = 5$,	$J(x) \in \{(5^2, 1), (5, 3^2), (4^2, 3)\}$;
		$p = 3$,	$J(x) \in \{(3^3, 1^2), (3^2, 2^2, 1), (3, 2^4)\}$;
d)	$G = D_6(K)$,	$p = 11, 13$,	$J(x) = (11, 1)$;
		$p = 11$,	$J(x) = (9, 3)$;
		$p = 7$,	$J(x) \in \{(7, 5), (7, 3, 1^2), (7, 2^2, 1), (6^2)\}$;
		$p = 5$,	$J(x) \in \{(5^2, 1^2), (5, 3^2, 1), (5, 3, 2^2),$ $(4^2, 3, 1), (4^2, 2^2)\}$;
		$p = 3$,	$J(x) \in \{(3^4), (3^3, 1^3), (3^2, 2^2, 1^2), (3, 2^4, 1), (2^6)\}$;
e)	$G = B_6(K)$,	$p = 13, 17, 19$,	$J(x) = (13)$;
		$p = 11, 13$,	$J(x) = (11, 1^2)$;
		$p = 11$,	$J(x) \in \{(9, 3, 1), (9, 2^2)\}$;
		$p = 7$,	$J(x) \in \{(7, 5, 1), (7, 3^2), (7, 3, 1^3), (7, 2^2, 1^2),$ $(6^2, 1)\}$;
		$p = 5, 7$,	$J(x) \in \{(5^2, 3), (5, 4^2)\}$;
		$p = 5$,	$J(x) \in \{(5, 3^2, 1^2), (5^2, 1^2), (5, 3, 2^2, 1), (5, 2^4),$ $(4^2, 3, 1^2), (4^2, 2^2, 1)\}$;
		$p = 3$,	$J(x) \in \{(3^4, 1), (3^3, 2^2), (3^3, 1^4), (3^2, 2^2, 1^3),$ $(3, 2^4, 1^2), (2^6, 1)\}$;
f)	$G = D_7(K)$,	$p = 13, 17, 19$,	$J(x) = (13, 1)$;

$p = 11, 13$, $J(x) \in \{(11, 3), (11, 1^3), (9, 5)\}$;
$p = 11$, $J(x) \in \{(9, 3, 1^2), (9, 2^2, 1)\}$;
$p = 7, 11$, $J(x) = (7^2)$;
$p = 7$, $J(x) \in \{(7, 5, 1^2), (7, 3^2, 1), (7, 3, 2^2), (7, 3, 1^4), (7, 2^2, 1^3), (6^2, 1^2)\}$;
$p = 5, 7$, $J(x) \in \{(5^2, 3, 1), (5^2, 2^2), (5, 4^2, 1)\}$;
$p = 5$, $J(x) \in \{(5^2, 1^4), (5, 3^3), (5, 3^2, 1^3), (5, 3, 2^2, 1^2), (5, 2^4, 1), (4^2, 3^2), (4^2, 3, 1^3), (4^2, 2^2, 1^2)\}$;
$p = 3$, $J(x) \in \{(3^4, 1^2), (3^3, 2^2, 1), (3^3, 1^5), (3^2, 2^4), (3^2, 2^2, 1^4), (3, 2^4, 1^3), (2^6, 1^2)\}$.

First we consider the behaviour of a regular unipotent element y for $G = B_n(K)$.

I.I. Let $G = B_5(K)$. Then $\dim \varphi = 32$ and $\sigma_y(\omega) = 15$. Taking into account the weight structure of M, it is not difficult to show that $\dim M_{15} = 1$, $\dim M_{13} = 1$, $\dim M_{11} = 1$, $\dim M_9 = 2$, $\dim M_7 = 2$, $\dim M_5 = 3$, $\dim M_3 = 3$, and $\dim M_1 = 3$.

Assume that $p = 11$ or 13. Then $I(M_y) = \{M(15), M(9), 2M(5)\}$ for $p = 11$ and $I(M_y) = \{M(15), 2M(9), M(5)\}$ for $p = 13$. Lemma 27 implies that $M_y = N \oplus M(9)$ for $p = 11$ and $M_y = N' \oplus M(5)$ for $p = 13$ where $I(N) = \{M(15), 2M(5)\}$ and $I(N') = \{M(15), 2M(9)\}$. Prove that N and $N' \cong T(15)$. Since $X_{-1}X_{-2}X_{-3}X_{-4}X_{-5}v \neq 0$, then $X^5_{-\alpha}v \neq 0$ and hence $X^3_{-\alpha}v \neq 0$. As N and N' are self-dual, Lemma 29 yields that N and $N' \cong T(15)$.

I.II. Let $G = B_6(K)$. Then $\dim \varphi = 64$ and $\sigma_y(\omega) = 21$. As we know the weight structure of M, we can conclude that $\dim M_{21} = 1$, $\dim M_{19} = 1$, $\dim M_{17} = 1$, $\dim M_{15} = 2$, $\dim M_{13} = 2$, $\dim M_{11} = 3$, $\dim M_9 = 4$, $\dim M_7 = 4$, $\dim M_5 = 4$, $\dim M_3 = 5$, and $\dim M_1 = 5$.

It is convenient to consider together the cases where $p = 17$ or 19. One easily observes that $I(M_y) = \{M(21), M(15), 2M(11), M(9), M(3)\}$ for $p = 17$ and $I(M_y) = \{M(21), 2M(15), M(11), M(9), M(3)\}$ for $p = 19$. Lemma 27 forces that $M_y = N \oplus M(15) \oplus M(9) \oplus M(3)$ for $p = 17$ and $M_y = N' \oplus M(11) \oplus M(9) \oplus M(3)$ for $p = 19$ where $I(N) = \{M(21), 2M(11)\}$ and $I(N') = \{M(21), 2M(15)\}$. Show that N and $N' \cong T(21)$. Since $X_{-2}X_{-3}X_{-4}X_{-5}X_{-6}v \neq 0$, then $X^5_{-\alpha}v \neq 0$ and $X^3_{-\alpha}v \neq 0$. As N and N' are self-dual, our assertion follows from Lemma 29.

Now let $p = 13$. Then $I(M_y) = \{M(21), M(15), M(11), 2M(9), 2M(3)\}$. By Lemma 27, $M_y = N_1 \oplus N_2 \oplus M(11)$ where $I(N_1) = \{M(21), 2M(3)\}$ and $I(N_2) = \{M(15), 2M(9)\}$. Prove that $N_1 \cong T(21)$ and $N_2 \cong T(15)$. Set $m_{15} = 4X_{-4}X_{-5}X_{-6}v + X_{-6}X_{-5}X_{-6}v$, $u_1 = v(3, 6, 1)$, $u_2 = X_{-4}X_{-5}X_{-6}u_1$, $u_3 = X_{-5}X_{-6}u_2$, $G_1 = G(4, 5, 6)$, and $G_2 = G(5, 6)$. Lemma 12 implies that the vector $u_1 \neq 0$ and is fixed by $\mathfrak{X}_i$ for $i \neq 3$. Hence u_1 generates an indecomposable G_1-module with highest weight $\omega(u_1)|G_1$. Applying Lemma 12 in this module, one can deduce that $u_2 \neq 0$ and generates an indecomposable G_2-module with highest weight $\omega(u_2)|G_2$. Applying Lemma 12 in this module once again, we show that $u_3 \neq 0$. One can directly verify that $m_{15} \in \operatorname{Inv} M_{15}$ and $X^9_{-\alpha}v$ has a nonzero weight component $4u_3$. Therefore $X^9_{-\alpha}v \neq 0$. Since $X_{-1}X_{-2}X_{-3}X_{-4}X_{-5}X_{-6}v \neq 0$, then $X^3_{-\alpha}m_{15} \neq 0$. As N_1 and N_2 are self-dual, Lemma 29 implies that $N_1 \cong T(21)$ and $N_2 \cong T(15)$.

For $\sigma_x(\omega) < p$ (in particular, for $G = B_3(K)$ or $G = D_4(K)$), this problem is being solved such as in other cases where M_x is a direct sum of p-restricted modules. Then the remaining cases for the groups $B_4(K)$, $D_5(K)$, $B_5(K)$, $D_6(K)$, $B_6(K)$, and $D_7(K)$ mentioned above are considered in turns. Here we use the fact that x is conjugate to an element from a proper semisimple subgroup H the restriction of M on which is completely reducible. After that we use the results for the groups of types B_r and D_r of smaller ranks, the results of Section 4, and Theorem 3. Below when describing the subgroup H and the restrictions $M|H$, we consider situations in which similar constructions are used, together.

I.III. Let $G = B_n(K)$ and one of the following hold:
a) x has a block of size $l \in \{2, 3, 5\}$ in the standard realization,
b) $J(x) = (11, 1^2)$.
Set $H = G(\alpha_1, \alpha_2, ..., \alpha_{n-2}, \varepsilon_{n-2} + \varepsilon_{n-1}, \alpha_n)$ for $l = 3$, $H = G(\alpha_1, \alpha_2, ..., \alpha_{n-3}, \varepsilon_{n-3} + \varepsilon_{n-2}, \alpha_{n-1}, \alpha_n)$ for $l = 5$, $H = G(\alpha_1, \varepsilon_1+\varepsilon_2, \alpha_3, ..., \alpha_{n-1}, \alpha_n)$ for $l = 2$, and $H \cong G(2, ..., n)$ in Case b). It is clear that $H \cong D_{n-1}(K) \times B_1(K)$ for $l = 3$, $D_{n-2}(K) \times B_2(K)$ for $l = 5$, $D_2(K) \times B_{n-2}(K)$ for $l = 2$, and $B_{n-1}(K)$ in Case b). We can assume that $x \in H$. If x satisfies the condition a) for several values of l, we choose any of them. Recall that $D_3(K) \cong A_3(K)$ and $D_2(K) \cong A_1(K) \times A_1(K)$. To describe the block structure of $\varphi(x)$, we use Lemma 30, results of Item II in Section 4, Lemma 21, and Theorem 3.

I.IV. Let $G = D_n(K)$ and x have a block of odd size in the standard realization of G. In this item l is a minimal size of such block. Then $l \leq 7$. We use Lemma 7 taking for H the subgroups from that lemma isomorphic to $B_{n-1}(K)$ for $l = 1$, $B_{n-2}(K) \times B_1(K)$ for $l = 3$, $B_{n-3}(K) \times B_2(K)$ for $l = 5$, and $B_3(K) \times B_3(K)$ for $l = 7$.

I.V. Let $G = D_6(K)$ and $J(x) \in \{(6^2), (4^2, 2^2), (2^6)\}$. Set $H_1 = G(1, 2, ..., 5)$ and $H_2 = G(1, ..., 4, 6)$. Using the facts on the unipotent conjugacy classes in $D_n(K)$ whose representatives have only blocks of even sizes in the standard realization mentioned at the beginning of this section, we can assume that $x \in H_1$ or $x \in H_2$.

Set $\Omega_i^1 = \{\mu \in \Lambda(M) | \mu = \omega - i\alpha_6 - \sum_{j=1}^{5} a_j\alpha_j\}$, $U_i^1 = \langle M_\mu | \mu \in \Omega_i^1 \rangle$, $\Omega_i^2 = \{\mu \in \Lambda(M) | \mu = \omega - i\alpha_5 - \sum_{j=1, j\neq 5}^{6} a_j\alpha_j\}$, $U_i^2 = \langle M_\mu | \mu \in \Omega_i^2 \rangle$, $m_1 = X_{-6}v$, and $m_2 = X_{-5}X_{-4}X_{-6}v$. It is well known that M is self-dual. One easily observes that $M|H_1 = U_0^1 \oplus U_1^1 \oplus U_2^1 \oplus U_3^1$ and $M|H_2 = U_0^2 \oplus U_1^2 \oplus U_2^2$. Theorem 1 and Lemma 11 imply that $U_0^1 \cong U_3^1 \cong M(0)$, $U_2^1 \cong (U_1^1)^*$, $U_0^2 \cong (U_2^2)^* \cong M(\omega_5)$, and $U_2^2 \cong M(\omega_1)$. By Lemma 6, $m_1 \neq 0$. It is clear that m_1 is fixed by $\mathfrak{X}_i$ for $i \neq 6$. Applying Lemma 12 in the $G(4, 5, 6)$-module generated by the vector v, we conclude that the vector $m_2 \neq 0$ and is fixed by $\mathfrak{X}_4$ and $\mathfrak{X}_6$. It is obvious that this vector is fixed by the subgroups $\mathfrak{X}_j$ for $j < 4$. Hence U_1^1 has a composition factor with highest weight $\omega(m_1)|H_1 = \omega_4$ and U_1^2 has a composition factor with highest weight $\omega(m_2)|H_2 = \omega_3$. Dimensional arguments yields that $U_1^1 \cong M(\omega_4)$, $U_2^1 \cong M(\omega_2)$, and $U_1^2 \cong M(\omega_3)$. Now we can use the results of Item IV of Section 4 to determine the block structure of $\varphi(x)$.

I.VI. Let $G = B_6(K)$ and $J(x) = (6^2, 1)$. Set $H = G(\alpha_1, ..., \alpha_5, \varepsilon_5 + \varepsilon_6)$. Then $H \cong D_6(K)$ and we can assume that $x \in H$. It is well known that $M|H \cong M(\omega_n) \oplus M(\omega_{n-1})$. Indeed, let N_1 (N_2) be the sum of all weight subspaces of M whose weights have the form $\{(\pm\varepsilon_1 \pm ... \pm \varepsilon_6)/2\}$ with an even (odd) number of minuses. It is easy to see that N_1 and N_2 are H-modules and to determine their highest weights. Then we use dimensional arguments. We apply the results of Item I.IX to determine the block structure of $\varphi(x)$. It does not matter which of the two classes of unipotent elements of $D_6(K)$ with the same $J(x)$ is considered since the restriction $M|H$ is the direct sum of different semispinor representations.

For the representations $\varphi(\omega_n)$ the problem is solved in all cases. So the problem is solved for groups $B_n(K)$ with $n > 3$ and $D_n(K)$ with $n > 4$.

II. Some representations of the groups $\mathbf{B_3(K)}$ and $\mathbf{D_4(K)}$

Now consider the remaining representations for the groups $B_3(K)$ and $D_4(K)$.

Let $G = B_3(K)$. Observe that the order of a regular unipotent element is equal to p only for $p \geq 7$ and an element x with $J(x) = (5, 1^2)$ has order p for $p > 3$. The arguments in Item I imply that we can assume that $\omega \notin \{2\omega_1, \omega_2, \omega_3\}$. Recall that $\sigma_x(\omega_3) < p$ for any element x of order p and hence $M(\omega_3)|A_x$ is a tilting module. By Lemma 14, $M|A_x$ is a tilting module in the following cases:

$\omega = 2\omega_3, \quad p \geq 3 \quad (\lambda_1 = \lambda_2 = \omega_3);$
$\omega = \omega_1 + \omega_3, \quad p \neq 7 \quad (\lambda_1 = \omega_1, \lambda_2 = \omega_3);$
$\omega = 3\omega_1, \quad p \neq 7 \quad (\lambda_1 = 2\omega_1, \lambda_2 = \omega_1)$

(here λ_1 and λ_2 are the weights from Lemma 14). One easily observes that $\sigma_y(\omega) = 6a_1 + 10a_2 + 6a_3$ for $\omega = a_1\omega_1 + a_2\omega_2 + a_3\omega_3$. Using the arguments of Section 3, we can suppose that for a regular element $X_\alpha = X_1 + 2X_2 + 3X_3$ and $X_{-\alpha} = 6X_{-1} + 5X_{-2} + 4X_{-3}$. The arguments above yield that for such element it remains to consider only the cases where $p = 7$ and $\omega = \omega_1 + \omega_3$ or $3\omega_1$.

II.I. Let $\omega = \omega_1 + \omega_3$ and $p = 7$. Then $\dim \varphi = 40$ and $\sigma_y(\omega) = 12$. One easily observes that $\dim M_{12} = 1$, $\dim M_{10} = 2$, and $\dim M_8 = 3$. Set $\lambda = \omega - \alpha_1 - \alpha_2 - \alpha_3$,

$$\Lambda_6 = \{\omega - \alpha_1 - \alpha_2 - \alpha_3,\ \omega - \alpha_2 - 2\alpha_3\};$$
$$\Lambda_4 = \{\omega - 2\alpha_1 - \alpha_2 - \alpha_3,\ \omega - \alpha_1 - \alpha_2 - 2\alpha_3,\ \omega - \alpha_1 - 2\alpha_2 - \alpha_3\};$$
$$\Lambda_2 = \{\omega - 2\alpha_1 - \alpha_2 - 2\alpha_3,\ \omega - 2\alpha_1 - 2\alpha_2 - \alpha_3,\ \omega - \alpha_1 - \alpha_2 - 3\alpha_3,$$
$$\omega - \alpha_1 - 2\alpha_2 - 2\alpha_3\};$$
$$\Lambda_0 = \{\omega - 2\alpha_1 - 2\alpha_2 - 2\alpha_3,\ \omega - \alpha_1 - 2\alpha_2 - 3\alpha_3\},$$
$$\Sigma = \{\omega - \alpha_1 - \alpha_2 - 2\alpha_3,\ \omega - \alpha_1 - 2\alpha_2 - 2\alpha_3,\ \omega - 2\alpha_1 - 2\alpha_2 - 2\alpha_3,$$
$$\omega - \alpha_1 - 2\alpha_2 - 3\alpha_3\}.$$

Then $\lambda = \omega_3$. It is not difficult to verify that $M_i = \langle M_\lambda | \lambda \in \Lambda_i \rangle$ for $i \in \{0, 2, 4, 6\}$. By [13, Part 2, § 8, Proposition 8.19], the maximal submodule in $V(\omega)$ is isomorphic to $M(\lambda)$. Using Freudenthal's formula for the weights of the module $V(\omega)$, one can show that $\dim M_\lambda = 2$. One easily observes that $\dim M_\mu = 2$ for $\mu \in \Sigma$ as μ lies in the same W-orbit with λ. The remaining weights from $\Lambda_6 \cup \Lambda_4 \cup \Lambda_2$ have multiplicity 1 since they lie in the same W-orbit with ω. Therefore $\dim M_6 = 3$, $\dim M_4 = 4$, $\dim M_2 = 5$, and $\dim M_0 = 4$. This implies that $I(M_y) = \{M(12), M(10), M(8), 2M(4), 2M(2)\}$. By Lemma 27, $M_y \cong M(12) \oplus N_1 \oplus N_2$ where $I(N_1) = \{M(10), 2M(2)\}$ and $I(N_2) = \{M(8), 2M(4)\}$. Show that $N_1 \cong T(10)$ and $N_2 \cong T(8)$.

Set $m_{10} = 2X_{-1}v + 3X_{-3}v$ and $m_8 = X_{-2}X_{-3}v + 5X_{-2}X_{-1}v$. One can directly verify that $m_{10} \in \operatorname{Inv} M_{10}$, $m_8 \in \operatorname{Inv} M_8$, and the vectors $X_{-\alpha}^4 m_{10}$ and $X_{-\alpha}^2 m_8$ have nonzero weight components $X_{-3}^3 X_{-2} X_{-1} v$ and $X_{-1}^2 X_{-2} X_{-3} v$, respectively. It is clear that $m_{10} \in N_1$ and $m_8 \in N_2$. As the modules N_1 and N_2 are self-dual, Lemma 29 yields that $N_1 \cong T(10)$ and $N_2 \cong T(8)$.

II.II. Let $\omega = 3\omega_1$ and $p = 7$. Then $\dim \varphi = 77$ and $\sigma_y(\omega) = 18$. One easily observes that $\dim M_{18} = 1$, $\dim M_{16} = 1$, and $\dim M_{14} = 2$. Set

$$\Lambda_{12} = \{\omega - 3\alpha_1,\ \omega - 2\alpha_1 - \alpha_2,\ \omega - \alpha_1 - \alpha_2 - \alpha_3\};$$
$$\Lambda_{10} = \{\omega - 3\alpha_1 - \alpha_2,\ \omega - 2\alpha_1 - \alpha_2 - \alpha_3, \omega - 2\alpha_1 - 2\alpha_2,\ \omega - \alpha_1 - \alpha_2 - 2\alpha_3\};$$
$$\Lambda_8 = \{\omega - 3\alpha_1 - \alpha_2 - \alpha_3,\ \omega - 3\alpha_1 - 2\alpha_2,\ \omega - 2\alpha_1 - \alpha_2 - 2\alpha_3,$$
$$\omega - 2\alpha_1 - 2\alpha_2 - \alpha_3,\ \omega - \alpha_1 - 2\alpha_2 - 2\alpha_3\};$$
$$\Lambda_6 = \{\omega - 3\alpha_1 - \alpha_2 - 2\alpha_3,\ \omega - 3\alpha_1 - 2\alpha_2 - \alpha_3,\ \omega - 3\alpha_1 - 3\alpha_2,\ \omega - 2\alpha_1 - 2\alpha_2 - 2\alpha_3\};$$
$$\Lambda_4 = \{\omega - 3\alpha_1 - 2\alpha_2 - 2\alpha_3,\ \omega - 3\alpha_1 - 3\alpha_2 - \alpha_3, \omega - 2\alpha_1 - 2\alpha_2 - 3\alpha_3,$$
$$\omega - 2\alpha_1 - 3\alpha_2 - 2\alpha_3\};$$
$$\Lambda_2 = \{\omega - 4\alpha_1 - 2\alpha_2 - 2\alpha_3,\ \omega - 3\alpha_1 - 2\alpha_2 - 3\alpha_3,\ \omega - 3\alpha_1 - 3\alpha_2 - 2\alpha_3,$$
$$\omega - 2\alpha_1 - 2\alpha_2 - 4\alpha_3,\ \omega - 2\alpha_1 - 3\alpha_2 - 3\alpha_3\};$$
$$\Lambda_0 = \{\omega - 4\alpha_1 - 3\alpha_2 - 2\alpha_3,\ \omega - 3\alpha_1 - 2\alpha_2 - 4\alpha_3,\ \omega - 3\alpha_1 - 3\alpha_2 - 3\alpha_3,$$
$$\omega - 3\alpha_1 - 4\alpha_2 - 2\alpha_3,\ \omega - 2\alpha_1 - 3\alpha_2 - 4\alpha_3\};$$
$$\Sigma = \{\lambda \in \Lambda(M) | \lambda = \omega - x\alpha_1 - y\alpha_2\},\ \delta = \omega - \alpha_1 - \alpha_2 - \alpha_3,\ \mu = \omega - 2\alpha_1 - \alpha_2 - \alpha_3,$$

$\tau = \omega - 2\alpha_1 - 2\alpha_2 - 2\alpha_3$, and $\eta = \omega - 3\alpha_1 - 3\alpha_2 - 3\alpha_3 = 0$. It is not difficult to conclude that $M_i = \langle M_\lambda | \lambda \in \Lambda_i \rangle$ for $i \in \{0, 2, 4, ..., 12\}$. It follows from [18, Table 6.23] that the module

$V(\omega)$ is irreducible. One easily deduces that $\dim M_\delta = \dim M_\mu = 1$. Using Freudenthal's formula, we can show that $\dim M_\tau = \dim M_\eta = 3$. Theorem 1 implies that $\dim M_\sigma = 1$ for $\sigma \in \Sigma$ since all the weight subspaces of the $A_2(K)$-module $M(3\omega_1)$ are one-dimensional. It is easy to see that $\dim M_\theta = 3$ for $\theta = \omega - 3\alpha_1 - 2\alpha_2 - 2\alpha_3$ or $\omega - 3\alpha_1 - 3\alpha_2 - 2\alpha_3$ as θ lies in the same W-orbit with τ. All other weights $\nu \in \Lambda_{10} \cup ... \cup \Lambda_0$ lie in the same W-orbit with weights from Σ, δ, or μ, hence $\dim M_\nu = 1$. Now it is not difficult to show that $\dim M_{12} = 3$, $\dim M_{10} = 4$, $\dim M_8 = 5$, $\dim M_6 = 6$, $\dim M_4 = 6$, $\dim M_2 = 7$, and $\dim M_0 = 7$. Then

$$I(M_y) = \{M(18), M(14), 2M(12), M(10), 2M(8), M(6), M(4), 2M(2), M(0)\}.$$

Proposition 8 and Lemma 27 yield that $M_y = N_1 \oplus N_2 \oplus N_3 \oplus M(6)$ where $I(N_1) = \{M(18), 2M(8), M(4)\}$, $I(N_2) = \{M(14), 2M(12), M(0)\}$, and $I(N_3) = \{M(10), 2M(2)\}$. Show that $N_1 \cong T(18)$, $N_2 \cong T(14)$, and $I(N_3) \cong T(10)$.

Set $u = X_{-2}X_{-3}^2X_{-2}X_{-1}v$. Then Lemmas 12 and 6 imply that $u \neq 0$. One can directly verify that $X_{-\alpha}^5 v$ has a weight component $6u$ and that the vectors $X_{-3}^3X_{-2}^2X_{-1}^2v$, $X_{-2}X_{-1}^2X_{-3}^2X_{-2}X_{-1}v$, and $X_{-1}^2X_{-2}X_{-3}^2X_{-2}X_{-1}v$ are linearly independent. Now Lemma 28 forces that $KA_yv \cong V(18)$. Set $\overline{N_1} = N_1/KA_yv$. It is clear that $I(\overline{N_1}) = \{M(8), M(4)\}$. Lemmas 12 and 6 imply that $X_{-2}^2X_{-1}X_{-3}^2X_{-2}X_{-1}v = X_{-2}^2X_{-1}v(3,1,1) \neq 0$. Now so we easily conclude that the vectors

$$X_{-3}^3X_{-2}^2X_{-1}^2v,\ X_{-2}X_{-1}^2X_{-3}^2X_{-2}X_{-1}v,\ X_{-1}^2X_{-2}X_{-3}^2X_{-2}X_{-1}v,$$
$$X_{-3}X_{-2}^3X_{-1}^3v,\ X_{-3}^3X_{-2}^2X_{-1}^2v,\ X_{-2}^2X_{-1}X_{-3}^2X_{-2}X_{-1}v$$

form a basis of M_4. Using this basis, we can show that $\mathrm{Inv}\, M_4 = 0$. Therefore the module N_1 has no submodules isomorphic to $M(4)$. As N_1 is self-dual, there are no such factor modules in N_1 and in $\overline{N_1}$. Lemma 23 yields that $\overline{N_1} \cong V(8)$ and N_1 has a filtration by Weyl modules. By Remark 4, N_1 is a tilting module. Knowing $I(N_1)$, we conclude that $N_1 \cong T(18)$.

Set $m_{14} = 3X_{-1}^2v + X_{-2}X_{-1}v$, $u_1 = X_{-2}X_{-1}^3X_{-2}X_{-3}^2X_{-2}X_{-1}v$, $u_2 = X_{-2}^2X_{-1}^2X_{-3}^2X_{-2}X_{-1}v$, and $u_3 = X_{-2}X_{-3}^4X_{-2}^2X_{-1}^2v$. One can directly verify that $m_{14} \in \mathrm{Inv}\, M_{14}$ and the vectors $X_{-3}^3X_{-2}^3X_{-1}^3v$, $X_{-1}X_{-2}X_{-3}^3X_{-2}^2X_{-1}^2v$, and $X_{-2}X_{-3}^3X_{-2}^2X_{-1}^3v$ are linearly independent. It is clear that $m_{14} \in N_2$. Since $X_{-1}^3v \neq 0$, then $X_{-\alpha}m_{14} \neq 0$. Then by Lemma 28, $KA_ym_{14} \cong V(14)$. Set $\overline{N_2} = N_2/KA_ym_{14}$. It is clear that $I(\overline{N_2}) = \{M(12), M(0)\}$. It is not difficult to check that $X_2X_1^3X_2u_1 = c_1v(3,1,1)$, $X_1^2X_2^2u_2 = c_2v(3,1,1)$, and $X_2u_3 = c_3v(3,1,2)$ where c_1, c_2, $c_3 \in K^*$. Therefore $u_i \neq 0$ by Lemma 12. Now one easily observes that the vectors

$$X_{-3}^3X_{-2}^3X_{-1}^3v, X_{-1}X_{-2}X_{-3}^3X_{-2}^2X_{-1}^2v, X_{-2}X_{-3}^3X_{-2}^2X_{-1}^3v, X_{-3}^4X_{-2}^2X_{-1}^3v, u_1, u_2, u_3$$

form a basis of M_0. Using this basis, we can show that $\mathrm{Inv}\, M_0 = 0$. As N_2 is self-dual, using Lemma 23 and Remark 4 and arguing as above for the analysis of the modules N_1 and $\overline{N_1}$, we conclude that $\overline{N_2}$ has no factor modules isomorphic to $M(0)$, $\overline{N_2} \cong V(12)$, N_2 has a filtration by Weyl modules and is a tilting module. Then it is easy to see that $N_2 \cong T(14)$.

Set $m_{10} = 2X_{-2}^2X_{-1}^2v + X_{-3}X_{-2}X_{-1}^2v + 4X_{-3}^2X_{-2}X_{-1}v$ and $w = X_{-2}X_{-3}^3X_{-2}^2X_{-1}^2v$. By Lemma 6, $X_2w = X_{-3}^3X_{-2}^2X_{-1}^2v \neq 0$. One can directly verify that $m_{10} \in \mathrm{Inv}\, M_{10}$ and the vector $X_{-\alpha}^4m_{10}$ has a weight component $4w$. It is clear that $m_{10} \in N_3$. As N_3 is self-dual, Lemma 29 forces that $N_3 \cong T(10)$.

II.III. Let x be not regular. By the results of Item I and the arguments at the beginning of Item II, it suffices to consider the following cases:
1) $\omega = \omega_1 + \omega_3$, $p = 7$;
2) $\omega = \omega_1 + \omega_2$, $p = 3$;
3) $\omega = \omega_2 + \omega_3$, $p = 5$;
4) $\omega = 3\omega_1$, $p = 7$.

Let $J(x) \in \{(5,1^2),(3,1^4),(2^2,1^3)\}$. Set $H = G(2,3)$. It is clear that $H \cong B_2(K)$. We can assume that $x \in H$. Show that the module $M|H$ is completely reducible, except the case where $p = 7$ and $\omega = 3\omega_1$. If $M|H$ is completely reducible, then to determine the block structure of x on M, we use the results of Item I of Section 5, taking into account that $B_2(K) \cong C_2(K)$. Below in this item $\Omega_i = \{\lambda \in \Lambda(M)|\lambda = \omega - i\alpha_1 - x\alpha_2 - y\alpha_3\}$, $U_i = \langle M_\lambda|\lambda \in \Omega_i\rangle$, $i(M)$ is the maximal i for which $\Omega_i \neq \varnothing$, and $\mu_0 = \omega|H$. Theorem 1 and Lemma 11 yield that $U_0 \cong U_{i(M)} \cong M(\mu_0)$ and $U_{i(M)-j} \cong U_j^*$.

II.III.1) Let $\omega = \omega_1 + \omega_3$ and $p = 7$. Then $\dim\varphi = 40$ and $\mu_0 = \omega_2$. It is not difficult to show that $i(M) = 3$. Set $v_1 = X_{-1}v$ and $\mu_1 = \omega(v_1)|H$. Then $\mu_1 = \omega_1+\omega_2$. By Lemma 12, $v_1 \neq 0$. It is clear that the vector v_1 is fixed by $\mathfrak{X}_2$ and $\mathfrak{X}_3$. This forces that the H-module U_1 has a composition factor $M(\mu_1)$.

II.III.2) Let $\omega = \omega_1+\omega_2$ and $p = 3$. Then $\dim\varphi = 63$ and $\mu_0 = \omega_1$. One easily concludes that $i(M) = 4$. Set $v_1 = X_{-1}v$, $v_2 = X_{-1}^2X_{-2}v$, and $\mu_i = \omega(v_i)|H$, $i = 1,2$. One easily observes that $\mu_1 = 2\omega_1$ and $\mu_2 = \omega_1 + 2\omega_2$. By Lemma 12, the vectors v_1 and $v_2 \neq 0$ and are fixed by $\mathfrak{X}_2$ and $\mathfrak{X}_3$. This implies that the H-modules U_i have composition factors $M(\mu_i)$ for $i = 1,2$.

II.III.3) Let $\omega = \omega_2 + \omega_3$ and $p = 5$. Then $\dim\varphi = 64$ and $\mu_0 = \omega_1 + \omega_2$. One easily concludes that $i(M) = 3$. Set $v_1 = X_{-1}X_{-2}v$ and $\mu_1 = \omega(v_1)|H$. It is not difficult to observe that $\mu_1 = 3\omega_2$. Using Lemma 12 and arguing as in Item II.III.2), we get that the H-module U_1 has a composition factor $M(\mu_1)$.

Taking into account the dimension of M and the composition factors of the H-modules U_i found above, we conclude that in all the cases from Items II.III.1)–II.III.3) the modules U_i are irreducible. Hence the restriction $M|H$ is completely reducible.

II.III.4) Let $\omega = 3\omega_1$ and $p = 7$. One easily observes that $\sigma_x(\omega) \geq p$ only for $J(x) = (5,1^2)$. Recall that $\dim\varphi = 77$ and $M(\omega) \cong V(\omega)$. So the weight multiplicities of M can be found with the use of Freudenthal's formula, several of them, are already found in Item II.II. Knowing these multiplicities, we determine the block structure of $\varphi(x)$ when $\sigma_x(\omega) < p$.

Let $J(x) = (5,1^2)$. As indicated at the end of Section 3, there exist a subgroup $A \subset H$ and a homomorphism $\sigma : \Lambda(H) \to \mathbb{Z}$ such that $A \cong A_1(K)$, $x \in A$, $\sigma(\alpha_i) = 2$ for $i = 2$ or 3, and σ is induced by the restriction of weights from a maximal torus in H to a maximal torus in A. In this item α is the positive root of A, X_α and $X_{-\alpha}$ are the root operators of the Lie algebra of this group. Using the formulae from Item I of Section 5, we can assume that $X_\alpha = X_2 + X_3$ and $X_{-\alpha} = 3X_{-2} + 4X_{-3}$. Set $\Omega_i^+ = \{\lambda \in \Omega_i|\sigma(\lambda) \geq 0\}$ and $U_{i,j} = \langle M_\lambda|\lambda \in \Omega_i, \sigma(\lambda) = j\rangle$. It is not difficult to show that $i(M) = 6$. Hence $U_0 \cong U_6 \cong M(0)$, $U_5 \cong U_1^*$, and $U_4 \cong U_2^*$. One easily observes that

$$\begin{aligned}
\Omega_1 =&\{\omega - \alpha_1,\ \omega - \alpha_1 - \alpha_2,\ \omega - \alpha_1 - \alpha_2 - \alpha_3,\ \ \omega - \alpha_1 - \alpha_2 - 3\alpha_3,\\
&\omega - \alpha_1 - 2\alpha_2 - 2\alpha_3\};\\
\Omega_2^+ =&\{\omega - 2\alpha_1, \omega - 2\alpha_1 - \alpha_2, \omega - 2\alpha_1 - 2\alpha_2, \omega - 2\alpha_1 - \alpha_2 - \alpha_3,\\
&\omega - 2\alpha_1 - \alpha_2 - 2\alpha_3, \omega - 2\alpha_1 - 2\alpha_2 - \alpha_3, \omega - 2\alpha_1 - 2\alpha_2 - 2\alpha_3\};\\
\Omega_3^+ =&\{\omega - 3\alpha_1, \omega - 3\alpha_1 - \alpha_2, \omega - 3\alpha_1 - 2\alpha_2, \omega - 3\alpha_1 - \alpha_2 - \alpha_3,\\
&\omega - 3\alpha_1 - \alpha_2 - 2\alpha_3, \omega - 3\alpha_1 - 2\alpha_2 - \alpha_3, \omega - 3\alpha_1 - 3\alpha_2, \omega - 3\alpha_1 - 2\alpha_2 - 2\alpha_3,\\
&\omega - 3\alpha_1 - 3\alpha_2 - \alpha_3, \omega - 3\alpha_1 - 2\alpha_2 - 3\alpha_3, \omega - 3\alpha_1 - 3\alpha_2 - 2\alpha_3,\\
&\omega - 3\alpha_1 - 2\alpha_2 - 4\alpha_3, \omega - 3\alpha_1 - 3\alpha_2 - 3\alpha_3, \omega - 3\alpha_1 - 4\alpha_2 - 2\alpha_3\}.
\end{aligned}$$

The dimensions of the weight subspaces M_μ with $\mu \in \Omega_1 \cup \Omega_2^+ \cup \Omega_3^+$ are found in Item II.II. Using this, it is not difficult to show that $\dim U_1 = 5$, $\dim U_{2,8} = 1$, $\dim U_{2,6} = 1$, $\dim U_{2,4} = 2$, $\dim U_{2,2} = 2$, $\dim U_{2,0} = 3$, $\dim U_{3,12} = 1$, $\dim U_{3,10} = 1$, $\dim U_{3,8} = 2$, $\dim U_{3,6} = 3$, $\dim U_{3,4} = 4$, $\dim U_{3,2} = 4$, and $\dim U_{3,0} = 5$.

Set $w = X_{-1}v$. Lemma 6 implies that $w \neq 0$. It is clear that the vector w is fixed by $\mathfrak{X}_2$ and $\mathfrak{X}_3$ and that $\omega(w)|H = \omega_1$. This yields that the H-module U_1 has a composition factor $F \cong M(\omega_1)$. Since $\dim U_1 = \dim F$, we conclude that the modules U_1 and U_5 are irreducible. So the block structure of x on the module $U_0 \oplus U_1 \oplus U_5 \oplus U_6$ is clear.

The arguments above imply that $I(U_2|A) = \{M(8), 2M(4), M(0)\}$ and $I(U_3|A) = \{M(12), M(8), M(6), 2M(4), 2M(0)\}$. By Lemma 27, $U_2|A \cong N \oplus M(0)$ and $U_3|A \cong N_1 \oplus N_2 \oplus M(6)$ where $I(N) = \{M(8), 2M(4)\}$, $I(N_1) = \{M(12), 2M(0)\}$ and $I(N_2) = \{M(8), 2M(4)\}$. Since $U_4 \cong U_2^*$, we have $U_4|A \cong N' \oplus M(0)$ where $N' \cong N^*$. Show that $N \cong N' \cong N_2 \cong T(8)$ and $N_1 \cong T(12)$.

Set $m_1 = X_{-1}^2 v$, $m_2 = X_{-1}^3 X_{-2} X_{-3}^2 X_{-2} X_{-1} v$, $u = X_{-3}^2 X_{-2} X_{-1} v$, and $\Gamma = G(1,2)$. Show that $X_{-2}^2 m_i \neq 0$ for $i = 1, 2$. By Lemma 12, $X_{-2}^2 m_1$ and $u \neq 0$ and the vector u is fixed by $\mathfrak{X}_1$ and $\mathfrak{X}_2$. Therefore it generates an indecomposable Γ-module with highest weight $\omega(u)|\Gamma$. One easily observes that m_2 coincides with the vector $u(1,2,1)$ constructed in this module. Applying Lemma 12 once again, we conclude that the vector $m_2 \neq 0$ and is fixed by $\mathfrak{X}_2$. Since $\langle \omega(m_2), \alpha_2 \rangle = 2$, then by Lemma 6, $X_{-2}^2 m_2 \neq 0$. This yields that $X_{-\alpha}^2 m_i \neq 0$ for $i = 1, 2$. Obviously, $m_1 \in U_{2,8}$ and $m_2 \in U_{4,8}$. Hence $m_1 \in N$, $m_2 \in N'$. By Lemma 28, $KAm_1 \cong KAm_2 \cong V(8)$. As $N \oplus N'$ is self-dual, Lemma 24 implies that $N \cong N' \cong T(8)$.

Set $m_3 = X_{-1}^3 v$ and $m_4 = X_{-3} X_{-2} X_{-1}^3 v + 6 X_{-2}^2 X_{-1}^3 v$. One can directly verify that $m_4 \in \operatorname{Inv} U_{3,8}$ and the vectors $X_{-\alpha}^6 m_3$ and $X_{-\alpha}^2 m_4$ have nonzero weight components $2 X_{-2}^3 X_{-3}^2 X_{-2} X_{-1}^3 v$ and $4 X_{-3} X_{-2}^3 X_{-1}^3 v$, respectively (we use Lemma 6 several times to show that these components are nonzero). It is clear that $m_3 \in N_1$ and $m_4 \in N_2$. As N_1 and N_2 are self-dual, Lemma 29 forces that $N_1 \cong T(12)$ and $N_2 \cong T(8)$.

II.IV. Let $J(x) = \{(3^2, 1), (3, 2^2)\}$. Then $\sigma_x(\omega) < p$ for $p = 7$ and $\omega = \omega_1 + \omega_3$ or $3\omega_1$. Recall that $M(\omega) \cong V(\omega)/M(\omega_3)$ for $p = 7$ and $\omega = \omega_1 + \omega_3$ and that the module $V(3\omega_1)$ is irreducible for $p = 7$. Taking into account that the weight subspaces of the module $M(\omega_3)$ are one-dimensional and using Freudenthal's formula, we can find the weight multiplicities of M in these two cases. Several of them are already found in Items II.I. and II.II. Knowing these multiplicities, we find the irreducible components of $M|A_x$. Now it remains to consider the following cases:
1) $\omega = \omega_1 + \omega_2$, $p = 3$;
2) $\omega = \omega_2 + \omega_3$, $p = 5$.

Let $J(x) = (3, 2^2)$. Set $H = G(1,3)$. It is clear that $H \cong A_1(K) \times A_1(K)$ and we can assume that x is a regular unipotent element of H. The arguments at the end of Section 3 imply that there exist a subgroup $A \subset H$ and a homomorphism $\sigma : \Lambda(H) \to \mathbb{Z}$ such that $A \cong A_1(K)$, $x \in A$, $\sigma(\alpha_i) = 2$ for $i = 1, 3$, and σ is induced by the restriction of weights from a maximal torus in H to a maximal torus in A. Recall that the set $\Lambda(H)$ can be identified in the standard way with the set $\mathbb{Z} \times \mathbb{Z}$. One easily observes that $\sigma((a,b)) = a + b$. Below in this item α is the positive root of A, X_α and $X_{-\alpha}$ are the root operators of its Lie algebra. Obviously, we can assume that $X_\alpha = X_1 + X_3$ and $X_{-\alpha} = X_{-1} + X_{-3}$.

1) Let $\omega = \omega_1 + \omega_2$ and $p = 3$. Then $\dim \varphi = 63$. Set $\Omega_i = \{\lambda \in \Lambda(M) | \lambda = \omega - a\alpha_1 - i\alpha_2 - b\alpha_3\}$ and $U_i = \langle M_\mu | \mu \in \Omega_i \rangle$. We define the parameter $i(M)$, the set Ω_i^+ and the subspaces $U_{i,j}$, as in Item II.III. It is not difficult to show that $i(M) = 6$. Corollary 4 and Lemma 11 imply that $U_0 \cong U_6 \cong M((1,0))$, $U_5 \cong U_1^*$, $U_4 \cong U_2^*$, and the H-module U_3 is self-dual. One easily observes that

$$\Omega_1 = \{\omega - a\alpha_1 - \alpha_2 - b\alpha_3 | 0 \leq a, b \leq 2\},$$

$$\Omega_2^+ = \{\omega - \alpha_1 - 2\alpha_2, \omega - 2\alpha_1 - 2\alpha_2, \omega - 2\alpha_2 - 2\alpha_3,$$
$$\omega - \alpha_1 - 2\alpha_2 - \alpha_3, \omega - 2\alpha_1 - 2\alpha_2 - \alpha_3, \omega - \alpha_1 - 2\alpha_2 - 2\alpha_3\},$$

$$\Omega_3^+ = \{\omega - 2\alpha_1 - 3\alpha_2 - \alpha_3, \omega - \alpha_1 - 3\alpha_2 - 2\alpha_3, \omega - 2\alpha_1 - 3\alpha_2 - 2\alpha_3,$$
$$\omega - \alpha_1 - 3\alpha_2 - 3\alpha_3, \omega - 3\alpha_1 - 3\alpha_2 - 2\alpha_3, \omega - 2\alpha_1 - 3\alpha_2 - 3\alpha_3, \omega - \alpha_1 - 3\alpha_2 - 4\alpha_3\}.$$

Set $\Sigma = \{\lambda \in \Lambda(M) | \lambda = \omega - b_1\alpha_1 - b_2\alpha_2 - b_3\alpha_3 | b_1 b_3 = 0\}$, $\mu = \omega - \alpha_1 - \alpha_2 - \alpha_3$, $\nu = \omega - \alpha_1 - 2\alpha_2 - 2\alpha_3$, and $\tau = \omega - 2\alpha_1 - 3\alpha_2 - 3\alpha_3 = 0$. As the weight subspaces of the $C_2(K)$-module $M(\omega_2)$ are one-dimensional, Theorem 1 and Proposition 4 yield that

$\dim M_\sigma = 1$ for $\sigma \in \Sigma$. Using the Jantzen filtration [13, Part 2, § 8, Proposition 8.19], we can check that the module $V(\omega)$ has three composition factors: $M(2\omega_3)$, $M(\omega_1)$, and $M(\omega)$. Taking into account the irreducibility of the modules $V(\mu)$ for $\mu \in \{2\omega_3, \omega_1\}$ and applying Freudenthal's formula, we can show that $\dim M_\mu = \dim M_\tau = 1$ and $\dim M_\nu = 2$. The weight $\delta = \omega - 2\alpha_1 - 3\alpha_2 - 2\alpha_3$ lies in the same W-orbit with ν, hence $\dim M_\delta = 2$. All other weights $\theta \in \Omega_1 \cup \Omega_2^+ \cup \Omega_3^+$ lie in the same W-orbit with weights from Σ or μ, therefore $\dim M_\theta = 1$. Now one easily deduces that $\dim U_1 = 9$, $\dim U_{2,5} = 1$, $\dim U_{2,3} = 3$, $\dim U_{2,1} = 3$, $\dim U_{3,4} = 2$, $\dim U_{3,2} = 3$, and $\dim U_{3,0} = 3$. Set $u = X_{-2}v$ and $\eta = \omega(u)|H$. It is clear that $\eta = (2,2)$. By Lemma 6, $u \neq 0$. Obviously, the vector u is fixed by $\mathfrak{X}_1$ and $\mathfrak{X}_3$. So the H-module U_1 has a composition factor $M(\eta)$. Since $\dim M(\eta) = \dim U_1$, we get that $U_1 \cong U_5 \cong M(\eta)$. To determine the block structure of x on the module $U_0 \oplus U_1 \oplus U_5 \oplus U_6$, we use Lemma 21 and Theorem 3.

The arguments above imply that $I(U_2|A) = \{M(5), 2M(3), 2M(1)\}$ and $I(U_3|A) = \{2M(4), M(2), 2M(0)\}$. By Proposition 8 and Lemma 27, $U_1|A \cong M(5) \oplus N_1$ and $U_3|A \cong N_2 \oplus M(2)$ where $I(N_1) = \{2M(3), 2M(1)\}$ and $I(N_2) = \{2M(4), 2M(0)\}$. Show that $N_1 \cong T(3) \oplus M(3)$ and $N_2 \cong T(4) \oplus M(4)$.

Set $m_2 = X_{-2}X_{-3}^2X_{-2}X_{-1}v + X_{-3}X_{-2}X_{-3}X_{-1}X_{-2}v$ and $m_3 = X_{-2}X_{-1,3}X_{-2}X_{-3}^2X_{-2}v + X_{-2}X_{-1}X_{-3,3}X_{-2}^2X_{-1}v + 2X_{-2}X_{-3,4}X_{-2}^2X_{-1}v$. One can directly verify that $X_{-\alpha}X_\alpha m_2$ and $X_{-\alpha}X_\alpha m_3$ have nonzero weight components $X_{-3,3}X_{-2}^2X_{-1}v$ and $X_{-1}X_{-2}X_{-3,4}X_{-2}^2X_{-1}v$, respectively, and $X_\alpha^2 m_2 = 0$. This implies that there exist vectors $m_2' \in N_1$ and $m_3' \in N_2$ such that $X_{-\alpha}X_\alpha m_2' = X_{-\alpha}X_\alpha m_2 \neq 0$ and $X_{-\alpha}X_\alpha m_3' = X_{-\alpha}X_\alpha m_3 \neq 0$. By Lemma 25, $N_1 \cong T(3) \oplus M(3)$ and $N_2 \cong T(4) \oplus M(4)$.

2) Let $\omega = \omega_2 + \omega_3$ and $p = 5$. Then $\dim \varphi = 64$. Using the notation Ω_i, Ω_i^+, U_i, $U_{i,j}$, $i(M)$, and Σ from the previous item, one easily observes that $i(M) = 6$. By Corollary 4 and Lemma 11, $U_0 \cong U_6 \cong M((0,1))$, $U_5 \cong U_1^*$, and $U_4 \cong U_2^*$. One easily checks that

$$\Omega_1 = \{\omega - a\alpha_1 - \alpha_2 - b\alpha_3 | 0 \leq a \leq 1, 0 \leq b \leq 3\},$$

$$\Omega_2^+ = \{\omega - 2\alpha_2 - \alpha_3, \omega - 2\alpha_2 - 2\alpha_3, \omega - \alpha_1 - 2\alpha_2 - \alpha_3, \omega - \alpha_1 - 2\alpha_2 - 2\alpha_3, \omega - 2\alpha_1 - 2\alpha_2 - \alpha_3, \omega - 2\alpha_2 - 3\alpha_3\},$$

$$\Omega_3^+ = \{\omega - \alpha_1 - 3\alpha_2 - 2\alpha_3, \omega - 3\alpha_2 - 3\alpha_3, \omega - 2\alpha_1 - 3\alpha_2 - 2\alpha_3, \omega - \alpha_1 - 3\alpha_2 - 3\alpha_3, \omega - 3\alpha_2 - 4\alpha_3, \omega - 2\alpha_1 - 3\alpha_2 - 3\alpha_3, \omega - \alpha_1 - 3\alpha_2 - 4\alpha_3\}.$$

Set $\mu = \omega_1 + \omega_3$, $\nu = \omega - \alpha_1 - \alpha_2 - \alpha_3$, and $\eta = \omega - \alpha_1 - 2\alpha_2 - 2\alpha_3$. Recall that all weight subspaces of the $A_2(K)$-module $M(\omega_2)$ are one-dimensional. Now Theorems 1 and 4 yield that $\dim M_\sigma = 1$ for $\sigma \in \Sigma$. Using the Jantzen filtration [13, Part 2, § 8, Proposition 8.19], we get that the maximal submodule in $V(\omega)$ is isomorphic to $M(\omega_1 + \omega_3)$ and $V(\omega_1 + \omega_3)$ is irreducible. Applying Freudenthal's formula, we can show that $\dim M_\nu = 1$ and $\dim M_\eta = 2$. One easily observes that $\dim M_\tau = 2$ for $\tau \in \{\omega - \alpha_1 - 3\alpha_2 - 3\alpha_3, \omega - 2\alpha_1 - 3\alpha_2 - 3\alpha_3, \omega - \alpha_1 - 3\alpha_2 - 4\alpha_3\}$ since τ lies in the same W-orbit with η. All other weights $\delta \in \Omega_1 \cup \Omega_2^+ \cup \Omega_3^+$ lie in the same W-orbit with weights from Σ or ν, hence $\dim M_\delta = 1$. Now one easily concludes that $\dim U_1 = 8$, $\dim U_{2,5} = 1$, $\dim U_{2,3} = 2$, $\dim U_{2,1} = 4$, $\dim U_{3,4} = 2$, $\dim U_{3,2} = 4$, and $\dim U_{3,0} = 4$. Arguing as for the analysis of the A-module U_1 in the previous item, we can show that $U_1 \cong U_5 \cong M((1,3))$. To determine the block structure of x on the module $U_0 \oplus U_1 \oplus U_5 \oplus U_6$, we use Lemma 21 and Theorem 3.

The arguments above imply that $I(U_2|A) = \{M(5), 2M(3), 2M(1)\}$ and $I(U_3|A) = \{2M(4), 2M(2)\}$. By Lemma 27, $U_3|A \cong 2M(4) \oplus 2M(2)$ and $U_2|A \cong N \oplus 2M(1)$ where $I(N) = \{M(5), 2M(3)\}$. As $U_4 \cong U_2^*$, we have $U_4 \cong N' \oplus 2M(1)$ where $N' \cong N^*$. Set $m_2 = X_{-2}^2X_{-3}v$, $w = X_{-3}^3X_{-2}v$, $m_4 = X_{-2}^3X_{-1}w$, and $\Gamma = G(1,2)$. It is clear that $m_2 \in N$ and $m_4 \in N'$. By Lemma 12, m_2 and $w \neq 0$ and the vector w is fixed by $\mathfrak{X}_1$ and $\mathfrak{X}_2$. Therefore w generates an indecomposable Γ-module S with highest weight $\omega(w)|\Gamma$. One easily observes that m_4 coincides with the vector $w(2,1,1)$ constructed in the module S. Applying Lemma 12 to S, we deduce that the vector $m_4 \neq 0$ and is fixed by $\mathfrak{X}_1$. One easily observes that $\langle \omega(m_4), \alpha_1 \rangle = 2$. By Lemma 6, $X_{-1}m_2$ and $X_{-1}m_4 \neq 0$.

Hence $X_{-\alpha}m_i \neq 0$ for $i = 2$ or 4. By Lemma 28, $KAm_2 \cong KAm_4 \cong V(5)$. As the module $N \oplus N'$ is self-dual, now Lemma 24 yields that $N \cong N' \cong T(5)$.

Let $J(x) = (3^2, 1)$. Set $H = G(1,2)$. It is clear that $H \cong A_2(K)$ and we can assume that $x \in H$. Let $\Omega_i = \{\lambda \in \Lambda(M) | \lambda = \omega - a\alpha_1 - b\alpha_2 - i\alpha_3\}$ and $U_i = \langle M_\mu | \mu \in \Omega_i \rangle$. We define the parameter $i(M)$ as above. Show that in both cases being considered the H-modules U_i are irreducible.

1) Let $\omega = \omega_1 + \omega_2$ and $p = 3$. Then $i(M) = 6$. Set $\nu_0 = \omega$, $\nu_1 = \omega - \alpha_2 - \alpha_3$, $\nu_2 = \omega - \alpha_2 - 2\alpha_3$, $\nu_3 = \omega - \alpha_1 - 2\alpha_2 - 3\alpha_3$, and $\mu_i = \nu_i | H$. It is clear that $\mu_i = \omega_1 + \omega_2$ for $i = 0$ and 3, $\mu_1 = 2\omega_1$, and $\mu_2 = 2\omega_1 + \omega_2$.

2) Let $\omega = \omega_2 + \omega_3$ and $p = 5$. Then $i(M) = 7$. Set $\nu_0 = \omega$, $\nu_1 = \omega - \alpha_3$, $\nu_2 = \omega - \alpha_2 - 2\alpha_3$, $\nu_3 = \omega - \alpha_2 - 3\alpha_3$, and $\mu_i = \nu_i | H$. We have $\mu_0 = \omega_2$, $\mu_1 = 2\omega_2$, $\mu_2 = \omega_1 + \omega_2$, and $\mu_3 = \omega_1 + 2\omega_2$.

Corollary 4 and Lemma 11 imply that in Case 1) $U_0 \cong U_6 \cong M(\mu_0)$, $U_5 \cong U_1^*$, and $U_4 \cong U_2^*$, in Case 2) $U_0 \cong U_7 \cong M(\mu_0)$, $U_6 \cong U_1^*$, $U_5 \cong U_2^*$, and $U_4 \cong U_3^*$. In both cases one easily concludes that $\nu_i \in \Omega_i$ and $\nu_i + k\alpha_j \notin \Lambda(M)$ for $j = 1, 2$ and $k > 0$. This yields that the H-modules U_i have composition factors $M(\mu_i)$ for the values of i indicated above. Knowing the dimensions of the modules M and $M(\mu_i)$, we can show that in both cases all modules U_i are irreducible. To determine the block structure of $\varphi(x)$, we use the results of Item I of Section 4.

So for $G = B_n(K)$ all possibilities have been considered.

II.V. Let $G = D_4(K)$. The arguments above imply that it suffices to consider the representations with highest weights $2\omega_4$, $\omega_1 + \omega_4$, and $\omega_3 + \omega_4$. Recall that $\sigma_x(\omega_4) < p$ for any element x of order p. Therefore by Lemma 14, $M | A_x$ is a tilting module for $\omega = 2\omega_4$.

Lemma 33. *The following formulae hold:*

$$M(\omega_1) \otimes M(\omega_4) = M(\omega_1 + \omega_4) \oplus M(\omega_3);$$
$$M(\omega_3) \otimes M(\omega_4) = M(\omega_3 + \omega_4) \oplus M(\omega_1).$$

Proof. Let $\omega = \omega_1 + \omega_4$ and $\lambda = \omega_3$. It is clear that $\lambda = \omega - \alpha_1 - \alpha_2 - \alpha_4$. Set $N = M(\omega_1) \otimes M(\omega_4)$. One easily observes that $\dim N_\lambda = 4$ and $\dim M(\omega)_\lambda \leq 3$. This implies that N has a composition factor isomorphic to $M(\lambda)$. The dimension arguments show that N has exactly two composition factors: $M(\omega)$ and $M(\lambda)$. Since the modules $V(\omega)$ and $V(\lambda)$ are irreducible, Lemma 3 forces that $N = M(\omega) \oplus M(\lambda)$.
Similar arguments are used to proof the second equality. □

To determine the block structure of x on $M(\omega)$ for $\omega = \omega_1 + \omega_4$ or $\omega_3 + \omega_4$, we apply Lemma 33, Theorem 3, and the results of Item I.

Now all p-restricted representations of the classical groups in odd characteristic whose dimensions ≤ 100 have been considered.

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Appendix

TABLES: THE BLOCK STRUCTURE OF THE IMAGES OF UNIPOTENT ELEMENTS OF SIMPLE ORDER IN SOME IRREDUCIBLE MODULAR REPRESENTATIONS OF CLASSICAL GROUPS

The tables below show the block structure of the images of unipotent elements in p-restricted irreducible representations of the classical algebraic groups whose dimensions are smaller than 100. If two representations can be obtained from each other with the help of a group automorphism, we indicate only one of them.

In what follows if $x \in GL(n, K)$ is a unipotent element having k_1 Jordan blocks of dimension d_1, k_2 blocks of dimension d_2, ..., k_t blocks of dimension d_t with $d_1 > d_2 > ... > d_t$ and $k_1d_1 + k_2d_2 + ... + k_td_t = n$, we write $J(x) = (d_1^{k_1}, ..., d_t^{k_t})$. In the tables n is the dimension and ω is the highest weight of a representation φ, $J(x)$ is determined by the standard realization of a group.

Table 1. $G = A_2(K)$

n	ω	p	$J(\varphi(x))$ for $J(x) = (3)$	$J(\varphi(x))$ for $J(x) = (2,1)$
6	$2\omega_1$	3	(3^2)	$(3,2,1)$
		>3	$(5,1)$	
7	$\omega_1+\omega_2$	3	$(3^2,1)$	$(3,2^2)$
8	$\omega_1+\omega_2$	≥ 5	(5,3)	$(3,2^2,1)$
10	$3\omega_1$	5	(5^2)	(4,3,2,1)
		>6	(7,3)	
15	$4\omega_1$	5	(5^3)	(5,4,3,2,1)
		7	$(7^2,1)$	
		≥ 11	(9,5,1)	
15	$2\omega_1+\omega_2$	3	(3^5)	$(3^4,2,1)$
		5	(5^3)	$(4,3^2,2^2,1)$
		≥ 7	$(7,5,3)$	
18	$3\omega_1+\omega_2$	5	$(5^3,3)$	$(5,4^2,3,2)$
19	$2\omega_1+2\omega_2$	5	$(5^3,3,1)$	$(5,4^2,3^2)$
21	$5\omega_1$	7	(7^3)	(6,5,4,3,2,1)
		≥ 11	(11,7,3)	
24	$3\omega_1+\omega_2$	7	$(7^3,3)$	$(5,4^2,3^2,2^2,1)$
		≥ 11	(9,7,5,3)	
27	$2\omega_1+2\omega_2$	3	(3^9)	(3^9)
		7	$(7^3,5,1)$	$(5, 4^2, 3^3, 2^2,1)$
		≥ 11	$(9,7,5^2,1)$	
28	$6\omega_1$	7	(7^4)	(7,6,5,4,3,2,1)
		11	$(11^2,5,1)$	
		≥ 13	(13,9,5,1)	
33	$5\omega_1+\omega_2$	7	$(7^4,5)$	$(7,6^2,5,4,3,2)$
35	$4\omega_1+\omega_2$	5	(5^7)	$(5^4,4,3^2,2^2,1)$
		7	(7^5)	$(6,5^2,4^2,3^2,2^2,1)$
		≥ 11	(11,9,7,5,3)	

Table 1 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(3)$	$J(\varphi(x))$ for $J(x)=(2,1)$
36	$7\omega_1$	11	$(11^3,3)$	(8,7,6,5,4,3,2,1)
		13	$(13^2,7,3)$	
		≥ 17	(15,11,7,3)	
36	$4\omega_1+2\omega_2$	7	$(7^4,5,3)$	$(7,6^2,5^2,4,3)$
37	$3\omega_1+3\omega_2$	7	$(7^4,5,3,1)$	$(7,6^2,5^2,4^2)$
39	$3\omega_1+2\omega_2$	5	$(5^7,3,1)$	$(5^4,4^2,3^3,2)$
42	$3\omega_1+2\omega_2$	7	(7^6)	$(6,5^2,4^3,3^3,2^2,1)$
		≥ 11	$(11,9,7^2,5,3)$	
45	$8\omega_1$	11	$(11^4,1)$	(9,8,7,6,5,4,3,2,1)
		13	$(13^3,5,1)$	
		≥ 17	(17,13,9,5,1)	
48	$5\omega_1+\omega_2$	11	$(11^3,7,5,3)$	$(7,6^2,5^2,4^2,3^2,2^2,1)$
		≥ 13	(13,11,9,7,5,3)	
55	$9\omega_1$	11	(11^5)	$(10,9,8,7,6,5,4,3,2,1)$
		13	$(13^4,3)$	
		17	$(17^2,11,7,3)$	
		≥ 19	$(19,15,11,7,3)$	
60	$4\omega_1+2\omega_2$	5	(5^{12})	$(5^9,4,3^2,2^2,1)$
		11	$(11^3,9,7,5^2,1)$	$(7,6^2,5^3,4^3,3^3,2^2,1)$
		≥ 13	$(13,11,9^2,7,5^2,1)$	
63	$6\omega_1+\omega_2$	7	(7^9)	$(7^4,6,5^2,4^2,3^2,2^2,1)$
		11	$(11^5,5,3)$	$(8,7^2,6^2,5^2,4^2,3^2,2^2,1)$
		13	$(13^3,9,7,5,3)$	
		≥ 17	$(15,13,11,9,7,5,3)$	
63	$3\omega_1+3\omega_2$	5	$(5^{12},3)$	$(5^9,4^2,3^2,2^2)$
64	$3\omega_1+3\omega_2$	11	$(11^3,9,7^2,5,3)$	$(7,6^2,5^3,4^4,3^3,2^2,1)$
		≥ 13	$(13,11,9^2,7^2,5,3)$	
66	$10\omega_1$	11	(11^6)	$(11,10,9,8,7,6,5,4,3,2,1)$
		13	$(13^5,1)$	
		17	$(17^3,9,5,1)$	
		19	$(19^2,13,9,5,1)$	
		≥ 23	$(21,17,13,9,5,1)$	
71	$5\omega_1+2\omega_2$	7	$(7^9,5,3)$	$(7^4,6^2,5^3,4^2,3^2,2)$
75	$9\omega_1+\omega_2$	11	$(11^6,9)$	$(11,10^2,9,8,7,6,5,4,3,2)$
75	$4\omega_1+3\omega_2$	7	$(7^9,5,3^2,1)$	$(7^4,6^2,5^4,4^3,3)$
78	$11\omega_1$	13	(13^6)	$(12,11,10,9,8,7,6,5,4,3,2,1)$
		17	$(17^4,7,3)$	
		19	$(19^3,11,7,3)$	
		≥ 23	$(23,19,15,11,7,3)$	
80	$7\omega_1+\omega_2$	11	$(11^7,3)$	$(9,8^2,7^2,6^2,5^2,4^2,3^2,2^2,1)$
		13	$(13^5,7,5,3)$	
		≥ 17	$(17,15,13,11,9,7,5,3)$	
81	$5\omega_1+2\omega_2$	11	$(11^6,7,5,3)$	$(8,7^2,6^3,5^3,4^3,3^3,2^2,1)$
		13	$(13^3,11,9,7^2,5,3)$	
		≥ 17	$(15,13,11^2,9,7^2,5,3)$	
82	$8\omega_1+2\omega_2$	11	$(11^6,9,7)$	$(11,10^2,9^2,8,7,6,5,4,3)$
87	$7\omega_1+3\omega_2$	11	$(11^6,9,7,5)$	$(11,10^2,9^2,8^2,7,6,5,4)$

Table 1 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(3)$	$J(\varphi(x))$ for $J(x)=(2,1)$
90	$4\omega_1+3\omega_2$	5	(5^{18})	$(5^{16},4,3,2,1)$
		11	$(11^6,9,7,5,3)$	$(8,7^2,6^3,5^4,4^4,3^3,2^2,1)$
		13	$(13^3,11,9^2,7^2,5,3)$	
		≥ 17	$(15,13,11^2,9^2,7^2,5,3)$	
90	$6\omega_1+4\omega_2$	11	$(11^6,9,7,5,3)$	$(11,10^2,9^2,8^2,7^2,6,5)$
91	$12\omega_1$	13	(13^7)	$(13,12,11,10,9,8,7,6,5,4,3,2,1)$
		17	$(17^5,5,1)$	
		19	$(19^4,9,5,1)$	
		23	$(23^2,17,13,9,5,1)$	
		≥ 29	$(25,21,17,13,9,5,1)$	
91	$5\omega_1+5\omega_2$	11	$(11^6,9,7,5,3,1)$	$(11,10^2,9^2,8^2,7^2,6^2)$
99	$8\omega_1+\omega_2$	11	(11^9)	$(10,9^2,8^2,7^2,6^2,5^2,4^2,3^2,2^2,1)$
		13	$(13^7,5,3)$	
		17	$(17^3,13,11,9,7,5,3)$	
		≥ 19	$(19,17,15,13,11,9,7,5,3)$	

TABLE 2. $G = A_3(K)$

n	ω	p	$J(\varphi(x))$ for $J(x) = (4)$	$J(\varphi(x))$ for $J(x) = (3,1)$	$J(\varphi(x))$ for $J(x) = (2^2)$	$J(\varphi(x))$ for $J(x) = (2,1^2)$
6	ω_2	≥ 3	-	-	$(3,1^3)$	$(2^2,1^2)$
		≥ 5	$(5,1)$	(3^2)		
10	$2\omega_1$	3	-	$(3^3,1)$	$(3^3,1)$	$(3,2^2,1^3)$
		5	(5^2)	$(5,3,1^2)$		
		≥ 7	$(7,3)$			
15	$\omega_1+\omega_3$	3	-	(3^5)	$(3^4,1^3)$	$(3,2^4,1^4)$
		5	(5^3)	$(5,3^3,1)$		
		≥ 7	$(7,5,3)$			
16	$\omega_1+\omega_2$	3	-	$(3^5,1)$	$(3^4,2^2)$	$(3^2,2^4,1^2)$
19	$2\omega_2$	3	-	$(3^6,1)$	$(3^5,1^4)$	$(3^3,2^4,1^2)$
20	$3\omega_1$	5	(5^4)	$(5^3,3,1^2)$	$(4^4,2^2)$	$(4,3^2,2^3,1^4)$
		7	$(7^2,6)$	$(7,5,3^2,1^2)$		
		≥ 11	$(10,6,4)$			
20	$\omega_1+\omega_2$	5	(5^4)	$(5^2,3^3,1)$	$(4^2,2^6)$	$(3^2,2^5,1^4)$
		7	$(7^2,4,2)$			
		≥ 11	$(8,6,4,2)$			
20	$2\omega_2$	5	(5^4)	$(5^3,3,1^2)$	$(5,3^3,1^6)$	$(3^3,2^4,1^3)$
		7	$(7^2,5,1)$			
		≥ 11	$(9,5^2,1)$			
32	$2\omega_1+\omega_3$	5	$(5^6,2)$	$(5^5,3^2,1)$	$(4^6,2^4)$	$(4,3^4,2^6,1^4)$
35	$4\omega_1$	5	(5^7)	$(5^6,3,1^2)$	$(5^5,3^3,1)$	$(5,4^2,3^3,2^4,1^5)$
		7	(7^5)	$(7^3,5,3^2,1^3)$		
		11	$(11^2,7,5,1)$	$(9,7,5^2,3^2,1^3)$		
		≥ 13	$(13,9,7,5,1)$			
36	$2\omega_1+\omega_3$	3	-	(3^{12})	(3^{12})	$(3^6,2^6,1^6)$
		7	$(7^4,6,2)$	$(7,5^3,3^4,1^2)$	$(4^6,2^6)$	$(4,3^4,2^7,1^6)$
		≥ 11	$(10,8,6^2,4,2)$			
44	$\omega_1+\omega_2+\omega_3$	3	-	$(3^{14},1^2)$	$(3^{13},1^5)$	$(3^{10},2^6,1^2)$

Table 2 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(4)$	$J(\varphi(x))$ for $J(x)=(3,1)$	$J(\varphi(x))$ for $J(x)=(2^2)$	$J(\varphi(x))$ for $J(x)=(2,1^2)$
45	$2\omega_1+\omega_2$	3	-	(3^{15})	(3^{15})	$(3^9,2^6,1^6)$
		5	(5^9)	$(5^7,3^3,1)$	$(5^3,3^9,1^3)$	$(4^2,3^5,2^8,1^6)$
		7	$(7^6,3)$	$(7^2,5^3,3^5,1)$		
		≥ 11	$(11,9,7^2,5,3^2)$			
50	$3\omega_2$	5	(5^{10})	(5^{10})	$(5^5,3^5,1^{10})$	$(4^4,3^6,2^6,1^4)$
		7	$(7^7,1)$	$(7^4,5^2,3^4)$	$(7,5^3,3^6,1^{10})$	
		11	$(11^2,9,7,5^2,1^2)$			
		≥ 13	$(13,9^2,7,5^2,1^2)$			
52	$3\omega_1+\omega_2$	5	$(5^{10},2)$	$(5^9,3^2,1)$	$(5^8,4^2,2^2)$	$(5^2,4^4,3^4,2^5,1^4)$
56	$5\omega_1$	7	(7^8)	$(7^6,5,3^2,1^3)$	$(6^6,4^4,2^2)$	$(6,5^2,4^3,3^4,2^5,1^6)$
		11	$(11^4,8,4)$	$(11,9,7^2,5^2,3^3,1^3)$		
		13	$(13^2,12,8,6,4)$			
		≥ 17	$(16,12,10,8,6,4)$			
58	$\omega_1+\omega_2+\omega_3$	5	$(5^{11},3)$	$(5^{10},3^2,1^2)$	$(5^4,3^{11},1^5)$	$(4^2,3^6,2^{14},1^4)$
60	$\omega_1+2\omega_2$	3	-	(3^{20})	$(3^{16},2^6)$	$(3^{14},2^6,1^6)$
		5	(5^{12})	$(5^{11},3,1^2)$	$(5^4,4^4,2^{12})$	$(4^3,3^8,2^9,1^6)$
		7	$(7^8,4)$	$(7^3,5^5,3^4,1^2)$	$(6^2,4^6,2^{12})$	
		11	$(11^2,8^2,6^2,4^2,2)$			
		≥ 13	$(12,10,8^2,6^2,4^2,2)$			
64	$\omega_1+\omega_2+\omega_3$	7	$(7^8,5,3)$	$(7^2,5^6,3^6,1^2)$	$(5^4,3^{12},1^8)$	$(4^2,3^8,2^{12},1^8)$
		≥ 11	$(11,9^2,7^2,5^3,3^2)$			
68	$2\omega_1+2\omega_2$	5	$(5^{13},3)$	$(5^{12},3^2,1^2)$	$(5^{11},3^4,1)$	$(5^3,4^6,3^6,2^4,1^3)$
69	$2\omega_1+2\omega_3$	3	-	(3^{23})	$(3^{22},1^3)$	$(3^{19},2^4,1^4)$
70	$3\omega_1+\omega_3$	5	(5^{14})	$(5^{12},3^3,1)$	$(5^8,3^9,1^3)$	$(5,4^4,3^7,2^{10},1^8)$
		7	(7^{10})	$(7^5,5^3,3^6,1^2)$		
		11	$(11^3,9,7^2,5^2,3,1)$	$(9,7^3,5^4,3^6,1^2)$		
		≥ 13	$(13,11,9^2,7^2,5^2,3,1)$			
80	$\omega_1+3\omega_2$	5	(5^{16})	$(5^{14},3^3,1)$	$(5^{12},4^2,2^6)$	$(5^4,4^8,3^6,2^4,1^2)$

Table 2 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(4)$	$J(\varphi(x))$ for $J(x)=(3,1)$	$J(\varphi(x))$ for $J(x)=(2^2)$	$J(\varphi(x))$ for $J(x)=(2,1^2)$
83	$2\omega_1+2\omega_3$	5	$(5^{16},3)$	$(5^{15},3^2,1^2)$	$(5^9,3^{11},1^5)$	$(5,4^4,3^{10},2^{12},1^8)$
84	$6\omega_1$	7	(7^{12})	$(7^{10},5,3^2,1^3)$	$(7^7,5^5,3^3,1)$	$(7,6^2,5^3,4^4,3^5,2^6,1^7)$
		11	$(11^7,7)$	$(11^3,9,7^2,5^3,3^3,1^4)$		
		13	$(13^5,9,7,3)$	$(13,11,9^2,7^2,5^3,3^3,1^4)$		
		17	$(17^2,13,11,9,7^2,3)$			
		≥ 19	$(19,15,13,11,9,7^2,3)$			
84	$3\omega_1+\omega_2$	7	(7^{12})	$(7^7,5^3,3^6,1^2)$	$(6^4,4^{12},2^6)$	$(5^2,4^5,3^8,2^{11},1^8)$
		11	$(11^4,10,8,6^2,4^2,2)$	$(9^2,7^3,5^5,3^6,1^2)$		
		13	$(13^2,10^2,8^2,6^2,4^2,2)$			
		≥ 17	$(14,12,10^2,8^2,6^2,4^2,2)$			
84	$2\omega_1+2\omega_3$	7	(7^{12})	$(7^5,5^6,3^5,1^4)$	$(5^9,3^{11},1^6)$	$(5,4^4,3^{10},2^{12},1^9)$
		11	$(11^3,9^2,7^2,5^3,3,1)$	$(9,7^3,5^7,3^5,1^4)$		
		≥ 13	$(13,11,9^3,7^2,5^3,3,1)$			
85	$4\omega_2$	5	(5^{17})	$(5^{15},3^3,1)$	$(5^{13},3^4,1^8)$	$(5^5,4^8,3^6,2^4,1^2)$
100	$4\omega_1+\omega_3$	7	$(7^{14},2)$	$(7^{11},5^2,3^4,1)$	$(6^{10},4^9,2^2)$	$(6,5^4,4^6,3^8,2^{10},1^6)$

TABLE 3. $G=A_4(K)$

n	ω	p	$J(\varphi(x))$ for $J(x)=(5)$	$J(\varphi(x))$ for $J(x)=(4,1)$	$J(\varphi(x))$ for $J(x)=(3,2)$
10	ω_2	3	-	-	$(3^3,1)$
		5	(5^2)	$(5,4,1)$	$(4,3,2,1)$
		≥ 7	$(7,3)$		
15	$2\omega_1$	3	-	-	(3^5)
		5	(5^3)	$(5^2,4,1)$	$(5,4,3,2,1)$
		7	$(7^2,1)$	$(7,4,3,1)$	
		≥ 11	$(9,5,1)$		

Table 3 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(5)$	$J(\varphi(x))$ for $J(x)=(4,1)$	$J(\varphi(x))$ for $J(x)=(3,2)$
23	$\omega_1+\omega_4$	5	$(5^4,3)$	$(5^3,4^2)$	$(5,4^2,3^2,2^2)$
24	$\omega_1+\omega_4$	3	-	-	(3^8)
		7	$(7^3,3)$	$(7,5,4^2,3,1)$	$(5,4^2,3^2,2^2,1)$
		≥ 11	$(9,7,5,3)$		
30	$\omega_1+\omega_2$	3	-	-	$(3^9,2,1)$
35	$3\omega_1$	5	(5^7)	$(5^6,4,1)$	$(5^5,4,3,2,1)$
		7	(7^5)	$(7^3,6,4,3,1)$	$(7,6,5,4^2,3^2,2,1)$
		11	$(11^2,7,5,1)$	$(10,7,6,4^2,3,1)$	
		≥ 13	$(13,9,7,5,1)$		
40	$\omega_1+\omega_2$	5	(5^8)	$(5^7,4,1)$	$(5^4,4,3^3,2^3,1)$
		7	$(7^5,5)$	$(7^3,5,4^2,3,2,1)$	$(6,5^2,4^2,3^3,2^3,1)$
		≥ 11	$(11,9,7,5^2,3)$	$(8,7,6,5,4^2,3,2,1)$	
45	$2\omega_2$	3	-	-	$(3^{14},2,1)$
45	$\omega_1+\omega_3$	3	-	-	(3^{15})
		5	(5^9)	$(5^8,4,1)$	$(5^4,4^2,3^3,2^3,1^2)$
		7	$(7^6,3)$	$(7^3,5^2,4^2,3,2,1)$	$(6,5^2,4^3,3^3,2^3,1^2)$
		≥ 11	$(11,9,7^2,5,3^2)$	$(8,7,6,5^2,4^2,3,2,1)$	
50	$2\omega_2$	5	(5^{10})	(5^{10})	$(5^6,4^2,3^2,2^2,1^2)$
		7	$(7^7,1)$	$(7^5,5,4,3,2,1)$	$(7,6,5^2,4^3,3^3,2^2,1^2)$
		11	$(11^2,9,7,5^2,1^2)$	$(9,8,7,6,5^2,4,3,2,1)$	
		≥ 13	$(13,9^2,7,5^2,1^2)$		
51	$\omega_2+\omega_3$	3	-	-	(3^{17})
65	$2\omega_1+\omega_4$	3	-	-	$(3^{19},2^4)$
70	$4\omega_1$	5	(5^{14})	$(5^{13},4,1)$	$(5^{13},3,2)$
		7	(7^{10})	$(7^8,6,4,3,1)$	$(7^5,6,5^2,4^2,3^2,2^2,1)$
		11	$(11^5,9,5,1)$	$(11^2,10,7^2,6,5,4^2,3,1^2)$	$(9,8,7,6^2,5^3,4^2,3^2,2^2,1)$
		13	$(13^3,11,9,5^2,1)$	$(13,10,9,7^2,6,5,4^2,3,1^2)$	
		≥ 17	$(17,13,11,9^2,5^2,1)$		

Table 3 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(5)$	$J(\varphi(x))$ for $J(x)=(4,1)$	$J(\varphi(x))$ for $J(x)=(3,2)$
70	$2\omega_1+\omega_4$	5	(5^{14})	$(5^{13},4,1)$	$(5^9,4^2,3^3,2^3,1^2)$
		7	(7^{10})	$(7^6,6,5,4^2,3^2,2,1)$	$(7,6^2,5^3,4^4,3^4,2^3,1^2)$
		11	$(11^3,9,7^2,5^2,3,1)$	$(10,8,7^2,6^2,5,4^3,3^2,2,1)$	
		≥ 13	$(13,11,9^2,7^2,5^2,3,1)$		
75	$\omega_2+\omega_3$	5	(5^{15})	(5^{15})	$(5^9,4^2,3^4,2^4,1^2)$
		7	$(7^{10},5)$	$(7^7,5^2,4^2,3,2^2,1)$	$(7,6^2,5^3,4^4,3^5,2^4,1^2)$
		11	$(11^3,9,7^2,5^3,3,1)$	$(9,8^2,7,6^2,5^3,4^2,3,2^2,1)$	
		≥ 13	$(13,11,9^2,7^2,5^3,3,1)$		

Table 3 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(3,1^2)$	$J(\varphi(x))$ for $J(x)=(2^2,1)$	$J(\varphi(x))$ for $J(x)=(2,1^3)$
10	ω_2	≥ 3	$(3^3,1)$	$(3,2^2,1^3)$	$(2^3,1^4)$
15	$2\omega_1$	3	$(3^4,1^3)$	$(3^3,2^2,1^2)$	$(3,2^3,1^6)$
		≥ 5	$(5,3^2,1^4)$		
23	$\omega_1+\omega_4$	5	$(5,3^5,1^3)$	$(3^4,2^4,1^3)$	$(3,2^6,1^8)$
24	$\omega_1+\omega_4$	3	$(3^7,1^3)$	$(3^4,2^4,1^4)$	$(3,2^6,1^9)$
		≥ 7	$(5,3^5,1^4)$		
30	$\omega_1+\omega_2$	3	$(3^9,1^3)$	$(3^5,2^6,1^3)$	$(3^3,2^7,1^7)$
35	$3\omega_1$	5	$(5^4,3^3,1^6)$	$(4^4,3^3,2^4,1^2)$	$(4,3^3,2^6,1^{10})$
		≥ 7	$(7,5^2,3^4,1^6)$		
40	$\omega_1+\omega_2$	≥ 5	$(5^3,3^7,1^4)$	$(4^2,3^4,2^8,1^4)$	$(3^3,2^{10},1^{11})$
45	$2\omega_2$	3	$(3^{14},1^3)$	$(3^{12},2^2,1^5)$	$(3^6,2^{10},1^7)$
45	$\omega_1+\omega_3$	3	(3^{15})	$(3^9,2^6,1^6)$	$(3^3,2^{12},1^{12})$
		≥ 5	$(5^3,3^9,1^3)$	$(4^2,3^5,2^8,1^6)$	
50	$2\omega_2$	$\neq 3$	$(5^6,3^5,1^5)$	$(5,4^2,3^6,2^6,1^7)$	$(3^6,2^{11},1^{10})$
51	$\omega_2+\omega_3$	3	$(3^{16},1^3)$	$(3^{13},2^4,1^4)$	$(3^7,2^{12},1^6)$

Table 3 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(3,1^2)$	$J(\varphi(x))$ for $J(x)=(2^2,1)$	$J(\varphi(x))$ for $J(x)=(2,1^3)$
65	$2\omega_1+\omega_4$	3	$(3^{19},1^8)$	$(3^{15},2^7,1^6)$	$(3^9,2^{12},1^{14})$
70	$4\omega_1$	5	$(5^{10},3^4,1^8)$	$(5^5,4^4,3^6,2^4,1^3)$	$(5,4^3,3^7,2^9,1^{14})$
		7	$(7^4,5^3,3^6,1^9)$		
		≥ 11	$(9,7^2,5^4,3^6,1^9)$		
70	$2\omega_1+\omega_4$	5	$(5^7,3^9,1^8)$	$(4^6,3^7,2^{10},1^5)$	$(4,3^6,2^{15},1^{18})$
		≥ 7	$(7,5^5,3^{10},1^8)$		
75	$\omega_2+\omega_3$	≥ 5	$(5^8,3^{10},1^5)$	$(5,4^4,3^7,2^{12},1^9)$	$(3^8,2^{18},1^{15})$

TABLE 4. $G=A_5(K)$

n	ω	p	$J(\varphi(x))$ for $J(x)=(6)$	$J(\varphi(x))$ for $J(x)=(5,1)$	$J(\varphi(x))$ for $J(x)=(4,2)$	$J(\varphi(x))$ for $J(x)=(4,1^2)$
15	ω_2	5	-	(5^3)	$(5^2,3,1^2)$	$(5,4^2,1^2)$
		7	$(7^2,1)$	$(7,5,3)$		
		≥ 11	$(9,5,1)$			
20	ω_3	5	-	(5^4)	$(5^2,4^2,2)$	$(5^2,4^2,1^2)$
		7	$(7^2,6)$	$(7^2,3^2)$	$(6,4^3,2)$	
		≥ 11	$(10,6,4)$			
21	$2\omega_1$	5	-	$(5^4,1)$	$(5^3,3^2)$	$(5^2,4^2,1^3)$
		7	(7^3)	$(7^2,5,1^2)$	$(7,5,3^3)$	$(7,4^2,3,1^3)$
		≥ 11	$(11,7,3)$	$(9,5^2,1^2)$		
35	$\omega_1+\omega_5$	5	-	(5^7)	$(5^5,3^3,1)$	$(5^3,4^4,1^4)$
		7	(7^5)	$(7^3,5^2,3,1)$	$(7,5^3,3^4,1)$	$(7,5,4^4,3,1^4)$
		≥ 11	$(11,9,7,5,3)$	$(9,7,5^3,3,1)$		

Table 4 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(6)$	$J(\varphi(x))$ for $J(x)=(5,1)$	$J(\varphi(x))$ for $J(x)=(4,2)$	$J(\varphi(x))$ for $J(x)=(4,1^2)$
56	$3\omega_1$	5	-	$(5^{11},1)$	$(5^{10},4,2)$	$(5^8,4^3,1^4)$
		7	(7^8)	$(7^7,5,1^2)$	$(7^4,6^2,4^3,2^2)$	$(7^4,6,4^3,3^2,1^4)$
		11	$(11^4,8,4)$	$(11^2,9,7,5^3,1^3)$	$(10,8,6^3,4^4,2^2)$	$(10,7^2,6,4^4,3^2,1^4)$
		13	$(13^2,12,8,6,4)$	$(13,9^2,7,5^3,1^3)$		
		≥ 17	$(16,12,10,8,6,4)$			
70	$\omega_1+\omega_2$	5	-	(5^{14})	$(5^{12},4,2^3)$	$(5^{10},4^4,1^4)$
		7	(7^{10})	$(7^8,5^2,3,1)$	$(7^4,6^2,4^5,2^5)$	$(7^4,5^2,4^5,3^2,2,1^4)$
		11	$(11^4,8,6^2,4,2)$	$(11,9^2,7^2,5^4,3^2,1)$	$(8^2,6^4,4^5,2^5)$	$(8,7^2,6,5^2,4^5,3^2,2,1^4)$
		13	$(13^2,10,8^2,6^2,4,2)$			
		≥ 17	$(14,12,10,8^2,6^2,4,2)$			
78	$\omega_1+\omega_4$	5	-	$(5^{15},3)$	$(5^{14},4,2^2)$	$(5^{12},4^4,1^2)$
84	$\omega_1+\omega_4$	7	(7^{12})	$(7^{10},5,3^3)$	$(7^4,6^3,4^7,2^5)$	$(7^4,5^4,4^6,3^2,2,1^4)$
		11	$(11^4,10,8,6^2,4^2,2)$	$(11,9^2,7^4,5^3,3^4)$	$(8^2,6^5,4^7,2^5)$	$(8,7^2,6,5^4,4^6,3^2,2,1^4)$
		13	$(13^2,10^2,8^2,6^2,4^2,2)$			
		≥ 17	$(14,12,10^2,8^2,6^2,4^2,2)$			

Table 4 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(3^2)$	$J(\varphi(x))$ for $J(x)=(3,2,1)$	$J(\varphi(x))$ for $J(x)=(3,1^3)$	$J(\varphi(x))$ for $J(x)=(2^3)$	$J(\varphi(x))$ for $J(x)=(2^2,1^2)$	$J(\varphi(x))$ for $J(x)=(2,1^4)$
15	ω_2	3	(3^5)	$(3^4,2,1)$	$(3^4,1^3)$	$(3^3,1^6)$	$(3,2^4,1^4)$	$(2^4,1^7)$
		≥ 5	$(5,3^3,1)$	$(4,3^2,2^2,1)$				
20	ω_3	3	$(3^6,1^2)$	$(3^6,1^2)$	$(3^6,1^2)$	$(3^2,2^7)$	$(3^2,2^4,1^6)$	$(2^6,1^8)$
		≥ 5	$(5^2,3^2,1^4)$	$(4^2,3^2,2^2,1^2)$		$(4,2^8)$		
21	$2\omega_1$	3	(3^7)	$(3^6,2,1)$	$(3^5,1^6)$	$(3^6,1^3)$	$(3^3,2^4,1^4)$	$(3,2^4,1^{10})$
		≥ 5	$(5^3,3,1^3)$	$(5,4,3^2,2^2,1^2)$	$(5,3^3,1^7)$			
34	$\omega_1+\omega_5$	3	$(3^{11},1)$	$(3^{10},2^2)$	$(3^9,1^7)$	$(3^9,1^7)$	$(3^4,2^8,1^6)$	$(3,2^8,1^{15})$

Table 4 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(3^2)$	$J(\varphi(x))$ for $J(x)=(3,2,1)$	$J(\varphi(x))$ for $J(x)=(3,1^3)$	$J(\varphi(x))$ for $J(x)=(2^3)$	$J(\varphi(x))$ for $J(x)=(2^2,1^2)$	$J(\varphi(x))$ for $J(x)=(2,1^4)$
35	$\omega_1+\omega_5$	≥ 5	$(5^4,3^4,1^3)$	$(5,4^2,3^4,2^4,1^2)$	$(5,3^7,1^9)$	$(3^9,1^8)$	$(3^4,2^8,1^7)$	$(3,2^8,1^{16})$
50	$\omega_1+\omega_2$	3	$(3^{16},1^2)$	$(3^{15},2^2,1)$	$(3^{14},1^8)$	$(3^{14},2^4)$	$(3^{10},2^8,1^4)$	$(3^4,2^{11},1^{16})$
56	$3\omega_1$	5	$(5^{10},3^2)$	$(5^6,4^2,3^3,2^3,1^3)$	$(5^5,3^6,1^{13})$	$(4^{10},2^8)$	$(4^4,3^6,2^8,1^6)$	$(4,3^4,2^{10},1^{20})$
		≥ 7	$(7^4,5^2,3^6)$	$(7,6,5^2,4^3,3^4,2^3,1^3)$	$(7,5^3,3^7,1^{13})$			
70	$\omega_1+\omega_2$	5	$(5^{10},3^6,1^2)$	$(5^5,4^3,3^6,2^6,1^3)$	$(5^4,3^{13},1^{11})$	$(4^8,2^{19})$	$(4^2,3^8,2^{14},1^{10})$	$(3^4,2^{17},1^{24})$
		≥ 7	$(7^2,5^6,3^8,1^2)$	$(6,5^3,4^4,3^6,2^6,1^3)$				
78	$\omega_1+\omega_4$	5	$(5^{10},3^8,1^4)$	$(5^6,3^{14},1^6)$	$(5^5,3^{15},1^8)$	$(4^9,2^{21})$	$(4^3,3^{14},2^{10},1^4)$	$(3^8,2^{18},1^{18})$
84	$\omega_1+\omega_4$	3	(3^{28})	$(3^{27},2,1)$	$(3^{26},1^6)$	$(3^{18},2^{15})$	$(3^{14},2^{14},1^{14})$	$(3^4,2^{22},1^{28})$
		≥ 7	$(7^2,5^8,3^8,1^6)$	$(6,5^3,4^6,3^7,2^7,1^4)$	$(5^4,3^{18},1^{10})$	$(4^9,2^{24})$	$(4^2,3^{10},2^{16},1^{14})$	
90	$2\omega_2$	3	(3^{30})	$(3^{28},2^2,1^2)$	$(3^{27},1^9)$	$(3^{27},1^9)$	$(3^{22},2^8,1^8)$	$(3^{10},2^{20},1^{20})$

Table 5. $G=A_6(K)$

n	ω	p	$J(\varphi(x))$ for $J(x)=(7)$	$J(\varphi(x))$ for $J(x)=(6,1)$	$J(\varphi(x))$ for $J(x)=(5,2)$
21	ω_2	5	-	-	$(5^4,1)$
		7	(7^3)	$(7^2,6,1)$	$(7,6,4,3,1)$
		≥ 11	$(11,7,3)$	$(9,6,5,1)$	
28	$2\omega_1$	5	-	-	$(5^5,3)$
		7	(7^4)	$(7^3,6,1)$	$(7^2,6,4,3,1)$
		11	$(11^2,5,1)$	$(11,7,6,3,1)$	$(9,6,5,4,3,1)$
		≥ 13	$(13,9,5,1)$		
35	ω_3	5	-	-	(5^7)
		7	(7^5)	$(7^4,6,1)$	$(7^3,5,4,3,2)$
		11	$(11^2,7,5,1)$	$(10,9,6,5,4,1)$	$(8,7,6,5,4,3,2)$
		≥ 13	$(13,9,7,5,1)$		

Table 5 continued

n	ω	p	$J(\varphi(x))$ for $J(x) = (7)$	$J(\varphi(x))$ for $J(x) = (6,1)$	$J(\varphi(x))$ for $J(x) = (5,2)$
47	$\omega_1 + \omega_6$	7	$(7^6, 5)$	$(7^5, 6^2)$	$(7^5, 3^4)$
48	$\omega_1 + \omega_6$	5	-	-	$(5^9, 3)$
		11	$(11^3, 7, 5, 3)$	$(11, 9, 7, 6^2, 5, 3, 1)$	$(9, 7, 6^2, 5, 4^2, 3^2, 1)$
		≥ 13	$(13, 11, 9, 7, 5, 3)$		
84	$3\omega_1$	5	-	-	$(5^{16}, 4)$
		7	(7^{12})	$(7^{11}, 6, 1)$	$(7^{10}, 5, 4, 3, 2)$
		11	$(11^7, 7)$	$(11^5, 8, 7, 6, 4, 3, 1)$	$(11^2, 10, 8, 7^2, 6, 5^2, 4^2, 3, 2, 1)$
		13	$(13^5, 9, 7, 3)$	$(13^2, 12, 11, 8, 7, 6^2, 4, 3, 1)$	$(13, 10, 9, 8, 7^2, 6, 5^2, 4^2, 3, 2, 1)$
		17	$(17^2, 13, 11, 9, 7^2, 3)$	$(16, 12, 11, 10, 8, 7, 6^2, 4, 3, 1)$	
		≥ 19	$(19, 15, 13, 11, 9, 7^2, 3)$		

Table 5 continued

n	ω	p	$J(\varphi(x))$ for $J(x) = (5,1^2)$	$J(\varphi(x))$ for $J(x) = (4,3)$	$J(\varphi(x))$ for $J(x) = (4,2,1)$
21	ω_2	5	$(5^4, 1)$	$(5^3, 3, 2, 1)$	$(5^2, 4, 3, 2, 1^2)$
		≥ 7	$(7, 5^2, 3, 1)$	$(6, 5, 4, 3, 2, 1)$	
28	$2\omega_1$	5	$(5^5, 1^3)$	$(5^5, 2, 1)$	$(5^3, 4, 3^2, 2, 1)$
		7	$(7^2, 5^2, 1^4)$	$(7, 6, 5, 4, 3, 2, 1)$	$(7, 5, 4, 3^3, 2, 1)$
		≥ 11	$(9, 5^3, 1^4)$		
35	ω_3	5	(5^7)	$(5^5, 4, 3, 2, 1)$	$(5^4, 4^2, 3, 2, 1^2)$
		≥ 7	$(7^3, 5, 3^3)$	$(7, 6, 5, 4^2, 3^2, 2, 1)$	$(6, 5^2, 4^3, 3, 2, 1^2)$
47	$\omega_1 + \omega_6$	7	$(7^3, 5^4, 3, 1^3)$	$(7, 6^2, 5^2, 4^2, 3^2, 2^2)$	$(7, 5^3, 4^2, 3^4, 2^2, 1)$
48	$\omega_1 + \omega_6$	5	$(5^9, 1^3)$	$(5^8, 3, 2^2, 1)$	$(5^5, 4^2, 3^3, 2^2, 1^2)$
		≥ 11	$(9, 7, 5^5, 3, 1^4)$	$(7, 6^2, 5^2, 4^2, 3^2, 2^2, 1)$	$(7, 5^3, 4^2, 3^4, 2^2, 1^2)$
84	$3\omega_1$	5	$(5^{16}, 1^4)$	$(5^{16}, 4)$	$(5^{13}, 4^2, 3^2, 2^2, 1)$
		7	$(7^9, 5^3, 1^6)$	$(7^8, 6, 5, 4^2, 3^2, 2, 1)$	$(7^5, 6^2, 5, 4^4, 3^3, 2^3, 1)$
		11	$(11^2, 9^2, 7, 5^6, 1^7)$	$(10, 9, 8, 7^2, 6^2, 5^2, 4^3, 3^2, 2, 1)$	$(10, 8, 7, 6^3, 5, 4^5, 3^3, 2^3, 1)$
		≥ 13	$(13, 9^3, 7, 5^6, 1^7)$		

Table 5 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(4,1^3)$	$J(\varphi(x))$ for $J(x)=(3^2,1)$	$J(\varphi(x))$ for $J(x)=(3,2^2)$	$J(\varphi(x))$ for $J(x)=(3,2,1^2)$
21	ω_2	3	-	(3^7)	$(3^6,1^3)$	$(3^5,2^2,1^2)$
		≥ 5	$(5,4^3,1^4)$	$(5,3^5,1)$	$(4^2,3^2,2^2,1^3)$	$(4,3^3,2^3,1^2)$
28	$2\omega_1$	3	-	$(3^9,1)$	$(3^9,1)$	$(3^7,2^2,1^3)$
		5	$(5^2,4^3,1^6)$	$(5^3,3^3,1^4)$	$(5,4^2,3^3,2^2,1^2)$	$(5,4,3^3,2^3,1^4)$
		≥ 7	$(7,4^3,3,1^6)$			
35	ω_3	3	-	$(3^{11},1^2)$	$(3^{10},2^2,1)$	$(3^{10},2,1^3)$
		≥ 5	$(5^3,4^4,1^4)$	$(5^3,3^5,1^5)$	$(5,4^2,3^4,2^4,1^2)$	$(4^3,3^4,2^4,1^3)$
47	$\omega_1+\omega_6$	7	$(7,5,4^6,3,1^8)$	$(5^4,3^8,1^3)$	$(5,4^4,3^5,2^4,1^3)$	$(5,4^2,3^6,2^6,1^4)$
48	$\omega_1+\omega_6$	3	-	(3^{16})	$(3^{15},1^3)$	$(3^{12},2^4,1^4)$
		5	$(5^3,4^6,1^9)$	$(5^4,3^8,1^4)$	$(5,4^4,3^5,2^4,1^4)$	$(5,4^2,3^6,2^6,1^5)$
		≥ 11	$(7,5,4^6,3,1^9)$			
77	$\omega_1+\omega_2$	3	-	$(3^{25},1^2)$	$(3^{24},2^2,1)$	$(3^{22},2^4,1^3)$
84	$3\omega_1$	5	$(5^{10},4^6,1^{10})$	$(5^{13},3^5,1^4)$	$(5^9,4^4,3^4,2^4,1^3)$	$(5^7,4^3,3^6,2^6,1^7)$
		7	$(7^5,6,4^6,3^3,1^{10})$	$(7^4,5^5,3^9,1^4)$	$(7,6^2,5^3,4^6,3^5,2^4,1^3)$	$(7,6,5^3,4^4,3^7,2^6,1^7)$
		≥ 11	$(10,7^3,6,4^7,3^3,1^{10})$			

Table 5 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(3,1^4)$	$J(\varphi(x))$ for $J(x)=(2^3,1)$	$J(\varphi(x))$ for $J(x)=(2^2,1^3)$	$J(\varphi(x))$ for $J(x)=(2,1^5)$
21	ω_2	≥ 3	$(3^5,1^6)$	$(3^3,2^3,1^6)$	$(3,2^6,1^6)$	$(2^5,1^{11})$
28	$2\omega_1$	3	$(3^6,1^{10})$	$(3^6,2^3,1^4)$	$(3^3,2^6,1^7)$	$(3,2^5,1^{15})$
		≥ 5	$(5,3^4,1^{11})$			
35	ω_3	3	$(3^{10},1^5)$	$(3^5,2^7,1^6)$	$(3^3,2^8,1^{10})$	$(2^{10},1^{15})$
		≥ 5		$(4,3^3,2^8,1^6)$		
47	$\omega_1+\omega_6$	7	$(5,3^9,1^{15})$	$(3^9,2^6,1^8)$	$(3^4,2^{12},1^{11})$	$(3,2^{10},1^{24})$
48	$\omega_1+\omega_6$	3	$(3^{11},1^{15})$	$(3^9,2^6,1^9)$	$(3^4,2^{12},1^{12})$	$(3,2^{10},1^{25})$
		≥ 5	$(5,3^9,1^{16})$			
77	$\omega_1+\omega_2$	3	$(3^{20},1^{17})$	$(3^{20},2^7,1^3)$	$(3^{13},2^{14},1^{10})$	$(3^5,2^{16},1^{30})$
84	$3\omega_1$	5	$(5^6,3^{10},1^{24})$	$(4^{10},3^6,2^{11},1^4)$	$(4^4,3^9,2^{14},1^{13})$	$(4,3^5,2^{15},1^{35})$
		≥ 7	$(7,5^4,3^{11},1^{24})$			

TABLE 6. $G=A_7(K)$

n	ω	p	$J(\varphi(x))$ for $J(x)=(8)$	$J(\varphi(x))$ for $J(x)=(7,1)$	$J(\varphi(x))$ for $J(x)=(6,2)$	$J(\varphi(x))$ for $J(x)=(6,1^2)$	$J(\varphi(x))$ for $J(x)=(5,3)$
28	ω_2	5	-	-	-	-	$(5^5,3)$
		7	-	(7^4)	$(7^3,5,1^2)$	$(7^2,6^2,1^2)$	$(7^2,5,3^3)$
		11	$(11^2,5,1)$	$(11,7^2,3)$	$(9,7,5^2,1^2)$	$(9,6^2,5,1^2)$	
		≥ 13	$(13,9,5,1)$				
36	$2\omega_1$	5	-	-	-	-	$(5^7,1)$
		7	-	$(7^5,1)$	$(7^4,5,3)$	$(7^3,6^2,1^3)$	$(7^3,5^2,3,1^2)$
		11	$(11^3,3)$	$(11^2,7,5,1^2)$	$(11,7^2,5,3^2)$	$(11,7,6^2,3,1^3)$	$(9,7,5^3,3,1^2)$
		13	$(13^2,7,3)$	$(13,9,7,5,1^2)$			
		≥ 17	$(15,11,7,3)$				

Table 6 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(8)$	$J(\varphi(x))$ for $J(x)=(7,1)$	$J(\varphi(x))$ for $J(x)=(6,2)$	$J(\varphi(x))$ for $J(x)=(6,1^2)$	$J(\varphi(x))$ for $J(x)=(5,3)$
56	ω_3	5	-	-	-	-	$(5^{11},1)$
		7	-	(7^8)	$(7^6,6^2,2)$	$(7^6,6^2,1^2)$	$(7^5,5^2,3^3,1^2)$
		11	$(11^4,8,4)$	$(11^3,7^2,5,3,1)$	$(10^2,8,6^3,4^2,2)$	$(10,9^2,6^2,5^2,4,1^2)$	$(9,7^3,5^3,3^3,1^2)$
		13	$(13^2,12,8,6,4)$	$(13,11,9,7^2,5,3,1)$			
		≥ 17	$(16,12,10,8,6,4)$				
63	$\omega_1+\omega_7$	5	-	-	-	-	$(5^{12},3)$
		7	-	(7^9)	$(7^7,5^2,3,1)$	$(7^5,6^4,1^4)$	$(7^5,5^3,3^4,1)$
		11	$(11^5,5,3)$	$(11^3,7^3,5,3,1)$	$(11,9,7^3,5^3,3^2,1)$	$(11,9,7,6^4,5,3,1^4)$	$(9,7^3,5^4,3^4,1)$
		13	$(13^3,9,7,5,3)$	$(13,11,9,7^3,5,3,1)$			
		≥ 17	$(15,13,11,9,7,5,3)$				
70	ω_4	5	-	-	-	-	(5^{14})
		7	-	(7^{10})	$(7^9,5,1^2)$	$(7^8,6^2,1^2)$	$(7^6,5^4,3^2,1^2)$
		11	$(11^5,9,5,1)$	$(11^4,7^2,5^2,1^2)$	$(11,9^3,7,5^4,3,1^2)$	$(10^2,9^2,6^2,5^2,4^2,1^2)$	$(9^2,7^2,5^6,3^2,1^2)$
		13	$(13^3,11,9,5^2,1)$	$(13^2,9^2,7^2,5^2,1^2)$			
		≥ 17	$(17,13,11,9^2,5^2,1)$				

Table 6 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(5,2,1)$	$J(\varphi(x))$ for $J(x)=(5,1^3)$	$J(\varphi(x))$ for $J(x)=(4^2)$	$J(\varphi(x))$ for $J(x)=(4,3,1)$
28	ω_2	5	$(5^5,2,1)$	$(5^5,1^3)$	$(5^5,1^3)$	$(5^3,4,3^2,2,1)$
		≥ 7	$(7,6,5,4,3,2,1)$	$(7,5^3,3,1^3)$	$(7,5^3,3,1^3)$	$(6,5,4^2,3^2,2,1)$
36	$2\omega_1$	5	$(5^6,3,2,1)$	$(5^6,1^6)$	$(5^7,1)$	$(5^5,4,3,2,1^2)$
		7	$(7^2,6,5,4,3,2,1^2)$	$(7^2,5^3,1^7)$	$(7^3,5,3^3,1)$	$(7,6,5,4^2,3^2,2,1^2)$
		≥ 11	$(9,6,5^2,4,3,2,1^2)$	$(9,5^4,1^7)$		

Table 6 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(5,2,1)$	$J(\varphi(x))$ for $J(x)=(5,1^3)$	$J(\varphi(x))$ for $J(x)=(4^2)$	$J(\varphi(x))$ for $J(x)=(4,3,1)$
56	ω_3	5	$(5^{11},1)$	$(5^{11},1)$	$(5^8,4^4)$	$(5^8,4,3^2,2^2,1^2)$
		7	$(7^4,6,5,4^2,3^2,2,1)$	$(7^4,5^3,3^4,1)$	$(7^4,4^6,2^2)$	$(7,6^2,5^2,4^3,3^3,2^2,1^2)$
		≥ 11	$(8,7^2,6^2,5,4^2,3^2,2,1)$			
63	$\omega_1+\omega_7$	5	$(5^{11},3,2^2,1)$	$(5^{11},1^8)$	$(5^{12},1^3)$	$(5^8,4^2,3^3,2^2,1^2)$
		7	$(7^3,6^2,5^2,4^2,3^2,2^2,1^2)$	$(7^3,5^6,3,1^9)$	$(7^4,5^4,3^4,1^3)$	$(7,6^2,5^2,4^4,3^4,2^2,1^2)$
		≥ 11	$(9,7,6^2,5^3,4^2,3^2,2^2,1^2)$	$(9,7,5^7,3,1^9)$		
70	ω_4	5	(5^{14})	(5^{14})	$(5^{13},1^5)$	$(5^{10},4^2,3^2,2^2,1^2)$
		7	$(7^6,5^2,4^2,3^2,2^2)$	$(7^6,5^2,3^6)$	$(7^5,5^4,3^3,1^6)$	$(7^2,6^2,5^2,4^4,3^4,2^2,1^2)$
		≥ 11	$(8^2,7^2,6^2,5^2,4^2,3^2,2^2)$		$(9,7^3,5^5,3^3,1^6)$	

Table 6 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(4,2^2)$	$J(\varphi(x))$ for $J(x)=(4,2,1^2)$	$J(\varphi(x))$ for $J(x)=(4,1^4)$	$J(\varphi(x))$ for $J(x)=(3^2,2)$
28	ω_2	3	-	-	-	$(3^9,1)$
		≥ 5	$(5^3,3^3,1^4)$	$(5^2,4^2,3,2^2,1^3)$	$(5,4^4,1^7)$	$(5,4^2,3^3,2^2,1^2)$
36	$2\omega_1$	3	-	-	-	(3^{12})
		5	$(5^4,3^5,1)$	$(5^3,4^2,3^2,2^2,1^3)$	$(5^2,4^4,1^{10})$	$(5^3,4^2,3^2,2^2,1^3)$
		≥ 7	$(7,5^2,3^6,1)$	$(7,5,4^2,3^3,2^2,1^3)$	$(7,4^4,3,1^{10})$	
56	ω_3	3	-	-	-	$(3^{18},1^2)$
		5	$(5^6,4^4,2^5)$	$(5^6,4^3,3^2,2^2,1^4)$	$(5^4,4^7,1^8)$	$(5^4,4^3,3^4,2^4,1^4)$
		≥ 7	$(6^3,4^7,2^5)$	$(6,5^4,4^4,3^2,2^2,1^4)$		$(6,5^2,4^4,3^4,2^4,1^4)$
63	$\omega_1+\omega_7$	3	-	-	-	(3^{21})
		5	$(5^7,3^8,1^4)$	$(5^5,4^4,3^3,2^4,1^5)$	$(5^3,4^8,1^{16})$	$(5^4,4^4,3^5,2^4,1^4)$
		≥ 7	$(7,5^5,3^9,1^4)$	$(7,5^3,4^4,3^4,2^4,1^5)$	$(7,5,4^8,3,1^{16})$	

Table 6 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(4,2^2)$	$J(\varphi(x))$ for $J(x)=(4,2,1^2)$	$J(\varphi(x))$ for $J(x)=(4,1^4)$	$J(\varphi(x))$ for $J(x)=(3^2,2)$
70	ω_4	3	-	-	-	$(3^{22},2^2)$
		5	$(5^{10},3^5,1^5)$	$(5^8,4^4,3^2,2^2,1^4)$	$(5^6,4^8,1^8)$	$(5^6,4^2,3^6,2^6,1^2)$
		≥ 7	$(7,5^8,3^6,1^5)$	$(6^2,5^4,4^6,3^2,2^2,1^4)$		$(6^2,5^2,4^4,3^6,2^6,1^2)$

Table 6 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(3^2,1^2)$	$J(\varphi(x))$ for $J(x)=(3,2^2,1)$	$J(\varphi(x))$ for $J(x)=(3,2,1^3)$	$J(\varphi(x))$ for $J(x)=(3,1^5)$
28	ω_2	3	$(3^9,1)$	$(3^7,2^2,1^3)$	$(3^6,2^3,1^4)$	$(3^6,1^{10})$
		≥ 5	$(5,3^7,1^2)$	$(4^2,3^3,2^4,1^3)$	$(4,3^4,2^4,1^4)$	
36	$2\omega_1$	3	$(3^{11},1^3)$	$(3^{10},2^2,1^2)$	$(3^8,2^3,1^6)$	$(3^7,1^{15})$
		≥ 5	$(5^3,3^5,1^6)$	$(5,4^2,3^4,2^4,1^3)$	$(5,4,3^4,2^4,1^7)$	$(5,3^5,1^{16})$
56	ω_3	3	$(3^{18},1^2)$	$(3^{16},2^2,1^4)$	$(3^{15},2^3,1^5)$	$(3^{15},1^{11})$
		≥ 5	$(5^3,3^{12},1^5)$	$(5,4^4,3^6,2^6,1^5)$	$(4^4,3^7,2^7,1^5)$	
63	$\omega_1+\omega_7$	3	$(3^{20},1^3)$	$(3^{17},2^4,1^4)$	$(3^{14},2^6,1^9)$	$(3^{13},1^{24})$
		≥ 5	$(5^4,3^{12},1^7)$	$(5,4^4,3^7,2^8,1^5)$	$(5,4^2,3^8,2^8,1^{10})$	$(5,3^{11},1^{25})$
70	ω_4	3	$(3^{22},1^4)$	$(3^{20},2^4,1^2)$	$(3^{20},2^2,1^6)$	$(3^{20},1^{10})$
		≥ 5	$(5^6,3^{10},1^{10})$	$(5^2,4^4,3^8,2^8,1^4)$	$(4^6,3^8,2^8,1^6)$	

Table 6 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(2^4)$	$J(\varphi(x))$ for $J(x)=(2^3,1^2)$	$J(\varphi(x))$ for $J(x)=(2^2,1^4)$	$J(\varphi(x))$ for $J(x)=(2,1^6)$
28	ω_2	≥ 3	$(3^6,1^{10})$	$(3^3,2^6,1^7)$	$(3,2^8,1^9)$	$(2^6,1^{16})$
36	$2\omega_1$	≥ 3	$(3^{10},1^6)$	$(3^6,2^6,1^6)$	$(3^3,2^8,1^{11})$	$(3,2^6,1^{21})$

Table 6 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(2^4)$	$J(\varphi(x))$ for $J(x)=(2^3,1^2)$	$J(\varphi(x))$ for $J(x)=(2^2,1^4)$	$J(\varphi(x))$ for $J(x)=(2,1^6)$
56	ω_3	3	$(3^8,2^{16})$	$(3^8,2^{10},1^{12})$	$(3^4,2^{14},1^{16})$	$(2^{15},1^{26})$
		≥ 5	$(4^4,2^{20})$	$(4,3^6,2^{11},1^{12})$		
63	$\omega_1+\omega_7$	≥ 3	$(3^{16},1^{15})$	$(3^9,2^{12},1^{12})$	$(3^4,2^{16},1^{19})$	$(3,2^{12},1^{36})$
70	ω_4	3	$(3^{17},1^{19})$	$(3^{10},2^{14},1^{12})$	$(3^6,2^{16},1^{20})$	$(2^{20},1^{30})$
		≥ 5	$(5,3^{15},1^{20})$	$(4^2,3^6,2^{16},1^{12})$		

TABLE 7. $G=A_8(K)$

n	ω	p	$J(\varphi(x))$ for $J(x)=(9)$	$J(\varphi(x))$ for $J(x)=(8,1)$	$J(\varphi(x))$ for $J(x)=(7,2)$	$J(\varphi(x))$ for $J(x)=(7,1^2)$
36	ω_2	7	-	-	$(7^5,1)$	$(7^5,1)$
		11	$(11^3,3)$	$(11^2,8,5,1)$	$(11,8,7,6,3,1)$	$(11,7^3,3,1)$
		13	$(13^2,7,3)$	$(13,9,8,5,1)$		
		≥ 17	$(15,11,7,3)$			
45	$2\omega_1$	7	-	-	$(7^6,3)$	$(7^6,1^3)$
		11	$(11^4,1)$	$(11^3,8,3,1)$	$(11^2,8,6,5,3,1)$	$(11^2,7^2,5,1^4)$
		13	$(13^3,5,1)$	$(13^2,8,7,3,1)$	$(13,9,8,6,5,3,1)$	$(13,9,7^2,5,1^4)$
		≥ 17	$(17,13,9,5,1)$	$(15,11,8,7,3,1)$		
80	$\omega_1+\omega_8$	7	-	-	$(7^{11},3)$	$(7^{11},1^3)$
		11	$(11^7,3)$	$(11^5,8^2,5,3,1)$	$(11^3,8^2,7,6^2,5,3^2,1)$	$(11^3,7^5,5,3,1^4)$
		13	$(13^5,7,5,3)$	$(13^3,9,8^2,7,5,3,1)$	$(13,11,9,8^2,7,6^2,5,3^2,1)$	$(13,11,9,7^5,5,3,1^4)$
		≥ 17	$(17,15,13,11,9,7,5,3)$	$(15,13,11,9,8^2,7,5,3,1)$		
84	ω_3	7	-	-	(7^{12})	(7^{12})
		11	$(11^7,7)$	$(11^6,8,5,4,1)$	$(11^4,8,7^2,6,5,4,2,1)$	$(11^4,7^4,5,3^2,1)$
		13	$(13^5,9,7,3)$	$(13^3,12,9,8,6,5,4,1)$	$(13,12,10,9,8,7^2,6,5,4,2,1)$	$(13,11^2,9,7^4,5,3^2,1)$
		17	$(17^2,13,11,9,7^2,3)$	$(16,13,12,10,9,8,6,5,4,1)$		
		≥ 19	$(19,15,13,11,9,7^2,3)$			

Table 7 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(6,3)$	$J(\varphi(x))$ for $J(x)=(6,2,1)$	$J(\varphi(x))$ for $J(x)=(6,1^3)$
36	ω_2	7	$(7^4,4,3,1)$	$(7^3,6,5,2,1^2)$	$(7^2,6^3,1^4)$
		≥ 11	$(9,8,6,5,4,3,1)$	$(9,7,6,5^2,2,1^2)$	$(9,6^3,5,1^4)$
45	$2\omega_1$	7	$(7^5,5,4,1)$	$(7^4,6,5,3,2,1)$	$(7^3,6^3,1^6)$
		≥ 11	$(11,8,7,6,5,4,3,1)$	$(11,7^2,6,5,3^2,2,1)$	$(11,7,6^3,3,1^6)$
80	$\omega_1+\omega_8$	7	$(7^9,5,4^2,3,1)$	$(7^7,6^2,5^2,3,2^2,1^2)$	$(7^5,6^6,1^9)$
		≥ 11	$(11,9,8^2,7,6^2,5^2,4^2,3^2,1)$	$(11,9,7^3,6^2,5^3,3^2,2^2,1^2)$	$(11,9,7,6^6,5,3,1^9)$
84	ω_3	7	$(7^{10},6,4,3,1)$	$(7^9,6^2,5,2,1^2)$	$(7^8,6^4,1^4)$
		≥ 11	$(11,10,9,8,7^2,6^2,5,4^2,3^2,1)$	$(10^2,9,8,7,6^3,5^2,4^2,2,1^2)$	$(10,9^3,6^4,5^3,4,1^4)$

Table 7 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(5,4)$	$J(\varphi(x))$ for $J(x)=(5,3,1)$	$J(\varphi(x))$ for $J(x)=(5,2^2)$	$J(\varphi(x))$ for $J(x)=(5,2,1^2)$	$J(\varphi(x))$ for $J(x)=(5,1^4)$
36	ω_2	5	$(5^7,1)$	$(5^6,3^2)$	$(5^6,3,1^3)$	$(5^6,2^2,1^2)$	$(5^6,1^6)$
		7	$(7^3,5,4,3,2,1)$	$(7^2,5^2,3^4)$	$(7,6^2,4^2,3^2,1^3)$	$(7,6,5^2,4,3,2^2,1^2)$	$(7,5^4,3,1^6)$
		≥ 11	$(8,7,6,5,4,3,2,1)$				
45	$2\omega_1$	5	(5^9)	$(5^8,3,1^2)$	$(5^7,3^3,1)$	$(5^7,3,2^2,1^3)$	$(5^7,1^{10})$
		7	$(7^5,4,3,2,1)$	$(7^3,5^3,3^2,1^3)$	$(7^2,6^2,4^2,3^3,1^2)$	$(7^2,6,5^2,4,3,2^2,1^4)$	$(7^2,5^4,1^{11})$
		≥ 11	$(9,8,7,6,5,4,3,2,1)$	$(9,7,5^4,3^2,1^3)$	$(9,6^2,5,4^2,3^3,1^2)$	$(9,6,5^3,4,3,2^2,1^4)$	$(9,5^5,1^{11})$
80	$\omega_1+\omega_8$	5	(5^{16})	$(5^{14},3^3,1)$	$(5^{13},3^4,1^3)$	$(5^{13},3,2^4,1^4)$	$(5^{13},1^{15})$
		7	$(7^8,5,4^2,3^2,2^2,1)$	$(7^5,5^5,3^6,1^2)$	$(7^3,6^4,4^4,3^5,1^4)$	$(7^3,6^2,5^4,4^2,3^2,2^4,1^5)$	$(7^3,5^8,3,1^{16})$
		≥ 11	$(9,8^2,7^2,6^2,5^2,4^2,3^2,2^2,1)$	$(9,7^3,5^6,3^6,1^2)$	$(9,7,6^4,5,4^4,3^5,1^4)$	$(9,7,6^2,5^5,4^2,3^2,2^4,1^5)$	$(9,7,5^9,3,1^{16})$
84	ω_3	5	$(5^{16},4)$	$(5^{16},3,1)$	$(5^{16},2^2)$	$(5^{16},2,1^2)$	$(5^{16},1^4)$
		7	$(7^8,6,5,4^2,3^2,2,1)$	$(7^7,5^3,3^6,1^2)$	$(7^6,5^4,4^2,3^2,2^4)$	$(7^5,6^2,5^2,4^3,3^3,2^2,1^2)$	$(7^5,5^6,3^5,1^4)$
		≥ 11	$(10,9,8,7^2,6^2,5^2,4^3,3^2,2,1)$	$(9,7^5,5^4,3^6,1^2)$	$(8^2,7^2,6^2,5^4,4^2,3^2,2^4)$	$(8,7^3,6^3,5^2,4^3,3^3,2^2,1^2)$	

Table 7 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(4^2,1)$	$J(\varphi(x))$ for $J(x)=(4,3,2)$	$J(\varphi(x))$ for $J(x)=(4,3,1^2)$	$J(\varphi(x))$ for $J(x)=(4,2^2,1)$	$J(\varphi(x))$ for $J(x)=(4,2,1^3)$
36	ω_2	5	$(5^5,4^2,1^3)$	$(5^4,4,3^2,2^2,1^2)$	$(5^3,4^2,3^3,2,1^2)$	$(5^3,4,3^3,2^2,1^4)$	$(5^2,4^3,3,2^3,1^5)$
		≥ 7	$(7,5^3,4^2,3,1^3)$	$(6,5^2,4^2,3^2,2^2,1^2)$	$(6,5,4^3,3^3,2,1^2)$		
45	$2\omega_1$	5	$(5^7,4^2,1^2)$	$(5^6,4,3^2,2^2,1)$	$(5^5,4^2,3^2,2,1^4)$	$(5^4,4,3^5,2^2,1^2)$	$(5^3,4^3,3^2,2^3,1^6)$
		≥ 7	$(7^3,5,4^2,3^3,1^2)$	$(7,6,5^2,4^2,3^3,2^2,1)$	$(7,6,5,4^3,3^3,2,1^4)$	$(7,5^2,4,3^6,2^2,1^2)$	$(7,5,4^3,3^3,2^3,1^6)$
80	$\omega_1+\omega_8$	5	$(5^{12},4^4,1^4)$	$(5^{10},4^2,3^4,2^4,1^2)$	$(5^8,4^4,3^5,2^2,1^5)$	$(5^7,4^2,3^8,2^4,1^5)$	$(5^5,4^6,3^3,2^6,1^{10})$
		≥ 7	$(7^4,5^4,4^4,3^4,1^4)$	$(7,6^2,5^4,4^4,3^5,2^4,1^2)$	$(7,6^2,5^2,4^6,3^6,2^2,1^5)$	$(7,5^5,4^2,3^9,2^4,1^5)$	$(7,5^3,4^6,3^4,2^6,1^{10})$
84	ω_3	5	$(5^{13},4^4,1^3)$	$(5^{11},4^3,3^3,2^3,1^2)$	$(5^{11},4^2,3^4,2^3,1^3)$	$(5^9,4^4,3^3,2^5,1^4)$	$(5^8,4^5,3^3,2^4,1^7)$
		7	$(7^5,5^3,4^6,3,2^2,1^3)$	$(7^2,6^2,5^3,4^5,3^5,2^3,1^2)$	$(7,6^3,5^3,4^5,3^5,2^3,1^3)$	$(6^3,5^3,4^7,3^3,2^5,1^4)$	$(6,5^6,4^6,3^3,2^4,1^7)$
		≥ 11	$(8^2,7,6^2,5^3,4^6,3,2^2,1^3)$				

Table 7 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(4,1^5)$	$J(\varphi(x))$ for $J(x)=(3^3)$	$J(\varphi(x))$ for $J(x)=(3^2,2,1)$	$J(\varphi(x))$ for $J(x)=(3^2,1^3)$	$J(\varphi(x))$ for $J(x)=(3,2^3)$	$J(\varphi(x))$ for $J(x)=(3,2^2,1^2)$
36	ω_2	3	-	(3^{12})	$(3^{11},2,1)$	$(3^{11},1^3)$	$(3^{10},1^6)$	$(3^8,2^4,1^4)$
		≥ 5	$(5,4^5,1^{11})$	$(5^3,3^6,1^3)$	$(5,4^2,3^5,2^3,1^2)$	$(5,3^9,1^4)$	$(4^3,3^4,2^3,1^6)$	$(4^2,3^4,2^6,1^4)$
45	$2\omega_1$	3	-	(3^{15})	$(3^{14},2,1)$	$(3^{13},1^6)$	$(3^{14},1^3)$	$(3^{11},2^4,1^4)$
		5	$(5^2,4^5,1^{15})$	$(5^6,3^3,1^6)$	$(5^3,4^2,3^4,2^3,1^4)$	$(5^3,3^7,1^9)$	$(5,4^3,3^6,2^3,1^4)$	$(5,4^2,3^5,2^6,1^5)$
		≥ 7	$(7,4^5,3,1^{15})$					
79	$\omega_1+\omega_8$	3	-	$(3^{26},1)$	$(3^{25},2^2)$	$(3^{24},1^7)$	$(3^{24},1^7)$	$(3^{19},2^8,1^6)$
80	$\omega_1+\omega_8$	5	$(5^3,4^{10},1^{25})$	$(5^9,3^9,1^8)$	$(5^4,4^4,3^9,2^6,1^5)$	$(5^4,3^{16},1^{12})$	$(5,4^6,3^{10},2^6,1^9)$	$(5,4^4,3^9,2^{12},1^8)$
		≥ 7	$(7,5,4^{10},3,1^{25})$					
84	ω_3	3	-	$(3^{27},1^3)$	$(3^{27},1^3)$	$(3^{27},1^3)$	$(3^{23},2^7,1)$	$(3^{23},2^4,1^7)$
		5	$(5^5,4^{11},1^{15})$	$(5^{10},3^8,1^{10})$	$(5^5,4^5,3^7,2^6,1^6)$	$(5^5,3^{17},1^8)$	$(5^3,4^4,3^9,2^{11},1^4)$	$(5,4^6,3^9,2^{10},1^8)$
		≥ 7		$(7,5^8,3^9,1^{10})$	$(6,5^3,4^6,3^7,2^6,1^6)$			

Table 7 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(3,2,1^4)$	$J(\varphi(x))$ for $J(x)=(3,1^6)$	$J(\varphi(x))$ for $J(x)=(2^4,1)$	$J(\varphi(x))$ for $J(x)=(2^3,1^3)$	$J(\varphi(x))$ for $J(x)=(2^2,1^5)$	$J(\varphi(x))$ for $J(x)=(2,1^7)$
36	ω_2	3	$(3^7,2^4,1^7)$	$(3^7,1^{15})$	$(3^6,2^4,1^{10})$	$(3^3,2^9,1^9)$	$(3,2^{10},1^{13})$	$(2^7,1^{22})$
		≥ 5	$(4,3^5,2^5,1^7)$					
45	$2\omega_1$	3	$(3^9,2^4,1^{10})$	$(3^8,1^{21})$	$(3^{10},2^4,1^7)$	$(3^6,2^9,1^9)$	$(3^3,2^{10},1^{16})$	$(3,2^7,1^{28})$
		≥ 5	$(5,4,3^5,2^5,1^{11})$	$(5,3^6,1^{22})$				
79	$\omega_1+\omega_8$	3	$(3^{16},2^8,1^{15})$	$(3^{15},1^{34})$	$(3^{16},2^8,1^{15})$	$(3^9,2^{18},1^{16})$	$(3^4,2^{20},1^{27})$	$(3,2^{14},1^{48})$
80	$\omega_1+\omega_8$	≥ 5	$(5,4^2,3^{10},2^{10},1^{17})$	$(5,3^{13},1^{36})$	$(3^{16},2^8,1^{16})$	$(3^9,2^{18},1^{17})$	$(3^4,2^{20},1^{28})$	$(3,2^{14},1^{49})$
84	ω_3	3	$(3^{21},2^6,1^9)$	$(3^{21},1^{21})$	$(3^{14},2^{16},1^{10})$	$(3^{11},2^{16},1^{19})$	$(3^5,2^{22},1^{25})$	$(2^{21},1^{42})$
		≥ 5	$(4^5,3^{11},2^{11},1^9)$		$(4^4,3^6,2^{20},1^{10})$	$(4,3^9,2^{17},1^{19})$		

TABLE 8. $G=A_9(K)$

n	ω	p	$J(\varphi(x))$ for $J(x)=(10)$	$J(\varphi(x))$ for $J(x)=(9,1)$	$J(\varphi(x))$ for $J(x)=(8,2)$	$J(\varphi(x))$ for $J(x)=(8,1^2)$
45	ω_2	11	$(11^4,1)$	$(11^3,9,3)$	$(11^2,9,7,5,1^2)$	$(11^2,8^2,5,1^2)$
		13	$(13^3,5,1)$	$(13^2,9,7,3)$	$(13,9^2,7,5,1^2)$	$(13,9,8^2,5,1^2)$
		≥ 17	$(17,13,9,5,1)$	$(15,11,9,7,3)$		
55	$2\omega_1$	11	(11^5)	$(11^4,9,1^2)$	$(11^3,9,7,3^2)$	$(11^3,8^2,3,1^3)$
		13	$(13^4,3)$	$(13^3,9,5,1^2)$	$(13^2,9,7^2,3^2)$	$(13^2,8^2,7,3,1^3)$
		17	$(17^2,11,7,3)$	$(17,13,9^2,5,1^2)$	$(15,11,9,7^2,3^2)$	$(15,11,8^2,7,3,1^3)$
		≥ 19	$(19,15,11,7,3)$			
99	$\omega_1+\omega_9$	11	(11^9)	$(11^7,9^2,3,1)$	$(11^5,9^2,7^2,5,3^2,1)$	$(11^5,8^4,5,3,1^4)$
		13	$(13^7,5,3)$	$(13^5,9^2,7,5,3,1)$	$(13^3,9^3,7^3,5,3^2,1)$	$(13^3,9,8^4,7,5,3,1^4)$
		17	$(17^3,13,11,9,7,5,3)$	$(17,15,13,11,9^3,7,5,3,1)$	$(15,13,11,9^3,7^3,5,3^2,1)$	$(15,13,11,9,8^4,7,5,3,1^4)$
		≥ 19	$(19,17,15,13,11,9,7,5,3)$			

Table 8 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(7,3)$	$J(\varphi(x))$ for $J(x)=(7,2,1)$	$J(\varphi(x))$ for $J(x)=(7,1^3)$	$J(\varphi(x))$ for $J(x)=(6,4)$
45	ω_2	7	$(7^6,3)$	$(7^6,2,1)$	$(7^6,1^3)$	$(7^5,5,3,1^2)$
		≥ 11	$(11,9,7^2,5,3^2)$	$(11,8,7^2,6,3,2,1)$	$(11,7^4,3,1^3)$	$(9^2,7,5^3,3,1^2)$
55	$2\omega_1$	7	$(7^7,5,1)$	$(7^7,3,2,1)$	$(7^7,1^6)$	$(7^7,3^2)$
		11	$(11^2,9,7,5^3,1^2)$	$(11^2,8,7,6,5,3,2,1^2)$	$(11^2,7^3,5,1^7)$	$(11,9,7^3,5,3^3)$
		≥ 13	$(13,9^2,7,5^3,1^2)$	$(13,9,8,7,6,5,3,2,1^2)$	$(13,9,7^3,5,1^7)$	
99	$\omega_1+\omega_9$	7	$(7^{13},5,3)$	$(7^{13},3,2^2,1)$	$(7^{13},1^8)$	$(7^{12},5,3^3,1)$
		11	$(11^3,9^2,7^3,5^4,3^2,1)$	$(11^3,8^2,7^3,6^2,5,3^2,2^2,1^2)$	$(11^3,7^7,5,3,1^9)$	$(11,9^3,7^4,5^4,3^4,1)$
		≥ 13	$(13,11,9^3,7^3,5^4,3^2,1)$	$(13,11,9,8^2,7^3,6^2,5,3^2,2^2,1^2)$	$(13,11,9,7^7,5,3,1^9)$	

Table 8 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(6,3,1)$	$J(\varphi(x))$ for $J(x)=(6,2^2)$	$J(\varphi(x))$ for $J(x)=(6,2,1^2)$	$J(\varphi(x))$ for $J(x)=(6,1^4)$	$J(\varphi(x))$ for $J(x)=(5^2)$
45	ω_2	5	-	-	-	-	(5^9)
		7	$(7^4,6,4,3^2,1)$	$(7^4,5^2,3,1^4)$	$(7^3,6^2,5,2^2,1^3)$	$(7^2,6^4,1^7)$	$(7^5,3^3,1)$
		≥ 11	$(9,8,6^2,5,4,3^2,1)$	$(9,7^2,5^3,3,1^4)$	$(9,7,6^2,5^2,2^2,1^3)$	$(9,6^4,5,1^7)$	$(9,7^3,5,3^3,1)$
55	$2\omega_1$	5	-	-	-	-	(5^{11})
		7	$(7^5,6,5,4,3,1^2)$	$(7^5,5^2,3^3,1)$	$(7^4,6^2,5,3,2^2,1^3)$	$(7^3,6^4,1^{10})$	$(7^7,3,1^3)$
		≥ 11	$(11,8,7,6^2,5,4,3^2,1^2)$	$(11,7^3,5^2,3^4,1)$	$(11,7^2,6^2,5,3^2,2^2,1^3)$	$(11,7,6^4,3,1^{10})$	$(9^3,7,5^3,3,1^3)$
98	$\omega_1+\omega_9$	5	-	-	-	-	$(5^{19},3)$
99	$\omega_1+\omega_9$	7	$(7^9,6^2,5,4^2,3^3,1^2)$	$(7^9,5^4,3^4,1^4)$	$(7^7,6^4,5^2,3,2^4,1^5)$	$(7^5,6^8,1^{16})$	$(7^{12},3^4,1^3)$
		≥ 11	$(11,9,8^2,7,6^4,5^2,4^2,3^4,1^2)$	$(11,9,7^5,5^5,3^5,1^4)$	$(11,9,7^3,6^4,5^3,3^2,2^4,1^5)$	$(11,9,7,6^8,5,3,1^{16})$	$(9^4,7^4,5^4,3^4,1^3)$

Table 8 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(5,4,1)$	$J(\varphi(x))$ for $J(x)=(5,3,2)$	$J(\varphi(x))$ for $J(x)=(5,3,1^2)$	$J(\varphi(x))$ for $J(x)=(5,2^2,1)$
45	ω_2	5	$(5^8,4,1)$	$(5^7,4,3,2,1)$	$(5^7,3^3,1)$	$(5^7,3,2^2,1^3)$
		7	$(7^3,5^2,4^2,3,2,1)$	$(7^2,6,5,4^2,3^3,2,1)$	$(7^2,5^3,3^5,1)$	$(7,6^2,5,4^2,3^2,2^2,1^3)$
		≥ 11	$(8,7,6,5^2,4^2,3,2,1)$			
55	$2\omega_1$	5	$(5^{10},4,1)$	$(5^9,4,3,2,1)$	$(5^9,3^2,1^4)$	$(5^8,3^3,2^2,1^2)$
		7	$(7^5,5,4^2,3,2,1^2)$	$(7^3,6,5^2,4^2,3^2,2,1^2)$	$(7^3,5^4,3^3,1^5)$	$(7^2,6^2,5,4^2,3^3,2^2,1^3)$
		≥ 11	$(9,8,7,6,5^2,4^2,3,2,1^2)$	$(9,7,6,5^3,4^2,3^2,2,1^2)$	$(9,7,5^5,3^3,1^5)$	$(9,6^2,5^2,4^2,3^3,2^2,1^3)$
98	$\omega_1+\omega_9$	5	$(5^{18},4^2)$	$(5^{16},4^2,3^2,2^2)$	$(5^{16},3^5,1^3)$	$(5^{15},3^4,2^4,1^3)$
99	$\omega_1+\omega_9$	7	$(7^8,5^3,4^4,3^2,2^2,1^2)$	$(7^5,6^2,5^3,4^4,3^5,2^2,1^2)$	$(7^5,5^7,3^8,1^5)$	$(7^3,6^4,5^2,4^4,3^5,2^4,1^5)$
		≥ 11	$(9,8^2,7^2,6^2,5^4,4^4,3^2,2^2,1^2)$	$(9,7^3,6^2,5^4,4^4,3^5,2^2,1^2)$	$(9,7^3,5^8,3^8,1^5)$	$(9,7,6^4,5^3,4^4,3^5,2^4,1^5)$

Table 8 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(5,2,1^3)$	$J(\varphi(x))$ for $J(x)=(5,1^5)$	$J(\varphi(x))$ for $J(x)=(4^2,2)$	$J(\varphi(x))$ for $J(x)=(4^2,1^2)$	$J(\varphi(x))$ for $J(x)=(4,3^2)$
45	ω_2	5	$(5^7,2^3,1^4)$	$(5^7,1^{10})$	$(5^7,3^2,1^4)$	$(5^5,4^4,1^4)$	$(5^6,3^3,2^2,1^2)$
		≥ 7	$(7,6,5^3,4,3,2^3,1^4)$	$(7,5^5,3,1^{10})$	$(7,5^5,3^3,1^4)$	$(7,5^3,4^4,3,1^4)$	$(6^2,5^2,4^2,3^3,2^2,1^2)$
55	$2\omega_1$	5	$(5^8,3,2^3,1^6)$	$(5^8,1^{15})$	$(5^9,3^3,1)$	$(5^7,4^4,1^4)$	$(5^9,3,2^2,1^3)$
		7	$(7^2,6,5^3,4,3,2^3,1^7)$	$(7^2,5^5,1^{16})$	$(7^3,5^3,3^6,1)$	$(7^3,5,4^4,3^3,1^4)$	$(7,6^2,5^3,4^2,3^2,2^2,1^3)$
		≥ 11	$(9,6,5^4,4,3,2^3,1^7)$	$(9,5^6,1^{16})$			
98	$\omega_1+\omega_9$	5	$(5^{15},3,2^6,1^8)$	$(5^{15},1^{23})$	$(5^{16},3^5,1^3)$	$(5^{12},4^8,1^6)$	$(5^{15},3^4,2^4,1^3)$
99	$\omega_1+\omega_9$	7	$(7^3,6^2,5^6,4^2,3^2,2^6,1^{10})$	$(7^3,5^{10},3,1^{25})$	$(7^4,5^8,3^9,1^4)$	$(7^4,5^4,4^8,3^4,1^7)$	$(7,6^4,5^5,4^4,3^5,2^4,1^4)$
		≥ 11	$(9,7,6^2,5^7,4^2,3^2,2^6,1^{10})$	$(9,7,5^{11},3,1^{25})$			

Table 8 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(4,3,2,1)$	$J(\varphi(x))$ for $J(x)=(4,3,1^3)$	$J(\varphi(x))$ for $J(x)=(4,2^3)$	$J(\varphi(x))$ for $J(x)=(4,2^2,1^2)$	$J(\varphi(x))$ for $J(x)=(4,2,1^4)$
45	ω_2	5	$(5^4,4^2,3^3,2^3,1^2)$	$(5^3,4^3,3^4,2,1^4)$	$(5^4,3^6,1^7)$	$(5^3,4^2,3^3,2^4,1^5)$	$(5^2,4^4,3,2^4,1^8)$
		≥ 7	$(6,5^2,4^3,3^3,2^3,1^2)$	$(6,5,4^4,3^4,2,1^4)$			
55	$2\omega_1$	5	$(5^6,4^2,3^3,2^3,1^2)$	$(5^5,4^3,3^3,2,1^7)$	$(5^5,3^9,1^3)$	$(5^4,4^2,3^5,2^4,1^4)$	$(5^3,4^4,3^2,2^4,1^{10})$
		≥ 7	$(7,6,5^2,4^3,3^4,2^3,1^2)$	$(7,6,5,4^4,3^4,2,1^7)$	$(7,5^3,3^{10},1^3)$	$(7,5^2,4^2,3^6,2^4,1^4)$	$(7,5,4^4,3^3,2^4,1^{10})$
98	$\omega_1+\omega_9$	5	$(5^{10},4^4,3^6,2^6,1^2)$	$(5^8,4^6,3^7,2^2,1^9)$	$(5^9,3^{15},1^8)$	$(5^7,4^4,3^8,2^8,1^7)$	$(5^5,4^8,3^3,2^8,1^{16})$
99	$\omega_1+\omega_9$	≥ 7	$(7,6^2,5^4,4^6,3^7,2^6,1^3)$	$(7,6^2,5^2,4^8,3^8,2^2,1^{10})$	$(7,5^7,3^{16},1^9)$	$(7,5^5,4^4,3^9,2^8,1^8)$	$(7,5^3,4^8,3^4,2^8,1^{17})$

Table 8 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(4,1^6)$	$J(\varphi(x))$ for $J(x)=(3^3,1)$	$J(\varphi(x))$ for $J(x)=(3^2,2^2)$	$J(\varphi(x))$ for $J(x)=(3^2,2,1^2)$	$J(\varphi(x))$ for $J(x)=(3^2,1^4)$
45	ω_2	3	-	(3^{15})	$(3^{14},1^3)$	$(3^{13},2^2,1^2)$	$(3^{13},1^6)$
		≥ 5	$(5,4^6,1^{16})$	$(5^3,3^9,1^3)$	$(5,4^4,3^4,2^4,1^4)$	$(5,4^2,3^7,2^4,1^3)$	$(5,3^{11},1^7)$
55	$2\omega_1$	3	-	$(3^{18},1)$	$(3^{18},1)$	$(3^{16},2^2,1^3)$	$(3^{15},1^{10})$
		5	$(5^2,4^6,1^{21})$	$(5^6,3^6,1^7)$	$(5^3,4^4,3^4,2^4,1^4)$	$(5^3,4^2,3^6,2^4,1^6)$	$(5^3,3^9,1^{13})$
		≥ 7	$(7,4^6,3,1^{21})$				
98	$\omega_1+\omega_9$	5	$(5^3,4^{12},1^{35})$	$(5^9,3^{15},1^8)$	$(5^4,4^8,3^8,2^8,1^6)$	$(5^4,4^4,3^{13},2^8,1^7)$	$(5^4,3^{20},1^{18})$
99	$\omega_1+\omega_9$	3	-	(3^{33})	$(3^{32},1^3)$	$(3^{29},2^4,1^4)$	$(3^{28},1^{15})$
		≥ 7	$(7,5,4^{12},3,1^{36})$	$(5^9,3^{15},1^9)$	$(5^4,4^8,3^8,2^8,1^7)$	$(5^4,4^4,3^{13},2^8,1^8)$	$(5^4,3^{20},1^{19})$

Table 8 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(3,2^3,1)$	$J(\varphi(x))$ for $J(x)=(3,2^2,1^3)$	$J(\varphi(x))$ for $J(x)=(3,2,1^5)$	$J(\varphi(x))$ for $J(x)=(3,1^7)$
45	ω_2	3	$(3^{11},2^3,1^6)$	$(3^9,2^6,1^6)$	$(3^8,2^5,1^{11})$	$(3^8,1^{21})$
		≥ 5	$(4^3,3^5,2^6,1^6)$	$(4^2,3^5,2^8,1^6)$	$(4,3^6,2^6,1^{11})$	
55	$2\omega_1$	3	$(3^{15},2^3,1^4)$	$(3^{12},2^6,1^7)$	$(3^{10},2^6,1^{13})$	$(3^9,1^{28})$
		≥ 5	$(5,4^3,3^7,2^6,1^5)$	$(5,4^2,3^6,2^8,1^8)$	$(5,4,3^6,2^7,1^{14})$	$(5,3^7,1^{29})$
98	$\omega_1+\omega_9$	5	$(5,4^6,3^{12},2^{12},1^9)$	$(5,4^4,3^{11},2^{16},1^{12})$	$(5,4^2,3^{12},2^{12},1^{25})$	$(5,3^{15},1^{48})$
99	$\omega_1+\omega_9$	3	$(3^{26},2^6,1^9)$	$(3^{21},2^{12},1^{12})$	$(3^{18},2^{10},1^{25})$	$(3^{17},1^{48})$
		≥ 7	$(5,4^6,3^{12},2^{12},1^{10})$	$(5,4^4,3^{11},2^{16},1^{13})$	$(5,4^2,3^{12},2^{12},1^{26})$	$(5,3^{15},1^{49})$

Table 8 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(2^5)$	$J(\varphi(x))$ for $J(x)=(2^4,1^2)$	$J(\varphi(x))$ for $J(x)=(2^3,1^4)$	$J(\varphi(x))$ for $J(x)=(2^2,1^6)$	$J(\varphi(x))$ for $J(x)=(2,1^8)$
45	ω_2	≥ 3	$(3^{10},1^{15})$	$(3^6,2^8,1^{11})$	$(3^3,2^{12},1^{12})$	$(3,2^{12},1^{18})$	$(2^8,1^{29})$
55	$2\omega_1$	≥ 3	$(3^{15},1^{10})$	$(3^{10},2^8,1^9)$	$(3^6,2^{12},1^{13})$	$(3^3,2^{12},1^{22})$	$(3,2^8,1^{36})$
98	$\omega_1+\omega_9$	5	$(3^{25},1^{23})$	$(3^{16},2^{16},1^{18})$	$(3^9,2^{24},1^{23})$	$(3^4,2^{24},1^{38})$	$(3,2^{16},1^{63})$
99	$\omega_1+\omega_9$	$3,\geq 7$	$(3^{25},1^{24})$	$(3^{16},2^{16},1^{19})$	$(3^9,2^{24},1^{24})$	$(3^4,2^{24},1^{39})$	$(3,2^{16},1^{64})$

TABLE 9. $G = A_{10}(K)$

n	ω	p	$J(\varphi(x))$ for $J(x) = (11)$	$J(\varphi(x))$ for $J(x) = (10,1)$	$J(\varphi(x))$ for $J(x) = (9,2)$	$J(\varphi(x))$ for $J(x) = (9,1^2)$	$J(\varphi(x))$ for $J(x) = (8,3)$
55	ω_2	11	(11^5)	$(11^4,10,1)$	$(11^3,10,8,3,1)$	$(11^3,9^2,3,1)$	$(11^2,10,8,6,5,3,1)$
		13	$(13^4,3)$	$(13^3,10,5,1)$	$(13^2,10,8,7,3,1)$	$(13^2,9^2,7,3,1)$	$(13,10,9,8,6,5,3,1)$
		17	$(17^2,11,7,3)$	$(17,13,10,9,5,1)$	$(15,11,10,8,7,3,1)$	$(15,11,9^2,7,3,1)$	
		≥ 19	$(19,15,11,7,3)$				
66	$2\omega_1$	11	(11^6)	$(11^5,10,1)$	$(11^4,10,8,3,1)$	$(11^4,9^2,1^4)$	$(11^3,10,8,6,5,3,1)$
		13	$(13^5,1)$	$(13^4,10,3,1)$	$(13^3,10,8,5,3,1)$	$(13^3,9^2,5,1^4)$	$(13^2,10,8,7,6,5,3,1)$
		17	$(17^3,9,5,1)$	$(17^2,11,10,7,3,1)$	$(17,13,10,9,8,5,3,1)$	$(17,13,9^3,5,1^4)$	$(15,11,10,8,7,6,5,3,1)$
		19	$(19^2,13,9,5,1)$	$(19,15,11,10,7,3,1)$			
		≥ 23	$(21,17,13,9,5,1)$				

Table 9 continued

n	ω	p	$J(\varphi(x))$ for $J(x) = (8,2,1)$	$J(\varphi(x))$ for $J(x) = (8,1^3)$	$J(\varphi(x))$ for $J(x) = (7,4)$	$J(\varphi(x))$ for $J(x) = (7,3,1)$	$J(\varphi(x))$ for $J(x) = (7,2^2)$
55	ω_2	7	-	-	$(7^7,5,1)$	$(7^7,3^2)$	$(7^7,3,1^3)$
		11	$(11^2,9,8,7,5,2,1^2)$	$(11^2,8^3,5,1^4)$	$(11,10,8,7,6,5,4,3,1)$	$(11,9,7^3,5,3^3)$	$(11,8^2,7,6^2,3^2,1^3)$
		≥ 13	$(13,9^2,8,7,5,2,1^2)$	$(13,9,8^3,5,1^4)$			
66	$2\omega_1$	7	-	-	$(7^9,3)$	$(7^8,5,3,1^2)$	$(7^8,3^3,1)$
		11	$(11^3,9,8,7,3^2,2,1)$	$(11^3,8^3,3,1^6)$	$(11^2,10,8,7,6,5,4,3,1)$	$(11^2,9,7^2,5^3,3,1^3)$	$(11^2,8^2,6^2,5,3^3,1^2)$
		13	$(13^2,9,8,7^2,3^2,2,1)$	$(13^2,8^3,7,3,1^6)$	$(13,10,9,8,7,6,5,4,3,1)$	$(13,9^2,7^2,5^3,3,1^3)$	$(13,9,8^2,6^2,5,3^3,1^2)$
		≥ 17	$(15,11,9,8,7^2,3^2,2,1)$	$(15,11,8^3,7,3,1^6)$			

Table 9 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(7,2,1^2)$	$J(\varphi(x))$ for $J(x)=(7,1^4)$	$J(\varphi(x))$ for $J(x)=(6,5)$	$J(\varphi(x))$ for $J(x)=(6,4,1)$	$J(\varphi(x))$ for $J(x)=(6,3,2)$
55	ω_2	7	$(7^7,2^2,1^2)$	$(7^7,1^6)$	$(7^7,3,2,1)$	$(7^5,6,5,4,3,1^2)$	$(7^5,5,4^2,3,2,1^2)$
		≥ 11	$(11,8,7^3,6,3,2^2,1^2)$	$(11,7^5,3,1^6)$	$(10,9,8,7,6,5,4,3,2,1)$	$(9^2,7,6,5^3,4,3,1^2)$	$(9,8,7,6,5^2,4^2,3,2,1^2)$
66	$2\omega_1$	7	$(7^8,3,2^2,1^3)$	$(7^8,1^{10})$	$(7^9,2,1)$	$(7^7,6,4,3^2,1)$	$(7^6,5^2,4^2,3,2,1)$
		11	$(11^2,8,7^2,6,5,3,2^2,1^4)$	$(11^2,7^4,5,1^{11})$	$(11,10,9,8,7,6,5,4,3,2,1)$	$(11,9,7^3,6,5,4,3^3,1)$	$(11,8,7^2,6,5^2,4^2,3^2,2,1)$
		≥ 13	$(13,9,8,7^2,6,5,3,2^2,1^4)$	$(13,9,7^4,5,1^{11})$			

Table 9 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(6,3,1^2)$	$J(\varphi(x))$ for $J(x)=(6,2^2,1)$	$J(\varphi(x))$ for $J(x)=(6,2,1^3)$	$J(\varphi(x))$ for $J(x)=(6,1^5)$	$J(\varphi(x))$ for $J(x)=(5^2,1)$
55	ω_2	5	-	-	-	-	(5^{11})
		7	$(7^4,6^2,4,3^3,1^2)$	$(7^4,6,5^2,3,2^2,1^4)$	$(7^3,6^3,5,2^3,1^5)$	$(7^2,6^5,1^{11})$	$(7^5,5^2,3^3,1)$
		≥ 11	$(9,8,6^3,5,4,3^3,1^2)$	$(9,7^2,6,5^3,3,2^2,1^4)$	$(9,7,6^3,5^2,2^3,1^5)$	$(9,6^5,5,1^{11})$	$(9,7^3,5^3,3^3,1)$
66	$2\omega_1$	5	-	-	-	-	$(5^{13},1)$
		7	$(7^5,6^2,5,4,3^2,1^4)$	$(7^5,6,5^2,3^3,2^2,1^2)$	$(7^4,6^3,5,3,2^3,1^6)$	$(7^3,6^5,1^{15})$	$(7^7,5^2,3,1^4)$
		≥ 11	$(11,8,7,6^3,5,4,3^3,1^4)$	$(11,7^3,6,5^2,3^4,2^2,1^2)$	$(11,7^2,6^3,5,3^2,2^3,1^6)$	$(11,7,6^5,3,1^{15})$	$(9^3,7,5^5,3,1^4)$

Table 9 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(5,4,2)$	$J(\varphi(x))$ for $J(x)=(5,4,1^2)$	$J(\varphi(x))$ for $J(x)=(5,3^2)$	$J(\varphi(x))$ for $J(x)=(5,3,2,1)$	$J(\varphi(x))$ for $J(x)=(5,3,1^3)$
55	ω_2	5	$(5^{10},3,1^2)$	$(5^9,4^2,1^2)$	$(5^9,3^3,1)$	$(5^8,4,3^2,2^2,1)$	$(5^8,3^4,1^3)$
		7	$(7^3,6,5^2,4^2,3^2,2,1^2)$	$(7^3,5^3,4^3,3,2,1^2)$	$(7^3,5^3,3^6,1)$	$(7^2,6,5^2,4^2,3^4,2^2,1)$	$(7^2,5^4,3^6,1^3)$
		≥ 11	$(8,7,6^2,5^2,4^2,3^2,2,1^2)$	$(8,7,6,5^3,4^3,3,2,1^2)$			
66	$2\omega_1$	5	$(5^{12},3^2)$	$(5^{11},4^2,1^3)$	$(5^{12},3,1^3)$	$(5^{10},4,3^2,2^2,1^2)$	$(5^{10},3^3,1^7)$
		7	$(7^5,6,5,4^2,3^3,2,1)$	$(7^5,5^2,4^3,3,2,1^4)$	$(7^4,5^5,3^3,1^4)$	$(7^3,6,5^3,4^2,3^3,2^2,1^3)$	$(7^3,5^5,3^4,1^8)$
		≥ 11	$(9,8,7,6^2,5^2,4^2,3^3,2,1)$	$(9,8,7,6,5^3,4^3,3,2,1^4)$	$(9,7^2,5^6,3^3,1^4)$	$(9,7,6,5^4,4^2,3^3,2^2,1^3)$	$(9,7,5^6,3^4,1^8)$

Table 9 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(5,2^3)$	$J(\varphi(x))$ for $J(x)=(5,2^2,1^2)$	$J(\varphi(x))$ for $J(x)=(5,2,1^4)$	$J(\varphi(x))$ for $J(x)=(5,1^6)$	$J(\varphi(x))$ for $J(x)=(4^2,3)$
55	ω_2	5	$(5^8,3^3,1^6)$	$(5^8,3,2^4,1^4)$	$(5^8,2^4,1^7)$	$(5^8,1^{15})$	$(5^9,3,2^2,1^3)$
		≥ 7	$(7,6^3,4^3,3^4,1^6)$	$(7,6^2,5^2,4^2,3^2,2^4,1^4)$	$(7,6,5^4,4,3,2^4,1^7)$	$(7,5^6,3,1^{15})$	$(7,6^2,5^3,4^2,3^2,2^2,1^3)$
66	$2\omega_1$	5	$(5^9,3^6,1^3)$	$(5^9,3^3,2^4,1^4)$	$(5^9,3,2^4,1^{10})$	$(5^9,1^{21})$	$(5^{12},2^2,1^2)$
		7	$(7^2,6^3,4^3,3^6,1^4)$	$(7^2,6^2,5^2,4^2,3^3,2^4,1^5)$	$(7^2,6,5^4,4,3,2^4,1^{11})$	$(7^2,5^6,1^{22})$	$(7^3,6^2,5^2,4^2,3^3,2^2,1^2)$
		≥ 11	$(9,6^3,5,4^3,3^6,1^4)$	$(9,6^2,5^3,4^2,3^3,2^4,1^5)$	$(9,6,5^5,4,3,2^4,1^{11})$	$(9,5^7,1^{22})$	

Table 9 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(4^2,2,1)$	$J(\varphi(x))$ for $J(x)=(4^2,1^3)$	$J(\varphi(x))$ for $J(x)=(4,3^2,1)$	$J(\varphi(x))$ for $J(x)=(4,3,2^2)$	$J(\varphi(x))$ for $J(x)=(4,3,2,1^2)$
55	ω_2	5	$(5^7,4^2,3^2,2,1^4)$	$(5^5,4^6,1^6)$	$(5^6,4,3^5,2^2,1^2)$	$(5^5,4^2,3^4,2^3,1^4)$	$(5^4,4^3,3^4,2^4,1^3)$
		≥ 7	$(7,5^5,4^2,3^3,2,1^4)$	$(7,5^3,4^6,3,1^6)$	$(6^2,5^2,4^3,3^5,2^2,1^2)$	$(6,5^3,4^3,3^4,2^3,1^4)$	$(6,5^2,4^4,3^4,2^4,1^3)$
66	$2\omega_1$	5	$(5^9,4^2,3^3,2,1^2)$	$(5^7,4^6,1^7)$	$(5^9,4,3^3,2^2,1^4)$	$(5^7,4^2,3^5,2^3,1^2)$	$(5^6,4^3,3^4,2^4,1^4)$
		≥ 7	$(7^3,5^3,4^2,3^6,2,1^2)$	$(7^3,5,4^6,3^3,1^7)$	$(7,6^2,5^3,4^3,3^4,2^2,1^4)$	$(7,6,5^3,4^3,3^6,2^3,1^2)$	$(7,6,5^2,4^4,3^5,2^4,1^4)$

Table 9 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(4,3,1^4)$	$J(\varphi(x))$ for $J(x)=(4,2^3,1)$	$J(\varphi(x))$ for $J(x)=(4,2^2,1^3)$	$J(\varphi(x))$ for $J(x)=(4,2,1^5)$	$J(\varphi(x))$ for $J(x)=(4,1^7)$
55	ω_2	5	$(5^3,4^4,3^5,2,1^7)$	$(5^4,4,3^6,2^3,1^7)$	$(5^3,4^3,3^3,2^6,1^7)$	$(5^2,4^5,3,2^5,1^{12})$	$(5,4^7,1^{22})$
		≥ 7	$(6,5,4^5,3^5,2,1^7)$				
66	$2\omega_1$	5	$(5^5,4^4,3^4,2,1^{11})$	$(5^5,4,3^9,2^3,1^4)$	$(5^4,4^3,3^5,2^6,1^7)$	$(5^3,4^5,3^2,2^5,1^{15})$	$(5^2,4^7,1^{28})$
		≥ 7	$(7,6,5,4^5,3^5,2,1^{11})$	$(7,5^3,4,3^{10},2^3,1^4)$	$(7,5^2,4^3,3^6,2^6,1^7)$	$(7,5,4^5,3^3,2^5,1^{15})$	$(7,4^7,3,1^{28})$

Table 9 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(3^3,2)$	$J(\varphi(x))$ for $J(x)=(3^3,1^2)$	$J(\varphi(x))$ for $J(x)=(3^2,2^2,1)$	$J(\varphi(x))$ for $J(x)=(3^2,2,1^3)$
55	ω_2	3	$(3^{18},1)$	$(3^{18},1)$	$(3^{16},2^2,1^3)$	$(3^{15},2^3,1^4)$
		≥ 5	$(5^3,4^3,3^6,2^3,1^4)$	$(5^3,3^{12},1^4)$	$(5,4^4,3^6,2^6,1^4)$	$(5,4^2,3^9,2^5,1^5)$
66	$2\omega_1$	3	(3^{22})	$(3^{21},1^3)$	$(3^{20},2^2,1^2)$	$(3^{18},2^3,1^6)$
		≥ 5	$(5^6,4^3,3^4,2^3,1^6)$	$(5^6,3^9,1^9)$	$(5^3,4^4,3^6,2^6,1^5)$	$(5^3,4^2,3^8,2^5,1^9)$

Table 9 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(3^2,1^5)$	$J(\varphi(x))$ for $J(x)=(3,2^4)$	$J(\varphi(x))$ for $J(x)=(3,2^3,1^2)$	$J(\varphi(x))$ for $J(x)=(3,2^2,1^4)$	$J(\varphi(x))$ for $J(x)=(3,2,1^6)$
55	ω_2	3	$(3^{15},1^{10})$	$(3^{15},1^{10})$	$(3^{12},2^6,1^7)$	$(3^{10},2^8,1^9)$	$(3^9,2^6,1^{16})$
		≥ 5	$(5,3^{13},1^{11})$	$(4^4,3^7,2^4,1^{10})$	$(4^3,3^6,2^9,1^7)$	$(4^2,3^6,2^{10},1^9)$	$(4,3^7,2^7,1^{16})$
66	$2\omega_1$	3	$(3^{17},1^{15})$	$(3^{20},1^6)$	$(3^{16},2^6,1^6)$	$(3^{13},2^8,1^{11})$	$(3^{11},2^6,1^{21})$
		≥ 5	$(5^3,3^{11},1^{18})$	$(5,4^4,3^{10},2^4,1^7)$	$(5,4^3,3^8,2^9,1^7)$	$(5,4^2,3^7,2^{10},1^{12})$	$(5,4,3^7,2^7,1^{22})$

Table 9 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(3,1^8)$	$J(\varphi(x))$ for $J(x)=(2^5,1)$	$J(\varphi(x))$ for $J(x)=(2^4,1^3)$	$J(\varphi(x))$ for $J(x)=(2^3,1^5)$	$J(\varphi(x))$ for $J(x)=(2^2,1^7)$	$J(\varphi(x))$ for $J(x)=(2,1^9)$
55	ω_2	≥ 3	$(3^9,1^{28})$	$(3^{10},2^5,1^{15})$	$(3^6,2^{12},1^{13})$	$(3^3,2^{15},1^{16})$	$(3,2^{14},1^{24})$	$(2^9,1^{37})$
66	$2\omega_1$	3	$(3^{10},1^{36})$	$(3^{15},2^5,1^{11})$	$(3^{10},2^{12},1^{12})$	$(3^6,2^{15},1^{18})$	$(3^3,2^{14},1^{29})$	$(3,2^9,1^{45})$
		≥ 5	$(5,3^8,1^{37})$					

TABLE 10. $G = A_{11}(K)$

n	ω	p	$J(\varphi(x))$ for $J(x) = (12)$	$J(\varphi(x))$ for $J(x) = (11,1)$	$J(\varphi(x))$ for $J(x) = (10,2)$	$J(\varphi(x))$ for $J(x) = (10,1^2)$	$J(\varphi(x))$ for $J(x) = (9,3)$
66	ω_2	11	-	(11^6)	$(11^5,9,1^2)$	$(11^4,10^2,1^2)$	$(11^4,9,7,3^2)$
		13	$(13^5,1)$	$(13^4,11,3)$	$(13^3,11,9,5,1^2)$	$(13^3,10^2,5,1^2)$	$(13^2,11,9,7^2,3^2)$
		17	$(17^3,9,5,1)$	$(17^2,11^2,7,3)$	$(17,13,11,9^2,5,1^2)$	$(17,13,10^2,9,5,1^2)$	$(15,11^2,9,7^2,3^2)$
		19	$(19^2,13,9,5,1)$	$(19,15,11^2,7,3)$			
		≥ 23	$(21,17,13,9,5,1)$				
78	$2\omega_1$	11	-	$(11^7,1)$	$(11^6,9,3)$	$(11^5,10^2,1^3)$	$(11^5,9,7,5,1^2)$
		13	(13^6)	$(13^5,11,1^2)$	$(13^4,11,9,3^2)$	$(13^4,10^2,3,1^3)$	$(13^3,11,9,7,5^2,1^2)$
		17	$(17^4,7,3)$	$(17^3,11,9,5,1^2)$	$(17^2,11^2,9,7,3^2)$	$(17^2,11,10^2,7,3,1^3)$	$(17,13,11,9^2,7,5^2,1^2)$
		19	$(19^3,11,7,3)$	$(19^2,13,11,9,5,1^2)$	$(19,15,11^2,9,7,3^2)$	$(19,15,11,10^2,7,3,1^3)$	
		≥ 23	$(23,19,15,11,7,3)$	$(21,17,13,11,9,5,1^2)$			

Table 10 continued

n	ω	p	$J(\varphi(x))$ for $J(x) = (9,2,1)$	$J(\varphi(x))$ for $J(x) = (9,1^3)$	$J(\varphi(x))$ for $J(x) = (8,4)$	$J(\varphi(x))$ for $J(x) = (8,3,1)$	$J(\varphi(x))$ for $J(x) = (8,2^2)$
66	ω_2	11	$(11^3,10,9,8,3,2,1)$	$(11^3,9^3,3,1^3)$	$(11^3,9,7,5^3,1^2)$	$(11^2,10,8^2,6,5,3^2,1)$	$(11^2,9^2,7^2,5,3,1^4)$
		13	$(13^2,10,9,8,7,3,2,1)$	$(13^2,9^3,7,3,1^3)$	$(13,11,9^2,7,5^3,1^2)$	$(13,10,9,8^2,6,5,3^2,1)$	$(13,9^3,7^2,5,3,1^4)$
		≥ 17	$(15,11,10,9,8,7,3,2,1)$	$(15,11,9^3,7,3,1^3)$			
78	$2\omega_1$	11	$(11^4,10,9,8,3,2,1^2)$	$(11^4,9^3,1^7)$	$(11^4,9,7^2,5,3^2)$	$(11^3,10,8^2,6,5,3^2,1^2)$	$(11^3,9^2,7^2,3^4,1)$
		13	$(13^3,10,9,8,5,3,2,1^2)$	$(13^3,9^3,5,1^7)$	$(13^2,11,9,7^3,5,3^2)$	$(13^2,10,8^2,7,6,5,3^2,1^2)$	$(13^2,9^2,7^3,3^4,1)$
		≥ 17	$(17,13,10,9^2,8,5,3,2,1^2)$	$(17,13,9^4,5,1^7)$	$(15,11^2,9,7^3,5,3^2)$	$(15,11,10,8^2,7,6,5,3^2,1^2)$	$(15,11,9^2,7^3,3^4,1)$

Table 10 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(8,2,1^2)$	$J(\varphi(x))$ for $J(x)=(8,1^4)$	$J(\varphi(x))$ for $J(x)=(7,5)$	$J(\varphi(x))$ for $J(x)=(7,4,1)$
66	ω_2	7	-	-	$(7^9,3)$	$(7^8,5,4,1)$
		11	$(11^2,9,8^2,7,5,2^2,1^3)$	$(11^2,8^4,5,1^7)$	$(11^2,9,7^3,5,3^3)$	$(11,10,8,7^2,6,5,4^2,3,1)$
		≥ 13	$(13,9^2,8^2,7,5,2^2,1^3)$	$(13,9,8^4,5,1^7)$		
78	$2\omega_1$	7	-	-	$(7^{11},1)$	$(7^{10},4,3,1)$
		11	$(11^3,9,8^2,7,3^2,2^2,1^3)$	$(11^3,8^4,3,1^{10})$	$(11^3,9^2,7,5^3,3,1^2)$	$(11^2,10,8,7^2,6,5,4^2,3,1^2)$
		13	$(13^2,9,8^2,7^2,3^2,2^2,1^3)$	$(13^2,8^4,7,3,1^{10})$	$(13,11,9^3,7,5^3,3,1^2)$	$(13,10,9,8,7^2,6,5,4^2,3,1^2)$
		≥ 17	$(15,11,9,8^2,7^2,3^2,2^2,1^3)$	$(15,11,8^4,7,3,1^{10})$		

Table 10 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(7,3,2)$	$J(\varphi(x))$ for $J(x)=(7,3,1^2)$	$J(\varphi(x))$ for $J(x)=(7,2^2,1)$	$J(\varphi(x))$ for $J(x)=(7,2,1^3)$	$J(\varphi(x))$ for $J(x)=(7,1^5)$
66	ω_2	7	$(7^8,4,3,2,1)$	$(7^8,3^3,1)$	$(7^8,3,2^2,1^3)$	$(7^8,2^3,1^4)$	$(7^8,1^{10})$
		≥ 11	$(11,9,8,7^2,6,5,4,3^2,2,1)$	$(11,9,7^4,5,3^4,1)$	$(11,8^2,7^2,6^2,3^2,2^2,1^3)$	$(11,8,7^4,6,3,2^3,1^4)$	$(11,7^6,3,1^{10})$
78	$2\omega_1$	7	$(7^9,5,4,3,2,1)$	$(7^9,5,3^2,1^4)$	$(7^9,3^3,2^2,1^2)$	$(7^9,3,2^3,1^6)$	$(7^9,1^{15})$
		11	$(11^2,9,8,7,6,5^3,4,3,2,1^2)$	$(11^2,9,7^3,5^3,3^2,1^5)$	$(11^2,8^2,7,6^2,5,3^3,2^2,1^3)$	$(11^2,8,7^3,6,5,3,2^3,1^7)$	$(11^2,7^5,5,1^{16})$
		≥ 13	$(13,9^2,8,7,6,5^3,4,3,2,1^2)$	$(13,9^2,7^3,5^3,3^2,1^5)$	$(13,9,8^2,7,6^2,5,3^3,2^2,1^3)$	$(13,9,8,7^3,6,5,3,2^3,1^7)$	$(13,9,7^5,5,1^{16})$

Table 10 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(6^2)$	$J(\varphi(x))$ for $J(x)=(6,5,1)$	$J(\varphi(x))$ for $J(x)=(6,4,2)$	$J(\varphi(x))$ for $J(x)=(6,4,1^2)$	$J(\varphi(x))$ for $J(x)=(6,3^2)$
66	ω_2	7	$(7^9,1^3)$	$(7^7,6,5,3,2,1)$	$(7^6,5^3,3^2,1^3)$	$(7^5,6^2,5,4^2,3,1^3)$	$(7^6,5,4^2,3^3,1^2)$
		≥ 11	$(11,9^3,7,5^3,3,1^3)$	$(10,9,8,7,6^2,5^2,4,3,2,1)$	$(9^2,7^2,5^5,3^2,1^3)$	$(9^2,7,6^2,5^3,4^2,3,1^3)$	$(9,8^2,6^2,5^2,4^2,3^3,1^2)$
78	$2\omega_1$	7	$(7^{11},1)$	$(7^9,6,5,2,1^2)$	$(7^8,5^2,3^4)$	$(7^7,6^2,4^2,3^2,1^3)$	$(7^7,5^3,4^2,3,1^3)$
		≥ 11	$(11^3,9,7^3,5,3^3,1)$	$(11,10,9,8,7,6^2,5^2,4,3,2,1^2)$	$(11,9,7^4,5^3,3^5)$	$(11,9,7^3,6^2,5,4^2,3^3,1^3)$	$(11,8^2,7,6^2,5^3,4^2,3^2,1^3)$

Table 10 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(6,3,2,1)$	$J(\varphi(x))$ for $J(x)=(6,3,1^3)$	$J(\varphi(x))$ for $J(x)=(6,2^3)$	$J(\varphi(x))$ for $J(x)=(6,2^2,1^2)$	$J(\varphi(x))$ for $J(x)=(6,2,1^4)$
66	ω_2	7	$(7^5,6,5,4^2,3^2,2^2,1^2)$	$(7^4,6^3,4,3^4,1^4)$	$(7^5,5^3,3^3,1^7)$	$(7^4,6^2,5^2,3,2^4,1^5)$	$(7^3,6^4,5,2^4,1^8)$
		≥ 11	$(9,8,7,6^2,5^2,4^2,3^2,2^2,1^2)$	$(9,8,6^4,5,4,3^4,1^4)$	$(9,7^3,5^4,3^3,1^7)$	$(9,7^2,6^2,5^3,3,2^4,1^5)$	$(9,7,6^4,5^2,2^4,1^8)$
78	$2\omega_1$	7	$(7^6,6,5^2,4^2,3^2,2^2,1^2)$	$(7^5,6^3,5,4,3^3,1^7)$	$(7^6,5^3,3^6,1^3)$	$(7^5,6^2,5^2,3^3,2^4,1^4)$	$(7^4,6^4,5,3,2^4,1^{10})$
		≥ 11	$(11,8,7^2,6^2,5^2,4^2,3^3,2^2,1^2)$	$(11,8,7,6^4,5,4,3^4,1^7)$	$(11,7^4,5^3,3^7,1^3)$	$(11,7^3,6^2,5^2,3^4,2^4,1^4)$	$(11,7^2,6^4,5,3^2,2^4,1^{10})$

Table 10 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(6,1^6)$	$J(\varphi(x))$ for $J(x)=(5^2,2)$	$J(\varphi(x))$ for $J(x)=(5^2,1^2)$	$J(\varphi(x))$ for $J(x)=(5,4,3)$	$J(\varphi(x))$ for $J(x)=(5,4,2,1)$
66	ω_2	5	-	$(5^{13},1)$	$(5^{13},1)$	$(5^{12},3,2,1)$	$(5^{11},4,3,2,1^2)$
		7	$(7^2,6^6,1^{16})$	$(7^5,6^2,4^2,3^3,1^2)$	$(7^5,5^4,3^3,1^2)$	$(7^4,6,5^2,4^2,3^3,2^2,1)$	$(7^3,6,5^3,4^3,3^2,2^2,1^2)$
		≥ 11	$(9,6^6,5,1^{16})$	$(9,7^3,6^2,5,4^2,3^3,1^2)$	$(9,7^3,5^5,3^3,1^2)$	$(8,7^2,6^2,5^2,4^2,3^3,2^2,1)$	$(8,7,6^2,5^3,4^3,3^2,2^2,1^2)$
78	$2\omega_1$	5	-	$(5^{15},3)$	$(5^{15},1^3)$	$(5^{15},2,1)$	$(5^{13},4,3^2,2,1)$
		7	$(7^3,6^6,1^{21})$	$(7^7,6^2,4^2,3^2,1^3)$	$(7^7,5^4,3,1^6)$	$(7^6,6,5^2,4^2,3^2,2^2,1^2)$	$(7^5,6,5^2,4^3,3^3,2^2,1^2)$
		≥ 11	$(11,7,6^6,3,1^{21})$	$(9^3,7,6^2,5^3,4^2,3^2,1^3)$	$(9^3,7,5^7,3,1^6)$	$(9,8,7^2,6^2,5^3,4^2,3^2,2^2,1^2)$	$(9,8,7,6^2,5^3,4^3,3^3,2^2,1^2)$

Table 10 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(5,4,1^3)$	$J(\varphi(x))$ for $J(x)=(5,3^2,1)$	$J(\varphi(x))$ for $J(x)=(5,3,2^2)$	$J(\varphi(x))$ for $J(x)=(5,3,2,1^2)$	$J(\varphi(x))$ for $J(x)=(5,3,1^4)$
66	ω_2	5	$(5^{10},4^3,1^4)$	$(5^{10},3^5,1)$	$(5^9,4^2,3^2,2^2,1^3)$	$(5^9,4,3^3,2^3,1^2)$	$(5^9,3^5,1^6)$
		7	$(7^3,5^4,4^4,3,2,1^4)$	$(7^3,5^4,3^8,1)$	$(7^2,6^2,5,4^4,3^4,2^2,1^3)$	$(7^2,6,5^3,4^2,3^5,2^3,1^2)$	$(7^2,5^5,3^7,1^6)$
		≥ 11	$(8,7,6,5^4,4^4,3,2,1^4)$				
78	$2\omega_1$	5	$(5^{12},4^3,1^6)$	$(5^{13},3^3,1^4)$	$(5^{11},4^2,3^3,2^2,1^2)$	$(5^{11},4,3^3,2^3,1^4)$	$(5^{11},3^4,1^{11})$
		7	$(7^5,5^3,4^4,3,2,1^7)$	$(7^4,5^6,3^5,1^5)$	$(7^3,6^2,5^2,4^4,3^4,2^2,1^3)$	$(7^3,6,5^4,4^2,3^4,2^3,1^5)$	$(7^3,5^6,3^5,1^{12})$
		≥ 11	$(9,8,7,6,5^4,4^4,3,2,1^7)$	$(9,7^2,5^7,3^5,1^5)$	$(9,7,6^2,5^3,4^4,3^4,2^2,1^3)$	$(9,7,6,5^5,4^2,3^4,2^3,1^5)$	$(9,7,5^7,3^5,1^{12})$

Table 10 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(5,2^3,1)$	$J(\varphi(x))$ for $J(x)=(5,2^2,1^3)$	$J(\varphi(x))$ for $J(x)=(5,2,1^5)$	$J(\varphi(x))$ for $J(x)=(5,1^7)$	$J(\varphi(x))$ for $J(x)=(4^3)$
66	ω_2	5	$(5^9,3^3,2^3,1^6)$	$(5^9,3,2^6,1^6)$	$(5^9,2^5,1^{11})$	$(5^9,1^{21})$	$(5^{12},1^6)$
		≥ 7	$(7,6^3,5,4^3,3^4,2^3,1^6)$	$(7,6^2,5^3,4^2,3^2,2^6,1^6)$	$(7,6,5^5,4,3,2^5,1^{11})$	$(7,5^7,3,1^{21})$	$(7^3,5^6,3^3,1^6)$
78	$2\omega_1$	5	$(5^{10},3^6,2^3,1^4)$	$(5^{10},3^3,2^6,1^7)$	$(5^{10},3,2^5,1^{15})$	$(5^{10},1^{28})$	$(5^{15},1^3)$
		7	$(7^2,6^3,5,4^3,3^6,2^3,1^5)$	$(7^2,6^2,5^3,4^2,3^3,2^6,1^8)$	$(7^2,6,5^5,4,3,2^5,1^{16})$	$(7^2,5^7,1^{29})$	$(7^6,5^3,3^6,1^3)$
		≥ 11	$(9,6^3,5^2,4^3,3^6,2^3,1^5)$	$(9,6^2,5^4,4^2,3^3,2^6,1^8)$	$(9,6,5^6,4,3,2^5,1^{16})$	$(9,5^8,1^{29})$	

Table 10 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(4^2,3,1)$	$J(\varphi(x))$ for $J(x)=(4^2,2^2)$	$J(\varphi(x))$ for $J(x)=(4^2,2,1^2)$	$J(\varphi(x))$ for $J(x)=(4^2,1^4)$	$J(\varphi(x))$ for $J(x)=(4,3^2,2)$
66	ω_2	5	$(5^9,4^2,3^2,2^2,1^3)$	$(5^9,3^5,1^6)$	$(5^7,4^4,3^2,2^2,1^5)$	$(5^5,4^8,1^9)$	$(5^7,4^2,3^4,2^4,1^3)$
		≥ 7	$(7,6^2,5^3,4^4,3^3,2^2,1^3)$	$(7,5^7,3^6,1^6)$	$(7,5^5,4^4,3^3,2^2,1^5)$	$(7,5^3,4^8,3,1^9)$	$(6^2,5^3,4^4,3^4,2^4,1^3)$
78	$2\omega_1$	5	$(5^{12},4^2,3,2^2,1^3)$	$(5^{11},3^7,1^2)$	$(5^9,4^4,3^3,2^2,1^4)$	$(5^7,4^8,1^{11})$	$(5^{10},4^2,3^3,2^4,1^3)$
		≥ 7	$(7^3,6^2,5^2,4^4,3^4,2^2,1^3)$	$(7^3,5^5,3^{10},1^2)$	$(7^3,5^3,4^4,3^6,2^2,1^4)$	$(7^3,5,4^8,3^3,1^{11})$	$(7,6^2,5^4,4^4,3^4,2^4,1^3)$

Table 10 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(4,3^2,1^2)$	$J(\varphi(x))$ for $J(x)=(4,3,2^2,1)$	$J(\varphi(x))$ for $J(x)=(4,3,2,1^3)$	$J(\varphi(x))$ for $J(x)=(4,3,1^5)$	$J(\varphi(x))$ for $J(x)=(4,2^4)$
66	ω_2	5	$(5^6,4^2,3^7,2^2,1^3)$	$(5^5,4^3,3^5,2^5,1^4)$	$(5^4,4^4,3^5,2^5,1^5)$	$(5^3,4^5,3^6,2,1^{11})$	$(5^5,3^{10},1^{11})$
		≥ 7	$(6^2,5^2,4^4,3^7,2^2,1^3)$	$(6,5^3,4^4,3^5,2^5,1^4)$	$(6,5^2,4^5,3^5,2^5,1^5)$	$(6,5,4^6,3^6,2,1^{11})$	
78	$2\omega_1$	5	$(5^9,4^2,3^5,2^2,1^6)$	$(5^7,4^3,3^6,2^5,1^3)$	$(5^6,4^4,3^5,2^5,1^7)$	$(5^5,4^5,3^5,2,1^{16})$	$(5^6,3^{14},1^6)$
		≥ 7	$(7,6^2,5^3,4^4,3^6,2^2,1^6)$	$(7,6,5^3,4^4,3^7,2^5,1^3)$	$(7,6,5^2,4^5,3^6,2^5,1^7)$	$(7,6,5,4^6,3^6,2,1^{16})$	$(7,5^4,3^{15},1^6)$

Table 10 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(4,2^3,1^2)$	$J(\varphi(x))$ for $J(x)=(4,2^2,1^4)$	$J(\varphi(x))$ for $J(x)=(4,2,1^6)$	$J(\varphi(x))$ for $J(x)=(4,1^8)$
66	ω_2	≥ 5	$(5^4,4^2,3^6,2^6,1^8)$	$(5^3,4^4,3^3,2^8,1^{10})$	$(5^2,4^6,3,2^6,1^{17})$	$(5,4^8,1^{29})$
78	$2\omega_1$	5	$(5^5,4^2,3^9,2^6,1^6)$	$(5^4,4^4,3^5,2^8,1^{11})$	$(5^3,4^6,3^2,2^6,1^{21})$	$(5^2,4^8,1^{36})$
		≥ 7	$(7,5^3,4^2,3^{10},2^6,1^6)$	$(7,5^2,4^4,3^6,2^8,1^{11})$	$(7,5,4^6,3^3,2^6,1^{21})$	$(7,4^8,3,1^{36})$

Table 10 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(3^4)$	$J(\varphi(x))$ for $J(x)=(3^3,2,1)$	$J(\varphi(x))$ for $J(x)=(3^3,1^3)$	$J(\varphi(x))$ for $J(x)=(3^2,2^3)$	$J(\varphi(x))$ for $J(x)=(3^2,2^2,1^2)$	$J(\varphi(x))$ for $J(x)=(3^2,2,1^4)$
66	ω_2	3	(3^{22})	$(3^{21},2,1)$	$(3^{21},1^3)$	$(3^{20},1^6)$	$(3^{18},2^4,1^4)$	$(3^{17},2^4,1^7)$
		≥ 5	$(5^6,3^{10},1^6)$	$(5^3,4^3,3^9,2^4,1^4)$	$(5^3,3^{15},1^6)$	$(5,4^6,3^6,2^6,1^7)$	$(5,4^4,3^8,2^8,1^5)$	$(5,4^2,3^{11},2^6,1^8)$
78	$2\omega_1$	3	(3^{26})	$(3^{25},2,1)$	$(3^{24},1^6)$	$(3^{25},1^3)$	$(3^{22},2^4,1^4)$	$(3^{20},2^4,1^{10})$
		≥ 5	$(5^{10},3^6,1^{10})$	$(5^6,4^3,3^7,2^4,1^7)$	$(5^6,3^{12},1^{12})$	$(5^3,4^6,3^7,2^6,1^6)$	$(5^3,4^4,3^8,2^8,1^7)$	$(5^3,4^2,3^{10},2^6,1^{13})$

Table 10 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(3^2,1^6)$	$J(\varphi(x))$ for $J(x)=(3,2^4,1)$	$J(\varphi(x))$ for $J(x)=(3,2^3,1^3)$	$J(\varphi(x))$ for $J(x)=(3,2^2,1^5)$	$J(\varphi(x))$ for $J(x)=(3,2,1^7)$	$J(\varphi(x))$ for $J(x)=(3,1^9)$
66	ω_2	3	$(3^{17},1^{15})$	$(3^{16},2^4,1^{10})$	$(3^{13},2^9,1^9)$	$(3^{11},2^{10},1^{13})$	$(3^{10},2^7,1^{22})$	$(3^{10},1^{36})$
		≥ 5	$(5,3^{15},1^{16})$	$(4^4,3^8,2^8,1^{10})$	$(4^3,3^7,2^{12},1^9)$	$(4^2,3^7,2^{12},1^{13})$	$(4,3^8,2^8,1^{22})$	
78	$2\omega_1$	3	$(3^{19},1^{21})$	$(3^{21},2^4,1^7)$	$(3^{17},2^9,1^9)$	$(3^{14},2^{10},1^{16})$	$(3^{12},2^7,1^{28})$	$(3^{11},1^{45})$
		≥ 5	$(5^3,3^{13},1^{24})$	$(5,4^4,3^{11},2^8,1^8)$	$(5,4^3,3^9,2^{12},1^{10})$	$(5,4^2,3^8,2^{12},1^{17})$	$(5,4,3^8,2^8,1^{29})$	$(5,3^9,1^{46})$

Table 10 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(2^6)$	$J(\varphi(x))$ for $J(x)=(2^5,1^2)$	$J(\varphi(x))$ for $J(x)=(2^4,1^4)$	$J(\varphi(x))$ for $J(x)=(2^3,1^6)$	$J(\varphi(x))$ for $J(x)=(2^2,1^8)$	$J(\varphi(x))$ for $J(x)=(2,1^{10})$
66	ω_2	≥ 3	$(3^{15},1^{21})$	$(3^{10},2^{10},1^{16})$	$(3^6,2^{16},1^{16})$	$(3^3,2^{18},1^{21})$	$(3,2^{16},1^{31})$	$(2^{10},1^{46})$
78	$2\omega_1$	≥ 3	$(3^{21},1^{15})$	$(3^{15},2^{10},1^{13})$	$(3^{10},2^{16},1^{16})$	$(3^6,2^{18},1^{24})$	$(3^3,2^{16},1^{37})$	$(3,2^{10},1^{55})$

TABLE 11. $G=A_{12}(K)$

n	ω	p	$J(\varphi(x))$ for $J(x)=(13)$	$J(\varphi(x))$ for $J(x)=(12,1)$	$J(\varphi(x))$ for $J(x)=(11,2)$	$J(\varphi(x))$ for $J(x)=(11,1^2)$
78	ω_2	11	-	-	$(11^7,1)$	$(11^7,1)$
		13	(13^6)	$(13^5,12,1)$	$(13^4,12,10,3,1)$	$(13^4,11^2,3,1)$
		17	$(17^4,7,3)$	$(17^3,12,9,5,1)$	$(17^2,12,11,10,7,3,1)$	$(17^2,11^3,7,3,1)$
		19	$(19^3,11,7,3)$	$(19^2,13,12,9,5,1)$	$(19,15,12,11,10,7,3,1)$	$(19,15,11^3,7,3,1)$
		≥ 23	$(23,19,15,11,7,3)$	$(21,17,13,12,9,5,1)$		
91	$2\omega_1$	11	-	-	$(11^8,3)$	$(11^8,1^3)$
		13	(13^7)	$(13^6,12,1)$	$(13^5,12,10,3,1)$	$(13^5,11^2,1^4)$
		17	$(17^5,5,1)$	$(17^4,12,7,3,1)$	$(17^3,12,10,9,5,3,1)$	$(17^3,11^2,9,5,1^4)$
		19	$(19^4,9,5,1)$	$(19^3,12,11,7,3,1)$	$(19^2,13,12,10,9,5,3,1)$	$(19^2,13,11^2,9,5,1^4)$
		23	$(23^2,17,13,9,5,1)$	$(23,19,15,12,11,7,3,1)$	$(21,17,13,12,10,9,5,3,1)$	$(21,17,13,11^2,9,5,1^4)$
		≥ 29	$(25,21,17,13,9,5,1)$			

Table 11 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(10,3)$	$J(\varphi(x))$ for $J(x)=(10,2,1)$	$J(\varphi(x))$ for $J(x)=(10,1^3)$	$J(\varphi(x))$ for $J(x)=(9,4)$
78	ω_2	11	$(11^6,8,3,1)$	$(11^5,10,9,2,1^2)$	$(11^4,10^3,1^4)$	$(11^5,8,6,5,3,1)$
		13	$(13^3,12,10,8,5,3,1)$	$(13^3,11,10,9,5,2,1^2)$	$(13^3,10^3,5,1^4)$	$(13^2,12,10,8,7,6,5,3,1)$
		≥ 17	$(17,13,12,10,9,8,5,3,1)$	$(17,13,11,10,9^2,5,2,1^2)$	$(17,13,10^3,9,5,1^4)$	$(15,12,11,10,8,7,6,5,3,1)$

Table 11 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(10,3)$	$J(\varphi(x))$ for $J(x)=(10,2,1)$	$J(\varphi(x))$ for $J(x)=(10,1^3)$	$J(\varphi(x))$ for $J(x)=(9,4)$
91	$2\omega_1$	11	$(11^7,8,5,1)$	$(11^6,10,9,3,2,1)$	$(11^5,10^3,1^6)$	$(11^6,8,7,6,3,1)$
		13	$(13^4,12,10,8,5,3,1)$	$(13^4,11,10,9,3^2,2,1)$	$(13^4,10^3,3,1^6)$	$(13^3,12,10,8,7,6,5,3,1)$
		17	$(17^2,12,11,10,8,7,5,3,1)$	$(17^2,11^2,10,9,7,3^2,2,1)$	$(17^2,11,10^3,7,3,1^6)$	$(17,13,12,10,9,8,7,6,5,3,1)$
		19	$(19,15,12,11,10,8,7,5,3,1)$	$(19,15,11^2,10,9,7,3^2,2,1)$	$(19,15,11,10^3,7,3,1^6)$	

Table 11 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(9,3,1)$	$J(\varphi(x))$ for $J(x)=(9,2^2)$	$J(\varphi(x))$ for $J(x)=(9,2,1^2)$	$J(\varphi(x))$ for $J(x)=(9,1^4)$
78	ω_2	11	$(11^4,9^2,7,3^3)$	$(11^3,10^2,8^2,3^2,1^3)$	$(11^3,10,9^2,8,3,2^2,1^2)$	$(11^3,9^4,3,1^6)$
		13	$(13^2,11,9^2,7^2,3^3)$	$(13^2,10^2,8^2,7,3^2,1^3)$	$(13^2,10,9^2,8,7,3,2^2,1^2)$	$(13^2,9^4,7,3,1^6)$
		≥ 17	$(15,11^2,9^2,7^2,3^3)$	$(15,11,10^2,8^2,7,3^2,1^3)$	$(15,11,10,9^2,8,7,3,2^2,1^2)$	$(15,11,9^4,7,3,1^6)$
91	$2\omega_1$	11	$(11^5,9^2,7,5,3,1^3)$	$(11^4,10^2,8^2,3^3,1^2)$	$(11^4,10,9^2,8,3,2^2,1^4)$	$(11^4,9^4,1^{11})$
		13	$(13^3,11,9^2,7,5^2,3,1^3)$	$(13^3,10^2,8^2,5,3^3,1^2)$	$(13^3,10,9^2,8,5,3,2^2,1^4)$	$(13^3,9^4,5,1^{11})$
		≥ 17	$(17,13,11,9^3,7,5^2,3,1^3)$	$(17,13,10^2,9,8^2,5,3^3,1^2)$	$(17,13,10,9^3,8,5,3,2^2,1^4)$	$(17,13,9^5,5,1^{11})$

Table 11 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(8,5)$	$J(\varphi(x))$ for $J(x)=(8,4,1)$	$J(\varphi(x))$ for $J(x)=(8,3,2)$	$J(\varphi(x))$ for $J(x)=(8,3,1^2)$
78	ω_2	11	$(11^4,8,7,6,5,4,3,1)$	$(11^3,9,8,7,5^3,4,1^2)$	$(11^2,10,9,8,7,6,5,4,3,2,1^2)$	$(11^2,10,8^3,6,5,3^3,1^2)$
		≥ 13	$(13,12,10,9,8,7,6,5,4,3,1)$	$(13,11,9^2,8,7,5^3,4,1^2)$	$(13,10,9^2,8,7,6,5,4,3,2,1^2)$	$(13,10,9,8^3,6,5,3^3,1^2)$
91	$2\omega_1$	11	$(11^5,9,8,6,5,4,3,1)$	$(11^4,9,8,7^2,5,4,3^2,1)$	$(11^3,10,9,8,7,6,5,4,3^2,2,1)$	$(11^3,10,8^3,6,5,3^3,1^4)$
		13	$(13^2,12,10,9,8,7,6,5,4,3,1)$	$(13^2,11,9,8,7^3,5,4,3^2,1)$	$(13^2,10,9,8,7^2,6,5,4,3^2,2,1)$	$(13^2,10,8^3,7,6,5,3^3,1^4)$
		≥ 17	$(15,12,11,10,9,8,7,6,5,4,3,1)$	$(15,11^2,9,8,7^3,5,4,3^2,1)$	$(15,11,10,9,8,7^2,6,5,4,3^2,2,1)$	$(15,11,10,8^3,7,6,5,3^3,1^4)$

Table 11 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(8,2^2,1)$	$J(\varphi(x))$ for $J(x)=(8,2,1^3)$	$J(\varphi(x))$ for $J(x)=(8,1^5)$	$J(\varphi(x))$ for $J(x)=(7,6)$
78	ω_2	7	-	-	-	$(7^{11},1)$
		11	$(11^2,9^2,8,7^2,5,3,2^2,1^4)$	$(11^2,9,8^3,7,5,2^3,1^5)$	$(11^2,8^5,5,1^{11})$	$(11^3,9,8,7,6,5,4,3,2,1)$
		≥ 13	$(13,9^3,8,7^2,5,3,2^2,1^4)$	$(13,9^2,8^3,7,5,2^3,1^5)$	$(13,9,8^5,5,1^{11})$	$(12,11,10,9,8,7,6,5,4,3,2,1)$
91	$2\omega_1$	7	-	-	-	(7^{13})
		11	$(11^3,9^2,8,7^2,3^4,2^2,1^2)$	$(11^3,9,8^3,7,3^2,2^3,1^6)$	$(11^3,8^5,3,1^{15})$	$(11^5,8,7,6,5,4,3,2,1)$
		13	$(13^2,9^2,8,7^3,3^4,2^2,1^2)$	$(13^2,9,8^3,7^2,3^2,2^3,1^6)$	$(13^2,8^5,7,3,1^{15})$	$(13,12,11,10,9,8,7,6,5,4,3,2,1)$
		≥ 17	$(15,11,9^2,8,7^3,3^4,2^2,1^2)$	$(15,11,9,8^3,7^2,3^2,2^3,1^6)$	$(15,11,8^5,7,3,1^{15})$	

Table 11 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(7,5,1)$	$J(\varphi(x))$ for $J(x)=(7,4,2)$	$J(\varphi(x))$ for $J(x)=(7,4,1^2)$	$J(\varphi(x))$ for $J(x)=(7,3^2)$
78	ω_2	7	$(7^{10},5,3)$	$(7^9,5^2,3,1^2)$	$(7^9,5,4^2,1^2)$	$(7^9,5,3^3,1)$
		≥ 11	$(11^2,9,7^4,5^2,3^3)$	$(11,10,8^2,7,6^2,5^2,4,3^2,1^2)$	$(11,10,8,7^3,6,5,4^3,3,1^2)$	$(11,9^2,7^3,5^3,3^4,1)$
91	$2\omega_1$	7	$(7^{12},5,1^2)$	$(7^{11},5,3^3)$	$(7^{11},4^2,3,1^3)$	$(7^{10},5^3,3,1^3)$
		11	$(11^3,9^2,7^2,5^4,3,1^3)$	$(11^2,10,8^2,7,6^2,5^2,4,3^3,1)$	$(11^2,10,8,7^3,6,5,4^3,3,1^4)$	$(11^2,9^2,7^2,5^6,3,1^4)$
		≥ 13	$(13,11,9^3,7^2,5^4,3,1^3)$	$(13,10,9,8^2,7,6^2,5^2,4,3^3,1)$	$(13,10,9,8,7^3,6,5,4^3,3,1^4)$	$(13,9^3,7^2,5^6,3,1^4)$

Table 11 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(7,3,2,1)$	$J(\varphi(x))$ for $J(x)=(7,3,1^3)$	$J(\varphi(x))$ for $J(x)=(7,2^3)$	$J(\varphi(x))$ for $J(x)=(7,2^2,1^2)$
78	ω_2	7	$(7^9,4,3^2,2^2,1)$	$(7^9,3^4,1^3)$	$(7^9,3^3,1^6)$	$(7^9,3,2^4,1^4)$
		≥ 11	$(11,9,8,7^3,6,5,4,3^3,2^2,1)$	$(11,9,7^5,5,3^5,1^3)$	$(11,8^3,7,6^3,3^4,1^6)$	$(11,8^2,7^3,6^2,3^2,2^4,1^4)$

Table 11 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(7,3,2,1)$	$J(\varphi(x))$ for $J(x)=(7,3,1^3)$	$J(\varphi(x))$ for $J(x)=(7,2^3)$	$J(\varphi(x))$ for $J(x)=(7,2^2,1^2)$
91	$2\omega_1$	7	$(7^{10},5,4,3^2,2^2,1^2)$	$(7^{10},5,3^3,1^7)$	$(7^{10},3^6,1^3)$	$(7^{10},3^3,2^4,1^4)$
		11	$(11^2,9,8,7^2,6,5^3,4,3^2,2^2,1^3)$	$(11^2,9,7^4,5^3,3^3,1^8)$	$(11^2,8^3,6^3,5,3^6,1^4)$	$(11^2,8^2,7^2,6^2,5,3^3,2^4,1^5)$
		≥ 13	$(13,9^2,8,7^2,6,5^3,4,3^2,2^2,1^3)$	$(13,9^2,7^4,5^3,3^3,1^8)$	$(13,9,8^3,6^3,5,3^6,1^4)$	$(13,9,8^2,7^2,6^2,5,3^3,2^4,1^5)$

Table 11 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(7,2,1^4)$	$J(\varphi(x))$ for $J(x)=(7,1^6)$	$J(\varphi(x))$ for $J(x)=(6^2,1)$	$J(\varphi(x))$ for $J(x)=(6,5,2)$
78	ω_2	7	$(7^9,2^4,1^7)$	$(7^9,1^{15})$	$(7^9,6^2,1^3)$	$(7^8,6,5,4,3,2,1^2)$
		≥ 11	$(11,8,7^5,6,3,2^4,1^7)$	$(11,7^7,3,1^{15})$	$(11,9^3,7,6^2,5^3,3,1^3)$	$(10,9,8,7^2,6^2,5^2,4^2,3,2,1^2)$
91	$2\omega_1$	7	$(7^{10},3,2^4,1^{10})$	$(7^{10},1^{21})$	$(7^{11},6^2,1^2)$	$(7^{10},6,5,4,3,2,1)$
		11	$(11^2,8,7^4,6,5,3,2^4,1^{11})$	$(11^2,7^6,5,1^{22})$	$(11^3,9,7^3,6^2,5,3^3,1^2)$	$(11,10,9,8,7^2,6^2,5^2,4^2,3^2,2,1)$
		≥ 13	$(13,9,8,7^4,6,5,3,2^4,1^{11})$	$(13,9,7^6,5,1^{22})$		

Table 11 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(6,5,1^2)$	$J(\varphi(x))$ for $J(x)=(6,4,3)$	$J(\varphi(x))$ for $J(x)=(6,4,2,1)$	$J(\varphi(x))$ for $J(x)=(6,4,1^3)$
78	ω_2	7	$(7^7,6^2,5^2,3,2,1^2)$	$(7^7,6,5,4^2,3^2,2,1^2)$	$(7^6,6,5^3,4,3^2,2,1^3)$	$(7^5,6^3,5,4^3,3,1^5)$
		≥ 11	$(10,9,8,7,6^3,5^3,4,3,2,1^2)$	$(9^2,8,7,6^2,5^3,4^2,3^2,2,1^2)$	$(9^2,7^2,6,5^5,4,3^2,2,1^3)$	$(9^2,7,6^3,5^3,4^3,3,1^5)$
91	$2\omega_1$	7	$(7^9,6^2,5^2,2,1^4)$	$(7^9,6,5,4^2,3^2,2,1)$	$(7^8,6,5^2,4,3^4,2,1)$	$(7^7,6^3,4^3,3^2,1^6)$
		≥ 11	$(11,10,9,8,7,6^3,5^3,4,3,2,1^4)$	$(11,9,8,7^3,6^2,5^2,4^2,3^3,2,1)$	$(11,9,7^4,6,5^3,4,3^5,2,1)$	$(11,9,7^3,6^3,5,4^3,3^3,1^6)$

Table 11 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(6,3^2,1)$	$J(\varphi(x))$ for $J(x)=(6,3,2^2)$	$J(\varphi(x))$ for $J(x)=(6,3,2,1^2)$	$J(\varphi(x))$ for $J(x)=(6,3,1^4)$
78	ω_2	7	$(7^6,6,5,4^2,3^5,1^2)$	$(7^6,5^2,4^3,3^2,2^2,1^4)$	$(7^5,6^2,5,4^2,3^3,2^3,1^3)$	$(7^4,6^4,4,3^5,1^7)$
		≥ 11	$(9,8^2,6^3,5^2,4^2,3^5,1^2)$	$(9,8,7^2,6,5^3,4^3,3^2,2^2,1^4)$	$(9,8,7,6^3,5^2,4^2,3^3,2^3,1^3)$	$(9,8,6^5,5,4,3^5,1^7)$
91	$2\omega_1$	7	$(7^7,6,5^3,4^2,3^3,1^4)$	$(7^7,5^3,4^3,3^3,2^2,1^2)$	$(7^6,6^2,5^2,4^2,3^3,2^3,1^4)$	$(7^5,6^4,5,4,3^4,1^{11})$
		≥ 11	$(11,8^2,7,6^3,5^3,4^2,3^4,1^4)$	$(11,8,7^3,6,5^3,4^3,3^4,2^2,1^2)$	$(11,8,7^2,6^3,5^2,4^2,3^4,2^3,1^4)$	$(11,8,7,6^5,5,4,3^5,1^{11})$

Table 11 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(6,2^3,1)$	$J(\varphi(x))$ for $J(x)=(6,2^2,1^3)$	$J(\varphi(x))$ for $J(x)=(6,2,1^5)$	$J(\varphi(x))$ for $J(x)=(6,1^7)$
78	ω_2	7	$(7^5,6,5^3,3^3,2^3,1^7)$	$(7^4,6^3,5^2,3,2^6,1^7)$	$(7^3,6^5,5,2^5,1^{12})$	$(7^2,6^7,1^{22})$
		≥ 11	$(9,7^3,6,5^4,3^3,2^3,1^7)$	$(9,7^2,6^3,5^3,3,2^6,1^7)$	$(9,7,6^5,5^2,2^5,1^{12})$	$(9,6^7,5,1^{22})$
91	$2\omega_1$	7	$(7^6,6,5^3,3^6,2^3,1^4)$	$(7^5,6^3,5^2,3^3,2^6,1^7)$	$(7^4,6^5,5,3,2^5,1^{15})$	$(7^3,6^7,1^{28})$
		≥ 11	$(11,7^4,6,5^3,3^7,2^3,1^4)$	$(11,7^3,6^3,5^2,3^4,2^6,1^7)$	$(11,7^2,6^5,5,3^2,2^5,1^{15})$	$(11,7,6^7,3,1^{28})$

Table 11 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(5^2,3)$	$J(\varphi(x))$ for $J(x)=(5^2,2,1)$	$J(\varphi(x))$ for $J(x)=(5^2,1^3)$	$J(\varphi(x))$ for $J(x)=(5,4^2)$
78	ω_2	5	$(5^{15},3)$	$(5^{15},2,1)$	$(5^{15},1^3)$	$(5^{15},1^3)$
		7	$(7^7,5^2,3^6,1)$	$(7^5,6^2,5^2,4^2,3^3,2,1^2)$	$(7^5,5^6,3^3,1^4)$	$(7^6,5^3,4^2,3^2,2^2,1^3)$
		≥ 11	$(9,7^5,5^3,3^6,1)$	$(9,7^3,6^2,5^3,4^2,3^3,2,1^2)$	$(9,7^3,5^7,3^3,1^4)$	$(8^2,7^2,6^2,5^3,4^2,3^2,2^2,1^3)$
91	$2\omega_1$	5	$(5^{18},1)$	$(5^{17},3,2,1)$	$(5^{17},1^6)$	$(5^{18},1)$
		7	$(7^9,5^3,3^3,1^4)$	$(7^7,6^2,5^2,4^2,3^2,2,1^4)$	$(7^7,5^6,3,1^9)$	$(7^9,5,4^2,3^3,2^2,1^2)$
		≥ 11	$(9^3,7^3,5^6,3^3,1^4)$	$(9^3,7,6^2,5^5,4^2,3^2,2,1^4)$	$(9^3,7,5^9,3,1^9)$	$(9,8^2,7^3,6^2,5^2,4^2,3^3,2^2,1^2)$

Table 11 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(5,4,3,1)$	$J(\varphi(x))$ for $J(x)=(5,4,2^2)$	$J(\varphi(x))$ for $J(x)=(5,4,2,1^2)$	$J(\varphi(x))$ for $J(x)=(5,4,1^4)$
78	ω_2	5	$(5^{13},4,3^2,2,1)$	$(5^{13},3^3,1^4)$	$(5^{12},4^2,3,2^2,1^3)$	$(5^{11},4^4,1^7)$
		7	$(7^4,6,5^3,4^3,3^4,2^2,1)$	$(7^3,6^2,5^3,4^3,3^4,2,1^4)$	$(7^3,6,5^4,4^4,3^2,2^3,1^3)$	$(7^3,5^5,4^5,3,2,1^7)$
		≥ 11	$(8,7^2,6^2,5^3,4^3,3^4,2^2,1)$	$(8,7,6^3,5^3,4^3,3^4,2,1^4)$	$(8,7,6^2,5^4,4^4,3^2,2^3,1^3)$	$(8,7,6,5^5,4^5,3,2,1^7)$
91	$2\omega_1$	5	$(5^{16},4,3,2,1^2)$	$(5^{15},3^5,1)$	$(5^{14},4^2,3^2,2^2,1^3)$	$(5^{13},4^4,1^{10})$
		7	$(7^6,6,5^3,4^3,3^3,2^2,1^3)$	$(7^5,6^2,5^2,4^3,3^6,2,1^2)$	$(7^5,6,5^3,4^4,3^3,2^3,1^4)$	$(7^5,5^4,4^5,3,2,1^{11})$
		≥ 11	$(9,8,7^2,6^2,5^4,4^3,3^3,2^2,1^3)$	$(9,8,7,6^3,5^3,4^3,3^6,2,1^2)$	$(9,8,7,6^2,5^4,4^4,3^3,2^3,1^4)$	$(9,8,7,6,5^5,4^5,3,2,1^{11})$

Table 11 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(5,3^2,2)$	$J(\varphi(x))$ for $J(x)=(5,3^2,1^2)$	$J(\varphi(x))$ for $J(x)=(5,3,2^2,1)$	$J(\varphi(x))$ for $J(x)=(5,3,2,1^3)$	$J(\varphi(x))$ for $J(x)=(5,3,1^5)$
78	ω_2	5	$(5^{11},4^2,3^3,2^2,1^2)$	$(5^{11},3^7,1^2)$	$(5^{10},4^2,3^3,2^4,1^3)$	$(5^{10},4,3^4,2^4,1^4)$	$(5^{10},3^6,1^{10})$
		≥ 7	$(7^3,6,5^3,4^3,3^6,2^2,1^2)$	$(7^3,5^5,3^{10},1^2)$	$(7^2,6^2,5^2,4^4,3^5,2^4,1^3)$	$(7^2,6,5^4,4^2,3^6,2^4,1^4)$	$(7^2,5^6,3^8,1^{10})$
91	$2\omega_1$	5	$(5^{14},4^2,3^2,2^2,1^3)$	$(5^{14},3^5,1^6)$	$(5^{12},4^2,3^4,2^4,1^3)$	$(5^{12},4,3^4,2^4,1^7)$	$(5^{12},3^5,1^{16})$
		7	$(7^4,6,5^5,4^3,3^4,2^2,1^4)$	$(7^4,5^7,3^7,1^7)$	$(7^3,6^2,5^3,4^4,3^5,2^4,1^4)$	$(7^3,6,5^5,4^2,3^5,2^4,1^8)$	$(7^3,5^7,3^6,1^{17})$
		≥ 11	$(9,7^2,6,5^6,4^3,3^4,2^2,1^4)$	$(9,7^2,5^8,3^7,1^7)$	$(9,7,6^2,5^4,4^4,3^5,2^4,1^4)$	$(9,7,6,5^6,4^2,3^5,2^4,1^8)$	$(9,7,5^8,3^6,1^{17})$

Table 11 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(5,2^4)$	$J(\varphi(x))$ for $J(x)=(5,2^3,1^2)$	$J(\varphi(x))$ for $J(x)=(5,2^2,1^4)$	$J(\varphi(x))$ for $J(x)=(5,2,1^6)$	$J(\varphi(x))$ for $J(x)=(5,1^8)$
78	ω_2	5	$(5^{10},3^6,1^{10})$	$(5^{10},3^3,2^6,1^7)$	$(5^{10},3,2^8,1^9)$	$(5^{10},2^6,1^{16})$	$(5^{10},1^{28})$
		≥ 7	$(7,6^4,4^4,3^7,1^{10})$	$(7,6^3,5^2,4^3,3^4,2^6,1^7)$	$(7,6^2,5^4,4^2,3^2,2^8,1^9)$	$(7,6,5^6,4,3,2^6,1^{16})$	$(7,5^8,3,1^{28})$
91	$2\omega_1$	5	$(5^{11},3^{10},1^6)$	$(5^{11},3^6,2^6,1^6)$	$(5^{11},3^3,2^8,1^{11})$	$(5^{11},3,2^6,1^{21})$	$(5^{11},1^{36})$
		7	$(7^2,6^4,4^4,3^{10},1^7)$	$(7^2,6^3,5^2,4^3,3^6,2^6,1^7)$	$(7^2,6^2,5^4,4^2,3^3,2^8,1^{12})$	$(7^2,6,5^6,4,3,2^6,1^{22})$	$(7^2,5^8,1^{37})$
		≥ 11	$(9,6^4,5,4^4,3^{10},1^7)$	$(9,6^3,5^3,4^3,3^6,2^6,1^7)$	$(9,6^2,5^5,4^2,3^3,2^8,1^{12})$	$(9,6,5^7,4,3,2^6,1^{22})$	$(9,5^9,1^{37})$

Table 11 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(4^3,1)$	$J(\varphi(x))$ for $J(x)=(4^2,3,2)$	$J(\varphi(x))$ for $J(x)=(4^2,3,1^2)$	$J(\varphi(x))$ for $J(x)=(4^2,2^2,1)$	$J(\varphi(x))$ for $J(x)=(4^2,2,1^3)$
78	ω_2	5	$(5^{12},4^3,1^6)$	$(5^{11},4,3^3,2^3,1^4)$	$(5^9,4^4,3^3,2^2,1^4)$	$(5^9,4^2,3^5,2^2,1^6)$	$(5^7,4^6,3^2,2^3,1^7)$
		≥ 7	$(7^3,5^6,4^3,3^3,1^6)$	$(7,6^2,5^5,4^3,3^4,2^3,1^4)$	$(7,6^2,5^3,4^6,3^4,2^2,1^4)$	$(7,5^7,4^2,3^6,2^2,1^6)$	$(7,5^5,4^6,3^3,2^3,1^7)$
91	$2\omega_1$	5	$(5^{15},4^3,1^4)$	$(5^{14},4,3^3,2^3,1^2)$	$(5^{12},4^4,3^2,2^2,1^5)$	$(5^{11},4^2,3^7,2^2,1^3)$	$(5^9,4^6,3^3,2^3,1^7)$
		≥ 7	$(7^6,5^3,4^3,3^6,1^4)$	$(7^3,6^2,5^4,4^3,3^6,2^3,1^2)$	$(7^3,6^2,5^2,4^6,3^5,2^2,1^5)$	$(7^3,5^5,4^2,3^{10},2^2,1^3)$	$(7^3,5^3,4^6,3^6,2^3,1^7)$

Table 11 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(4^2,1^5)$	$J(\varphi(x))$ for $J(x)=(4,3^3)$	$J(\varphi(x))$ for $J(x)=(4,3^2,2,1)$	$J(\varphi(x))$ for $J(x)=(4,3^2,1^3)$	$J(\varphi(x))$ for $J(x)=(4,3,2^3)$
78	ω_2	5	$(5^5,4^{10},1^{13})$	$(5^{10},3^6,2^3,1^4)$	$(5^7,4^3,3^6,2^5,1^3)$	$(5^6,4^3,3^9,2^2,1^5)$	$(5^6,4^3,3^7,2^4,1^7)$
		≥ 7	$(7,5^3,4^{10},3,1^{13})$	$(6^3,5^4,4^3,3^6,2^3,1^4)$	$(6^2,5^3,4^5,3^6,2^5,1^3)$	$(6^2,5^2,4^5,3^9,2^2,1^5)$	$(6,5^4,4^4,3^7,2^4,1^7)$
91	$2\omega_1$	5	$(5^7,4^{10},1^{16})$	$(5^{14},3^3,2^3,1^6)$	$(5^{10},4^3,3^5,2^5,1^4)$	$(5^9,4^3,3^7,2^2,1^9)$	$(5^8,4^3,3^9,2^4,1^4)$
		≥ 7	$(7^3,5,4^{10},3^3,1^{16})$	$(7,6^3,5^6,4^3,3^4,2^3,1^6)$	$(7,6^2,5^4,4^5,3^6,2^5,1^4)$	$(7,6^2,5^3,4^5,3^8,2^2,1^9)$	$(7,6,5^4,4^4,3^{10},2^4,1^4)$

Table 11 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(4,3,2^2,1^2)$	$J(\varphi(x))$ for $J(x)=(4,3,2,1^4)$	$J(\varphi(x))$ for $J(x)=(4,3,1^6)$	$J(\varphi(x))$ for $J(x)=(4,2^4,1)$	$J(\varphi(x))$ for $J(x)=(4,2^3,1^3)$
78	ω_2	5	$(5^5,4^4,3^6,2^7,1^5)$	$(5^4,4^5,3^6,2^6,1^8)$	$(5^3,4^6,3^7,2,1^{16})$	$(5^5,4,3^{10},2^4,1^{11})$	$(5^4,4^3,3^6,2^9,1^{10})$
		≥ 7	$(6,5^3,4^5,3^6,2^7,1^5)$	$(6,5^2,4^6,3^6,2^6,1^8)$	$(6,5,4^7,3^7,2,1^{16})$		
91	$2\omega_1$	5	$(5^7,4^4,3^7,2^7,1^5)$	$(5^6,4^5,3^6,2^6,1^{11})$	$(5^5,4^6,3^6,2,1^{22})$	$(5^6,4,3^{14},2^4,1^7)$	$(5^5,4^3,3^9,2^9,1^9)$
		≥ 7	$(7,6,5^3,4^5,3^8,2^7,1^5)$	$(7,6,5^2,4^6,3^7,2^6,1^{11})$	$(7,6,5,4^7,3^7,2,1^{22})$	$(7,5^4,4,3^{15},2^4,1^7)$	$(7,5^3,4^3,3^{10},2^9,1^9)$

Table 11 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(4,2^2,1^5)$	$J(\varphi(x))$ for $J(x)=(4,2,1^7)$	$J(\varphi(x))$ for $J(x)=(4,1^9)$	$J(\varphi(x))$ for $J(x)=(3^4,1)$	$J(\varphi(x))$ for $J(x)=(3^3,2^2)$
78	ω_2	3	-	-	-	(3^{26})	$(3^{25},1^3)$
		≥ 5	$(5^3,4^5,3^3,2^{10},1^{14})$	$(5^2,4^7,3,2^7,1^{23})$	$(5,4^9,1^{37})$	$(5^6,3^{14},1^6)$	$(5^3,4^6,3^7,2^6,1^6)$
91	$2\omega_1$	3	-	-	-	$(3^{30},1)$	$(3^{30},1)$
		5	$(5^4,4^5,3^5,2^{10},1^{16})$	$(5^3,4^7,3^2,2^7,1^{28})$	$(5^2,4^9,1^{45})$	$(5^{10},3^{10},1^{11})$	$(5^6,4^6,3^6,2^6,1^7)$
		≥ 7	$(7,5^2,4^5,3^6,2^{10},1^{16})$	$(7,5,4^7,3^3,2^7,1^{28})$	$(7,4^9,3,1^{45})$		

Table 11 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(3^3,2,1^2)$	$J(\varphi(x))$ for $J(x)=(3^3,1^4)$	$J(\varphi(x))$ for $J(x)=(3^2,2^3,1)$	$J(\varphi(x))$ for $J(x)=(3^2,2^2,1^3)$	$J(\varphi(x))$ for $J(x)=(3^2,2,1^5)$	$J(\varphi(x))$ for $J(x)=(3^2,1^7)$
78	ω_2	3	$(3^{24},2^2,1^2)$	$(3^{24},1^6)$	$(3^{22},2^3,1^6)$	$(3^{20},2^6,1^6)$	$(3^{19},2^5,1^{11})$	$(3^{19},1^{21})$
		≥ 5	$(5^3,4^3,3^{12},2^5,1^5)$	$(5^3,3^{18},1^9)$	$(5,4^6,3^8,2^9,1^7)$	$(5,4^4,3^{10},2^{10},1^7)$	$(5,4^2,3^{13},2^7,1^{12})$	$(5,3^{17},1^{22})$
91	$2\omega_1$	3	$(3^{28},2^2,1^3)$	$(3^{27},1^{10})$	$(3^{27},2^3,1^4)$	$(3^{24},2^6,1^7)$	$(3^{22},2^5,1^{15})$	$(3^{21},1^{28})$
		≥ 5	$(5^6,4^3,3^{10},2^5,1^9)$	$(5^6,3^{15},1^{16})$	$(5^3,4^6,3^9,2^9,1^7)$	$(5^3,4^4,3^{10},2^{10},1^{10})$	$(5^3,4^2,3^{12},2^7,1^{18})$	$(5^3,3^{15},1^{31})$

Table 11 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(3,2^5)$	$J(\varphi(x))$ for $J(x)=(3,2^4,1^2)$	$J(\varphi(x))$ for $J(x)=(3,2^3,1^4)$	$J(\varphi(x))$ for $J(x)=(3,2^2,1^6)$	$J(\varphi(x))$ for $J(x)=(3,2,1^8)$	$J(\varphi(x))$ for $J(x)=(3,1^{10})$
78	ω_2	3	$(3^{21},1^{15})$	$(3^{17},2^8,1^{11})$	$(3^{14},2^{12},1^{12})$	$(3^{12},2^{12},1^{18})$	$(3^{11},2^8,1^{29})$	$(3^{11},1^{45})$
		≥ 5	$(4^5,3^{11},2^5,1^{15})$	$(4^4,3^9,2^{12},1^{11})$	$(4^3,3^8,2^{15},1^{12})$	$(4^2,3^8,2^{14},1^{18})$	$(4,3^9,2^9,1^{29})$	
91	$2\omega_1$	3	$(3^{27},1^{10})$	$(3^{22},2^8,1^9)$	$(3^{18},2^{12},1^{13})$	$(3^{15},2^{12},1^{22})$	$(3^{13},2^8,1^{36})$	$(3^{12},1^{55})$
		≥ 5	$(5,4^5,3^{15},2^5,1^{11})$	$(5,4^4,3^{12},2^{12},1^{10})$	$(5,4^3,3^{10},2^{15},1^{14})$	$(5,4^2,3^9,2^{14},1^{23})$	$(5,4,3^9,2^9,1^{37})$	$(5,3^{10},1^{56})$

Table 11 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(2^6,1)$	$J(\varphi(x))$ for $J(x)=(2^5,1^3)$	$J(\varphi(x))$ for $J(x)=(2^4,1^5)$	$J(\varphi(x))$ for $J(x)=(2^3,1^7)$	$J(\varphi(x))$ for $J(x)=(2^2,1^9)$	$J(\varphi(x))$ for $J(x)=(2,1^{11})$
78	ω_2	≥ 3	$(3^{15},2^6,1^{21})$	$(3^{10},2^{15},1^{18})$	$(3^6,2^{20},1^{20})$	$(3^3,2^{21},1^{27})$	$(3,2^{18},1^{39})$	$(2^{11},1^{56})$
91	$2\omega_1$	≥ 3	$(3^{21},2^6,1^{16})$	$(3^{15},2^{15},1^{16})$	$(3^{10},2^{20},1^{21})$	$(3^6,2^{21},1^{31})$	$(3^3,2^{18},1^{46})$	$(3,2^{11},1^{66})$

TABLE 12. $G=A_{13}(K)$

n	ω	p	$J(\varphi(x))$ for $J(x)=(14)$	$J(\varphi(x))$ for $J(x)=(13,1)$	$J(\varphi(x))$ for $J(x)=(12,2)$	$J(\varphi(x))$ for $J(x)=(12,1^2)$	$J(\varphi(x))$ for $J(x)=(11,3)$
91	ω_2	11	-	-	-	-	$(11^8,3)$
		13	-	(13^7)	$(13^6,11,1^2)$	$(13^5,12^2,1^2)$	$(13^5,11,9,3^2)$
		17	$(17^5,5,1)$	$(17^4,13,7,3)$	$(17^3,13,11,9,5,1^2)$	$(17^3,12^2,9,5,1^2)$	$(17^2,13,11^2,9,7,3^2)$
		19	$(19^4,9,5,1)$	$(19^3,13,11,7,3)$	$(19^2,13^2,11,9,5,1^2)$	$(19^2,13,12^2,9,5,1^2)$	$(19,15,13,11^2,9,7,3^2)$
		23	$(23^2,17,13,9,5,1)$	$(23,19,15,13,11,7,3)$	$(21,17,13^2,11,9,5,1^2)$	$(21,17,13,12^2,9,5,1^2)$	
		≥ 29	$(25,21,17,13,9,5,1)$				

Table 12 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(11,2,1)$	$J(\varphi(x))$ for $J(x)=(11,1^3)$	$J(\varphi(x))$ for $J(x)=(10,4)$	$J(\varphi(x))$ for $J(x)=(10,3,1)$
91	ω_2	11	$(11^8,2,1)$	$(11^8,1^3)$	$(11^7,7,5,1^2)$	$(11^6,10,8,3^2,1)$
		13	$(13^4,12,11,10,3,2,1)$	$(13^4,11^3,3,1^3)$	$(13^4,11,9,7,5^2,1^2)$	$(13^3,12,10^2,8,5,3^2,1)$
		17	$(17^2,12,11^2,10,7,3,2,1)$	$(17^2,11^4,7,3,1^3)$	$(17,13^2,11,9^2,7,5^2,1^2)$	$(17,13,12,10^2,9,8,5,3^2,1)$
		≥ 19	$(19,15,12,11^2,10,7,3,2,1)$	$(19,15,11^4,7,3,1^3)$		

Table 12 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(10,2^2)$	$J(\varphi(x))$ for $J(x)=(10,2,1^2)$	$J(\varphi(x))$ for $J(x)=(10,1^4)$	$J(\varphi(x))$ for $J(x)=(9,5)$
91	ω_2	11	$(11^6,9^2,3,1^4)$	$(11^5,10^2,9,2^2,1^3)$	$(11^4,10^4,1^7)$	$(11^6,7^2,5,3^2)$
		13	$(13^3,11^2,9^2,5,3,1^4)$	$(13^3,11,10^2,9,5,2^2,1^3)$	$(13^3,10^4,5,1^7)$	$(13^3,11,9,7^3,5,3^2)$
		≥ 17	$(17,13,11^2,9^3,5,3,1^4)$	$(17,13,11,10^2,9^2,5,2^2,1^3)$	$(17,13,10^4,9,5,1^7)$	$(15,13,11^2,9,7^3,5,3^2)$

Table 12 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(9,4,1)$	$J(\varphi(x))$ for $J(x)=(9,3,2)$	$J(\varphi(x))$ for $J(x)=(9,3,1^2)$	$J(\varphi(x))$ for $J(x)=(9,2^2,1)$
91	ω_2	11	$(11^5,9,8,6,5,4,3,1)$	$(11^4,10,9,8,7,4,3^2,2,1)$	$(11^4,9^3,7,3^4,1)$	$(11^3,10^2,9,8^2,3^2,2^2,1^3)$
		13	$(13^2,12,10,9,8,7,6,5,4,3,1)$	$(13^2,11,10,9,8,7^2,4,3^2,2,1)$	$(13^2,11,9^3,7^2,3^4,1)$	$(13^2,10^2,9,8^2,7,3^2,2^2,1^3)$
		≥ 17	$(15,12,11,10,9,8,7,6,5,4,3,1)$	$(15,11^2,10,9,8,7^2,4,3^2,2,1)$	$(15,11^2,9^3,7^2,3^4,1)$	$(15,11,10^2,9,8^2,7,3^2,2^2,1^3)$

Table 12 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(9,2,1^3)$	$J(\varphi(x))$ for $J(x)=(9,1^5)$	$J(\varphi(x))$ for $J(x)=(8,6)$	$J(\varphi(x))$ for $J(x)=(8,5,1)$
91	ω_2	11	$(11^3,10,9^3,8,3,2^3,1^4)$	$(11^3,9^5,3,1^{10})$	$(11^5,9,7,5^3,3,1^2)$	$(11^4,8^2,7,6,5^2,4,3,1)$
		13	$(13^2,10,9^3,8,7,3,2^3,1^4)$	$(13^2,9^5,7,3,1^{10})$	$(13^2,11,9^3,7,5^3,3,1^2)$	$(13,12,10,9,8^2,7,6,5^2,4,3,1)$
		≥ 17	$(15,11,10,9^3,8,7,3,2^3,1^4)$	$(15,11,9^5,7,3,1^{10})$		

Table 12 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(8,4,2)$	$J(\varphi(x))$ for $J(x)=(8,4,1^2)$	$J(\varphi(x))$ for $J(x)=(8,3^2)$	$J(\varphi(x))$ for $J(x)=(8,3,2,1)$
91	ω_2	11	$(11^3,9^2,7^2,5^4,3,1^3)$	$(11^3,9,8^2,7,5^3,4^3,1^3)$	$(11^2,10^2,8^2,6^2,5^2,3^3,1^2)$	$(11^2,10,9,8^2,7,6,5,4,3^2,2^2,1^2)$
		≥ 13	$(13,11,9^3,7^2,5^4,3,1^3)$	$(13,11,9^2,8^2,7,5^3,4^2,1^3)$	$(13,10^2,9,8^2,6^2,5^2,3^3,1^2)$	$(13,10,9^2,8^2,7,6,5,4,3^2,2^2,1^2)$

Table 12 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(8,3,1^3)$	$J(\varphi(x))$ for $J(x)=(8,2^3)$	$J(\varphi(x))$ for $J(x)=(8,2^2,1^2)$	$J(\varphi(x))$ for $J(x)=(8,2,1^4)$
91	ω_2	11	$(11^2,10,8^4,6,5,3^4,1^4)$	$(11^2,9^3,7^3,5,3^3,1^7)$	$(11^2,9^2,8^2,7^2,5,3,2^4,1^5)$	$(11^2,9,8^4,7,5,2^4,1^8)$
		≥ 13	$(13,10,9,8^4,6,5,3^4,1^4)$	$(13,9^4,7^3,5,3^3,1^7)$	$(13,9^3,8^2,7^2,5,3,2^4,1^5)$	$(13,9^2,8^4,7,5,2^4,1^8)$

Table 12 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(8,1^6)$	$J(\varphi(x))$ for $J(x)=(7^2)$	$J(\varphi(x))$ for $J(x)=(7,6,1)$	$J(\varphi(x))$ for $J(x)=(7,5,2)$
91	ω_2	7	-	(7^{13})	$(7^{12},6,1)$	$(7^{11},6,4,3,1)$
		11	$(11^2,8^6,5,1^{16})$	$(11^5,7^3,5,3^3,1)$	$(11^3,9,8,7^2,6^2,5,4,3,2,1)$	$(11^2,9,8,7^3,6^2,5,4,3^3,1)$
		≥ 13	$(13,9,8^6,5,1^{16})$	$(13,11^3,9,7^3,5,3^3,1)$	$(12,11,10,9,8,7^2,6^2,5,4,3,2,1)$	

Table 12 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(7,5,1^2)$	$J(\varphi(x))$ for $J(x)=(7,4,3)$	$J(\varphi(x))$ for $J(x)=(7,4,2,1)$	$J(\varphi(x))$ for $J(x)=(7,4,1^3)$
91	ω_2	7	$(7^{11},5^2,3,1)$	$(7^{10},6,5,4,3,2,1)$	$(7^{10},5^2,4,3,2,1^2)$	$(7^{10},5,4^3,1^4)$
		≥ 11	$(11^2,9,7^5,5^3,3^3,1)$	$(11,10,9,8,7^2,6^2,5^2,4^2,3^2,2,1)$	$(11,10,8^2,7^2,6^2,5^2,4^2,3^2,2,1^2)$	$(11,10,8,7^4,6,5,4^4,3,1^4)$

Table 12 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(7,3^2,1)$	$J(\varphi(x))$ for $J(x)=(7,3,2^2)$	$J(\varphi(x))$ for $J(x)=(7,3,2,1^2)$	$J(\varphi(x))$ for $J(x)=(7,3,1^4)$	$J(\varphi(x))$ for $J(x)=(7,2^3,1)$
91	ω_2	7	$(7^{10},5,3^5,1)$	$(7^{10},4^2,3^2,2^2,1^3)$	$(7^{10},4,3^3,2^3,1^2)$	$(7^{10},3^5,1^6)$	$(7^{10},3^3,2^3,1^6)$
		≥ 11	$(11,9^2,7^4,5^3,3^6,1)$	$(11,9,8^2,7^2,6^2,5,4^2,3^3,2^2,1^3)$	$(11,9,8,7^4,6,5,4,3^4,2^3,1^2)$	$(11,9,7^6,5,3^6,1^6)$	$(11,8^3,7^2,6^3,3^4,2^3,1^6)$

Table 12 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(7,2^2,1^3)$	$J(\varphi(x))$ for $J(x)=(7,2,1^5)$	$J(\varphi(x))$ for $J(x)=(7,1^7)$	$J(\varphi(x))$ for $J(x)=(6^2,2)$	$J(\varphi(x))$ for $J(x)=(6^2,1^2)$
91	ω_2	7	$(7^{10},3,2^6,1^6)$	$(7^{10},2^5,1^{11})$	$(7^{10},1^{21})$	$(7^{11},5^2,1^4)$	$(7^9,6^4,1^4)$
		≥ 11	$(11,8^2,7^4,6^2,3^2,2^6,1^6)$	$(11,8,7^6,6,3,2^5,1^{11})$	$(11,7^8,3,1^{21})$	$(11,9^3,7^3,5^5,3,1^4)$	$(11,9^3,7,6^4,5^3,3,1^4)$

Table 12 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(6,5,3)$	$J(\varphi(x))$ for $J(x)=(6,5,2,1)$	$J(\varphi(x))$ for $J(x)=(6,5,1^3)$	$J(\varphi(x))$ for $J(x)=(6,4^2)$
91	ω_2	7	$(7^{10},5,4,3^3,2,1)$	$(7^8,6^2,5^2,4,3,2^2,1^2)$	$(7^7,6^3,5^3,3,2,1^4)$	$(7^9,5^3,3^3,1^4)$
		≥ 11	$(10,9,8^2,7^2,6^2,5^2,4^2,3^3,2,1)$	$(10,9,8,7^2,6^3,5^3,4^2,3,2^2,1^2)$	$(10,9,8,7,6^4,5^4,4,3,2,1^4)$	$(9^3,7^3,5^6,3^3,1^4)$

Table 12 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(6,4,3,1)$	$J(\varphi(x))$ for $J(x)=(6,4,2^2)$	$J(\varphi(x))$ for $J(x)=(6,4,2,1^2)$	$J(\varphi(x))$ for $J(x)=(6,4,1^4)$
91	ω_2	7	$(7^7,6^2,5,4^3,3^3,2,1^2)$	$(7^7,5^5,3^4,1^5)$	$(7^6,6^2,5^3,4^2,3^2,2^2,1^4)$	$(7^5,6^4,5,4^4,3,1^8)$
		≥ 11	$(9^2,8,7,6^3,5^3,4^3,3^3,2,1^2)$	$(9^2,7^3,5^7,3^4,1^5)$	$(9^2,7^2,6^2,5^5,4^2,3^2,2^2,1^4)$	$(9^2,7,6^4,5^3,4^4,3,1^8)$

Table 12 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(6,3^2,2)$	$J(\varphi(x))$ for $J(x)=(6,3^2,1^2)$	$J(\varphi(x))$ for $J(x)=(6,3,2^2,1)$	$J(\varphi(x))$ for $J(x)=(6,3,2,1^3)$
91	ω_2	7	$(7^7,5^2,4^4,3^3,2^2,1^3)$	$(7^6,6^2,5,4^2,3^7,1^3)$	$(7^6,6,5^2,4^3,3^3,2^4,1^4)$	$(7^5,6^3,5,4^2,3^4,2^4,1^5)$
		≥ 11	$(9,8^2,7,6^2,5^3,4^4,3^3,2^2,1^3)$	$(9,8^2,6^4,5^2,4^2,3^7,1^3)$	$(9,8,7^2,6^2,5^3,4^3,3^3,2^4,1^4)$	$(9,8,7,6^4,5^2,4^2,3^4,2^4,1^5)$

Table 12 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(6,3,1^5)$	$J(\varphi(x))$ for $J(x)=(6,2^4)$	$J(\varphi(x))$ for $J(x)=(6,2^3,1^2)$	$J(\varphi(x))$ for $J(x)=(6,2^2,1^4)$	$J(\varphi(x))$ for $J(x)=(6,2,1^6)$
91	ω_2	7	$(7^4,6^5,4,3^6,1^{11})$	$(7^6,5^4,3^6,1^{11})$	$(7^5,6^2,5^3,3^3,2^6,1^8)$	$(7^4,6^4,5^2,3,2^8,1^{10})$	$(7^3,6^6,5,2^6,1^{17})$
		≥ 11	$(9,8,6^6,5,4,3^6,1^{11})$	$(9,7^4,5^5,3^6,1^{11})$	$(9,7^3,6^2,5^4,3^3,2^6,1^8)$	$(9,7^2,6^4,5^3,3,2^8,1^{10})$	$(9,7,6^6,5^2,2^6,1^{17})$

Table 12 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(6,1^8)$	$J(\varphi(x))$ for $J(x)=(5^2,4)$	$J(\varphi(x))$ for $J(x)=(5^2,3,1)$	$J(\varphi(x))$ for $J(x)=(5^2,2^2)$	$J(\varphi(x))$ for $J(x)=(5^2,2,1^2)$
91	ω_2	5	-	$(5^{18},1)$	$(5^{17},3^2)$	$(5^{17},3,1^3)$	$(5^{17},2^2,1^2)$
		7	$(7^2,6^8,1^{29})$	$(7^9,5,4^2,3^3,2^2,1^2)$	$(7^7,5^4,3^7,1)$	$(7^5,6^4,4^4,3^4,1^4)$	$(7^5,6^2,5^4,4^2,3^3,2^2,1^3)$
		≥ 11	$(9,6^8,5,1^{29})$	$(9,8^2,7^3,6^2,5^2,4^2,3^3,2^2,1^2)$	$(9,7^5,5^5,3^7,1)$	$(9,7^3,6^4,5,4^4,3^4,1^4)$	$(9,7^3,6^2,5^5,4^2,3^3,2^2,1^3)$

Table 12 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(5^2,1^4)$	$J(\varphi(x))$ for $J(x)=(5,4^2,1)$	$J(\varphi(x))$ for $J(x)=(5,4,3,2)$	$J(\varphi(x))$ for $J(x)=(5,4,3,1^2)$	$J(\varphi(x))$ for $J(x)=(5,4,2^2,1)$
91	ω_2	5	$(5^{17},1^6)$	$(5^{16},4^2,1^3)$	$(5^{15},4,3^2,2^2,1^2)$	$(5^{14},4^2,3^3,2,1^2)$	$(5^{14},4,3^3,2^2,1^4)$
		7	$(7^5,5^8,3^3,1^7)$	$(7^6,5^4,4^4,3^2,2^2,1^3)$	$(7^4,6^2,5^3,4^4,3^4,2^3,1^2)$	$(7^4,6,5^4,4^4,3^5,2^2,1^2)$	$(7^3,6^2,5^4,4^4,3^4,2^3,1^4)$
		≥ 11	$(9,7^3,5^9,3^3,1^7)$	$(8^2,7^2,6^2,5^4,4^4,3^2,2^2,1^3)$	$(8,7^2,6^3,5^3,4^4,3^4,2^3,1^2)$	$(8,7^2,6^2,5^4,4^4,3^5,2^2,1^2)$	$(8,7,6^3,5^4,4^4,3^4,2^3,1^4)$

Table 12 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(5,4,2,1^3)$	$J(\varphi(x))$ for $J(x)=(5,4,1^5)$	$J(\varphi(x))$ for $J(x)=(5,3^3)$	$J(\varphi(x))$ for $J(x)=(5,3^2,2,1)$	$J(\varphi(x))$ for $J(x)=(5,3^2,1^3)$
91	ω_2	5	$(5^{13},4^3,3,2^3,1^5)$	$(5^{12},4^5,1^{11})$	$(5^{14},3^6,1^3)$	$(5^{12},4^2,3^5,2^3,1^2)$	$(5^{12},3^9,1^4)$
		7	$(7^3,6,5^5,4^5,3^2,2^4,1^5)$	$(7^3,5^6,4^6,3,2,1^{11})$	$(7^4,5^6,3^{10},1^3)$	$(7^3,6,5^4,4^3,3^8,2^3,1^2)$	$(7^3,5^6,3^{12},1^4)$
		≥ 11	$(8,7,6^2,5^5,4^5,3^2,2^4,1^5)$	$(8,7,6,5^6,4^6,3,2,1^{11})$			

Table 12 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(5,3,2^3)$	$J(\varphi(x))$ for $J(x)=(5,3,2^2,1^2)$	$J(\varphi(x))$ for $J(x)=(5,3,2,1^4)$	$J(\varphi(x))$ for $J(x)=(5,3,1^6)$	$J(\varphi(x))$ for $J(x)=(5,2^4,1)$
91	ω_2	5	$(5^{11},4^3,3^4,2^3,1^6)$	$(5^{11},4^2,3^4,2^6,1^4)$	$(5^{11},4,3^5,2^5,1^7)$	$(5^{11},3^7,1^{15})$	$(5^{11},3^6,2^4,1^{10})$
		≥ 7	$(7^2,6^3,5,4^6,3^6,2^3,1^6)$	$(7^2,6^2,5^3,4^4,3^6,2^6,1^4)$	$(7^2,6,5^5,4^2,3^7,2^5,1^7)$	$(7^2,5^7,3^9,1^{15})$	$(7,6^4,5,4^4,3^7,2^4,1^{10})$

Table 12 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(5,2^3,1^3)$	$J(\varphi(x))$ for $J(x)=(5,2^2,1^5)$	$J(\varphi(x))$ for $J(x)=(5,2,1^7)$	$J(\varphi(x))$ for $J(x)=(5,1^9)$	$J(\varphi(x))$ for $J(x)=(4^3,2)$	$J(\varphi(x))$ for $J(x)=(4^3,1^2)$
91	ω_2	5	$(5^{11},3^3,2^9,1^9)$	$(5^{11},3,2^{10},1^{13})$	$(5^{11},2^7,1^{22})$	$(5^{11},1^{36})$	$(5^{15},3^3,1^7)$	$(5^{12},4^6,1^7)$
		≥ 7	$(7,6^3,5^3,4^3,3^4,2^9,1^9)$	$(7,6^2,5^5,4^2,3^2,2^{10},1^{13})$	$(7,6,5^7,4,3,2^7,1^{22})$	$(7,5^9,3,1^{36})$	$(7^3,5^9,3^6,1^7)$	$(7^3,5^6,4^6,3^3,1^7)$

Table 12 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(4^2,3^2)$	$J(\varphi(x))$ for $J(x)=(4^2,3,2,1)$	$J(\varphi(x))$ for $J(x)=(4^2,3,1^3)$	$J(\varphi(x))$ for $J(x)=(4^2,2^3)$	$J(\varphi(x))$ for $J(x)=(4^2,2^2,1^2)$	$J(\varphi(x))$ for $J(x)=(4^2,2,1^4)$
91	ω_2	5	$(5^{14},3^3,2^4,1^4)$	$(5^{11},4^3,3^4,2^4,1^4)$	$(5^9,4^6,3^4,2^2,1^6)$	$(5^{11},3^9,1^9)$	$(5^9,4^4,3^5,2^4,1^7)$	$(5^7,4^8,3^2,2^4,1^{10})$
		≥ 7	$(7,6^4,5^4,4^4,3^4,2^4,1^4)$	$(7,6^2,5^5,4^5,3^5,2^4,1^4)$	$(7,6^2,5^3,4^8,3^5,2^2,1^6)$	$(7,5^9,3^{10},1^9)$	$(7,5^7,4^4,3^6,2^4,1^7)$	$(7,5^5,4^8,3^3,2^4,1^{10})$

Table 12 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(4^2,1^6)$	$J(\varphi(x))$ for $J(x)=(4,3^3,1)$	$J(\varphi(x))$ for $J(x)=(4,3^2,2^2)$	$J(\varphi(x))$ for $J(x)=(4,3^2,2,1^2)$	$J(\varphi(x))$ for $J(x)=(4,3^2,1^4)$
91	ω_2	5	$(5^5,4^{12},1^{18})$	$(5^{10},4,3^9,2^3,1^4)$	$(5^8,4^4,3^6,2^6,1^5)$	$(5^7,4^4,3^8,2^6,1^4)$	$(5^6,4^4,3^{11},2^2,1^8)$
		≥ 7	$(7,5^3,4^{12},3,1^{18})$	$(6^3,5^4,4^4,3^9,2^3,1^4)$	$(6^2,5^4,4^6,3^6,2^6,1^5)$	$(6^2,5^3,4^6,3^8,2^6,1^4)$	$(6^2,5^2,4^6,3^{11},2^2,1^8)$

Table 12 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(4,3,2^3,1)$	$J(\varphi(x))$ for $J(x)=(4,3,2^2,1^3)$	$J(\varphi(x))$ for $J(x)=(4,3,2,1^5)$	$J(\varphi(x))$ for $J(x)=(4,3,1^7)$	$J(\varphi(x))$ for $J(x)=(4,2^5)$	$J(\varphi(x))$ for $J(x)=(4,2^4,1^2)$
91	ω_2	5	$(5^6,4^4,3^8,2^7,1^7)$	$(5^5,4^5,3^7,2^9,1^7)$	$(5^4,4^6,3^7,2^7,1^{12})$	$(5^3,4^7,3^8,2,1^{22})$	$(5^6,3^{15},1^{16})$	$(5^5,4^2,3^{10},2^8,1^{12})$
		≥ 7	$(6,5^4,4^5,3^8,2^7,1^7)$	$(6,5^3,4^6,3^7,2^9,1^7)$	$(6,5^2,4^7,3^7,2^7,1^{12})$	$(6,5,4^8,3^8,2,1^{22})$		

Table 12 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(4,2^3,1^4)$	$J(\varphi(x))$ for $J(x)=(4,2^2,1^6)$	$J(\varphi(x))$ for $J(x)=(4,2,1^8)$	$J(\varphi(x))$ for $J(x)=(4,1^{10})$	$J(\varphi(x))$ for $J(x)=(3^4,2)$	$J(\varphi(x))$ for $J(x)=(3^4,1^2)$	$J(\varphi(x))$ for $J(x)=(3^3,2^2,1)$
91	ω_2	3	-	-	-	-	$(3^{30},1)$	$(3^{30},1)$	$(3^{28},2^2,1^3)$
		≥ 5	$(5^4,4^4,3^6,2^{12},1^{13})$	$(5^3,4^6,3^3,2^{12},1^{19})$	$(5^2,4^8,3,2^8,1^{30})$	$(5,4^{10},1^{46})$	$(5^6,4^4,3^{10},2^4,1^7)$	$(5^6,3^{18},1^7)$	$(5^3,4^6,3^{10},2^8,1^6)$

Table 12 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(3^3,2,1^3)$	$J(\varphi(x))$ for $J(x)=(3^3,1^5)$	$J(\varphi(x))$ for $J(x)=(3^2,2^4)$	$J(\varphi(x))$ for $J(x)=(3^2,2^3,1^2)$	$J(\varphi(x))$ for $J(x)=(3^2,2^2,1^4)$	$J(\varphi(x))$ for $J(x)=(3^2,2,1^6)$	$J(\varphi(x))$ for $J(x)=(3^2,1^8)$
91	ω_2	3	$(3^{27},2^3,1^4)$	$(3^{27},1^{10})$	$(3^{27},1^{10})$	$(3^{24},2^6,1^7)$	$(3^{22},2^8,1^9)$	$(3^{21},2^6,1^{16})$	$(3^{21},1^{28})$
		≥ 5	$(5^3,4^3,3^{15},2^6,1^7)$	$(5^3,3^{21},1^{13})$	$(5,4^8,3^9,2^8,1^{11})$	$(5,4^6,3^{10},2^{12},1^8)$	$(5,4^4,3^{12},2^{12},1^{10})$	$(5,4^2,3^{15},2^8,1^{17})$	$(5,3^{19},1^{29})$

Table 12 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(3,2^5,1)$	$J(\varphi(x))$ for $J(x)=(3,2^4,1^3)$	$J(\varphi(x))$ for $J(x)=(3,2^3,1^5)$	$J(\varphi(x))$ for $J(x)=(3,2^2,1^7)$	$J(\varphi(x))$ for $J(x)=(3,2,1^9)$	$J(\varphi(x))$ for $J(x)=(3,1^{11})$
91	ω_2	3	$(3^{22},2^5,1^{15})$	$(3^{18},2^{12},1^{13})$	$(3^{15},2^{15},1^{16})$	$(3^{13},2^{14},1^{24})$	$(3^{12},2^9,1^{37})$	$(3^{12},1^{55})$
		≥ 5	$(4^5,3^{12},2^{10},1^{15})$	$(4^4,3^{10},2^{16},1^{13})$	$(4^3,3^9,2^{18},1^{16})$	$(4^2,3^9,2^{16},1^{24})$	$(4,3^{10},2^{10},1^{37})$	

Table 12 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(2^7)$	$J(\varphi(x))$ for $J(x)=(2^6,1^2)$	$J(\varphi(x))$ for $J(x)=(2^5,1^4)$	$J(\varphi(x))$ for $J(x)=(2^4,1^6)$	$J(\varphi(x))$ for $J(x)=(2^3,1^8)$	$J(\varphi(x))$ for $J(x)=(2^2,1^{10})$	$J(\varphi(x))$ for $J(x)=(2,1^{12})$
91	ω_2	≥ 3	$(3^{21},1^{28})$	$(3^{15},2^{12},1^{22})$	$(3^{10},2^{20},1^{21})$	$(3^6,2^{24},1^{25})$	$(3^3,2^{24},1^{34})$	$(3,2^{20},1^{48})$	$(2^{12},1^{67})$

TABLE 13. $G = B_3(K)$

n	ω	p	$J(\varphi(x))$ for $J(x)=(7)$	$J(\varphi(x))$ for $J(x)=(5,1^2)$	$J(\varphi(x))$ for $J(x)=(3^2,1)$	$J(\varphi(x))$ for $J(x)=(3,2^2)$	$J(\varphi(x))$ for $J(x)=(3,1^4)$	$J(\varphi(x))$ for $J(x)=(2^2,1^3)$
8	ω_3	3	-	-	$(3^2,1^2)$	$(3,2^2,1)$	(2^4)	$(2^2,1^4)$
		5	-	(4^2)				
		≥ 7	$(7,1)$					
21	ω_2	3	-	-	(3^7)	$(3^6,1^3)$	$(3^5,1^6)$	$(3,2^6,1^6)$
		5	-	$(5^4,1)$	$(5,3^5,1)$	$(4^2,3^2,2^2,1^3)$		
		7	(7^3)	$(7,5^2,3,1)$				
		≥ 11	$(11,7,3)$					
26	$2\omega_1$	7	$(7^3,5)$	$(7^2,5^2,1^2)$	$(5^3,3^3,1^2)$	$(5,4^2,3^3,2^2)$	$(5,3^4,1^9)$	$(3^3,2^6,1^5)$
27	$2\omega_1$	3	-	-	(3^9)	(3^9)	$(3^6,1^9)$	$(3^3,2^6,1^6)$
		5	-	$(5^5,1^2)$	$(5^3,3^3,1^3)$	$(5,4^2,3^3,2^2,1)$	$(5,3^4,1^{10})$	
		11	$(11^2,5)$	$(9,5^3,1^3)$				
		≥ 13	$(13,9,5)$					
35	$2\omega_3$	3	-	-	$(3^{11},1^2)$	$(3^{10},2^2,1)$	$(3^{10},1^5)$	$(3^3,2^8,1^{10})$
		5	-	(5^7)	$(5^3,3^5,1^5)$	$(5,4^2,3^4,2^4,1^2)$		
		7	(7^5)	$(7^3,5,3^3)$				
		11	$(11^2,7,5,1)$					
		≥ 13	$(13,9,7,5,1)$					

Table 13 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(7)$	$J(\varphi(x))$ for $J(x)=(5,1^2)$	$J(\varphi(x))$ for $J(x)=(3^2,1)$	$J(\varphi(x))$ for $J(x)=(3,2^2)$	$J(\varphi(x))$ for $J(x)=(3,1^4)$	$J(\varphi(x))$ for $J(x)=(2^2,1^3)$
40	$\omega_1+\omega_3$	7	$(7^5,5)$	$(7^4,4^2,2^2)$	$(5^4,3^6,1^2)$	$(5,4^4,3^4,2^2,1^3)$	$(4^4,2^{12})$	$(3^4,2^{10},1^8)$
48	$\omega_1+\omega_3$	3	-	-	(3^{16})	$(3^{15},1^3)$	$(3^8,2^{12})$	$(3^4,2^{12},1^{12})$
		5	-	$(5^8,4^2)$	$(5^4,3^8,1^4)$	$(5,4^4,3^5,2^4,1^4)$	$(4^4,2^{16})$	
		11	$(11^3,7,5,3)$	$(8^2,6^2,4^4,2^2)$				
		≥ 13	$(13,11,9,7,5,3)$					
63	$\omega_1+\omega_2$	3	-	-	$(3^{20},1^3)$	$(3^{18},2^4,1)$	$(3^{18},1^9)$	$(3^{13},2^{10},1^4)$
64	$\omega_2+\omega_3$	5	-	$(5^{12},2^2)$	$(5^{10},3^4,1^2)$	$(5^8,3^4,2^6)$	$(4^{12},2^8)$	$(4^2,3^8,2^{12},1^8)$
77	$3\omega_1$	5	-	$(5^{15},1^2)$	$(5^{13},3^3,1^3)$	$(5^9,4^4,3^3,2^2,1^3)$	$(5^6,3^9,1^{20})$	$(4^4,3^9,2^{12},1^{10})$
		7	(7^{11})	$(7^9,5^2,1^4)$	$(7^4,5^5,3^7,1^3)$	$(7,6^2,5^3,4^6,3^4,2^2,1^3)$	$(7,5^4,3^{10},1^{20})$	
		11	(11^7)	$(11^2,9^2,7,5^5,1^5)$				
		13	$(13^5,9,3)$	$(13,9^3,7,5^5,1^5)$				
		17	$(17^2,13,11,9,7,3)$					
		≥ 19	$(19,15,13,11,9,7,3)$					

TABLE 14. $G = B_4(K)$

n	ω	p	$J(\varphi(x))$ for $J(x) = (9)$	$J(\varphi(x))$ for $J(x) = (7, 1^2)$	$J(\varphi(x))$ for $J(x) = (5, 3, 1)$	$J(\varphi(x))$ for $J(x) = (5, 2^2)$	$J(\varphi(x))$ for $J(x) = (5, 1^4)$
16	ω_4	5	-	-	$(5^2, 3^2)$	$(5, 4^2, 3)$	(4^4)
		7	-	$(7^2, 1^2)$			
		≥ 11	$(11, 5)$				
36	ω_2	5	-	-	$(5^6, 3^2)$	$(5^6, 3, 1^3)$	$(5^6, 1^6)$
		7	-	$(7^5, 1)$	$(7^2, 5^2, 3^4)$	$(7, 6^2, 4^2, 3^2, 1^3)$	$(7, 5^4, 3, 1^6)$
		11	$(11^3, 3)$	$(11, 7^3, 3, 1)$			
		13	$(13^2, 7, 3)$				
		≥ 17	$(15, 11, 7, 3)$				
44	$2\omega_1$	5	-	-	$(5^8, 3, 1)$	$(5^7, 3^3)$	$(5^7, 1^9)$
		7	-	$(7^6, 1^2)$	$(7^3, 5^3, 3^2, 1^2)$	$(7^2, 6^2, 4^2, 3^3, 1)$	$(7^2, 5^4, 1^{10})$
		11	(11^4)	$(11^2, 7^2, 5, 1^3)$	$(9, 7, 5^4, 3^2, 1^2)$	$(9, 6^2, 5, 4^2, 3^3, 1)$	$(9, 5^5, 1^{10})$
		13	$(13^3, 5)$	$(13, 9, 7^2, 5, 1^3)$			
		≥ 17	$(17, 13, 9, 5)$				
84	ω_3	5	-	-	$(5^{16}, 3, 1)$	$(5^{16}, 2^2)$	$(5^{16}, 1^4)$
		7	-	(7^{12})	$(7^7, 5^3, 3^6, 1^2)$	$(7^6, 5^4, 4^2, 3^2, 2^4)$	$(7^5, 5^6, 3^5, 1^4)$
		11	$(11^7, 7)$	$(11^4, 7^4, 5, 3^2, 1)$	$(9, 7^5, 5^4, 3^6, 1^2)$	$(8^2, 7^2, 6^2, 5^4, 4^2, 3^2, 2^4)$	
		13	$(13^5, 9, 7, 3)$	$(13, 11^2, 9, 7^4, 5, 3^2, 1)$			
		17	$(17^2, 13, 11, 9, 7^2, 3)$				
		≥ 19	$(19, 15, 13, 11, 9, 7^2, 3)$				

Table 14 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(4^2,1)$	$J(\varphi(x))$ for $J(x)=(3^3)$	$J(\varphi(x))$ for $J(x)=(3^2,1^3)$	$J(\varphi(x))$ for $J(x)=(3,2^2,1^2)$	$J(\varphi(x))$ for $J(x)=(3,1^6)$	$J(\varphi(x))$ for $J(x)=(2^4,1)$	$J(\varphi(x))$ for $J(x)=(2^2,1^5)$
16	ω_4	3	-	$(3^4,2^2)$	$(3^4,1^4)$	$(3^2,2^4,1^2)$	(2^8)	$(3,2^4,1^5)$	$(2^4,1^8)$
		≥ 5	$(5,4^2,1^3)$	$(4^2,2^4)$					
36	ω_2	3	-	(3^{12})	$(3^{11},1^3)$	$(3^8,2^4,1^4)$	$(3^7,1^{15})$	$(3^6,2^4,1^{10})$	$(3,2^{10},1^{13})$
		5	$(5^5,4^2,1^3)$	$(5^3,3^6,1^3)$	$(5,3^9,1^4)$	$(4^2,3^4,2^6,1^4)$			
		≥ 7	$(7,5^3,4^2,3,1^3)$						
43	$2\omega_1$	3	-	$(3^{14},1)$	$(3^{13},1^4)$	$(3^{11},2^4,1^2)$	$(3^8,1^{19})$	$(3^{10},2^4,1^5)$	$(3^3,2^{10},1^{14})$
44	$2\omega_1$	≥ 5	$(5^7,4^2,1)$	$(5^6,3^3,1^5)$	$(5^3,3^7,1^8)$	$(5,4^2,3^5,2^6,1^4)$	$(5,3^6,1^{21})$	$(3^{10},2^4,1^6)$	$(3^3,2^{10},1^{15})$
		≥ 7	$(7^3,5,4^2,3^3,1)$						
84	ω_3	3	-	$(3^{27},1^3)$	$(3^{27},1^3)$	$(3^{23},2^4,1^7)$	$(3^{21},1^{21})$	$(3^{14},2^{16},1^{10})$	$(3^5,2^{22},1^{25})$
		5	$(5^{13},4^4,1^3)$	$(5^{10},3^8,1^{10})$	$(5^5,3^{17},1^8)$	$(5,4^6,3^9,2^{10},1^8)$		$(4^4,3^6,2^{20},1^{10})$	
		7	$(7^5,5^3,4^6,3,2^2,1^3)$	$(7,5^8,3^9,1^{10})$					
		≥ 11	$(8^2,7,6^2,5^3,4^6,3,2^2,1^3)$						

TABLE 15. $G=B_5(K)$

n	ω	p	$J(\varphi(x))$ for $J(x)=(11)$	$J(\varphi(x))$ for $J(x)=(9,1^2)$	$J(\varphi(x))$ for $J(x)=(7,3,1)$	$J(\varphi(x))$ for $J(x)=(7,2^2)$	$J(\varphi(x))$ for $J(x)=(7,1^4)$
32	ω_5	7	-	-	$(7^4,2^2)$	$(7^4,2,1^2)$	$(7^4,1^4)$
		11	$(11^2,10)$	$(11^2,5^2)$	$(8^2,6^2,2^2)$	$(8,7^2,6,2,1^2)$	
		13	$(13^2,6)$				
		≥ 17	$(16,10,6)$				
55	ω_2	7	-	-	$(7^7,3^2)$	$(7^7,3,1^3)$	$(7^7,1^6)$
		11	(11^5)	$(11^3,9^2,3,1)$	$(11,9,7^3,5,3^3)$	$(11,8^2,7,6^2,3^2,1^3)$	$(11,7^5,3,1^6)$
		13	$(13^4,3)$	$(13^2,9^2,7,3,1)$			
		17	$(17^2,11,7,3)$	$(15,11,9^2,7,3,1)$			
		≥ 19	$(19,15,11,7,3)$				

Table 15 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(11)$	$J(\varphi(x))$ for $J(x)=(9,1^2)$	$J(\varphi(x))$ for $J(x)=(7,3,1)$	$J(\varphi(x))$ for $J(x)=(7,2^2)$	$J(\varphi(x))$ for $J(x)=(7,1^4)$
64	$2\omega_1$	11	$(11^5,9)$	$(11^4,9^2,1^2)$	$(11^2,9,7^2,5^3,3,1)$	$(11^2,8^2,6^2,5,3^3)$	$(11^2,7^4,5,1^9)$
65	$2\omega_1$	7	-	-	$(7^8,5,3,1)$	$(7^8,3^3)$	$(7^8,1^9)$
		13	(13^5)	$(13^3,9^2,5,1^3)$	$(13,9^2,7^2,5^3,3,1^2)$	$(13,9,8^2,6^2,5,3^3,1)$	$(13,9,7^4,5,1^{10})$
		17	$(17^3,9,5)$	$(17,13,9^3,5,1^3)$			
		19	$(19^2,13,9,5)$				
		≥ 23	$(21,17,13,9,5)$				

Table 15 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(5^2,1)$	$J(\varphi(x))$ for $J(x)=(5,3^2)$	$J(\varphi(x))$ for $J(x)=(5,3,1^3)$	$J(\varphi(x))$ for $J(x)=(5,2^2,1^2)$	$J(\varphi(x))$ for $J(x)=(5,1^6)$
32	ω_5	5	$(5^6,1^2)$	$(5^4,4^2,2^2)$	$(5^4,3^4)$	$(5^2,4^4,3^2)$	(4^8)
		≥ 7	$(7^2,5^2,3^2,1^2)$	$(6^2,4^4,2^2)$			
55	ω_2	5	(5^{11})	$(5^9,3^3,1)$	$(5^8,3^4,1^3)$	$(5^8,3,2^4,1^4)$	$(5^8,1^{15})$
		7	$(7^5,5^2,3^3,1)$	$(7^3,5^3,3^6,1)$	$(7^2,5^4,3^6,1^3)$	$(7,6^2,5^2,4^2,3^2,2^4,1^4)$	$(7,5^6,3,1^{15})$
		≥ 11	$(9,7^3,5^3,3^3,1)$				
64	$2\omega_1$	11	$(9^3,7,5^5,3,1^2)$	$(9,7^2,5^6,3^3,1^2)$	$(9,7,5^6,3^4,1^6)$	$(9,6^2,5^3,4^2,3^3,2^4,1^3)$	$(9,5^7,1^{20})$
65	$2\omega_1$	5	(5^{13})	$(5^{12},3,1^2)$	$(5^{10},3^3,1^6)$	$(5^9,3^3,2^4,1^3)$	$(5^9,1^{20})$
		7	$(7^7,5^2,3,1^3)$	$(7^4,5^5,3^3,1^3)$	$(7^3,5^5,3^4,1^7)$	$(7^2,6^2,5^2,4^2,3^3,2^4,1^4)$	$(7^2,5^6,1^{21})$
		≥ 13	$(9^3,7,5^5,3,1^3)$	$(9,7^2,5^6,3^3,1^3)$	$(9,7,5^6,3^4,1^7)$	$(9,6^2,5^3,4^2,3^3,2^4,1^4)$	$(9,5^7,1^{21})$

Table 15 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(4^2,3)$	$J(\varphi(x))$ for $J(x)=(4^2,1^3)$	$J(\varphi(x))$ for $J(x)=(3^3,1^2)$	$J(\varphi(x))$ for $J(x)=(3^2,2^2,1)$	$J(\varphi(x))$ for $J(x)=(3,2^4)$	$J(\varphi(x))$ for $J(x)=(3^2,1^5)$
32	ω_5	3	-	-	$(3^8,2^4)$	$(3^8,2^2,1^4)$	$(3^6,2^5,1^4)$	$(3^8,1^8)$
		5	$(5^4,3^2,2^3)$	$(5^2,4^4,1^6)$	$(4^4,2^8)$	$(4^2,3^4,2^4,1^4)$	$(4,3^4,2^6,1^4)$	
		≥ 7	$(6,5^2,4,3^2,2^3)$					
55	ω_2	3	-	-	$(3^{18},1)$	$(3^{16},2^2,1^3)$	$(3^{15},1^{10})$	$(3^{15},1^{10})$
		5	$(5^9,3,2^2,1^3)$	$(5^5,4^6,1^6)$	$(5^3,3^{12},1^4)$	$(5,4^4,3^6,2^6,1^4)$	$(4^4,3^7,2^4,1^{10})$	$(5,3^{13},1^{11})$
		≥ 7	$(7,6^2,5^3,4^2,3^2,2^2,1^3)$	$(7,5^3,4^6,3,1^6)$				
64	$2\omega_1$	11	$(7^3,6^2,5^2,4^2,3^3,2^2)$	$(7^3,5,4^6,3^3,1^5)$	$(5^6,3^9,1^7)$	$(5^3,4^4,3^6,2^6,1^3)$	$(5,4^4,3^{10},2^4,1^5)$	$(5^3,3^{11},1^{16})$
65	$2\omega_1$	3	-	-	$(3^{21},1^2)$	$(3^{20},2^2,1)$	$(3^{20},1^5)$	$(3^{17},1^{14})$
		5	$(5^{12},2^2,1)$	$(5^7,4^6,1^6)$	$(5^6,3^9,1^8)$	$(5^3,4^4,3^6,2^6,1^4)$	$(5,4^4,3^{10},2^4,1^6)$	$(5^3,3^{11},1^{17})$
		≥ 7	$(7^3,6^2,5^2,4^2,3^3,2^2,1)$	$(7^3,5,4^6,3^3,1^6)$				

Table 15 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(3,2^2,1^4)$	$J(\varphi(x))$ for $J(x)=(3,1^8)$	$J(\varphi(x))$ for $J(x)=(2^4,1^3)$	$J(\varphi(x))$ for $J(x)=(2^2,1^7)$
32	ω_5	≥ 3	$(3^4,2^8,1^4)$	(2^{16})	$(3^2,2^8,1^{10})$	$(2^8,1^{16})$
55	ω_2	3	$(3^{10},2^8,1^9)$	$(3^9,1^{28})$	$(3^6,2^{12},1^{13})$	$(3,2^{14},1^{24})$
		≥ 5	$(4^2,3^6,2^{10},1^9)$			
64	$2\omega_1$	11	$(5,4^2,3^7,2^{10},1^{10})$	$(5,3^8,1^{35})$	$(3^{10},2^{12},1^{10})$	$(3^3,2^{14},1^{27})$
65	$2\omega_1$	3	$(3^{13},2^8,1^{10})$	$(3^{10},1^{35})$	$(3^{10},2^{12},1^{11})$	$(3^3,2^{14},1^{28})$
		≥ 5	$(5,4^2,3^7,2^{10},1^{11})$	$(5,3^8,1^{36})$		

TABLE 16. $G = B_6(K)$

n	ω	p	$J(\varphi(x))$ for $J(x) = (13)$	$J(\varphi(x))$ for $J(x) = (11, 1^2)$	$J(\varphi(x))$ for $J(x) = (9, 3, 1)$	$J(\varphi(x))$ for $J(x) = (9, 2^2)$	$J(\varphi(x))$ for $J(x) = (9, 1^4)$
64	ω_6	11	-	$(11^4, 10^2)$	$(11^4, 6^2, 4^2)$	$(11^2, 6, 5^2, 4)$	$(11^4, 5^4)$
		13	$(13^4, 12)$	$(13^4, 6^2)$	$(12^2, 10^2, 6^2, 4^2)$	$(12, 11^2, 10, 6, 5^2, 4)$	
		17	$(17^2, 16, 10, 4)$	$(16^2, 10^2, 6^2)$			
		19	$(19^2, 12, 10, 4)$				
		≥ 23	$(22, 16, 12, 10, 4)$				
78	ω_2	11	-	$(11^7, 1)$	$(11^4, 9^2, 7, 3^3)$	$(11^3, 10^2, 8^2, 3^2, 1^3)$	$(11^3, 9^4, 3, 1^6)$
		13	(13^6)	$(13^4, 11^2, 3, 1)$	$(13^2, 11, 9^2, 7^2, 3^3)$	$(13^2, 10^2, 8^2, 7, 3^2, 1^3)$	$(13^2, 9^4, 7, 3, 1^6)$
		17	$(17^4, 7, 3)$	$(17^2, 11^3, 7, 3, 1)$	$(15, 11^2, 9^2, 7^2, 3^3)$	$(15, 11, 10^2, 8^2, 7, 3^2, 1^3)$	$(15, 11, 9^4, 7, 3, 1^6)$
		19	$(19^3, 11, 7, 3)$	$(19, 15, 11^3, 7, 3, 1)$			
		≥ 23	$(23, 19, 15, 11, 7, 3)$				
89	$2\omega_1$	13	$(13^6, 11)$	$(13^5, 11^2, 1^2)$	$(13^3, 11, 9^2, 7, 5^2, 3, 1)$	$(13^3, 10^2, 8^2, 5, 3^3)$	$(13^3, 9^4, 5, 1^9)$
90	$2\omega_1$	11	-	$(11^8, 1^2)$	$(11^5, 9^2, 7, 5, 3, 1^2)$	$(11^4, 10^2, 8^2, 3^3, 1)$	$(11^4, 9^4, 1^{10})$
		17	$(17^5, 5)$	$(17^3, 11^2, 9, 5, 1^3)$	$(17, 13, 11, 9^3, 7, 5^2, 3, 1^2)$	$(17, 13, 10^2, 9, 8^2, 5, 3^3, 1)$	$(17, 13, 9^5, 5, 1^{10})$
		19	$(19^4, 9, 5)$	$(19^2, 13, 11^2, 9, 5, 1^3)$			
		23	$(23^2, 17, 13, 9, 5)$	$(21, 17, 13, 11^2, 9, 5, 1^3)$			
		≥ 27	$(25, 21, 17, 13, 9, 5)$				

Table 16 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(7,5,1)$	$J(\varphi(x))$ for $J(x)=(7,3^2)$	$J(\varphi(x))$ for $J(x)=(7,3,1^3)$	$J(\varphi(x))$ for $J(x)=(7,2^2,1^2)$	$J(\varphi(x))$ for $J(x)=(7,1^6)$
64	ω_6	7	$(7^8,4^2)$	$(7^8,3^2,1^2)$	$(7^8,2^4)$	$(7^8,2^2,1^4)$	$(7^8,1^8)$
		≥ 11	$(10^2,8^2,6^2,4^4)$	$(9^2,7^4,5^2,3^2,1^2)$	$(8^4,6^4,2^4)$	$(8^2,7^4,6^2,2^2,1^4)$	
78	ω_2	7	$(7^{10},5,3)$	$(7^9,5,3^3,1)$	$(7^9,3^4,1^3)$	$(7^9,3,2^4,1^4)$	$(7^9,1^{15})$
		≥ 11	$(11^2,9,7^4,5^2,3^3)$	$(11,9^2,7^3,5^3,3^4,1)$	$(11,9,7^5,5,3^5,1^3)$	$(11,8^2,7^3,6^2,3^2,2^4,1^4)$	$(11,7^7,3,1^{15})$
89	$2\omega_1$	13	$(13,11,9^3,7^2,5^4,3,1)$	$(13,9^3,7^2,5^6,3,1^2)$	$(13,9^2,7^4,5^3,3^3,1^6)$	$(13,9,8^2,7^2,6^2,5,3^3,2^4,1^3)$	$(13,9,7^6,5,1^{20})$
90	$2\omega_1$	7	$(7^{12},5,1)$	$(7^{10},5^3,3,1^2)$	$(7^{10},5,3^3,1^6)$	$(7^{10},3^3,2^4,1^3)$	$(7^{10},1^{20})$
		11	$(11^3,9^2,7^2,5^4,3,1^2)$	$(11^2,9^2,7^2,5^6,3,1^3)$	$(11^2,9,7^4,5^3,3^3,1^7)$	$(11^2,8^2,7^2,6^2,5,3^3,2^4,1^4)$	$(11^2,7^6,5,1^{21})$
		≥ 17	$(13,11,9^3,7^2,5^4,3,1^2)$	$(13,9^3,7^2,5^6,3,1^3)$	$(13,9^2,7^4,5^3,3^3,1^7)$	$(13,9,8^2,7^2,6^2,5,3^3,2^4,1^4)$	$(13,9,7^6,5,1^{21})$

Table 16 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(6^2,1)$	$J(\varphi(x))$ for $J(x)=(5^2,3)$	$J(\varphi(x))$ for $J(x)=(5^2,1^3)$	$J(\varphi(x))$ for $J(x)=(5,4^2)$	$J(\varphi(x))$ for $J(x)=(5,3^2,1^2)$
64	ω_6	5	-	$(5^{12},2^2)$	$(5^{12},1^4)$	$(5^{10},4^3,1^2)$	$(5^8,4^4,2^4)$
		7	$(7^6,6^3,1^4)$	$(7^4,6^2,4^4,2^4)$	$(7^4,5^4,3^4,1^4)$	$(7^4,5^2,4^4,3^2,2,1^2)$	$(6^4,4^8,2^4)$
		≥ 11	$(10,9^2,6^3,5^2,4,1^4)$	$(8^2,6^4,4^4,2^4)$		$(8,7^2,6,5^2,4^4,3^2,2,1^2)$	
78	ω_2	5	-	$(5^{15},3)$	$(5^{15},1^3)$	$(5^{15},1^3)$	$(5^{11},3^7,1^2)$
		7	$(7^9,6^2,1^3)$	$(7^7,5^2,3^6,1)$	$(7^5,5^6,3^3,1^4)$	$(7^6,5^3,4^2,3^2,2^2,1^3)$	$(7^3,5^5,3^{10},1^2)$
		≥ 11	$(11,9^3,7,6^2,5^3,3,1^3)$	$(9,7^5,5^3,3^6,1)$	$(9,7^3,5^7,3^3,1^4)$	$(8^2,7^2,6^2,5^3,4^2,3^2,2^2,1^3)$	
89	$2\omega_1$	13	$(11^3,9,7^3,6^2,5,3^3)$	$(9^3,7^3,5^6,3^3,1^2)$	$(9^3,7,5^9,3,1^7)$	$(9,8^2,7^3,6^2,5^2,4^2,3^3,2^2)$	$(9,7^2,5^8,3^7,1^5)$
90	$2\omega_1$	5	-	(5^{18})	$(5^{17},1^5)$	(5^{18})	$(5^{14},3^5,1^5)$
		7	$(7^{11},6^2,1)$	$(7^9,5^3,3^3,1^3)$	$(7^7,5^6,3,1^8)$	$(7^9,5,4^2,3^3,2^2,1)$	$(7^4,5^7,3^7,1^6)$
		$11,\geq 17$	$(11^3,9,7^3,6^2,5,3^3,1)$	$(9^3,7^3,5^6,3^3,1^3)$	$(9^3,7,5^9,3,1^8)$	$(9,8^2,7^3,6^2,5^2,4^2,3^3,2^2,1)$	$(9,7^2,5^8,3^7,1^6)$

Table 16 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(5,3,2^2,1)$	$J(\varphi(x))$ for $J(x)=(5,3,1^5)$	$J(\varphi(x))$ for $J(x)=(5,2^4)$	$J(\varphi(x))$ for $J(x)=(5,2^2,1^4)$	$J(\varphi(x))$ for $J(x)=(5,1^8)$
64	ω_6	5	$(5^8,4^2,3^4,2^2)$	$(5^8,3^8)$	$(5^6,4^5,3^4,2)$	$(5^4,4^8,3^4)$	(4^{16})
		≥ 7	$(6^2,5^4,4^4,3^4,2^2)$		$(6,5^4,4^6,3^4,2)$		
78	ω_2	5	$(5^{10},4^2,3^3,2^4,1^3)$	$(5^{10},3^6,1^{10})$	$(5^{10},3^6,1^{10})$	$(5^{10},3,2^8,1^9)$	$(5^{10},1^{28})$
		≥ 7	$(7^2,6^2,5^2,4^4,3^5,2^4,1^3)$	$(7^2,5^6,3^8,1^{10})$	$(7,6^4,4^4,3^7,1^{10})$	$(7,6^2,5^4,4^2,3^2,2^8,1^9)$	$(7,5^8,3,1^{28})$
89	$2\omega_1$	13	$(9,7,6^2,5^4,4^4,3^5,2^4,1^2)$	$(9,7,5^8,3^6,1^{15})$	$(9,6^4,5,4^4,3^{10},1^5)$	$(9,6^2,5^5,4^2,3^3,2^8,1^{10})$	$(9,5^9,1^{35})$
90	$2\omega_1$	5	$(5^{12},4^2,3^4,2^4,1^2)$	$(5^{12},3^5,1^{15})$	$(5^{11},3^{10},1^5)$	$(5^{11},3^3,2^8,1^{10})$	$(5^{11},1^{35})$
		7	$(7^3,6^2,5^3,4^4,3^5,2^4,1^3)$	$(7^3,5^7,3^6,1^{16})$	$(7^2,6^4,4^4,3^{10},1^6)$	$(7^2,6^2,5^4,4^2,3^3,2^8,1^{11})$	$(7^2,5^8,1^{36})$
		$11,\geq 17$	$(9,7,6^2,5^4,4^4,3^5,2^4,1^3)$	$(9,7,5^8,3^6,1^{16})$	$(9,6^4,5,4^4,3^{10},1^6)$	$(9,6^2,5^5,4^2,3^3,2^8,1^{11})$	$(9,5^9,1^{36})$

Table 16 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(4^2,3,1^2)$	$J(\varphi(x))$ for $J(x)=(4^2,2^2,1)$	$J(\varphi(x))$ for $J(x)=(4^2,1^5)$	$J(\varphi(x))$ for $J(x)=(3^4,1)$	$J(\varphi(x))$ for $J(x)=(3^3,2^2)$
64	ω_6	3	-	-	-	$(3^{20},1^4)$	$(3^{18},2^4,1^2)$
		5	$(5^8,3^4,2^6)$	$(5^6,4^4,3^2,2^3,1^6)$	$(5^4,4^8,1^{12})$	$(5^4,3^{12},1^8)$	$(5^2,4^4,3^6,2^8,1^4)$
		≥ 7	$(6^2,5^4,4^2,3^4,2^6)$	$(6,5^4,4^5,3^2,2^3,1^6)$			
78	ω_2	3	-	-	-	(3^{26})	$(3^{25},1^3)$
		5	$(5^9,4^4,3^3,2^2,1^4)$	$(5^9,4^2,3^5,2^2,1^6)$	$(5^5,4^{10},1^{13})$	$(5^6,3^{14},1^6)$	$(5^3,4^6,3^7,2^6,1^6)$
		≥ 7	$(7,6^2,5^3,4^6,3^4,2^2,1^4)$	$(7,5^7,4^2,3^6,2^2,1^6)$	$(7,5^3,4^{10},3,1^{13})$		
89	$2\omega_1$	13	$(7^3,6^2,5^2,4^6,3^5,2^2,1^3)$	$(7^3,5^5,4^2,3^{10},2^2,1)$	$(7^3,5,4^{10},3^3,1^{14})$	$(5^{10},3^{10},1^9)$	$(5^6,4^6,3^6,2^6,1^5)$
90	$2\omega_1$	3	-	-	-	(3^{30})	(3^{30})
		5	$(5^{12},4^4,3^2,2^2,1^4)$	$(5^{11},4^2,3^7,2^2,1^2)$	$(5^7,4^{10},1^{15})$	$(5^{10},3^{10},1^{10})$	$(5^6,4^6,3^6,2^6,1^6)$
		≥ 7	$(7^3,6^2,5^2,4^6,3^5,2^2,1^4)$	$(7^3,5^5,4^2,3^{10},2^2,1^2)$	$(7^3,5,4^{10},3^3,1^{15})$		

Table 16 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(3^3,1^4)$	$J(\varphi(x))$ for $J(x)=(3^2,2^2,1^3)$	$J(\varphi(x))$ for $J(x)=(3^2,1^7)$	$J(\varphi(x))$ for $J(x)=(3,2^4,1^2)$	$J(\varphi(x))$ for $J(x)=(3,2^2,1^6)$
64	ω_6	3	$(3^{16},2^8)$	$(3^{16},2^4,1^8)$	$(3^{16},1^{16})$	$(3^{12},2^{10},1^8)$	$(3^8,2^{16},1^8)$
		≥ 5	$(4^8,2^{16})$	$(4^4,3^8,2^8,1^8)$		$(4^2,3^8,2^{12},1^8)$	
78	ω_2	3	$(3^{24},1^6)$	$(3^{20},2^6,1^6)$	$(3^{19},1^{21})$	$(3^{17},2^8,1^{11})$	$(3^{12},2^{12},1^{18})$
		≥ 5	$(5^3,3^{18},1^9)$	$(5,4^4,3^{10},2^{10},1^7)$	$(5,3^{17},1^{22})$	$(4^4,3^9,2^{12},1^{11})$	$(4^2,3^8,2^{14},1^{18})$
89	$2\omega_1$	13	$(5^6,3^{15},1^{14})$	$(5^3,4^4,3^{10},2^{10},1^8)$	$(5^3,3^{15},1^{29})$	$(5,4^4,3^{12},2^{12},1^8)$	$(5,4^2,3^9,2^{14},1^{21})$
90	$2\omega_1$	3	$(3^{27},1^9)$	$(3^{24},2^6,1^6)$	$(3^{21},1^{27})$	$(3^{22},2^8,1^8)$	$(3^{15},2^{12},1^{21})$
		≥ 5	$(5^6,3^{15},1^{15})$	$(5^3,4^4,3^{10},2^{10},1^9)$	$(5^3,3^{15},1^{30})$	$(5,4^4,3^{12},2^{12},1^9)$	$(5,4^2,3^9,2^{14},1^{22})$

Table 16 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(3,1^{10})$	$J(\varphi(x))$ for $J(x)=(2^6,1)$	$J(\varphi(x))$ for $J(x)=(2^4,1^5)$	$J(\varphi(x))$ for $J(x)=(2^2,1^9)$
64	ω_6	3	(2^{32})	$(3^8,2^{13},1^{14})$	$(3^4,2^{16},1^{20})$	$(2^{16},1^{32})$
		≥ 5		$(4,3^6,2^{14},1^{14})$		
78	ω_2	≥ 3	$(3^{11},1^{45})$	$(3^{15},2^6,1^{21})$	$(3^6,2^{20},1^{20})$	$(3,2^{18},1^{39})$
89	$2\omega_1$	13	$(5,3^{10},1^{54})$	$(3^{21},2^6,1^{14})$	$(3^{10},2^{20},1^{19})$	$(3^3,2^{18},1^{44})$
90	$2\omega_1$	3	$(3^{12},1^{54})$	$(3^{21},2^6,1^{15})$	$(3^{10},2^{20},1^{20})$	$(3^3,2^{18},1^{45})$
		≥ 5	$(5,3^{10},1^{55})$			

TABLE 17. $G = C_2(K)$

n	ω	p	$J(\varphi(x))$ for $J(x) = (4)$	$J(\varphi(x))$ for $J(x) = (2^2)$	$J(\varphi(x))$ for $J(x) = (2, 1^2)$
5	ω_2	3	-	$(3, 1^2)$	$(2^2, 1)$
		≥ 5	(5)		
10	$2\omega_1$	3	-	$(3^3, 1)$	$(3, 2^2, 1^3)$
		5	(5^2)		
		≥ 7	$(7, 3)$		
12	$\omega_1 + \omega_2$	5	$(5^2, 2)$	$(4^2, 2^2)$	$(3^2, 2^3)$
13	$2\omega_2$	5	$(5^2, 3)$	$(5, 3^2, 1^2)$	$(3^3, 2^2)$
14	$2\omega_2$	3	-	$(3^4, 1^2)$	$(3^3, 2^2, 1)$
		7	(7^2)	$(5, 3^2, 1^3)$	
		≥ 11	$(9, 5)$		
16	$\omega_1 + \omega_2$	3	-	$(3^4, 2^2)$	$(3^2, 2^4, 1^2)$
		7	$(7^2, 2)$	$(4^2, 2^4)$	
		≥ 11	$(8, 6, 2)$		
20	$3\omega_1$	5	(5^4)	$(4^4, 2^2)$	$(4, 3^2, 2^3, 1^4)$
		7	$(7^2, 6)$		
		≥ 11	$(10, 6, 4)$		
24	$\omega_1 + 2\omega_2$	7	$(7^2, 6, 4)$	$(6^2, 4^2, 2^2)$	$(4^3, 3^4)$
25	$3\omega_2$	7	$(7^3, 3, 1)$	$(7, 5^2, 3^2, 1^2)$	$(4^4, 3^3)$
25	$2\omega_1 + \omega_2$	3	-	$(3^8, 1)$	$(3^7, 2^2)$
30	$3\omega_2$	5	(5^6)	$(5^4, 3^2, 1^4)$	$(4^4, 3^3, 2^2, 1)$
		11	$(11^2, 7, 1)$	$(7, 5^2, 3^3, 1^4)$	
		≥ 13	$(13, 9, 7, 1)$		
35	$2\omega_1 + \omega_2$	5	(5^7)	$(5^3, 3^6, 1^2)$	$(4^2, 3^4, 2^6, 1^3)$
		7	(7^5)		
		≥ 11	$(11, 9, 7, 5, 3)$		
35	$4\omega_1$	5	(5^7)	$(5^5, 3^3, 1)$	$(5, 4^2, 3^3, 2^4, 1^5)$
		7	(7^5)		
		11	$(11^2, 7, 5, 1)$		
		≥ 13	$(13, 9, 7, 5, 1)$		
40	$\omega_1 + 2\omega_2$	3	-	$(3^{12}, 2^2)$	$(3^{12}, 2, 1^2)$
		5	(5^8)	$(5^4, 4^2, 2^6)$	$(4^3, 3^6, 2^4, 1^2)$
		11	$(11^2, 8, 6, 4)$	$(6^2, 4^4, 2^6)$	
		≥ 13	$(12, 10, 8, 6, 4)$		
44	$3\omega_1 + \omega_2$	7	$(7^6, 2)$	$(6^4, 4^4, 2^2)$	$(5^2, 4^3, 3^4, 2^5)$
52	$3\omega_1 + \omega_2$	5	$(5^{10}, 2)$	$(5^8, 4^2, 2^2)$	$(5^2, 4^4, 3^4, 2^5, 1^4)$
54	$4\omega_2$	7	$(7^7, 5)$	$(7^4, 5^2, 3^4, 1^4)$	$(5^5, 4^4, 3^3, 2^2)$
55	$4\omega_2$	5	(5^{11})	$(5^9, 3^2, 1^4)$	$(5^5, 4^4, 3^3, 2^2, 1)$
		11	(11^5)	$(9, 7^2, 5^3, 3^4, 1^5)$	
		13	$(13^3, 11, 5)$		
		≥ 17	$(17, 13, 11, 9, 5)$		
56	$5\omega_1$	7	(7^8)	$(6^6, 4^4, 2^2)$	$(6, 5^2, 4^3, 3^4, 2^5, 1^6)$
		11	$(11^4, 8, 4)$		
		13	$(13^2, 12, 8, 6, 4)$		
		≥ 17	$(16, 12, 10, 8, 6, 4)$		
60	$\omega_1 + 4\omega_2$	11	$(11^4, 8, 6, 2)$	$(10^2, 8^2, 6^2, 4^2, 2^2)$	$(6^5, 5^6)$
61	$5\omega_2$	11	$(11^4, 9, 5, 3)$	$(11, 9^2, 7^2, 5^2, 3^2, 1^2)$	$(6^6, 5^5)$
64	$3\omega_1 + \omega_2$	11	$(11^4, 8, 6, 4, 2)$	$(6^4, 4^8, 2^4)$	$(5^2, 4^4, 3^6, 2^8, 1^4)$
		13	$(13^2, 10, 8^2, 6, 4, 2)$		
		≥ 17	$(14, 12, 10, 8^2, 6, 4, 2)$		
68	$2\omega_1 + 2\omega_2$	5	$(5^{13}, 3)$	$(5^{11}, 3^4, 1)$	$(5^3, 4^6, 3^6, 2^4, 1^3)$
71	$2\omega_1 + 2\omega_2$	7	$(7^9, 5, 3)$	$(7^3, 5^6, 3^6, 1^2)$	$(5^3, 4^6, 3^8, 2^4)$
76	$\omega_1 + 3\omega_2$	7	$(7^{10}, 4, 2)$	$(7^4, 6^2, 4^6, 2^6)$	$(5^4, 4^8, 3^6, 2^3)$
80	$\omega_1 + 3\omega_2$	5	(5^{16})	$(5^{12}, 4^2, 2^6)$	$(5^4, 4^8, 3^6, 2^4, 1^2)$
		11	$(11^6, 10, 4)$	$(8^2, 6^4, 4^6, 2^8)$	
		13	$(13^4, 10, 8, 6, 4)$		
		≥ 17	$(16, 14, 12, 10^2, 8, 6, 4)$		
81	$2\omega_1 + 2\omega_2$	3	-	(3^{27})	(3^{27})
		11	$(11^6, 7, 5, 3)$	$(7^3, 5^6, 3^9, 1^3)$	$(5^3, 4^6, 3^9, 2^6, 1^3)$
		13	$(13^3, 11, 9, 7^2, 5, 3)$		
		≥ 17	$(15, 13, 11^2, 9, 7^2, 5, 3)$		
84	$\omega_1 + 5\omega_2$	13	$(13^4, 12, 10, 6, 4)$	$(12^2, 10^2, 8^2, 6^2, 4^2, 2^2)$	$(7^6, 6^7)$

TABLE 17. $G = C_2(K)$

n	ω	p	$J(\varphi(x))$ for $J(x) = (4)$	$J(\varphi(x))$ for $J(x) = (2^2)$	$J(\varphi(x))$ for $J(x) = (2, 1^2)$
84	$6\omega_1$	7	(7^{12})	$(7^7, 5^5, 3^3, 1)$	$(7, 6^2, 5^3, 4^4, 3^5, 2^6, 1^7)$
		11	$(11^7, 7)$		
		13	$(13^5, 9, 7, 3)$		
		17	$(17^2, 13, 11, 9, 7^2, 3)$		
		≥ 19	$(19, 15, 13, 11, 9, 7^2, 3)$		
85	$6\omega_2$	13	$(13^5, 9, 7, 3, 1)$	$(13, 11^2, 9^2, 7^2, 5^2, 3^2, 1^2)$	$(7^7, 6^6)$
86	$2\omega_1 + 3\omega_2$	5	$(5^{17}, 1)$	$(5^{15}, 3^3, 1^2)$	$(5^{13}, 4^2, 3^3, 2^2)$
91	$5\omega_2$	7	(7^{13})	$(7^9, 5^2, 3^4, 1^6)$	$(6^6, 5^5, 4^4, 3^3, 2^2, 1)$
		13	(13^7)	$(11, 9^2, 7^3, 5^4, 3^5, 1^6)$	
		17	$(17^3, 15, 11, 9, 5)$		
		19	$(19^2, 15, 13, 11, 9, 5)$		
		≥ 23	$(21, 17, 15, 13, 11, 9, 5)$		

TABLE 18. $G = C_3(K)$

n	ω	p	$J(\varphi(x))$ for $J(x) = (6)$	$J(\varphi(x))$ for $J(x) = (4, 2)$	$J(\varphi(x))$ for $J(x) = (4, 1^2)$
14	ω_2	5	-	$(5^2, 3, 1)$	$(5, 4^2, 1)$
		7	(7^2)		
		≥ 11	$(9, 5)$		
14	ω_3	5	-	$(5^2, 4)$	$(5^2, 4)$
		7	(7^2)	$(6, 4^2)$	
		≥ 11	$(10, 4)$		
21	$2\omega_1$	5	-	$(5^3, 3^2)$	$(5^2, 4^2, 1^3)$
		7	(7^3)	$(7, 5, 3^3)$	$(7, 4^2, 3, 1^3)$
		≥ 11	$(11, 7, 3)$		
56	$3\omega_1$	5	-	$(5^{10}, 4, 2)$	$(5^8, 4^3, 1^4)$
		7	(7^8)	$(7^4, 6^2, 4^3, 2^2)$	$(7^4, 6, 4^3, 3^2, 1^4)$
		11	$(11^4, 8, 4)$	$(10, 8, 6^3, 4^4, 2^2)$	$(10, 7^2, 6, 4^4, 3^2, 1^4)$
		13	$(13^2, 12, 8, 6, 4)$		
		≥ 17	$(16, 12, 10, 8, 6, 4)$		
58	$\omega_1 + \omega_2$	7	$(7^8, 2)$	$(7^4, 6^2, 4^3, 2^3)$	$(7^4, 5^2, 4^3, 3^2, 2)$
62	$\omega_2 + \omega_3$	5	-	$(5^{10}, 4^2, 2^2)$	$(5^{10}, 3^2, 2^3)$
63	$2\omega_3$	5	-	$(5^{11}, 3^2, 1^2)$	$(5^{10}, 3^3, 2^2)$
64	$\omega_1 + \omega_2$	5	-	$(5^{12}, 2^2)$	$(5^{10}, 4^3, 1^2)$
		11	$(11^4, 8, 6, 4, 2)$	$(8^2, 6^4, 4^4, 2^4)$	$(8, 7^2, 6, 5^2, 4^4, 3^2, 2, 1^2)$
		13	$(13^2, 10, 8^2, 6, 4, 2)$		
		≥ 17	$(14, 12, 10, 8^2, 6, 4, 2)$		
70	$\omega_1 + \omega_3$	5	-	(5^{14})	(5^{14})
		7	(7^{10})	$(7^6, 5^3, 3^4, 1)$	$(7^5, 5^4, 4^2, 3, 2^2)$
		11	$(11^5, 7, 5, 3)$	$(9, 7^4, 5^4, 3^4, 1)$	$(8^2, 7, 6^2, 5^4, 4^2, 3, 2^2)$
		13	$(13^3, 9, 7^2, 5, 3)$		
		≥ 17	$(15, 13, 11, 9, 7^2, 5, 3)$		
84	$2\omega_3$	7	(7^{12})	$(7^{11}, 3^2, 1)$	$(7^{11}, 3, 2^2)$
		11	$(11^7, 7)$	$(11, 9^2, 7^5, 5^2, 3^3, 1)$	$(9^3, 8^2, 7, 6^2, 5^3, 3, 2^2)$
		13	$(13^5, 9, 7, 3)$		
		17	$(17^2, 13, 11, 9, 7^2, 3)$		
		≥ 19	$(19, 15, 13, 11, 9, 7^2, 3)$		
89	$2\omega_2$	7	$(7^{12}, 5)$	$(7^9, 5^3, 3^3, 1^2)$	$(7^9, 5, 4^2, 3^3, 2^2)$
90	$2\omega_2$	5	-	(5^{18})	(5^{18})
		11	$(11^7, 7, 5, 1)$	$(9^3, 7^3, 5^6, 3^3, 1^3)$	$(9, 8^2, 7^3, 6^2, 5^2, 4^2, 3^3, 2^2, 1)$
		13	$(13^4, 11, 9, 7, 5^2, 1)$		
		≥ 17	$(17, 13^2, 11, 9^2, 7, 5^2, 1)$		

Table 18 continued

n	ω	p	$J(\varphi(x))$ for $J(x) = (3^2)$	$J(\varphi(x))$ for $J(x) = (2^3)$	$J(\varphi(x))$ for $J(x) = (2^2, 1^2)$	$J(\varphi(x))$ for $J(x) = (2, 1^4)$
13	ω_2	3	$(3^4, 1)$	$(3^3, 1^4)$	$(3, 2^4, 1^2)$	$(2^4, 1^5)$
14	ω_2	≥ 5	$(5, 3^3)$	$(3^3, 1^5)$	$(3, 2^4, 1^3)$	$(2^4, 1^6)$
14	ω_3	3	$(3^4, 1^2)$	$(3^2, 2^4)$	$(3^2, 2^2, 1^4)$	$(2^5, 1^4)$
		≥ 5	$(5^2, 1^4)$	$(4, 2^5)$		
21	$2\omega_1$	3	(3^7)	$(3^6, 1^3)$	$(3^3, 2^4, 1^4)$	$(3, 2^4, 1^{10})$
		≥ 5	$(5^3, 3, 1^3)$			
50	$\omega_1 + \omega_2$	3	$(3^{14}, 1^8)$	$(3^{14}, 2^4)$	$(3^{10}, 2^8, 1^4)$	$(3^4, 2^{11}, 1^{16})$
56	$3\omega_1$	5	$(5^{10}, 3^2)$	$(4^{10}, 2^8)$	$(4^4, 3^6, 2^8, 1^6)$	$(4, 3^4, 2^{10}, 1^{20})$
		≥ 7	$(7^4, 5^2, 3^6)$			
57	$\omega_1 + \omega_3$	3	(3^{19})	$(3^{18}, 1^3)$	$(3^{14}, 2^4, 1^7)$	$(3^5, 2^{16}, 1^{10})$
58	$\omega_1 + \omega_2$	7	$(7^2, 5^6, 3^4, 1^2)$	$(4^8, 2^{13})$	$(4^2, 3^8, 2^{10}, 1^6)$	$(3^4, 2^{15}, 1^{16})$
62	$\omega_2 + \omega_3$	5	$(5^{10}, 3^4)$	$(5^6, 4^4, 2^8)$	$(5^2, 4^6, 3^4, 2^6, 1^4)$	$(3^{12}, 2^{13})$
63	$2\omega_3$	5	$(5^{11}, 3, 1^5)$	$(5^7, 3^8, 1^4)$	$(5^3, 4^4, 3^6, 2^4, 1^6)$	$(3^{13}, 2^{12})$
64	$\omega_1 + \omega_2$	5	$(5^{10}, 3^4, 1^2)$	$(4^8, 2^{16})$	$(4^2, 3^8, 2^{12}, 1^8)$	$(3^4, 2^{16}, 1^{20})$
		≥ 11	$(7^2, 5^6, 3^6, 1^2)$			
70	$\omega_1 + \omega_3$	5	$(5^{11}, 3^5)$	$(5^3, 3^{15}, 1^{10})$	$(4^4, 3^7, 2^{12}, 1^9)$	$(3^5, 2^{20}, 1^{15})$
		≥ 7	$(7^4, 5^3, 3^9)$			
84	$2\omega_3$	3	$(3^{27}, 1^3)$	$(3^{27}, 1^3)$	$(3^{23}, 2^4, 1^7)$	$(3^{14}, 2^{16}, 1^{10})$
		7	$(7^7, 5^5, 1^{10})$	$(7, 5^5, 3^{15}, 1^7)$	$(5^3, 4^4, 3^9, 2^8, 1^{10})$	
		≥ 11	$(9^3, 7, 5^8, 1^{10})$			
89	$2\omega_2$	7	$(7^5, 5^8, 3^3, 1^5)$	$(5^6, 3^{15}, 1^{14})$	$(5, 4^4, 3^{12}, 2^{12}, 1^8)$	$(3^{10}, 2^{20}, 1^{19})$
90	$2\omega_2$	3	(3^{30})	$(3^{27}, 1^9)$	$(3^{22}, 2^8, 1^8)$	$(3^{10}, 2^{20}, 1^{20})$
		5	$(5^{17}, 1^5)$	$(5^6, 3^{15}, 1^{15})$	$(5, 4^4, 3^{12}, 2^{12}, 1^9)$	
		≥ 11	$(9, 7^3, 5^9, 3^3, 1^6)$			

TABLE 19. $G = C_4(K)$

n	ω	p	$J(\varphi(x))$ for $J(x) = (8)$	$J(\varphi(x))$ for $J(x) = (6, 2)$	$J(\varphi(x))$ for $J(x) = (6, 1^2)$
27	ω_2	7	-	$(7^3, 5, 1)$	$(7^2, 6^2, 1)$
		11	$(11^2, 5)$	$(9, 7, 5^2, 1)$	$(9, 6^2, 5, 1)$
		≥ 13	$(13, 9, 5)$		
36	$2\omega_1$	7	-	$(7^4, 5, 3)$	$(7^3, 6^2, 1^3)$
		11	$(11^3, 3)$	$(11, 7^2, 5, 3^2)$	$(11, 7, 6^2, 3, 1^3)$
		13	$(13^2, 7, 3)$		
		≥ 17	$(15, 11, 7, 3)$		
42	ω_4	7	-	(7^6)	(7^6)
		11	$(11^3, 9)$	$(11, 9^2, 5^2, 3)$	$(10^2, 9, 5, 4^2)$
		13	$(13^2, 11, 5)$		
		≥ 17	$(17, 11, 9, 5)$		
48	ω_3	7	-	$(7^6, 6)$	$(7^6, 6)$
		11	$(11^4, 4)$	$(10^2, 8, 6^2, 4^2)$	$(10, 9^2, 6, 5^2, 4)$
		13	$(13^2, 12, 6, 4)$		
		≥ 17	$(16, 12, 10, 6, 4)$		

Table 19 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(4^2)$	$J(\varphi(x))$ for $J(x)=(4,2^2)$	$J(\varphi(x))$ for $J(x)=(4,2,1^2)$
27	ω_2	5	$(5^5,1^2)$	$(5^3,3^3,1^3)$	$(5^2,4^2,3,2^2,1^2)$
		≥ 7	$(7,5^3,3,1^2)$		
36	$2\omega_1$	5	$(5^7,1)$	$(5^4,3^5,1)$	$(5^3,4^2,3^2,2^2,1^3)$
		≥ 7	$(7^3,5,3^3,1)$	$(7,5^2,3^6,1)$	$(6^2,5^2,4^4,3,1)$
42	ω_4	5	$(5^8,1^2)$	$(5^7,3^2,1)$	$(5^6,4^2,3,1)$
		7	$(7^4,5,3^2,1^3)$	$(7,5,4^2,3^3,2^2,1^3)$	$(7,5^5,3^3,1)$
		≥ 11	$(9,7^2,5^2,3^2,1^3)$		
48	ω_3	5	$(5^8,4^2)$	$(5^6,4^3,2^3)$	$(5^6,4^2,3^2,2,1^2)$
		7	$(7^4,4^4,2^2)$	$(6^3,4^6,2^3)$	$(6,5^4,4^3,3^2,2,1^2)$
		≥ 11	$(8^2,6^2,4^4,2^2)$		

Table 19 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(4,1^4)$	$J(\varphi(x))$ for $J(x)=(3^2,2)$	$J(\varphi(x))$ for $J(x)=(3^2,1^2)$
27	ω_2	3	-	(3^9)	(3^9)
		≥ 5	$(5,4^4,1^6)$	$(5,4^2,3^3,2^2,1)$	$(5,3^7,1)$
36	$2\omega_1$	3	-	(3^{12})	$(3^{11},1^3)$
		5	$(5^2,4^4,1^{10})$	$(5^3,4^2,3^2,2^2,1^3)$	$(5^3,3^5,1^6)$
		≥ 7	$(7,4^4,3,1^{10})$		
40	ω_3	3	-	$(3^{12},2,1^2)$	$(3^{12},1^4)$
41	ω_4	3	-	$(3^{12},2^2,1)$	$(3^{12},1^5)$
42	ω_4	5	$(5^5,4^4,1)$	$(5^5,3^3,2^4)$	$(5^5,3^3,1^8)$
		≥ 7		$(6^2,5,4^2,3^3,2^4)$	
48	ω_3	5	$(5^4,4^6,1^4)$	$(5^4,4^3,3^2,2^3,1^4)$	$(5^4,3^8,1^4)$
		≥ 7		$(6,5^2,4^4,3^2,2^3,1^4)$	

Table 19 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(2^4)$	$J(\varphi(x))$ for $J(x)=(2^3,1^2)$	$J(\varphi(x))$ for $J(x)=(2^2,1^4)$	$J(\varphi(x))$ for $J(x)=(2,1^6)$
27	ω_2	≥ 3	$(3^6,1^9)$	$(3^3,2^6,1^6)$	$(3,2^8,1^8)$	$(2^6,1^{15})$
36	$2\omega_1$	≥ 3	$(3^{10},1^6)$	$(3^6,2^6,1^6)$	$(3^3,2^8,1^{11})$	$(3,2^6,1^{21})$
40	ω_3	3	$(3^8,2^8)$	$(3^8,2^4,1^8)$	$(3^4,2^{10},1^8)$	$(2^{13},1^{14})$
41	ω_4	3	$(3^{11},1^8)$	$(3^7,2^8,1^4)$	$(3^5,2^8,1^{10})$	$(2^{14},1^{13})$
42	ω_4	≥ 5	$(5,3^9,1^{10})$	$(4^2,3^3,2^{10},1^5)$	$(3^5,2^8,1^{11})$	$(2^{14},1^{14})$
48	ω_3	≥ 5	$(4^4,2^{16})$	$(4,3^6,2^8,1^{10})$	$(3^4,2^{12},1^{12})$	$(2^{14},1^{20})$

TABLE 20. $G = C_5(K)$

n	ω	p	$J(\varphi(x))$ for $J(x) = (10)$	$J(\varphi(x))$ for $J(x) = (8, 2)$	$J(\varphi(x))$ for $J(x) = (8, 1^2)$
44	ω_2	11	(11^4)	$(11^2, 9, 7, 5, 1)$	$(11^2, 8^2, 5, 1)$
		13	$(13^3, 5)$	$(13, 9^2, 7, 5, 1)$	$(13, 9, 8^2, 5, 1)$
		≥ 17	$(17, 13, 9, 5)$		
55	$2\omega_1$	11	(11^5)	$(11^3, 9, 7, 3^2)$	$(11^3, 8^2, 3, 1^3)$
		13	$(13^4, 3)$	$(13^2, 9, 7^2, 3^2)$	$(13^2, 8^2, 7, 3, 1^3)$
		17	$(17^2, 11, 7, 3)$	$(15, 11, 9, 7^2, 3^2)$	$(15, 11, 8^2, 7, 3, 1^3)$
		≥ 19	$(19, 15, 11, 7, 3)$		

Table 20 continued

n	ω	p	$J(\varphi(x))$ for $J(x) = (6, 4)$	$J(\varphi(x))$ for $J(x) = (6, 2^2)$	$J(\varphi(x))$ for $J(x) = (6, 2, 1^2)$
44	ω_2	7	$(7^5, 5, 3, 1)$	$(7^4, 5^2, 3, 1^3)$	$(7^3, 6^2, 5, 2^2, 1^2)$
		≥ 11	$(9^2, 7, 5^3, 3, 1)$	$(9, 7^2, 5^3, 3, 1^3)$	$(9, 7, 6^2, 5^2, 2^2, 1^2)$
55	$2\omega_1$	7	$(7^7, 3^2)$	$(7^5, 5^2, 3^3, 1)$	$(7^4, 6^2, 5, 3, 2^2, 1^3)$
		≥ 11	$(11, 9, 7^3, 5, 3^3)$	$(11, 7^3, 5^2, 3^4, 1)$	$(11, 7^2, 6^2, 5, 3^2, 2^2, 1^3)$

Table 20 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(6,1^4)$	$J(\varphi(x))$ for $J(x)=(5^2)$	$J(\varphi(x))$ for $J(x)=(4^2,2)$	$J(\varphi(x))$ for $J(x)=(4^2,1^2)$	$J(\varphi(x))$ for $J(x)=(4,3^2)$
43	ω_2	5	-	$(5^8,3)$	$(5^7,3^2,1^2)$	$(5^5,4^4,1^2)$	$(5^6,3^3,2^2)$
44	ω_2	7	$(7^2,6^4,1^6)$	$(7^5,3^3)$	$(7,5^5,3^3,1^3)$	$(7,5^3,4^4,3,1^3)$	$(6^2,5^2,4^2,3^3,2^2,1)$
		≥ 11	$(9,6^4,5,1^6)$	$(9,7^3,5,3^3)$			
55	$2\omega_1$	5	-	(5^{11})	$(5^9,3^3,1)$	$(5^7,4^4,1^4)$	$(5^9,3,2^2,1^3)$
		7	$(7^3,6^4,1^{10})$	$(7^7,3,1^3)$	$(7^3,5^3,3^6,1)$	$(7^3,5,4^4,3^3,1^4)$	$(7,6^2,5^3,4^2,3^2,2^2,1^3)$
		≥ 11	$(11,7,6^4,3,1^{10})$	$(9^3,7,5^3,3,1^3)$			

Table 20 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(4,2^3)$	$J(\varphi(x))$ for $J(x)=(4,2^2,1^2)$	$J(\varphi(x))$ for $J(x)=(4,2,1^4)$	$J(\varphi(x))$ for $J(x)=(4,1^6)$	$J(\varphi(x))$ for $J(x)=(3^2,2^2)$	$J(\varphi(x))$ for $J(x)=(3^2,2,1^2)$
43	ω_2	5	$(5^4,3^6,1^5)$	$(5^3,4^2,3^3,2^4,1^3)$	$(5^2,4^4,3,2^4,1^6)$	$(5,4^6,1^{14})$	$(5,4^4,3^4,2^4,1^2)$	$(5,4^2,3^7,2^4,1)$
44	ω_2	3	-	-	-	-	$(3^{14},1^2)$	$(3^{13},2^2,1)$
		≥ 7	$(5^4,3^6,1^6)$	$(5^3,4^2,3^3,2^4,1^4)$	$(5^2,4^4,3,2^4,1^7)$	$(5,4^6,1^{15})$	$(5,4^4,3^4,2^4,1^3)$	$(5,4^2,3^7,2^4,1^2)$
55	$2\omega_1$	3	-	-	-	-	$(3^{18},1)$	$(3^{16},2^2,1^3)$
		5	$(5^5,3^9,1^3)$	$(5^4,4^2,3^5,2^4,1^4)$	$(5^3,4^4,3^2,2^4,1^{10})$	$(5^2,4^6,1^{21})$	$(5^3,4^4,3^4,2^4,1^4)$	$(5^3,4^2,3^6,2^4,1^6)$
		≥ 7	$(7,5^3,3^{10},1^3)$	$(7,5^2,4^2,3^6,2^4,1^4)$	$(7,5,4^4,3^3,2^4,1^{10})$	$(7,4^6,3,1^{21})$		

Table 20 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(3^2,1^4)$	$J(\varphi(x))$ for $J(x)=(2^5)$	$J(\varphi(x))$ for $J(x)=(2^4,1^2)$	$J(\varphi(x))$ for $J(x)=(2^3,1^4)$	$J(\varphi(x))$ for $J(x)=(2^2,1^6)$	$J(\varphi(x))$ for $J(x)=(2,1^8)$
43	ω_2	5	$(5,3^{11},1^5)$	$(3^{10},1^{13})$	$(3^6,2^8,1^9)$	$(3^3,2^{12},1^{10})$	$(3,2^{12},1^{16})$	$(2^8,1^{27})$
44	ω_2	3	$(3^{13},1^5)$	$(3^{10},1^{14})$	$(3^6,2^8,1^{10})$	$(3^3,2^{12},1^{11})$	$(3,2^{12},1^{17})$	$(2^8,1^{28})$
		≥ 7	$(5,3^{11},1^6)$					
55	$2\omega_1$	3	$(3^{15},1^{10})$	$(3^{15},1^{10})$	$(3^{10},2^8,1^9)$	$(3^6,2^{12},1^{13})$	$(3^3,2^{12},1^{22})$	$(3,2^8,1^{36})$
		≥ 5	$(5^3,3^9,1^{13})$					

TABLE 21. $G = C_6(K)$

n	ω	p	$J(\varphi(x))$ for $J(x) = (12)$	$J(\varphi(x))$ for $J(x) = (10,2)$	$J(\varphi(x))$ for $J(x) = (10,1^2)$	$J(\varphi(x))$ for $J(x) = (8,4)$	$J(\varphi(x))$ for $J(x) = (8,2^2)$
65	ω_2	11	-	$(11^5,9,1)$	$(11^4,10^2,1)$	$(11^3,9,7,5^3,1)$	$(11^2,9^2,7^2,5,3,1^3)$
		13	(13^5)	$(13^3,11,9,5,1)$	$(13^3,10^2,5,1)$	$(13,11,9^2,7,5^3,1)$	$(13,9^3,7^2,5,3,1^3)$
		17	$(17^3,9,5)$	$(17,13,11,9^2,5,1)$	$(17,13,10^2,9,5,1)$		
		19	$(19^2,13,9,5)$				
		≥ 23	$(21,17,13,9,5)$				
78	$2\omega_1$	11	-	$(11^6,9,3)$	$(11^5,10^2,1^3)$	$(11^4,9,7^2,5,3^2)$	$(11^3,9^2,7^2,3^4,1)$
		13	(13^6)	$(13^4,11,9,3^2)$	$(13^4,10^2,3,1^3)$	$(13^2,11,9,7^3,5,3^2)$	$(13^2,9^2,7^3,3^4,1)$
		17	$(17^4,7,3)$	$(17^2,11^2,9,7,3^2)$	$(17^2,11,10^2,7,3,1^3)$	$(15,11^2,9,7^3,5,3^2)$	$(15,11,9^2,7^3,3^4,1)$
		19	$(19^3,11,7,3)$	$(19,15,11^2,9,7,3^2)$	$(19,15,11,10^2,7,3,1^3)$		
		≥ 23	$(23,19,15,11,7,3)$				

Table 21 continued

n	ω	p	$J(\varphi(x))$ for $J(x) = (8,2,1^2)$	$J(\varphi(x))$ for $J(x) = (8,1^4)$	$J(\varphi(x))$ for $J(x) = (6^2)$	$J(\varphi(x))$ for $J(x) = (6,4,2)$	$J(\varphi(x))$ for $J(x) = (6,4,1^2)$
65	ω_2	7	-	-	$(7^9,1^2)$	$(7^6,5^3,3^2,1^2)$	$(7^5,6^2,5,4^2,3,1^2)$
		11	$(11^2,9,8^2,7,5,2^2,1^2)$	$(11^2,8^4,5,1^6)$	$(11,9^3,7,5^3,3,1^2)$	$(9^2,7^2,5^5,3^2,1^2)$	$(9^2,7,6^2,5^3,4^2,3,1^2)$
		≥ 13	$(13,9^2,8^2,7,5,2^2,1^2)$	$(13,9,8^4,5,1^6)$			
78	$2\omega_1$	7	-	-	$(7^{11},1)$	$(7^8,5^2,3^4)$	$(7^7,6^2,4^2,3^2,1^3)$
		11	$(11^3,9,8^2,7,3^2,2^2,1^3)$	$(11^3,8^4,3,1^{10})$	$(11^3,9,7^3,5,3^3,1)$	$(11,9,7^4,5^3,3^5)$	$(11,9,7^3,6^2,5,4^2,3^3,1^3)$
		13	$(13^2,9,8^2,7^2,3^2,2^2,1^3)$	$(13^2,8^4,7,3,1^{10})$			
		≥ 17	$(15,11,9,8^2,7^2,3^2,2^2,1^3)$	$(15,11,8^4,7,3,1^{10})$			

Table 21 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(6,3^2)$	$J(\varphi(x))$ for $J(x)=(6,2^3)$	$J(\varphi(x))$ for $J(x)=(6,2^2,1^2)$	$J(\varphi(x))$ for $J(x)=(6,2,1^4)$	$J(\varphi(x))$ for $J(x)=(6,1^6)$
65	ω_2	7	$(7^6,5,4^2,3^3,1)$	$(7^5,5^3,3^3,1^6)$	$(7^4,6^2,5^2,3,2^4,1^4)$	$(7^3,6^4,5,2^4,1^7)$	$(7^2,6^6,1^{15})$
		≥ 11	$(9,8^2,6^2,5^2,4^2,3^3,1)$	$(9,7^3,5^4,3^3,1^6)$	$(9,7^2,6^2,5^3,3,2^4,1^4)$	$(9,7,6^4,5^2,2^4,1^7)$	$(9,6^6,5,1^{15})$
78	$2\omega_1$	7	$(7^7,5^3,4^2,3,1^3)$	$(7^6,5^3,3^6,1^3)$	$(7^5,6^2,5^2,3^3,2^4,1^4)$	$(7^4,6^4,5,3,2^4,1^{10})$	$(7^3,6^6,1^{21})$
		≥ 11	$(11,8^2,7,6^2,5^3,4^2,3^2,1^3)$	$(11,7^4,5^3,3^7,1^3)$	$(11,7^3,6^2,5^2,3^4,2^4,1^4)$	$(11,7^2,6^4,5,3^2,2^4,1^{10})$	$(11,7,6^6,3,1^{21})$

n	ω	p	$J(\varphi(x))$ for $J(x)=(5^2,2)$	$J(\varphi(x))$ for $J(x)=(5^2,1^2)$	$J(\varphi(x))$ for $J(x)=(4^3)$	$J(\varphi(x))$ for $J(x)=(4^2,2^2)$	$J(\varphi(x))$ for $J(x)=(4^2,2,1^2)$	$J(\varphi(x))$ for $J(x)=(4^2,1^4)$
65	ω_2	5	(5^{13})	(5^{13})	$(5^{12},1^5)$	$(5^9,3^5,1^5)$	$(5^7,4^4,3^2,2^2,1^4)$	$(5^5,4^8,1^8)$
		7	$(7^5,6^2,4^2,3^2,1)$	$(7^5,5^4,3^3,1)$	$(7^3,5^6,3^3,1^5)$	$(7,5^7,3^6,1^5)$	$(7,5^5,4^4,3^3,2^2,1^4)$	$(7,5^3,4^8,3,1^8)$
		≥ 11	$(9,7^3,6^2,5,4^2,3^3,1)$	$(9,7^3,5^5,3^3,1)$				
78	$2\omega_1$	5	$(5^{15},3)$	$(5^{15},1^3)$	$(5^{15},1^3)$	$(5^{11},3^7,1^2)$	$(5^9,4^4,3^3,2^2,1^4)$	$(5^7,4^8,1^{11})$
		7	$(7^7,6^2,4^2,3^2,1^3)$	$(7^7,5^4,3,1^6)$	$(7^6,5^3,3^6,1^3)$	$(7^3,5^5,3^{10},1^2)$	$(7^3,5^3,4^4,3^6,2^2,1^4)$	$(7^3,5,4^8,3^3,1^{11})$
		≥ 11	$(9^3,7,6^2,5^3,4^2,3^2,1^3)$	$(9^3,7,5^7,3,1^6)$				

Table 21 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(4,3^2,2)$	$J(\varphi(x))$ for $J(x)=(4,3^2,1^2)$	$J(\varphi(x))$ for $J(x)=(4,2^4)$	$J(\varphi(x))$ for $J(x)=(4,2^3,1^2)$	$J(\varphi(x))$ for $J(x)=(4,2^2,1^4)$	$J(\varphi(x))$ for $J(x)=(4,2,1^6)$
65	ω_2	5	$(5^7,4^2,3^4,2^4,1^2)$	$(5^6,4^2,3^7,2^2,1^2)$	$(5^5,3^{10},1^{10})$	$(5^4,4^2,3^6,2^6,1^7)$	$(5^3,4^4,3^3,2^8,1^9)$	$(5^2,4^6,3,2^6,1^{16})$
		≥ 7	$(6^2,5^3,4^4,3^4,2^4,1^2)$	$(6^2,5^2,4^4,3^7,2^2,1^2)$				
78	$2\omega_1$	5	$(5^{10},4^2,3^3,2^4,1^3)$	$(5^9,4^2,3^5,2^2,1^6)$	$(5^6,3^{14},1^6)$	$(5^5,4^2,3^9,2^6,1^6)$	$(5^4,4^4,3^5,2^8,1^{11})$	$(5^3,4^6,3^2,2^6,1^{21})$
		≥ 7	$(7,6^2,5^4,4^4,3^4,2^4,1^3)$	$(7,6^2,5^3,4^4,3^6,2^2,1^6)$	$(7,5^4,3^{15},1^6)$	$(7,5^3,4^2,3^{10},2^6,1^6)$	$(7,5^2,4^4,3^6,2^8,1^{11})$	$(7,5,4^6,3^3,2^6,1^{21})$

Table 21 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(4,1^8)$	$J(\varphi(x))$ for $J(x)=(3^4)$	$J(\varphi(x))$ for $J(x)=(3^2,2^3)$	$J(\varphi(x))$ for $J(x)=(3^2,2^2,1^2)$	$J(\varphi(x))$ for $J(x)=(3^2,2,1^4)$	$J(\varphi(x))$ for $J(x)=(3^2,1^6)$
64	ω_2	3	-	$(3^{21},1)$	$(3^{20},1^4)$	$(3^{18},2^4,1^2)$	$(3^{17},2^4,1^5)$	$(3^{17},1^{13})$
65	ω_2	≥ 5	$(5,4^8,1^{28})$	$(5^6,3^{10},1^5)$	$(5,4^6,3^6,2^6,1^6)$	$(5,4^4,3^8,2^8,1^4)$	$(5,4^2,3^{11},2^6,1^7)$	$(5,3^{15},1^{15})$
78	$2\omega_1$	3	-	(3^{26})	$(3^{25},1^3)$	$(3^{22},2^4,1^4)$	$(3^{20},2^4,1^{10})$	$(3^{19},1^{21})$
		5	$(5^2,4^8,1^{36})$	$(5^{10},3^6,1^{10})$	$(5^3,4^6,3^7,2^6,1^6)$	$(5^3,4^4,3^8,2^8,1^7)$	$(5^3,4^2,3^{10},2^6,1^{13})$	$(5^3,3^{13},1^{24})$
		≥ 7	$(7,4^8,3,1^{36})$					

Table 21 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(2^6)$	$J(\varphi(x))$ for $J(x)=(2^5,1^2)$	$J(\varphi(x))$ for $J(x)=(2^4,1^4)$	$J(\varphi(x))$ for $J(x)=(2^3,1^6)$	$J(\varphi(x))$ for $J(x)=(2^2,1^8)$	$J(\varphi(x))$ for $J(x)=(2,1^{10})$
64	ω_2	3	$(3^{15},1^{19})$	$(3^{10},2^{10},1^{14})$	$(3^6,2^{16},1^{14})$	$(3^3,2^{18},1^{19})$	$(3,2^{16},1^{29})$	$(2^{10},1^{44})$
65	ω_2	≥ 5	$(3^{15},1^{20})$	$(3^{10},2^{10},1^{15})$	$(3^6,2^{16},1^{15})$	$(3^3,2^{18},1^{20})$	$(3,2^{16},1^{30})$	$(2^{10},1^{45})$
78	$2\omega_1$	≥ 3	$(3^{21},1^{15})$	$(3^{15},2^{10},1^{13})$	$(3^{10},2^{16},1^{16})$	$(3^6,2^{18},1^{24})$	$(3^3,2^{16},1^{37})$	$(3,2^{10},1^{55})$

TABLE 22. $G=C_7(K)$

n	ω	p	$J(\varphi(x))$ for $J(x)=(14)$	$J(\varphi(x))$ for $J(x)=(12,2)$	$J(\varphi(x))$ for $J(x)=(12,1^2)$	$J(\varphi(x))$ for $J(x)=(10,4)$	$J(\varphi(x))$ for $J(x)=(10,2^2)$
90	ω_2	11	-	-	-	$(11^7,7,5,1)$	$(11^6,9^2,3,1^3)$
		13	-	$(13^6,11,1)$	$(13^5,12^2,1)$	$(13^4,11,9,7,5^2,1)$	$(13^3,11^2,9^2,5,3,1^3)$
		17	$(17^5,5)$	$(17^3,13,11,9,5,1)$	$(17^3,12^2,9,5,1)$	$(17,13^2,11,9^2,7,5^2,1)$	$(17,13,11^2,9^3,5,3,1^3)$
		19	$(19^4,9,5)$	$(19^2,13^2,11,9,5,1)$	$(19^2,13,12^2,9,5,1)$		
		23	$(23^2,17,13,9,5)$	$(21,17,13^2,11,9,5,1)$	$(21,17,13,12^2,9,5,1)$		
		≥ 29	$(25,21,17,13,9,5)$				

Table 22 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(10,2,1^2)$	$J(\varphi(x))$ for $J(x)=(10,1^4)$	$J(\varphi(x))$ for $J(x)=(8,6)$	$J(\varphi(x))$ for $J(x)=(8,4,2)$	$J(\varphi(x))$ for $J(x)=(8,4,1^2)$
90	ω_2	11	$(11^5,10^2,9,2^2,1^2)$	$(11^4,10^4,1^6)$	$(11^5,9,7,5^3,3,1)$	$(11^3,9^2,7^2,5^4,3,1^2)$	$(11^3,9,8^2,7,5^3,4^2,1^2)$
		13	$(13^3,11,10^2,9,5,2^2,1^2)$	$(13^3,10^4,5,1^6)$	$(13^2,11,9^3,7,5^3,3,1)$	$(13,11,9^3,7^2,5^4,3,1^2)$	$(13,11,9^2,8^2,7,5^3,4^2,1^2)$
		≥ 17	$(17,13,11,10^2,9^2,5,2^2,1^2)$	$(17,13,10^4,9,5,1^6)$			

Table 22 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(8,3^2)$	$J(\varphi(x))$ for $J(x)=(8,2^3)$	$J(\varphi(x))$ for $J(x)=(8,2^2,1^2)$	$J(\varphi(x))$ for $J(x)=(8,2,1^4)$	$J(\varphi(x))$ for $J(x)=(8,1^6)$
90	ω_2	11	$(11^2,10^2,8^2,6^2,5^2,3^3,1)$	$(11^2,9^3,7^3,5,3^3,1^6)$	$(11^2,9^2,8^2,7^2,5,3,2^4,1^4)$	$(11^2,9,8^4,7,5,2^4,1^7)$	$(11^2,8^6,5,1^{15})$
		≥ 13	$(13,10^2,9,8^2,6^2,5^2,3^3,1)$	$(13,9^4,7^3,5,3^3,1^6)$	$(13,9^3,8^2,7^2,5,3,2^4,1^4)$	$(13,9^2,8^4,7,5,2^4,1^7)$	$(13,9,8^6,5,1^{15})$

Table 22 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(7^2)$	$J(\varphi(x))$ for $J(x)=(6^2,2)$	$J(\varphi(x))$ for $J(x)=(6^2,1^2)$	$J(\varphi(x))$ for $J(x)=(6,4^2)$	$J(\varphi(x))$ for $J(x)=(6,4,2^2)$
89	ω_2	7	$(7^{12},5)$	$(7^{11},5^2,1^2)$	$(7^9,6^4,1^2)$	$(7^9,5^3,3^3,1^2)$	$(7^7,5^5,3^4,1^3)$
90	ω_2	11	$(11^5,7^3,5,3^3)$	$(11,9^3,7^3,5^5,3,1^3)$	$(11,9^3,7,6^4,5^3,3,1^3)$	$(9^3,7^3,5^6,3^3,1^3)$	$(9^2,7^3,5^7,3^4,1^4)$
		≥ 13	$(13,11^3,9,7^3,5,3^3)$				

Table 22 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(6,4,2,1^2)$	$J(\varphi(x))$ for $J(x)=(6,4,1^4)$	$J(\varphi(x))$ for $J(x)=(6,3^2,2)$	$J(\varphi(x))$ for $J(x)=(6,3^2,1^2)$	$J(\varphi(x))$ for $J(x)=(6,2^4)$
89	ω_2	7	$(7^6,6^2,5^3,4^2,3^2,2^2,1^2)$	$(7^5,6^4,5,4^4,3,1^6)$	$(7^7,5^2,4^4,3^3,2^2,1)$	$(7^6,6^2,5,4^2,3^7,1)$	$(7^6,5^4,3^6,1^9)$
90	ω_2	≥ 11	$(9^2,7^2,6^2,5^5,4^2,3^2,2^2,1^3)$	$(9^2,7,6^4,5^3,4^4,3,1^7)$	$(9,8^2,7,6^2,5^3,4^4,3^3,2^2,1^2)$	$(9,8^2,6^4,5^2,4^2,3^7,1^2)$	$(9,7^4,5^5,3^6,1^{10})$

Table 22 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(6,2^3,1^2)$	$J(\varphi(x))$ for $J(x)=(6,2^2,1^4)$	$J(\varphi(x))$ for $J(x)=(6,2,1^6)$	$J(\varphi(x))$ for $J(x)=(6,1^8)$
89	ω_2	7	$(7^5,6^2,5^3,3^3,2^6,1^6)$	$(7^4,6^4,5^2,3,2^8,1^8)$	$(7^3,6^6,5,2^6,1^{15})$	$(7^2,6^8,1^{27})$
90	ω_2	≥ 11	$(9,7^3,6^2,5^4,3^3,2^6,1^7)$	$(9,7^2,6^4,5^3,3,2^8,1^9)$	$(9,7,6^6,5^2,2^6,1^{16})$	$(9,6^8,5,1^{28})$

Table 22 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(5^2,4)$	$J(\varphi(x))$ for $J(x)=(5^2,2^2)$	$J(\varphi(x))$ for $J(x)=(5^2,2,1^2)$	$J(\varphi(x))$ for $J(x)=(5^2,1^4)$	$J(\varphi(x))$ for $J(x)=(4^3,2)$
89	ω_2	7	$(7^9,5,4^2,3^3,2^2)$	$(7^5,6^4,4^4,3^4,1^2)$	$(7^5,6^2,5^4,4^2,3^3,2^2,1)$	$(7^5,5^8,3^3,1^5)$	$(7^3,5^9,3^6,1^5)$
90	ω_2	5	(5^{18})	$(5^{17},3,1^2)$	$(5^{17},2^2,1)$	$(5^{17},1^5)$	$(5^{15},3^3,1^6)$
		≥ 11	$(9,8^2,7^3,6^2,5^2,4^2,3^3,2^2,1)$	$(9,7^3,6^4,5,4^4,3^4,1^3)$	$(9,7^3,6^2,5^5,4^2,3^3,2^2,1^2)$	$(9,7^3,5^9,3^3,1^6)$	$(7^3,5^9,3^6,1^6)$

Table 22 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(4^3,1^2)$	$J(\varphi(x))$ for $J(x)=(4^2,3^2)$	$J(\varphi(x))$ for $J(x)=(4^2,2^3)$	$J(\varphi(x))$ for $J(x)=(4^2,2^2,1^2)$	$J(\varphi(x))$ for $J(x)=(4^2,2,1^4)$	$J(\varphi(x))$ for $J(x)=(4^2,1^6)$
89	ω_2	7	$(7^3,5^6,4^6,3^3,1^5)$	$(7,6^4,5^4,4^4,3^4,2^4,1^2)$	$(7,5^9,3^{10},1^7)$	$(7,5^7,4^4,3^6,2^4,1^5)$	$(7,5^5,4^8,3^3,2^4,1^8)$	$(7,5^3,4^{12},3,1^{16})$
90	ω_2	5	$(5^{12},4^6,1^6)$	$(5^{14},3^3,2^4,1^3)$	$(5^{11},3^9,1^8)$	$(5^9,4^4,3^5,2^4,1^6)$	$(5^7,4^8,3^2,2^4,1^9)$	$(5^5,4^{12},1^{17})$
		≥ 11	$(7^3,5^6,4^6,3^3,1^6)$	$(7,6^4,5^4,4^4,3^4,2^4,1^3)$	$(7,5^9,3^{10},1^8)$	$(7,5^7,4^4,3^6,2^4,1^6)$	$(7,5^5,4^8,3^3,2^4,1^9)$	$(7,5^3,4^{12},3,1^{17})$

Table 22 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(4,3^2,2^2)$	$J(\varphi(x))$ for $J(x)=(4,3^2,2,1^2)$	$J(\varphi(x))$ for $J(x)=(4,3^2,1^4)$	$J(\varphi(x))$ for $J(x)=(4,2^5)$	$J(\varphi(x))$ for $J(x)=(4,2^4,1^2)$
89	ω_2	7	$(6^2,5^4,4^6,3^6,2^6,1^3)$	$(6^2,5^3,4^6,3^8,2^6,1^2)$	$(6^2,5^2,4^6,3^{11},2^2,1^6)$	$(5^6,3^{15},1^{14})$	$(5^5,4^2,3^{10},2^8,1^{10})$
90	ω_2	5	$(5^8,4^4,3^6,2^6,1^4)$	$(5^7,4^4,3^8,2^6,1^3)$	$(5^6,4^4,3^{11},2^2,1^7)$	$(5^6,3^{15},1^{15})$	$(5^5,4^2,3^{10},2^8,1^{11})$
		≥ 11	$(6^2,5^4,4^6,3^6,2^6,1^4)$	$(6^2,5^3,4^6,3^8,2^6,1^3)$	$(6^2,5^2,4^6,3^{11},2^2,1^7)$		

Table 22 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(4,2^3,1^4)$	$J(\varphi(x))$ for $J(x)=(4,2^2,1^6)$	$J(\varphi(x))$ for $J(x)=(4,2,1^8)$	$J(\varphi(x))$ for $J(x)=(4,1^{10})$	$J(\varphi(x))$ for $J(x)=(3^4,2)$	$J(\varphi(x))$ for $J(x)=(3^4,1^2)$
89	ω_2	7	$(5^4,4^4,3^6,2^{12},1^{11})$	$(5^3,4^6,3^3,2^{12},1^{17})$	$(5^2,4^8,3,2^8,1^{28})$	$(5,4^{10},1^{44})$	$(5^6,4^4,3^{10},2^4,1^5)$	$(5^6,3^{18},1^5)$
90	ω_2	3	-	-	-	-	(3^{30})	(3^{30})
		≥ 5	$(5^4,4^4,3^6,2^{12},1^{12})$	$(5^3,4^6,3^3,2^{12},1^{18})$	$(5^2,4^8,3,2^8,1^{29})$	$(5,4^{10},1^{45})$	$(5^6,4^4,3^{10},2^4,1^6)$	$(5^6,3^{18},1^6)$

Table 22 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(3^2,2^4)$	$J(\varphi(x))$ for $J(x)=(3^2,2^3,1^2)$	$J(\varphi(x))$ for $J(x)=(3^2,2^2,1^4)$	$J(\varphi(x))$ for $J(x)=(3^2,2,1^6)$	$J(\varphi(x))$ for $J(x)=(3^2,1^8)$
89	ω_2	7	$(5,4^8,3^9,2^8,1^9)$	$(5,4^6,3^{10},2^{12},1^6)$	$(5,4^4,3^{12},2^{12},1^8)$	$(5,4^2,3^{15},2^8,1^{15})$	$(5,3^{19},1^{27})$
90	ω_2	3	$(3^{27},1^9)$	$(3^{24},2^6,1^6)$	$(3^{22},2^8,1^8)$	$(3^{21},2^6,1^{15})$	$(3^{21},1^{27})$
		≥ 5	$(5,4^8,3^9,2^8,1^{10})$	$(5,4^6,3^{10},2^{12},1^7)$	$(5,4^4,3^{12},2^{12},1^9)$	$(5,4^2,3^{15},2^8,1^{16})$	$(5,3^{19},1^{28})$

Table 22 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(2^7)$	$J(\varphi(x))$ for $J(x)=(2^6,1^2)$	$J(\varphi(x))$ for $J(x)=(2^5,1^4)$	$J(\varphi(x))$ for $J(x)=(2^4,1^6)$	$J(\varphi(x))$ for $J(x)=(2^3,1^8)$	$J(\varphi(x))$ for $J(x)=(2^2,1^{10})$	$J(\varphi(x))$ for $J(x)=(2,1^{12})$
89	ω_2	7	$(3^{21},1^{26})$	$(3^{15},2^{12},1^{20})$	$(3^{10},2^{20},1^{19})$	$(3^6,2^{24},1^{23})$	$(3^3,2^{24},1^{32})$	$(3,2^{20},1^{46})$	$(2^{12},1^{65})$
90	ω_2	$\neq 7$	$(3^{21},1^{27})$	$(3^{15},2^{12},1^{21})$	$(3^{10},2^{20},1^{20})$	$(3^6,2^{24},1^{24})$	$(3^3,2^{24},1^{33})$	$(3,2^{20},1^{47})$	$(2^{12},1^{66})$

In Table 23 and 25 we have $H_1 = G(1, 2, ..., n-1)$ and $H_2 = G(1, 2, ..., n-2, n)$.

TABLE 23. $G = D_4(K)$

n	ω	p	$J(\varphi(x))$ for $J(x) = (7,1)$	$J(\varphi(x))$ for $J(x) = (5,3)$	$J(\varphi(x))$ for $J(x) = (5,1^3)$	$J(\varphi(x))$ for $J(x) = (4^2)$, $x \in H_1$	$J(\varphi(x))$ for $J(x) = (4^2)$, $x \in H_2$	$J(\varphi(x))$ for $J(x) = (3^2,1^2)$
8	ω_4	3	-	-	-	-	-	$(3^2,1^2)$
		5	-	$(5,3)$	(4^2)	$(5,1^3)$	(4^2)	
		≥ 7	$(7,1)$					
28	ω_2	3	-	-	-	-	-	$(3^9,1)$
		5	-	$(5^5,3)$	$(5^5,1^3)$	$(5^5,1^3)$	$(5^5,1^3)$	$(5,3^7,1^2)$
		7	(7^4)	$(7^2,5,3^3)$	$(7,5^3,3,1^3)$	$(7,5^3,3,1^3)$	$(7,5^3,3,1^3)$	
		≥ 11	$(11,7^2,3)$					
35	$2\omega_4$	3	-	-	-	-	-	$(3^{11},1^2)$
		5	-	(5^7)	(5^7)	$(5^6,1^5)$	(5^7)	$(5^3,3^5,1^5)$
		7	(7^5)	$(7^3,5^2,3,1)$	$(7^3,5,3^3)$	$(7^2,5^3,1^6)$	$(7^3,5,3^3)$	
		11	$(11^2,7,5,1)$	$(9,7,5^3,3,1)$		$(9,5^4,1^6)$		
		≥ 13	$(13,9,7,5,1)$					
56	$\omega_1+\omega_4$	3	-	-	-	-	-	$(3^{18},1^2)$
		5	-	$(5^{11},1)$	$(5^8,4^4)$	$(5^8,4^4)$	$(5^{11},1)$	$(5^4,3^{10},1^6)$
		7	(7^8)	$(7^5,5^2,3^3,1^2)$	$(7^4,4^6,2^2)$	$(7^4,4^6,2^2)$	$(7^4,5^3,3^4,1)$	
		11	$(11^3,7^2,5,3,1)$	$(9,7^3,5^3,3^3,1^2)$	$(8^2,6^2,4^6,2^2)$	$(8^2,6^2,4^6,2^2)$		
		≥ 13	$(13,11,9,7^2,5,3,1)$					
56	$\omega_3+\omega_4$	3	-	-	-	-	-	$(3^{18},1^2)$
		5	-	$(5^{11},1)$	$(5^{11},1)$	$(5^8,4^4)$	$(5^8,4^4)$	$(5^4,3^{10},1^6)$
		7	(7^8)	$(7^5,5^2,3^3,1^2)$	$(7^4,5^3,3^4,1)$	$(7^4,4^6,2^2)$	$(7^4,4^6,2^2)$	
		11	$(11^3,7^2,5,3,1)$	$(9,7^3,5^3,3^3,1^2)$		$(8^2,6^2,4^6,2^2)$	$(8^2,6^2,4^6,2^2)$	
		≥ 13	$(13,11,9,7^2,5,3,1)$					

Table 23 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(3,2^2,1)$	$J(\varphi(x))$ for $J(x)=(3,1^5)$	$J(\varphi(x))$ for $J(x)=(2^4)$, $x\in H_1$	$J(\varphi(x))$ for $J(x)=(2^4)$, $x\in H_2$	$J(\varphi(x))$ for $J(x)=(2^2,1^4)$
8	ω_4	≥ 3	$(3,2^2,1)$	(2^4)	$(3,1^5)$	(2^4)	$(2^2,1^4)$
28	ω_2	3	$(3^7,2^2,1^3)$	$(3^6,1^{10})$	$(3^6,1^{10})$	$(3^6,1^{10})$	$(3,2^8,1^9)$
		≥ 5	$(4^2,3^3,2^4,1^3)$				
35	$2\omega_4$	3	$(3^{10},2^2,1)$	$(3^{10},1^5)$	$(3^7,1^{14})$	$(3^{10},1^5)$	$(3^3,2^8,1^{10})$
		≥ 5	$(5,4^2,3^4,2^4,1^2)$		$(5,3^5,1^{15})$		
56	$\omega_1+\omega_4$	3	$(3^{16},2^2,1^4)$	$(3^8,2^{16})$	$(3^8,2^{16})$	$(3^{15},1^{11})$	$(3^4,2^{14},1^{16})$
		≥ 7	$(5,4^4,3^6,2^6,1^5)$	$(4^4,2^{20})$	$(4^4,2^{20})$		
56	$\omega_3+\omega_4$	3	$(3^{16},2^2,1^4)$	$(3^{15},1^{11})$	$(3^8,2^{16})$	$(3^8,2^{16})$	$(3^4,2^{14},1^{16})$
		≥ 7	$(5,4^4,3^6,2^6,1^5)$		$(4^4,2^{20})$	$(4^4,2^{20})$	

TABLE 24. $G=D_5(K)$

n	ω	p	$J(\varphi(x))$ for $J(x)=(9,1)$	$J(\varphi(x))$ for $J(x)=(7,3)$	$J(\varphi(x))$ for $J(x)=(7,1^3)$	$J(\varphi(x))$ for $J(x)=(5^2)$	$J(\varphi(x))$ for $J(x)=(5,3,1^2)$	$J(\varphi(x))$ for $J(x)=(5,1^5)$
16	ω_5	5	-	-	-	$(5^3,1)$	$(5^2,3^2)$	(4^4)
		7	-	$(7^2,2)$	$(7^2,1^2)$	$(7,5,3,1)$		
		≥ 11	$(11,5)$	$(8,6,2)$				
45	ω_2	5	-	-	-	(5^9)	$(5^7,3^3,1)$	$(5^7,1^{10})$
		7	-	$(7^6,3)$	$(7^6,1^3)$	$(7^5,3^3,1)$	$(7^2,5^3,3^5,1)$	$(7,5^5,3,1^{10})$
		11	$(11^3,9,3)$	$(11,9,7^2,5,3^2)$	$(11,7^4,3,1^3)$	$(9,7^3,5,3^3,1)$		
		13	$(13^2,9,7,3)$					
		≥ 17	$(15,11,9,7,3)$					
53	$2\omega_1$	5	-	-	-	$(5^{10},3)$	$(5^9,3^2,1^2)$	$(5^8,1^{13})$
54	$2\omega_1$	7	-	$(7^7,5)$	$(7^7,1^5)$	$(7^7,3,1^2)$	$(7^3,5^4,3^3,1^4)$	$(7^2,5^5,1^{15})$
		11	$(11^4,9,1)$	$(11^2,9,7,5^3,1)$	$(11^2,7^3,5,1^6)$	$(9^3,7,5^3,3,1^2)$	$(9,7,5^5,3^3,1^4)$	$(9,5^6,1^{15})$
		13	$(13^3,9,5,1)$	$(13,9^2,7,5^3,1)$	$(13,9,7^3,5,1^6)$			
		≥ 17	$(17,13,9^2,5,1)$					

Table 24 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(5,2^2,1)$	$J(\varphi(x))$ for $J(x)=(4^2,1^2)$	$J(\varphi(x))$ for $J(x)=(3^3,1)$	$J(\varphi(x))$ for $J(x)=(3^2,2^2)$	$J(\varphi(x))$ for $J(x)=(3^2,1^4)$
16	ω_5	3	-	-	$(3^4,2^2)$	$(3^4,2,1^2)$	$(3^4,1^4)$
		≥ 5	$(5,4^2,3)$	$(5,4^2,1^3)$	$(4^2,2^4)$	$(4,3^2,2^2,1^2)$	
45	ω_2	3	-	-	(3^{15})	$(3^{14},1^3)$	$(3^{13},1^6)$
		5	$(5^7,3,2^2,1^3)$	$(5^5,4^4,1^4)$	$(5^3,3^9,1^3)$	$(5,4^4,3^4,2^4,1^4)$	$(5,3^{11},1^7)$
		≥ 7	$(7,6^2,5,4^2,3^2,2^2,1^3)$	$(7,5^3,4^4,3,1^4)$			
53	$2\omega_1$	5	$(5^9,3^2,1^2)$	$(5^7,4^4,1^2)$	$(5^6,3^6,1^5)$	$(5^3,4^4,3^4,2^4,1^2)$	$(5^3,3^9,1^{11})$
54	$2\omega_1$	3	-	-	(3^{18})	(3^{18})	$(3^{15},1^9)$
		7	$(7^2,6^2,5,4^2,3^3,2^2,1^2)$	$(7^3,5,4^4,3^3,1^3)$	$(5^6,3^6,1^6)$	$(5^3,4^4,3^4,2^4,1^3)$	$(5^3,3^9,1^{12})$
		≥ 11	$(9,6^2,5^2,4^2,3^3,2^2,1^2)$				

Table 24 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(3,2^2,1^3)$	$J(\varphi(x))$ for $J(x)=(3,1^7)$	$J(\varphi(x))$ for $J(x)=(2^4,1^2)$	$J(\varphi(x))$ for $J(x)=(2^2,1^6)$
16	ω_5	≥ 3	$(3^2,2^4,1^2)$	(2^8)	$(3,2^4,1^5)$	$(2^4,1^8)$
45	ω_2	3	$(3^9,2^6,1^6)$	$(3^8,1^{21})$	$(3^6,2^8,1^{11})$	$(3,2^{12},1^{18})$
		≥ 5	$(4^2,3^5,2^8,1^6)$			
53	$2\omega_1$	5	$(5,4^2,3^6,2^8,1^6)$	$(5,3^7,1^{27})$	$(3^{10},2^8,1^7)$	$(3^3,2^{12},1^{20})$
54	$2\omega_1$	3	$(3^{12},2^6,1^6)$	$(3^9,1^{27})$	$(3^{10},2^8,1^8)$	$(3^3,2^{12},1^{21})$
		≥ 7	$(5,4^2,3^6,2^8,1^7)$	$(5,3^7,1^{28})$		

TABLE 25. $G = D_6(K)$

n	ω	p	$J(\varphi(x))$ for $J(x) = (11, 1)$	$J(\varphi(x))$ for $J(x) = (9, 3)$	$J(\varphi(x))$ for $J(x) = (9, 1^3)$	$J(\varphi(x))$ for $J(x) = (7, 5)$
32	ω_6	7	-	-	-	$(7^4, 4)$
		11	$(11^2, 10)$	$(11^2, 6, 4)$	$(11^2, 5^2)$	$(10, 8, 6, 4^2)$
		13	$(13^2, 6)$	$(12, 10, 6, 4)$		
		≥ 17	$(16, 10, 6)$			
66	ω_2	7	-	-	-	$(7^9, 3)$
		11	(11^6)	$(11^4, 9, 7, 3^2)$	$(11^3, 9^3, 3, 1^3)$	$(11^2, 9, 7^3, 5, 3^3)$
		13	$(13^4, 11, 3)$	$(13^2, 11, 9, 7^2, 3^2)$	$(13^2, 9^3, 7, 3, 1^3)$	
		17	$(17^2, 11^2, 7, 3)$	$(15, 11^2, 9, 7^2, 3^2)$	$(15, 11, 9^3, 7, 3, 1^3)$	
		≥ 19	$(19, 15, 11^2, 7, 3)$			
77	$2\omega_1$	7	-	-	-	(7^{11})
		11	(11^7)	$(11^5, 9, 7, 5, 1)$	$(11^4, 9^3, 1^6)$	$(11^3, 9^2, 7, 5^3, 3, 1)$
		13	$(13^5, 11, 1)$	$(13^3, 11, 9, 7, 5^2, 1)$	$(13^3, 9^3, 5, 1^6)$	$(13, 11, 9^3, 7, 5^3, 3, 1)$
		17	$(17^3, 11, 9, 5, 1)$	$(17, 13, 11, 9^2, 7, 5^2, 1)$	$(17, 13, 9^4, 5, 1^6)$	
		19	$(19^2, 13, 11, 9, 5, 1)$			
		≥ 23	$(21, 17, 13, 11, 9, 5, 1)$			

Table 25 continued

n	ω	p	$J(\varphi(x))$ for $J(x) = (7, 3, 1^2)$	$J(\varphi(x))$ for $J(x) = (7, 2^2, 1)$	$J(\varphi(x))$ for $J(x) = (7, 1^5)$	$J(\varphi(x))$ $J(x) = (6^2)$, $x \in H_1$	$J(\varphi(x))$ for $J(x) = (6^2)$, $x \in H_2$
32	ω_6	7	$(7^4, 2^2)$	$(7^4, 2, 1^2)$	$(7^4, 1^4)$	$(7^4, 1^4)$	$(7^2, 6^3)$
		≥ 11	$(8^2, 6^2, 2^2)$	$(8, 7^2, 6, 2, 1^2)$		$(9^2, 5^2, 1^4)$	$(10, 6^3, 4)$
66	ω_2	7	$(7^8, 3^3, 1)$	$(7^8, 3, 2^2, 1^3)$	$(7^8, 1^{10})$	$(7^9, 1^3)$	$(7^9, 1^3)$
		≥ 11	$(11, 9, 7^4, 5, 3^4, 1)$	$(11, 8^2, 7^2, 6^2, 3^2, 2^2, 1^3)$	$(11, 7^6, 3, 1^{10})$	$(11, 9^3, 7, 5^3, 3, 1^3)$	$(11, 9^3, 7, 5^3, 3, 1^3)$
77	$2\omega_1$	7	$(7^9, 5, 3^2, 1^3)$	$(7^9, 3^3, 2^2, 1)$	$(7^9, 1^{14})$	(7^{11})	(7^{11})
		11	$(11^2, 9, 7^3, 5^3, 3^2, 1^4)$	$(11^2, 8^2, 7, 6^2, 5, 3^3, 2^2, 1^2)$	$(11^2, 7^5, 5, 1^{15})$	$(11^3, 9, 7^3, 5, 3^3)$	$(11^3, 9, 7^3, 5, 3^3)$
		≥ 13	$(13, 9^2, 7^3, 5^3, 3^2, 1^4)$	$(13, 9, 8^2, 7, 6^2, 5, 3^3, 2^2, 1^2)$	$(13, 9, 7^5, 5, 1^{15})$		

Table 25 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(5^2,1^2)$	$J(\varphi(x))$ for $J(x)=(5,3^2,1)$	$J(\varphi(x))$ for $J(x)=(5,3,2^2)$	$J(\varphi(x))$ for $J(x)=(5,3,1^4)$	$J(\varphi(x))$ for $J(x)=(5,2^2,1^3)$
32	ω_6	5	$(5^6,1^2)$	$(5^4,4^2,2^2)$	$(5^4,4,3^2,2)$	$(5^4,3^4)$	$(5^2,4^4,3^2)$
		≥ 7	$(7^2,5^2,3^2,1^2)$	$(6^2,4^4,2^2)$	$(6,5^2,4^2,3^2,2)$		
66	ω_2	5	$(5^{13},1)$	$(5^{10},3^5,1)$	$(5^9,4^2,3^2,2^2,1^3)$	$(5^9,3^5,1^6)$	$(5^9,3,2^6,1^6)$
		7	$(7^5,5^4,3^3,1^2)$	$(7^3,5^4,3^8,1)$	$(7^2,6^2,5,4^4,3^4,2^2,1^3)$	$(7^2,5^5,3^7,1^6)$	$(7,6^2,5^3,4^2,3^2,2^6,1^6)$
		≥ 11	$(9,7^3,5^5,3^3,1^2)$				
77	$2\omega_1$	5	$(5^{15},1^2)$	$(5^{13},3^3,1^3)$	$(5^{11},4^2,3^3,2^2,1)$	$(5^{11},3^4,1^{10})$	$(5^{10},3^3,2^6,1^6)$
		7	$(7^7,5^4,3,1^5)$	$(7^4,5^6,3^5,1^4)$	$(7^3,6^2,5^2,4^4,3^4,2^2,1^2)$	$(7^3,5^6,3^5,1^{11})$	$(7^2,6^2,5^3,4^2,3^3,2^6,1^7)$
		≥ 11	$(9^3,7,5^7,3,1^5)$	$(9,7^2,5^7,3^5,1^4)$	$(9,7,6^2,5^3,4^4,3^4,2^2,1^2)$	$(9,7,5^7,3^5,1^{11})$	$(9,6^2,5^4,4^2,3^3,2^6,1^7)$

Table 25 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(5,1^7)$	$J(\varphi(x))$ for $J(x)=(4^2,3,1)$	$J(\varphi(x))$ for $J(x)=(4^2,2^2)$, $x\in H_1$	$J(\varphi(x))$for $J(x)=(4^2,2^2)$, $x\in H_2$	$J(\varphi(x))$ for $J(x)=(4^2,1^4)$
32	ω_6	5	(4^8)	$(5^4,3^2,2^3)$	$(5^4,3^2,1^6)$	$(5^2,4^4,2^3)$	$(5^2,4^4,1^6)$
		≥ 7		$(6,5^2,4,3^2,2^3)$		$(6,4^5,2^3)$	
66	ω_2	5	$(5^9,1^{21})$	$(5^9,4^2,3^2,2^2,1^3)$	$(5^9,3^5,1^6)$	$(5^9,3^5,1^6)$	$(5^5,4^8,1^9)$
		≥ 7	$(7,5^7,3,1^{21})$	$(7,6^2,5^3,4^4,3^3,2^2,1^3)$	$(7,5^7,3^6,1^6)$	$(7,5^7,3^6,1^6)$	$(7,5^3,4^8,3,1^9)$
77	$2\omega_1$	5	$(5^{10},1^{27})$	$(5^{12},4^2,3,2^2,1^2)$	$(5^{11},3^7,1)$	$(5^{11},3^7,1)$	$(5^7,4^8,1^{10})$
		7	$(7^2,5^7,1^{28})$	$(7^3,6^2,5^2,4^4,3^4,2^2,1^2)$	$(7^3,5^5,3^{10},1)$	$(7^3,5^5,3^{10},1)$	$(7^3,5,4^8,3^3,1^{10})$
		≥ 11	$(9,5^8,1^{28})$				

Table 25 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(3^4)$	$J(\varphi(x))$ for $J(x)=(3^3,1^3)$	$J(\varphi(x))$ for $J(x)=(3^2,2^2,1^2)$	$J(\varphi(x))$ for $J(x)=(3^2,1^6)$	$J(\varphi(x))$ for $J(x)=(3,2^4,1)$	$J(\varphi(x))$ for $J(x)=(3,2^2,1^5)$
32	ω_6	3	$(3^{10},1^2)$	$(3^8,2^4)$	$(3^8,2^2,1^4)$	$(3^8,1^8)$	$(3^6,2^5,1^4)$	$(3^4,2^8,1^4)$
		≥ 5	$(5^2,3^6,1^4)$	$(4^4,2^8)$	$(4^2,3^4,2^4,1^4)$		$(4,3^4,2^6,1^4)$	
66	ω_2	3	(3^{22})	$(3^{21},1^3)$	$(3^{18},2^4,1^4)$	$(3^{17},1^{15})$	$(3^{16},2^4,1^{10})$	$(3^{11},2^{10},1^{13})$
		≥ 5	$(5^6,3^{10},1^6)$	$(5^3,3^{15},1^6)$	$(5,4^4,3^8,2^8,1^5)$	$(5,3^{15},1^{16})$	$(4^4,3^8,2^8,1^{10})$	$(4^2,3^7,2^{12},1^{13})$
76	$2\omega_1$	3	$(3^{25},1)$	$(3^{24},1^4)$	$(3^{22},2^4,1^2)$	$(3^{19},1^{19})$	$(3^{21},2^4,1^5)$	$(3^{14},2^{10},1^{14})$
77	$2\omega_1$	≥ 5	$(5^{10},3^6,1^9)$	$(5^6,3^{12},1^{11})$	$(5^3,4^4,3^8,2^8,1^6)$	$(5,3^{13},1^{23})$	$(5,4^4,3^{11},2^8,1^7)$	$(5,4^2,3^8,2^{12},1^{16})$

Table 25 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(3,1^9)$	$J(\varphi(x))$ for $J(x)=(2^6)$, $x\in H_1$	$J(\varphi(x))$ for $J(x)=(2^6)$, $x\in H_2$	$J(\varphi(x))$ for $J(x)=(2^4,1^4)$	$J(\varphi(x))$ for $J(x)=(2^2,1^8)$
32	ω_6	3	(2^{16})	$(3^6,2^{14})$	$(3^2,1^{13})$	$(3^2,2^8,1^{10})$	$(2^8,1^{16})$
		≥ 5			$(4,2^{14})$		
66	ω_2	≥ 3	$(3^{10},1^{36})$	$(3^{15},1^{21})$	$(3^{15},1^{21})$	$(3^6,2^{16},1^{16})$	$(3,2^{16},1^{31})$
76	$2\omega_1$	3	$(3^{11},1^{43})$	$(3^{21},1^{13})$	$(3^{21},1^{13})$	$(3^{10},2^{16},1^{14})$	$(3^3,2^{16},1^{35})$
77	$2\omega_1$	≥ 5	$(5,3^9,1^{45})$	$(3^{21},1^{14})$	$(3^{21},1^{14})$	$(3^{10},2^{16},1^{15})$	$(3^3,2^{16},1^{36})$

TABLE 26. $G=D_7(K)$

n	ω	p	$J(\varphi(x))$ for $J(x)=(13,1)$	$J(\varphi(x))$ for $J(x)=(11,3)$	$J(\varphi(x))$ for $J(x)=(11,1^3)$	$J(\varphi(x))$ for $J(x)=(9,5)$	$J(\varphi(x))$ for $J(x)=(9,3,1^2)$
64	ω_7	11	-	$(11^5,9)$	$(11^4,10^2)$	$(11^4,8,6,4,2)$	$(11^4,6^2,4^2)$
		13	$(13^4,12)$	$(13^4,7,5)$	$(13^4,6^2)$	$(13^2,10,8^2,6,4,2)$	$(12^2,10^2,6^2,4^2)$
		17	$(17^2,16,10,4)$	$(17,15,11,9,7,5)$	$(16^2,10^2,6^2)$	$(14,12,10,8^2,6,4,2)$	
		19	$(19^2,12,10,4)$				
		≥ 23	$(22,16,12,10,4)$				

Table 26 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(13,1)$	$J(\varphi(x))$ for $J(x)=(11,3)$	$J(\varphi(x))$ for $J(x)=(11,1^3)$	$J(\varphi(x))$ for $J(x)=(9,5)$	$J(\varphi(x))$ for $J(x)=(9,3,1^2)$
91	ω_2	11	-	$(11^8,3)$	$(11^8,1^3)$	$(11^6,7^2,5,3^2)$	$(11^4,9^3,7,3^4,1)$
		13	(13^7)	$(13^5,11,9,3^2)$	$(13^4,11^3,3,1^3)$	$(13^3,11,9,7^3,5,3^2)$	$(13^2,11,9^3,7^2,3^4,1)$
		17	$(17^4,13,7,3)$	$(17^2,13,11^2,9,7,3^2)$	$(17^2,11^4,7,3,1^3)$	$(15,13,11^2,9,7^3,5,3^2)$	$(15,11^2,9^3,7^2,3^4,1)$
		19	$(19^3,13,11,7,3)$	$(19,15,13,11^2,9,7,3^2)$	$(19,15,11^4,7,3,1^3)$		
		≥ 23	$(23,19,15,13,11,7,3)$				

Table 26 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(9,2^2,1)$	$J(\varphi(x))$ for $J(x)=(9,1^5)$	$J(\varphi(x))$ for $J(x)=(7^2)$	$J(\varphi(x))$ for $J(x)=(7,5,1^2)$	$J(\varphi(x))$ for $J(x)=(7,3^2,1)$
64	ω_7	7	-	-	$(7^9,1)$	$(7^8,4^2)$	$(7^8,3^2,1^2)$
		11	$(11^4,6,5^2,4)$	$(11^4,5^4)$	$(11^3,7^3,5,3,1^2)$	$(10^2,8^2,6^2,4^4)$	$(9^2,7^4,5^2,3^2,1^2)$
		≥ 13	$(12,11^2,10,6,5^2,4)$		$(13,11,9,7^3,5,3,1^2)$		
91	ω_2	7	-	-	(7^{13})	$(7^{11},5^2,3,1)$	$(7^{10},5,3^5,1)$
		11	$(11^3,10^2,9,8^2,3^2,2^2,1^3)$	$(11^3,9^5,3,1^{10})$	$(11^5,7^3,5,3^3,1)$	$(11^2,9,7^5,5^3,3^3,1)$	$(11,9^2,7^4,5^3,3^6,1)$
		13	$(13^2,10^2,9,8^2,7,3^2,2^2,1^3)$	$(13^2,9^5,7,3,1^{10})$	$(13,11^3,9,7^3,5,3^3,1)$		
		≥ 17	$(15,11,10^2,9,8^2,7,3^2,2^2,1^3)$	$(15,11,9^5,7,3,1^{10})$			

Table 26 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(7,3,2^2)$	$J(\varphi(x))$ for $J(x)=(7,3,1^4)$	$J(\varphi(x))$ for $J(x)=(7,2^2,1^3)$	$J(\varphi(x))$ for $J(x)=(7,1^7)$	$J(\varphi(x))$ for $J(x)=(6^2,1^2)$
64	ω_7	7	$(7^8,3,2^2,1)$	$(7^8,2^4)$	$(7^8,2^2,1^4)$	$(7^8,1^8)$	$(7^6,6^3,1^4)$
		≥ 11	$(9,8^2,7^2,6^2,5,3,2^2,1)$	$(8^4,6^4,2^4)$	$(8^2,7^4,6^2,2^2,1^4)$		$(10,9^2,6^3,5^2,4,1^4)$
91	ω_2	7	$(7^{10},4^2,3^2,2^2,1^3)$	$(7^{10},3^5,1^6)$	$(7^{10},3,2^6,1^6)$	$(7^{10},1^{21})$	$(7^9,6^4,1^4)$
		≥ 11	$(11,9,8^2,7^2,6^2,5,4^2,3^3,2^2,1^3)$	$(11,9,7^6,5,3^6,1^6)$	$(11,8^2,7^4,6^2,3^2,2^6,1^6)$	$(11,7^8,3,1^{21})$	$(11,9^3,7,6^4,5^3,3,1^4)$

Table 26 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(5^2,3,1)$	$J(\varphi(x))$ for $J(x)=(5^2,2^2)$	$J(\varphi(x))$ for $J(x)=(5^2,1^4)$	$J(\varphi(x))$ for $J(x)=(5,4^2,1)$	$J(\varphi(x))$ for $J(x)=(5,3^3)$
64	ω_7	5	$(5^{12},2^2)$	$(5^{12},2,1^2)$	$(5^{12},1^4)$	$(5^{10},4^3,1^2)$	$(5^{10},3^4,1^2)$
		7	$(7^4,6^2,4^4,2^4)$	$(7^4,6,5^2,4^2,3^2,2^2,1^2)$	$(7^4,5^4,3^4,1^4)$	$(7^4,5^2,4^4,3^2,2,1^2)$	$(7^2,5^6,3^6,1^2)$
		≥ 11	$(8^2,6^4,4^4,2^4)$	$(8,7^2,6^2,5^2,4^2,3^2,2^2,1^2)$		$(8,7^2,6,5^2,4^4,3^2,2,1^2)$	
91	ω_2	5	$(5^{17},3^2)$	$(5^{17},3,1^3)$	$(5^{17},1^6)$	$(5^{16},4^2,1^3)$	$(5^{14},3^6,1^3)$
		7	$(7^7,5^4,3^7,1)$	$(7^5,6^4,4^4,3^4,1^4)$	$(7^5,5^8,3^3,1^7)$	$(7^6,5^4,4^4,3^2,2^2,1^3)$	$(7^4,5^6,3^{10},1^3)$
		≥ 11	$(9,7^5,5^5,3^7,1)$	$(9,7^3,6^4,5,4^4,3^4,1^4)$	$(9,7^3,5^9,3^3,1^7)$	$(8^2,7^2,6^2,5^4,4^4,3^2,2^2,1^3)$	

Table 26 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(5,3^2,1^3)$	$J(\varphi(x))$ for $J(x)=(5,3,2^2,1^2)$	$J(\varphi(x))$ for $J(x)=(5,3,1^6)$	$J(\varphi(x))$ for $J(x)=(5,2^4,1)$	$J(\varphi(x))$ for $J(x)=(5,2^2,1^5)$
64	ω_7	5	$(5^8,4^4,2^4)$	$(5^8,4^2,3^4,2^2)$	$(5^8,3^8)$	$(5^6,4^5,3^4,2)$	$(5^4,4^8,3^4)$
		≥ 7	$(6^4,4^8,2^4)$	$(6^2,5^4,4^4,3^4,2^2)$		$(6,5^4,4^6,3^4,2)$	
91	ω_2	5	$(5^{12},3^9,1^4)$	$(5^{11},4^2,3^4,2^6,1^4)$	$(5^{11},3^7,1^{15})$	$(5^{11},3^6,2^4,1^{10})$	$(5^{11},3,2^{10},1^{13})$
		≥ 7	$(7^3,5^6,3^{12},1^4)$	$(7^2,6^2,5^3,4^4,3^6,2^6,1^4)$	$(7^2,5^7,3^9,1^{15})$	$(7,6^4,5,4^4,3^7,2^4,1^{10})$	$(7,6^2,5^5,4^2,3^2,2^{10},1^{13})$

Table 26 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(5,1^9)$	$J(\varphi(x))$ for $J(x)=(4^2,3^2)$	$J(\varphi(x))$ for $J(x)=(4^2,3,1^3)$	$J(\varphi(x))$ for $J(x)=(4^2,2^2,1^2)$	$J(\varphi(x))$ for $J(x)=(4^2,1^6)$
64	ω_7	5	(4^{16})	$(5^8,4^2,3^3,2^2,1^3)$	$(5^8,3^4,2^6)$	$(5^6,4^4,3^2,2^3,1^6)$	$(5^4,4^8,1^{12})$
		≥ 7		$(7,6^2,5^2,4^4,3^4,2^2,1^3)$	$(6^2,5^4,4^2,3^4,2^6)$	$(6,5^4,4^5,3^2,2^3,1^6)$	
91	ω_2	5	$(5^{11},1^{36})$	$(5^{14},3^3,2^4,1^4)$	$(5^9,4^6,3^4,2^2,1^6)$	$(5^9,4^4,3^5,2^4,1^7)$	$(5^5,4^{12},1^{18})$
		≥ 7	$(7,5^9,3,1^{36})$	$(7,6^4,5^4,4^4,3^4,2^4,1^4)$	$(7,6^2,5^3,4^8,3^5,2^2,1^6)$	$(7,5^7,4^4,3^6,2^4,1^7)$	$(7,5^3,4^{12},3,1^{18})$

Table 26 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(3^4,1^2)$	$J(\varphi(x))$ for $J(x)=(3^3,2^2,1)$	$J(\varphi(x))$ for $J(x)=(3^3,1^5)$	$J(\varphi(x))$ for $J(x)=(3^2,2^4)$	$J(\varphi(x))$ for $J(x)=(3^2,2^2,1^4)$	$J(\varphi(x))$ for $J(x)=(3^2,1^8)$
64	ω_7	3	$(3^{20},1^4)$	$(3^{18},2^4,1^2)$	$(3^{16},2^8)$	$(3^{17},2^4,1^5)$	$(3^{16},2^4,1^8)$	$(3^{16},1^{16})$
		≥ 5	$(5^4,3^{12},1^8)$	$(5^2,4^4,3^6,2^8,1^4)$	$(4^8,2^{16})$	$(5,4^4,3^7,2^8,1^6)$	$(4^4,3^8,2^8,1^8)$	
91	ω_2	3	$(3^{30},1)$	$(3^{28},2^2,1^3)$	$(3^{27},1^{10})$	$(3^{27},1^{10})$	$(3^{22},2^8,1^9)$	$(3^{21},1^{28})$
		≥ 5	$(5^6,3^{18},1^7)$	$(5^3,4^6,3^{10},2^8,1^6)$	$(5^3,3^{21},1^{13})$	$(5,4^8,3^9,2^8,1^{11})$	$(5,4^4,3^{12},2^{12},1^{10})$	$(5,3^{19},1^{29})$

Table 26 continued

n	ω	p	$J(\varphi(x))$ for $J(x)=(3,2^4,1^3)$	$J(\varphi(x))$ for $J(x)=(3,2^2,1^7)$	$J(\varphi(x))$ for $J(x)=(3,1^{11})$	$J(\varphi(x))$ for $J(x)=(2^6,1^2)$	$J(\varphi(x))$ for $J(x)=(2^4,1^6)$	$J(\varphi(x))$ for $J(x)=(2^2,1^{10})$
64	ω_7	3	$(3^{12},2^{10},1^8)$	$(3^8,2^{16},1^8)$	(2^{32})	$(3^8,2^{13},1^{14})$	$(3^4,2^{16},1^{20})$	$(2^{16},1^{32})$
		≥ 5	$(4^2,3^8,2^{12},1^8)$			$(4,3^6,2^{14},1^{14})$		
91	ω_2	3	$(3^{18},2^{12},1^{13})$	$(3^{13},2^{14},1^{24})$	$(3^{12},1^{55})$	$(3^{15},2^{12},1^{22})$	$(3^6,2^{24},1^{25})$	$(3,2^{20},1^{48})$
		≥ 5	$(4^4,3^{10},2^{16},1^{13})$	$(4^2,3^9,2^{16},1^{24})$				

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