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ON THE DISSYMMETRIZATION THEOREM

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Abstract. A new property of the previously proposed dissymmetrization of functions is established. The conjecture about the capacity of condensers in a circular ring with plates in the form of circles or radial cuts is discussed. The connection of this conjecture with the well-known Gonchar-Baernstein problem of a harmonic measure is shown.

Keywords: dissymmetrization, harmonic measure, Dirichlet integral, condenser capacity.

1. Introduction and statement of result

Let $n \geq 2$ be a natural number and let

$$L_j^* = \{z: \ \arg z = 2\pi j/n\}, \quad j = 1, ..., n.$$

Denote by Φ the group of symmetries of $\overline{\mathbb{C}}$ formed by the composites of the reflections in the rays $L_j^*,\ j = 1, ..., n$, and in the bisectors of the angles formed by these rays. Throughout this paper symmetry means Φ-invariance. We say that a set $A \subset \overline{\mathbb{C}}$ is symmetric if $\varphi(A) = A$ for any isometry $\varphi \in \Phi$. A real function v on a symmetric set Ω is said to be symmetric if $v(z) \equiv v(\varphi(z))$ for any $\varphi \in \Phi$. We call a system of closed sectors with vertices at origin a decomposition of the sphere $\overline{\mathbb{C}}$ if no two sectors have common interior points and their union is $\overline{\mathbb{C}}$. It has been shown in [1] (see also [2], [3, Sec. 4.4]) that for any different rays $L_j,\ j = 1, ..., n$, starting from the origin there exists a decomposition $\{P_j\}_{j=1}^{j_0},\ j_0 \geq n$, of the plane $\overline{\mathbb{C}}$ and a set of rotations $\{\lambda_j\}_{j=1}^{j_0}$ of the form $\lambda_j(z) = e^{i\theta_j} z,\ j = 1, ..., j_0$ (dissymmetrization)

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such that the set $\{\lambda_j(P_j)\}_{j=1}^{j_0}$ is also a decomposition $\overline{\mathbb{C}}$, $\operatorname{Dis} L_j^* = L_j,\ j = 1, ..., n$, and for any the symmetric function $v,\ z \in \Omega$, the function $\operatorname{Dis} v$ is well defined:

$$\operatorname{Dis} v(z) := v(\lambda_j^{-1}(z)),\ z \in \lambda_j(P_j) \cap \operatorname{Dis}\Omega, \quad j = 1, ..., j_0.$$

Here $\operatorname{Dis} A$ means $\bigcup_{j=1}^{j_0} \lambda_j(A \cap P_j)$. Conceived as a method for solving the Gonchar problem on harmonic measure [1], dissymmetrization has found application in other questions of function theory (see, for example, [3]–[8]). In this paper, we consider a new property of this transformation. Let us introduce the notation

$$B(s,t) = \{z :\ s < |z| < t\}, \quad 0 < s < t < \infty,$$
$$T(r) = \{z :\ |z| = r\}, \quad T = T(1).$$

The following theorem is true.

Theorem 1. *Let E^* be a symmetric compact set on $\bigcup_{j=1}^{n} L_j^* \cap B(s,1)$ and let f^* be a symmetric continuous function on $\overline{B(s,1)}$, constant on T, harmonic in $B(s,1) \setminus E^*$, and f is a continuous function on $\overline{B(s,1)}$, coinciding with the function $\operatorname{Dis} f^*$ on $\partial(B(s,1) \setminus \operatorname{Dis} E^*)$ and harmonic in $B(s,1) \setminus \operatorname{Dis} E^*$. Then*

$$\int_T \left[\left(\frac{\partial f}{\partial n}\right)^2 - \left(\frac{\partial f^*}{\partial n}\right)^2\right] ds \geq \frac{1}{\pi} \int_T \frac{\partial f^*}{\partial n} ds \int_T \left(\frac{\partial f}{\partial n} - \frac{\partial f^*}{\partial n}\right) ds, \tag{1}$$

where $\partial/\partial n$ denotes differentiation along the inward normal.

It is known that the Dirichlet integral does not increase under the dissymmetrization [3, Sec. 4.4]. Inequality [1] gives information about the behavior of the variation of the Dirichlet integral under dissymmetrization (see Lemma 1). In particular, when the right-hand side of (1) is equal to zero, the variation of the Dirichlet integral of a symmetric function does not exceed the variation of such an integral of a function that does not have such symmetry (cf. [8]) This statement, together with [8], leads to a new conjecture about the behavior of the capacities of some condensers during dissymmetrization which is closely related to the well-known Gonchar-Baernstein problem on the harmonic measure of radial cuts [9]. The next section is auxiliary.

2. Variation of the Dirichlet integral

Everywhere below, the notation

$$I(f, B) := \iint_B |\nabla f|^2 dxdy$$

is adopted. In the unit disk $|z| < 1$, consider a finitely connected domain B whose boundary consists of analytic Jordan curves, including the circle T. Let f be a function twice continuously differentiable in $\overline{B}$, harmonic in B and equal to zero on T, and let f_t be a function harmonic in $B_t := B \cap \{z :\ |z| < t\}$, $t < 1$, twice continuously differentiable in $\overline{B}_t$, taking boundary values f on $(\partial B) \setminus T$ and equal to zero on $T(t)$.

Lemma 1. *The following asymptotic formula holds:*

$$I(f_t, B_t) = I(f, B) + (1-t) \int_T \left(\frac{\partial f}{\partial n}\right)^2 ds + O((1-t)^2), \quad t \to 1.$$

Proof. On the circle $T(t)$ the uniform estimate

$$f(z) = f(z) - f_t(z) = (1-t)\frac{\partial f}{\partial n}(z/t) + O((1-t)^2)$$

is satisfied. Applying Green's formula, we obtain the equality

$$I(f-f_t, B_t) = -\int_{\partial B_t} (f-f_t)\frac{\partial (f-f_t)}{\partial n} ds = -\int_{T(t)} (f-f_t)\frac{\partial (f-f_t)}{\partial n} ds = O((1-t)^2).$$

On the other hand, again by Green's formula

$$I(f-f_t, B_t) = I(f, B_t) + I(f_t, B_t) + 2\int_{T(t)} (f_t - f + f)\frac{\partial f}{\partial n} ds$$

$$= I(f_t, B_t) - I(f, B_t) + 2\int_{T(t)} (f_t - f)\frac{\partial f}{\partial n} ds = I(f_t, B_t) - I(f, B) - \int_{T(t)} f\frac{\partial f}{\partial n} ds.$$

Therefore,

$$I(f_t, B_t) - I(f, B) - \int_{T(t)} f\frac{\partial f}{\partial n} ds = O((1-t)^2),$$

which completes the proof of Lemma 1.

We note that in the case when the function f takes constant values on the connected components of the boundary of the domain B, our formula follows from the classical variational formula [10, (A3.12)].

3. Proof of the theorem 1

It suffices to establish inequality (1) in the new formulation of the problem. Namely, to replace the set E^* in the hypothesis of Theorem 1 by a symmetric set $\mathcal{E}^*$ located sufficiently close to it, bounded by a finite number of analytic Jordan curves, and the function f^* to be assumed to be three times continuously differentiable on $\partial(B(s,1)\setminus\mathcal{E}^*)$ and equal to zero on T. We fix t such that $0 < t < 1$, $\sup\{|z| : z \in \mathcal{E}^*\} < 1 - 2\Delta t$ ($\Delta t = 1 - t$) and consider functions

$$b(z) := \log|z|, \quad u^* = b + \varepsilon f^*,$$

where $\varepsilon > 0$ is sufficiently small. It is easy to see that the Hausdorff distance between the curve

$$\gamma_t^* : u^* = b(t)$$

and the circle $T(t)$ is the quantity[1] $O'(\varepsilon\Delta t)$, $\varepsilon \to 0$. Further, the notation (γ, Γ) means a doubly connected domain on the plane $\mathbb{C}$, bounded by closed curves γ, Γ, and mod (γ, Γ) is the module of the domain (γ, Γ). Let the quantities $t(\varepsilon)$ and $R(\varepsilon)$ be defined by the relations:

$$\text{mod }(\gamma_\tau^*, \gamma_t^*) = \text{mod }(\gamma_\tau^*, T(t(\varepsilon))), \quad \tau = 1 - 2\Delta t,$$

$$\text{mod }(\gamma_t^*, T) = \text{mod }(T(t(\varepsilon)), T(R(\varepsilon))).$$

Following the arguments in [8] from formula (3.1) to (3.5), where it is necessary to set the $-1/\log r = \varepsilon$, we see that

$$R(\varepsilon) \geq 1 \text{ and } R(\varepsilon) - 1 = O'((\varepsilon\Delta t)^2). \tag{2}$$

[1]Here and below, the prime at the symbol O – large means that the corresponding quantity admits a uniform estimate for all sufficiently small Δt at $\varepsilon \to 0$.

By definition,

$$\operatorname{mod}(\gamma_t^*, T) = \frac{1}{2\pi}\log\frac{R(\varepsilon)}{t(\varepsilon)} = \frac{1}{2\pi}\log\frac{1}{t(\varepsilon)} + O'((\varepsilon\Delta t)^2).$$

On the other hand, the Hadamard formula [10 (A3.11)], [8, (2.2)] gives

$$\operatorname{mod}(\gamma_t^*, T) = \frac{1}{2\pi}\log\frac{1}{t} - \left(\frac{1}{2\pi}\log\frac{1}{t}\right)^2 \int\limits_{T(t)} \left(\frac{\partial\omega}{\partial n}\right)^2 [-\varepsilon t f^*(te^{i\theta})]ds + O'((\varepsilon\Delta t)^2),$$

where $\omega(z) = (\log|z|)/\log t$. Hence

$$\log\frac{t}{t(\varepsilon)} = \frac{\varepsilon}{2\pi}\int\limits_0^{2\pi} f^*(te^{i\theta})d\theta + O'((\varepsilon\Delta t)^2). \tag{3}$$

Let F^- be some conformal mapping of the domain $(T(t(\varepsilon)), T(R(\varepsilon)))$ onto (γ_t^*, T) and let F^+ be a conformal mapping of $(\gamma_\tau^*, T(t(\varepsilon)))$ onto $(\gamma_\tau^*, \gamma_t^*)$ in such a way that $F^-(T(t(\varepsilon))) = F^+(T(t(\varepsilon))) = \gamma_t^*$. Following the proof of Theorem 1 [8], we set

$$v^*(z) = \begin{cases} u^*(F^-(z)), & z \in \overline{(T(t(\varepsilon)), T(R(\varepsilon)))}, \\ u^*(F^+(z)), & z \in \overline{(\gamma_\tau^*, T(t(\varepsilon)))}, \\ u^*(z), & z \in \overline{(T(s), \gamma_\tau^*) \setminus \mathcal{E}^*}. \end{cases}$$

Note that in the ring $(T(t(\varepsilon)), T(R(\varepsilon)))$

$$v^*(z) = \frac{\log|z/R(\varepsilon)|}{\log|t(\varepsilon)/R(\varepsilon)|}\log t$$

is fulfilled.

The conformal invariance of the Dirichlet integral and the Dirichlet principle implies

$$\begin{aligned} I(u^*, (T(s), T) \setminus \mathcal{E}^*) &= I(v^*, (T(s), T(R(\varepsilon))) \setminus \mathcal{E}^*) \\ &\geq I(h^*, (T(s), T(t(\varepsilon))) \setminus \mathcal{E}^*) + I(v^*, (T(t(\varepsilon))), T(R(\varepsilon))). \end{aligned}$$

Here h^* is a harmonic function on the set $(T(s), T(t(\varepsilon))) \setminus \mathcal{E}^*$, continuous in the closure of this set and taking the following boundary values:

$$h^*(z) = \begin{cases} b(t), & z \in T(t(\varepsilon)), \\ b(z) + \varepsilon f^*(z), & z \in T(s) \cup \partial\mathcal{E}^*. \end{cases}$$

Note the symmetry of the function h^* and the fact that the circles $T(s)$, $T(t(\varepsilon))$ do not change under dissymmetrization Dis (Sec.1). Using the Dirichlet principle again, we obtain

$$\begin{aligned} I(h^*, (T(s), T(t(\varepsilon))) \setminus \mathcal{E}^*) &= I(\operatorname{Dis} h^*, (T(s), T(t(\varepsilon))) \setminus \operatorname{Dis}\mathcal{E}^*) \\ &\geq I(h, (T(s), T(t(\varepsilon))) \setminus \operatorname{Dis}\mathcal{E}^*), \end{aligned}$$

where h is a harmonic function on the set $(T(s), T(t(\varepsilon))) \setminus \operatorname{Dis}\mathcal{E}^*$, continuous in the closure of this set and taking boundary values:

$$h(z) = \begin{cases} b(t), & z \in (T(t(\varepsilon)), \\ b(z) + \varepsilon \operatorname{Dis} f^*(z), & z \in T(s) \cup \partial\operatorname{Dis}\mathcal{E}^*. \end{cases}$$

Again, the Dirichlet principle gives

$$I(h, (T(s), T(t(\varepsilon))) \setminus \operatorname{Dis}\mathcal{E}^*) + I(\tilde{v}^*, (T(t(\varepsilon)), T)) \geq I(u, (T(s), T) \setminus \operatorname{Dis}\mathcal{E}^*)$$

for

$$\tilde{v}^*(z) = \frac{\log|z|}{\log t(\varepsilon)} \log t$$

and the function u, harmonic in $(T(s), T) \setminus \operatorname{Dis} \mathcal{E}^*$ continuous in the closure of this set and taking boundary values:

$$u(z) = \begin{cases} 0, & z \in T, \\ b(z) + \varepsilon \operatorname{Dis} f^*(z), & z \in T(s) \cup \partial \operatorname{Dis} \mathcal{E}^*. \end{cases}$$

The main inequality in the proof of Theorem 1 follows from the inequalities written above:

$$I(u^*, (T(s), T) \setminus \mathcal{E}^*) - I(u, (T(s), T) \setminus \operatorname{Dis} \mathcal{E}^*) \geq I(h^*, (T(s), T(t(\varepsilon))) \setminus \mathcal{E}^*)$$

$$(4) \quad -I(h, (T(s), T(t(\varepsilon))) \setminus \operatorname{Dis} \mathcal{E}^*) + I(v^*, (T(t(\varepsilon)), T(R(\varepsilon)))) - I(\tilde{v}^*, (T(t(s)), T)).$$

Let us now turn to estimates of the integrals in (4). According to Green's formula, the first integral on the left in (4) is

$$I(u^*, (T(s), T) \setminus \mathcal{E}^*) = I(b, (T(s), T) \setminus \mathcal{E}^*) + \varepsilon^2 I(f^*, (T(s), T) \setminus \mathcal{E}^*)$$

$$(5) \qquad -2\varepsilon \int\limits_{T(s) \cup \partial \mathcal{E}^*} f^* \frac{\partial b}{\partial n} ds.$$

By the uniqueness theorem for harmonic functions in $(T(s), T) \setminus \operatorname{Dis} \mathcal{E}^*$, the equality

$$u = b + \varepsilon f$$

is true. Therefore,

$$I(u, (T(s), T) \setminus \operatorname{Dis} \mathcal{E}^*) = I(b, (T(s), T) \setminus \operatorname{Dis} \mathcal{E}^*)$$

$$(6) \qquad +\varepsilon^2 I(f, (T(s), T) \setminus \operatorname{Dis} \mathcal{E}^*) - 2\varepsilon \int\limits_{T(s) \cup \partial \operatorname{Dis} \mathcal{E}^*} f \frac{\partial b}{\partial n} ds.$$

To calculate the first integral of the right-hand side of (4), we represent the function h^* in the form

$$h^* = b + \varepsilon f^*_{t(\varepsilon)} + \sigma^*,$$

where the functions $f^*_{t(\varepsilon)}$, σ^* are harmonic on the set $(T(s), T(t(\varepsilon))) \setminus \mathcal{E}^*$, continuous in the closure of this set and take boundary values

$$f^*_{t(\varepsilon)}(z) = \begin{cases} 0, & z \in T(t(\varepsilon)), \\ f^*(z), & z \in T(s) \cup \partial \mathcal{E}^*, \end{cases} \qquad \sigma^*(z) = \begin{cases} \log(t/t(\varepsilon)), & z \in T(t(\varepsilon)), \\ 0, & z \in T(s) \cup \partial \mathcal{E}^*. \end{cases}$$

By successively applying Green's formula, we obtain

$$I(h^*, (T(s), T(t(\varepsilon))) \setminus \mathcal{E}^*) = I(b, (T(s), T(t(\varepsilon))) \setminus \mathcal{E}^*) + \varepsilon^2 I(f^*_{t(\varepsilon)}, (T(s), T(t(\varepsilon))) \setminus \mathcal{E}^*)$$

$$+I(\sigma^*, (T(s), T(t(\varepsilon))) \setminus \mathcal{E}^*) - 2\varepsilon \int\limits_{T(s) \cup \partial \mathcal{E}^*} f^* \frac{\partial b}{\partial n} ds$$

$$(7) \qquad -2 \log \frac{t}{t(\varepsilon)} \int\limits_{T(t(\varepsilon))} \frac{\partial b}{\partial n} ds - 2\varepsilon \log \frac{t}{t(\varepsilon)} \int\limits_{T(t(\varepsilon))} \frac{\partial f^*_{t(\varepsilon)}}{\partial n} ds.$$

Similar to (7)

$$h = b + \varepsilon f_{t(\varepsilon)} + \sigma$$

where the functions $f_{t(\varepsilon)}$, σ are harmonic in $(T(s),T(t(\varepsilon)))\setminus \operatorname{Dis}\mathcal{E}^*$, continuous in the closure of this set, and take the boundary values

$$h_{t(\varepsilon)}(z)=\begin{cases}0, & z\in T(t(\varepsilon)),\\ \operatorname{Dis} f^*, & z\in T(s)\cup\partial\operatorname{Dis}\mathcal{E}^*,\end{cases}\quad \sigma(z)=\begin{cases}\log(t/t(\varepsilon)), & z\in T(t(\varepsilon)),\\ 0, & z\in T(s)\cup\partial\operatorname{Dis}\mathcal{E}^*,\end{cases}$$

$$I(h,(T(s),T(t(\varepsilon)))\setminus \operatorname{Dis}\mathcal{E}^*)=I(b,(T(s),T(t(\varepsilon)))\setminus \operatorname{Dis}\mathcal{E}^*)$$
$$+\varepsilon^2 I(f_{t(\varepsilon)},(T(s),T(t(\varepsilon)))\setminus \operatorname{Dis}\mathcal{E}^*)+I(\sigma,(T(s),T(t(\varepsilon)))\setminus \operatorname{Dis}\mathcal{E}^*)$$

$$(8)\quad -2\varepsilon\int\limits_{T(s)\cup\partial\operatorname{Dis}\mathcal{E}^*} f\frac{\partial b}{\partial n}ds-2\log\frac{t}{t(\varepsilon)}\int\limits_{T(t(\varepsilon))}\frac{\partial b}{\partial n}ds-2\varepsilon\log\frac{t}{t(\varepsilon)}\int\limits_{T(t(\varepsilon))}\frac{\partial f_{t(\varepsilon)}}{\partial n}ds.$$

Finally, taking into account (2) and (3), we have

$$I(v^*,(T(t(\varepsilon)),T(R(\varepsilon))))-I(\tilde{v}^*,(T(t(\varepsilon)),T))=\frac{2\pi(\log t)^2}{\log\frac{R(\varepsilon)}{t(\varepsilon)}}-\frac{2\pi(\log t)^2}{\log\frac{1}{t(\varepsilon)}}$$

$$(9)\qquad =2\pi(\log t)^2\left[\frac{1}{O'((\varepsilon\Delta t)^2)-\log t(\varepsilon)}-\frac{1}{-\log t(\varepsilon)}\right]=O'((\varepsilon\Delta t)^2).$$

Substituting (5) - (9) into inequality (4), after obvious reductions and $\varepsilon\to 0$ we arrive at the relation:

$$I(f^*,(T(s),T)\setminus\mathcal{E}^*)-I(f,(T(s),T)\setminus\operatorname{Dis}\mathcal{E}^*)\ge I(f_t^*,(T(s),T(t))\setminus\mathcal{E}^*)$$

(10)

$$-I(f_t,(T(s),T(t))\setminus\operatorname{Dis}\mathcal{E}^*)-\frac{\Delta t}{\pi}\int\limits_0^{2\pi}\frac{\partial f^*}{\partial n}(e^{i\theta})d\theta\int\limits_{T(t)}\left(\frac{\partial f_t^*}{\partial n}-\frac{\partial f_t}{\partial n}\right)ds+O((\Delta t)^2).$$

Here the function f_t^* is harmonic in $(T(s),T(t))\setminus\mathcal{E}^*$, continuous in the closure of this set and

$$f_t^*(z)=\begin{cases}0, & z\in T(t),\\ f^*(z), & z\in T(s)\cup\partial\mathcal{E}^*.\end{cases}$$

The function f_t is harmonic in $(T(s),T(t))\setminus\operatorname{Dis}\mathcal{E}^*$, continuous in the closure of this set and

$$f_t(z)=\begin{cases}0, & z\in T(t),\\ f(z), & z\in T(s)\cup\partial\operatorname{Dis}\mathcal{E}^*.\end{cases}$$

According to Kellogg's theorem, the functions f_t^* and f_t are twice continuously differentiable in the closure of the corresponding domains. By virtue of Lemma 1, inequality (10) can be rewritten in the form of

$$\Delta t\int\limits_T\left(\frac{\partial f}{\partial n}\right)^2ds-\Delta t\int\limits_T\left(\frac{\partial f^*}{\partial n}\right)^2ds\ge\frac{\Delta t}{\pi}\int\limits_T\frac{\partial f^*}{\partial n}ds\int\limits_T\left(\frac{\partial f_t}{\partial n}-\frac{\partial f_t^*}{\partial n}\right)ds+O((\Delta t)^2).$$

It remains to divide both parts of the last inequality by Δt and pass to the limit at $\Delta t\to 0$ Theorem 1 is proved.

4. Another look at the Gonchar-Baernstein conjecture

Let $a = (a_1, ..., a_n)$ be a set of arbitrary distinct points on the circle T, and $a^* = (a_1^*, ..., a_n^*)$ be a set of symmetric points $a_j^* = \exp(i2\pi j/n),\ j = 1, ..., n$. Let K be an arbitrary fixed compact set on the half-interval $(0, 1]$. Denote by $\omega(z)$ the harmonic measure of the circle T with respect to the domain of $D(a, K) = \{z : |z| < 1\} \setminus \bigcup_{j=1}^{n} \{z = ta_j :\ t \in K\}$, and let $\omega^*(z)$ be the harmonic measure of T with respect to $D(a^*, K)$. Gonchar suggested that the inequality

$$\omega(0) \geq \omega^*(0) \tag{11}$$

holds for $K = [t, 1],\ 0 < t < 1$ (see, for example, [11, Problem 7.45]). The solution of the Gonchar problem is given in paper [1]. This solution is implicitly related to numerous unsolved problems of function theory, where the assumed extremal configuration has n-fold symmetry [3] – [5]. In 1987, Baernstein published a statement of the generalized problem of the validity of inequality (11) for any compact $K \subset (0, 1]$ [9]. Such a natural generalization caused serious difficulties for experts. Solynin [12] achieved the greatest result by proving (11) in the case of $K = [t_1, t_2],\ 0 < t_1 < t_2 \leq 1$. Theorem 1 in [8] and the proof of Theorem 1 in this article lead us to the following conjecture about the condenser capacity. Let $C = (E_0, E_1)$ be the simplest condenser considered as a pair of non-empty non-intersecting closed sets E_0 and E_1 of the plane $\mathbb{C}$ [3, Sec. 1.2]. The capacity $\operatorname{cap} C$ of the condenser C is defined as the greatest lower bound of the integrals $I(v, \mathbb{C} \setminus (E_0 \cup E_1))$ over all functions v, continuous in $\overline{\mathbb{C}}$, satisfying the Lipschitz condition locally in $\mathbb{C} \setminus (E_0 \cup E_1)$, equal to zero on E_0 and to unity on E_1. Under the condition of the existence of a potential function u of the condenser C, continuous in $\overline{\mathbb{C}}$, harmonic in $\mathbb{C} \setminus (E_0 \cup E_1)$, equal to zero on E_0 and unity on E_1, the equality

$$\operatorname{cap} C = I(u, \mathbb{C} \setminus (E_0 \cup E_1))$$

is fulfilled. Let K be an arbitrary compact set on $(0, 1)$, and let s be an arbitrary number, $0 < s < \inf K$. In the notation adopted above, we put

$$C(s) = (T(s) \cup \bigcup_{j=1}^{n} \{z = ta_j :\ t \in K\}, T),$$

$$C^*(s) = (T(s) \cup \bigcup_{j=1}^{n} \{z = ta_j^* :\ t \in K\}, T).$$

Conjecture. For any $s_1,\ s_2,\ 0 \leq s_1 < s_2 < \inf K$ the inequality

$$\operatorname{cap} C^*(s_1) - \operatorname{cap} C(s_1) \geq \operatorname{cap} C^*(s_2) - \operatorname{cap} C(s_2) \tag{12}$$

is valid.

Since the capacity of the condenser does not increase with dissymmetrization [3, Sec. 4.4], then both differences in (12) are non-negative, and our Conjecture (just like Theorem 1 [8]) defines a new level of complexity in this matter. The Hadamard formula [10,(A3.11)] (see also Lemma 1) implies the "differential" form of Conjecture (12):

$$\int\limits_{T(s)} \left(\frac{\partial u}{\partial n}\right)^2 ds \geq \int\limits_{T(s)} \left(\frac{\partial u^*}{\partial n}\right)^2 ds$$

where $0 < s < \inf K$ and $u,\ u^*$ are the potential functions of the condensers $C(s)$, and $C^*(c)$ respectively.

Theorem 2. *If Conjecture* (12) *is true, then Conjecture* (11) *is also true.*

Proof. Consider the functions $u = \omega - g$ and

$$g(z) = -\frac{w(0)}{\log r}\left(1 + \frac{\log r(D(a,K),0)}{\log r}\right) g_{D(a,K)}(z,0)$$

where $g_{D(a,K)}(z,0)$ is the Green's function of the domain $D(a,K)$, and $r(D(a,K),0)$ is the inner radius of this domain with respect to the origin, and the parameter $r > 0$ is sufficiently small (cf. [3, proof of Theorem 2.1]). The set $E(r) = \{z:\ |z| < \sqrt{r},\ u(z) \leq 0|\}$ is an "almost disk" centered at the origin and radius r. Let $\varphi(s)$ be an infinitesimal quantity at $s \to 0$ such that the set $E(r),\ r = s(1+\varphi(s))$ contains the disk $|z| \leq s$. In view of the monotonicity of the capacity [3, Theorem 1.15],

$$\operatorname{cap} C(s) \leq \operatorname{cap}(\{z:\ u(z) = 0\}, \{z:\ u(z) = 1\}).$$

The capacity of the last condenser is

$$I(u, D(a,K)\setminus E(r)) = I(\omega, D(a,K)\setminus E(r)) + I(g, D(a,K)\setminus E(r)) + 2\int\limits_{\partial E(r)} g\frac{\partial\omega}{\partial n}ds$$

$$= I(\omega, D(a,K)\setminus E(r)) - \frac{2\pi\omega^2(0)}{\log(s(1+\varphi(s)))} + O\left(\left(\frac{1}{\log s}\right)^2\right)$$

$$= I(\omega, D(a,K)) - \frac{2\pi\omega^2(0)}{\log s} + O\left(\left(\frac{1}{\log s}\right)^2\right), \quad s \to 0.$$

We have used the formula [3, (2.12)]. Therefore, the inequality

$$\operatorname{cap} C(s) \leq I(\omega, D(a,K)) - \frac{2\pi\omega^2(0)}{\log s} + O\left(\left(\frac{1}{\log s}\right)^2\right), \quad s \to 0, \tag{13}$$

holds. Similarly, inequality is shown in the other direction. Thus, in (13) there is an equal sign. Writing this equality for the capacities

$$\operatorname{cap} C^*(s^2),\ \operatorname{cap} C(s^2),\ \operatorname{cap} C^*(s) \text{ and } \operatorname{cap} C(s)$$

and substituting these asymptotics into inequality (12), we obtain (11). Theorem 2 is proved.

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