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THE COMPLEXITY OF QUASIVARIETY LATTICES. II

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Abstract. We prove that if a quasivariety $\mathbf{K}$ contains a finite B^*-class relative to some subquasivariety and some variety possessing some additional property, then $\mathbf{K}$ contains continuum many Q-universal non-profinite subquasivarieties having an independent quasi-equational basis as well as continuum many Q-universal non-profinite subquasivarieties having no such basis.

Keywords: inverse limit, quasi-equational basis, quasivariety, profinite structure, profinite quasivariety.

1. Introduction

Under certain sufficient condition, quasivarieties and their subquasivariety lattices turn out highly complicated structure — from the structural, syntactical, algorithmic, as well as topological point of view. First condition of this type (which ensures the existence of a countable subclass possessing certain properties) were found by M.E. Adams and W. Dziobiak [1]; a bit later, similar conditions were proposed by V.A. Gorbunov [8]. Relatively recently, A.V. Kravchenko *et al.* [9] suggested another sufficient condition, the existence of so-called B-class in a quasivariety, which implies that the quasivariety has complicated inner structure, see [9]–[12] and [13, 14]. However, it turned out that this condition was very strong as very similar complexity results hold for certain concrete quasivarieties of algebraic structures possessing no B-class, see [15]–[17], [2, 32].

In [30], a weaker sufficient condition, the existence of a B^*-class in a quasivariety, was introduced, see Definition 1. It was established in [30, 31] that quasivarieties having B^*-classes possess many of the properties of quasivarieties with B-classes.

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The present paper continues [28]. We establish here that if a quasivariety $\mathbf{K}$ contains a finite B*-class relative to some subquasivariety and some variety possessing some additional property which we call (B*), then $\mathbf{K}$ contains continuum many Q-universal non-profinite subquasivarieties having an independent quasi-equational basis as well as continuum many Q-universal non-profinite subquasivarieties having no such basis, see Theorems 9 and 10. The main results of [12] and [17] turn out to be corollaries of these two theorems. We note that the notion of profiniteness was intensively studied for [quasi]varieties by D. M. Clark *et al.* in [3]–[7], see also A. M. Nurakunov and M. M. Stronkowski [25] as well as A. V. Kravchenko *et al.* [12] and [24].

In Section 4, we present some applications of our main results to concrete quasivarieties of algebraic structures.

2. Main definitions and auxiliary results

This section contains all the necessary definitions and results established earlier that we use in the present paper. For all the definitions and notation which are note defined here, we refer to [12, 31].

2.1. B***-classes.** Here, we present our central definition, the definition of B*-class, as it appeared in [30].

We denote by $\mathcal{P}_{fin}(\omega)$ the set of finite subsets of the set ω of natural numbers. Let $J \subseteq \omega$ be an infinite set. We may assume that $J = \{j_n \mid n < \omega\}$, where $j_m < j_n$ if and only if $m < n$. For a set $T \subseteq \omega$, we put $J(T) = \{j_n \in J \mid n \in T\}$.

Definition 1. Let $\mathbf{M} \subseteq \mathbf{K}(\sigma)$ be a quasivariety of a finite similarity type σ and let $\mathbf{V} \subseteq \mathbf{K}(\sigma)$ be a nonempty homomorphically closed class. A class $\mathbf{A} = \{\mathcal{A}_F \mid F \in \mathcal{P}_{fin}(\omega)\} \subseteq \mathbf{M}$ is called a B*-*class with respect to* $\mathbf{M}$ *and* $\mathbf{V}$, if $\mathbf{A}$ satisfies the following conditions:

- (B_0) for each nonempty $F \in \mathcal{P}_{fin}(\omega)$, the structure $\mathcal{A}_F$ is finitely presented in $\mathbf{M}$; $\mathcal{A}_\varnothing$ is a trivial structure;
- (B_1) if $F = G \cup H$ in $\mathcal{P}_{fin}(\omega)$ then $\mathcal{A}_F \in \mathbf{Q}(\mathcal{A}_G, \mathcal{A}_H)$;
- (B_2^*) for each F, $G \in \mathcal{P}_{fin}(\omega)$, if $F \neq \varnothing$ and $\mathcal{A}_F \in \mathbf{Q}(\mathcal{A}_G, \mathbf{V})$ then $F = G$;
- (B_3^*) for every $F \in \mathcal{P}_{fin}(\omega)$ and every $i < \omega$, if $f \in \mathrm{Hom}(\mathcal{A}_F, \mathcal{A}_{\{i\}})$ then either $f(\mathcal{A}_F) \in \mathbf{V}$ or $i \in F$;
- (B_4^*) for all $F \in \mathcal{P}_{fin}(\omega)$, $\big(\mathbf{H}(\mathcal{A}_F) \cap \mathbf{M}\big) \setminus \mathbf{V} \subseteq \mathbf{A}$.

If $\mathcal{A}_F$ is a finite structure for every nonempty set $F \in \mathcal{P}_{fin}(\omega)$, then we call $\mathbf{A}$ a *finite* B*-*class with respect to* $\mathbf{M}$ *and* $\mathbf{V}$.

Definition 1 was inspired by M. E. Adams and W. Dziobiak [1] as well as by A. V. Kravchenko *et al.* [9, Definition 2.1]. The next statement is a straightforward corollary of the definition of B-class given in [9].

Corollary 1. *If* $\mathbf{A}$ *is a* B*-class with respect to some quasivariety* $\mathbf{M}$ *then* $\mathbf{A}$ *is a* B**-class with respect to* $\mathbf{M}$ *and the trivial variety* $\mathbf{T}$.

2.2. **Properties** (B_Θ^*) **and** (B*)**.** Let $\mathbf{M} \subseteq \mathbf{K}(\sigma)$ be a quasivariety of a finite similarity type σ, let $\mathbf{V} \subseteq \mathbf{K}(\sigma)$ be a nonempty homomorphically closed class, and let a class $\mathbf{A} = \{\mathcal{A}_F \mid F \in \mathcal{P}_{fin}(\omega)\} \subseteq \mathbf{M}$ be a B*-class with respect to $\mathbf{M}$ and $\mathbf{V}$. Consider a nonempty set $F \in \mathcal{P}_{fin}(\omega)$. According to condition (B_0) from Definition 1, there are a finite set X_F of variables and a finite set Δ_F of atomic formulas with free variables belonging to X_F such that $\mathcal{A}_F \cong \mathcal{F}_{\mathbf{M}}(X_F, \Delta_F)$. Let $\gamma_F\colon X_F \to A_F$ be the corresponding interpretation of the variables from X_F.

Furthermore, let $G \subseteq F$. By Lemma 2(ii) below, there is a homomorphism from $\mathcal{A}_F$ onto $\mathcal{A}_G$. We fix one such homomorphism and denote it by f_G^F. By [9, Lemma 1.2], there is a finite set Δ_G^F of atomic formulas and an interpretation $\gamma_G^F\colon X_F \to A_G$ such that $\mathcal{A}_G \cong \mathcal{F}_{\mathbf{M}}(X_F, \Delta_G^F)$ with respect to γ_G^F. In this case, we have also $\Delta_G^F \models_{\mathbf{M}} \Delta_F$ and $f_G^F\big(\gamma_F(x)\big) = \gamma_G^F(x)$ for all $x \in X_F$. We assume that $\Delta_F^F = \Delta_F$ and $\gamma_F^F = \gamma_F$. We put

$$\begin{aligned}
&\Theta = \big(\Delta(\Theta), \Gamma(\Theta), \Omega(\Theta)\big), \quad \text{where}\\
&\Delta(\Theta) = \big\{(X_F, \Delta_F, \gamma_F) \mid F \in \mathcal{P}_{fin}(\omega)\backslash\{\varnothing\}\big\};\\
&\Gamma(\Theta) = \{\gamma_G^F \mid F, G \in \mathcal{P}_{fin}(\omega)\backslash\{\varnothing\},\ G \subseteq F\};\\
&\Omega(\Theta) = \{f_G^F \mid F, G \in \mathcal{P}_{fin}(\omega)\backslash\{\varnothing\},\ G \subseteq F\}.
\end{aligned}$$

We consider also the following condition on the class $\mathbf{A}$ which depends on Θ:

(B_Θ^*) for sets F, $G \in \mathcal{P}_{fin}(\omega)$ such that $\varnothing \neq G \subseteq F$, for a structure $\mathcal{B} \in \mathbf{V}$ and a homomorphism $f \in \mathrm{Hom}(\mathcal{A}_F, \mathcal{B})$, there is a homomorphism $g \in \mathrm{Hom}(\mathcal{A}_G, \mathcal{B})$ such that $g\big(\gamma_G^F(x)\big) = f\big(\gamma_F(x)\big)$ for all $x \in X_F$.

We also consider the following condition on the class $\mathbf{A}$:

(B^*) for sets F, $G \in \mathcal{P}_{fin}(\omega)$ such that $\varnothing \neq G \subseteq F$, for a structure $\mathcal{B} \in \mathbf{V}$ and homomorphisms $f \in \mathrm{Hom}(\mathcal{A}_F, \mathcal{B})$ and $g \in \mathrm{Hom}(\mathcal{A}_F, \mathcal{A}_G)$, there is a homomorphism $h \in \mathrm{Hom}(\mathcal{A}_G, \mathcal{B})$ such that $f = hg$.

Properties (B_Θ^*) and (B^*) were introduced and considered in [30].

2.3. **Auxiliary results on** B^***-classes.** We present here some results from [30, 31] which we need.

Lemma 2. [30, Lemma 1.3] *Let $\mathbf{M} \subseteq \mathbf{K}(\sigma)$ be a quasivariety of a finite similarity type σ, let $\mathbf{V} \subseteq \mathbf{K}(\sigma)$ be a nonempty homomorphically closed class, and let a class $\mathbf{A} = \{\mathcal{A}_F \mid F \in \mathcal{P}_{fin}(\omega)\} \subseteq \mathbf{M}$ satisfy conditions* (B_0) *and* (B_2^*) *with respect to $\mathbf{M}$ and $\mathbf{V}$. The following statements hold.*

(i) *If $\mathcal{A}_F \in \mathbf{V}$ for some $F \in \mathcal{P}_{fin}(\omega)$ then $F = \varnothing$.*

(ii) *Let $\mathbf{A}$ satisfy in addition* (B_1) *and* (B_4^*) *with respect to $\mathbf{M}$ and $\mathbf{V}$. If $G \subseteq F \in \mathcal{P}_{fin}(\omega)$ then $\mathcal{A}_G \in \mathbf{H}(\mathcal{A}_F)$.*

(iii) *Let $\mathbf{A}$ be a* B^**-class with respect to $\mathbf{M}$ and $\mathbf{V}$. If $f \in \mathrm{Hom}(\mathcal{A}_F, \mathcal{A}_G)$ for some F, $G \in \mathcal{P}_{fin}(\omega)$ then either $f(\mathcal{A}_F) \in \mathbf{V}$ or $G \subseteq F$ and $f(\mathcal{A}_F) \cong \mathcal{A}_G$.*

In the context of Definition 1, let $\mathbf{M}^* = (\mathbf{M}\backslash\mathbf{V}) \cup \{\mathcal{E}\}$, where $\mathcal{E}$ denotes the trivial structure.

Lemma 3. [30, Lemma 1.4] *Let $\mathbf{M} \subseteq \mathbf{K}(\sigma)$ be a quasivariety, let $\mathbf{V} \subseteq \mathbf{K}(\sigma)$ be a nonempty homomorphically closed class, let a class $\mathbf{A} = \{\mathcal{A}_F \mid F \in \mathcal{P}_{fin}(\omega)\} \subseteq \mathbf{M}$ be a* B^**-class with respect to $\mathbf{M}$ and $\mathbf{V}$, and let $i < \omega$. If $1_{\mathcal{A}_{\{i\}}}$ has a lower cover in the poset $\mathrm{Con}_{\mathbf{M}^*} \mathcal{A}_{\{i\}}$, then $\mathcal{A}_{\{i\}}$ is an $\mathbf{M}^*$-simple structure.*

Remark 1. We notice that Lemma 3 applies when σ is a finite similarity type and the structure $\mathcal{A}_{\{i\}}$ is finite for all $i < \omega$. In particular, if $\mathbf{A}$ is a finite B^*-class with respect to $\mathbf{M}$, $\mathbf{V} \subseteq \mathbf{K}(\sigma)$, then $\mathbf{A}$ satisfies the following condition:

(B_5^*) for all $i < \omega$, $\mathcal{A}_{\{i\}}$ is an $\mathbf{M}^*$-simple structure.

Lemma 4. [30, Lemma 2.3] *Let* $\mathbf{M} \subseteq \mathbf{K}(\sigma)$ *be a quasivariety of a finite similarity type* σ*, let* $\mathbf{V} \subseteq \mathbf{K}(\sigma)$ *be a variety, and let a class* $\mathbf{A} = \{\mathcal{A}_F \mid F \in \mathcal{P}_{fin}(\omega)\} \subseteq \mathbf{M}$ *be a* B^**-class satisfying* (B^*_Θ) *with respect to* $\mathbf{M}$*,* $\mathbf{V}$*, and some* Θ*. Then condition* (B^*_5) *is equivalent to each one of the following five conditions.*

(i) *For arbitrary sets* $F, G \in \mathcal{P}_{fin}(\omega)$ *such that* $G \subseteq F$ *and for arbitrary homomorphisms* $f_0, f_1 \in \mathrm{Hom}(\mathcal{A}_F, \mathcal{A}_G)$*, if* $f_0(\mathcal{A}_F) \notin \mathbf{V}$ *then there is* $g \in \mathrm{Hom}\big(f_0(\mathcal{A}_F), \mathcal{A}_G\big)$ *such that* $f_1 = gf_0$*;*

(ii) *For arbitrary sets* $F, G, H \in \mathcal{P}_{fin}(\omega)$ *such that* $F \subseteq G \subseteq H$ *and for arbitrary homomorphisms* $f \in \mathrm{Hom}(\mathcal{A}_H, \mathcal{A}_F)$ *and* $g \in \mathrm{Hom}(\mathcal{A}_H, \mathcal{A}_G)$ *such that* $g(\mathcal{A}_H) \notin \mathbf{V}$*, there is* $h \in \mathrm{Hom}\big(g(\mathcal{A}_H), \mathcal{A}_F\big)$ *such that* $f = hg$*.*

(iii) *For arbitrary sets* $F, G \in \mathcal{P}_{fin}(\omega)$ *such that* $G \subseteq F$ *and for arbitrary homomorphisms* $f_0, f_1 \in \mathrm{Hom}(\mathcal{A}_F, \mathcal{A}_G)$ *such that* f_0 *is an onto homomorphism and* $f_1(\mathcal{A}_F) \notin \mathbf{V}$*, there is an embedding* $g \in \mathrm{Hom}(\mathcal{A}_G, \mathcal{A}_G)$ *such that* $f_1 = gf_0$*.*

(iv) *If* $F, G \in \mathcal{P}_{fin}(\omega)$ *and* $G \subseteq F$ *then there exists a unique* $\mathbf{M}$*-congruence* θ *on* $\mathcal{A}_F$ *such that* $\mathcal{A}_G \cong \mathcal{A}_F/\theta$*.*

(v) *For an arbitrary set* $F \in \mathcal{P}_{fin}(\omega)$*,* $\mathrm{Con}_{\mathbf{M}^*} \mathcal{A}_F \cong 2^F$*, where* 2 *denotes a two-element lattice.*

The next statement is a corollary of Lemma 4.

Corollary 5. [30, Corollary 2.4] *Let* $\mathbf{M} \subseteq \mathbf{K}(\sigma)$ *be a quasivariety of a finite similarity type* σ*, let* $\mathbf{V} \subseteq \mathbf{K}(\sigma)$ *be a variety, and let a class* $\mathbf{A} = \{\mathcal{A}_F \mid F \in \mathcal{P}_{fin}(\omega)\} \subseteq \mathbf{M}$ *be a* B^**-class satisfying* (B^*_5) *with respect to* $\mathbf{M}$ *and* $\mathbf{V}$*. Then the class* $\mathbf{A}$ *satisfies* (B^*) *if and only if* $\mathbf{A}$ *satisfies* (B^*_Θ) *for some* Θ*.*

The next statements follows from the proof of [30, Proposition 2.5].

Proposition 6. *Let a quasivariety* $\mathbf{M} \subseteq \mathbf{K}(\sigma)$ *contain a* B^**-class* $\mathbf{A} = \big\{\mathcal{A}_F \mid F \in \mathcal{P}_{fin}(\omega)\big\}$ *satisfying* (B^*) *with respect to* $\mathbf{M}$ *and some variety* $\mathbf{V} \subseteq \mathbf{K}(\sigma)$*. If* $J \subseteq \omega$ *is an infinite set then*

$$\mathbf{B}_J = \big\{\mathcal{A}_{J(F)} \mid F \in \mathcal{P}_{fin}(\omega)\big\}$$

is a B^**-class with respect to* $\mathbf{M}$ *and* $\mathbf{V}$ *which also satisfies* (B^*)*.*

Theorem 7. [31, Theorem 2.1] *Let a class* $\mathbf{A} = \{\mathcal{A}_F \mid F \in \mathcal{P}_{fin}(\omega)\}$ *of algebraic structures of a finite similarity type* σ *be a* B^**-class satisfying* (B^*_5) *and* (B^*) *with respect to a quasivariety* $\mathbf{M} \subseteq \mathbf{K}(\sigma)$ *and a variety* $\mathbf{V} \subseteq \mathbf{K}(\sigma)$*. Then there are continuum many quasivarieties* $\mathbf{K} \subseteq \mathbf{M}$ *with the following properties:*

(i) $\mathbf{K}$ *has a finitely partitionable* ω*-independent quasi-equational basis relative to* $\mathbf{M}$*;*

(ii) *there are quasivarieties* $\mathbf{K}_I \subseteq \mathbf{M}$*,* $I \in \mathfrak{I}$*, such that* $\mathbf{K} = \bigcap_{I \in \mathfrak{I}} \mathbf{K}_I$ *and the quasivariety* $\mathbf{K}_I$ *has no finitely partitionable* ω*-independent quasi-equational basis relative to* $\mathbf{M}$*, but has an* ω*-independent quasi-equational basis relative to* $\mathbf{M}$ *for all* $I \in \mathfrak{I}$*.*

If $\mathbf{A}$ *is a finite* B^**-class then* $\mathbf{K}$ *above can be chosen Q-universal.*

2.4. **Profinite quasivarieties.** For a structure $\mathcal{A} \in \mathbf{K}(\sigma)$, we say that $\mathbb{A} = \langle \mathcal{A}, \mathcal{T}\rangle$ is a *topological structure*, if $\mathcal{T}$ is a topology on A and all the basic operations of $\mathcal{A}$ are continuous and all the basic relations on $\mathcal{A}$ are closed with respect to $\mathcal{T}$. For a topological structure $\mathbb{A}$, we denote its algebraic reduct by $\mathcal{A}$ and its topology by $\mathcal{T}(\mathbb{A})$. A topology $\mathcal{T}$ on a set A is *Boolean* if the topological space $\langle A, \mathcal{T}\rangle$ is Hausdorff, compact and has a base of clopen sets. A topological structure $\mathbb{A}$ is *Boolean* if $\mathcal{T}(\mathbb{A})$ is a Boolean topology.

A structure $\mathcal{A} \in \mathbf{K}(\sigma)$ is *profinite* if it is isomorphic to an inverse limit of finite structures. Profinite structures are naturally equipped with Boolean topologies which are in this case product topologies of discrete topologies on finite spaces. A prevariety $\mathbf{K}$ is *profinite* if each profinite structure belonging to $\mathbf{K}$ is *profinite with respect to* $\mathbf{K}$; that is, is isomorphic to an inverse limit of finite structures belonging to $\mathbf{K}$.

Lemma 8. [6, Lemma 3.2] *Let σ contain only finitely many relation symbols, let $\mathcal{A} = \varprojlim_{i\in I} \mathcal{A}_i$ be a surjective inverse limit of finite structures, let $\mathcal{B}$ be a finite structure, and let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be a [continuous] homomorphism. Then there exist $i \in I$ and a [continuous] homomorphism $\psi\colon \mathcal{A}_i \to \mathcal{B}$ such that $\varphi = \psi\pi_i$.*

3. Main results

In the proof of the following theorem, we use ideas from the proof of [12, Theorem 4] as well as certain statements from the proof of [30, Theorem 5.1].

Theorem 9. *Let a class $\mathbf{A} = \{\mathcal{A}_F \mid F \in \mathcal{P}_{fin}(\omega)\}$ of structures of a finite similarity type σ be a finite* B**-class with respect to a quasivariety $\mathbf{M} \subseteq \mathbf{K}(\sigma)$ and a variety $\mathbf{V} \subseteq \mathbf{K}(\sigma)$ which satisfies* (B*)*. Then there are continuum many Q-universal subquasivarieties in $\mathbf{M}$ which are not profinite and have no finitely partitionable ω-independent quasi-equational basis relative to $\mathbf{M}$.*

Proof. By Remark 1 and Corollary 5, $\mathbf{A}$ satisfies (B$^*_\Theta$) for some Θ with

$$\begin{aligned}
&\Theta = \big(\Delta(\Theta), \Gamma(\Theta), \Omega(\Theta)\big), \quad \text{where}\\
&\Delta(\Theta) = \big\{(X_F, \Delta_F, \gamma_F) \mid F \in \mathcal{P}_{fin}(\omega)\backslash\{\varnothing\}\big\};\\
&\Gamma(\Theta) = \{\gamma^F_G \mid F, G \in \mathcal{P}_{fin}(\omega)\backslash\{\varnothing\},\ G \subseteq F\};\\
&\Omega(\Theta) = \{f^F_G \mid F, G \in \mathcal{P}_{fin}(\omega)\backslash\{\varnothing\},\ G \subseteq F\}.
\end{aligned}$$

Following the proof of [30, Theorem 5.1], we fix one such Θ. Moreover, we fix an infinite set $I \subseteq \omega$ such that $\omega\backslash I \neq \varnothing$ and assume that $I = \{i_n \mid n < \omega\}$, where $i_n \leqslant i_m$ if and only if $n \leqslant m$ for all $m, n < \omega$. As in the proof of [12, Theorem 4], we also fix an element $k \in \omega\backslash I$ and we put

$$\begin{aligned}
&F_n = \{i_0, \dots, i_n\} \quad \text{for all } n < \omega;\\
&\mathcal{A} = \mathcal{A}_{\{k\}}, \quad \mathcal{A}_n = \mathcal{A}_{F_n} \quad \text{and} \quad \mathcal{B}_n = \mathcal{A}_{\{k\}\cup F_n} \quad \text{for all } n < \omega.
\end{aligned}$$

As in the proof of Theorem 5.1 in [30], we consider the following sentence φ^I_F which is equivalent to a finite set of quasi-identities:

$$\forall\, \overline{x}\ \ \&\Delta_F(\overline{x}) \ \longrightarrow\ \&\Delta^F_{F\cap I}(\overline{x}),$$

where $\mathcal{A}_F \cong \mathcal{F}_{\mathbf{K}}(X_F, \Delta_F)$ with a canonical interpretation γ_F for all $F \in \mathcal{P}_{fin}(\omega)$. We put also

$$\Phi_I = \big\{\varphi^I_F \mid F \in \mathcal{P}_{fin}(\omega),\ F \cap I \neq \varnothing\big\} \quad \text{and} \quad \mathbf{K}_I = \operatorname{Mod}\Phi_I \cap \mathbf{M}.$$

By [30, Theorem 5.1], $\mathbf{K}_I$ is Q-universal and has no finitely partitionable ω-independent quasi-equational basis relative to $\mathbf{M}$. The following statement is the content of Claim 5.2 from the proof of [30, Theorem 5.1].

Claim 1. For each $F \in \mathcal{P}_{fin}(\omega)$, $\mathcal{A}_F \in \mathbf{K}_I$ if and only if $F \subseteq I$.

We construct an inverse spectrum $\Lambda = \langle \omega, \mathcal{B}_j, \pi_{ij} \rangle$ as in the proof of [12, Theorem 4]. By Lemma 2(ii), for each $n < \omega$, there is a surjective homomorphism $\pi_{n,n+1} \colon \mathcal{B}_{n+1} \to \mathcal{B}_n$. We put for all $i \leqslant n < \omega$:

$$\pi_{in} = \begin{cases} \mathrm{id}_{\mathcal{B}_i}, & \text{if } i = n; \\ \pi_{i,i+1} \dots \pi_{n-1,n}, & \text{if } i < n. \end{cases}$$

We obtain therefore

Claim 2. $\Lambda = \langle \omega, \mathcal{B}_j, \pi_{ij} \rangle$ is a surjective inverse limit.

We put $\mathcal{B} = \varprojlim \Lambda$. Since B_n is a finite set for all $n < \omega$, we conclude that $B \neq \varnothing$. Moreover, $\mathcal{B} \in \mathbf{SP}(\mathbf{A}) \subseteq \mathbf{Q}(\mathbf{A}) \subseteq \mathbf{M}$.

Claim 3. $\mathcal{B}$ is an infinite structure and $\mathcal{B} \in \mathbf{K}_I$.

Proof of Claim. We show first that $\mathcal{B} \in \mathbf{K}_I$. As $\mathcal{B} \in \mathbf{M}$, it suffices to establish that $\mathcal{B} \models \Phi_I$. We consider therefore an arbitrary sentence $\varphi_F^I \in \Phi_I$; then $F \cap I \neq \varnothing$. Let $\gamma \colon X_F \to B$ be an interpretation such that $\mathcal{B} \models \Delta_F[\gamma]$. This means that there is a homomorphism $f \colon \mathcal{A}_F \to \mathcal{B}$ such that $f\big(\gamma_F(x)\big) = \gamma(x)$ for each $x \in X_F$. For each $n < \omega$, let $f_n = \pi_n f$, where $\pi_n \colon \mathcal{B} \to \mathcal{B}_n$ denotes the canonical projection from $\mathcal{B}$ onto $\mathcal{B}_n$. Two cases are possible.

Case 1: $f(\mathcal{A}_F) \in \mathbf{V}$.

Case 2: $f(\mathcal{A}_F) \notin \mathbf{V}$. In this case, we obtain by Lemma 2(iii) that $f_n(\mathcal{A}_F) \cong \mathcal{A}_G \leq \mathcal{B}_n$ for some nonempty set $G \subseteq F$. Therefore, (B_2^*) yields the equality $G = \{k\} \cup F_n$. This implies that $f_n(\mathcal{A}_F) \cong \mathcal{B}_n$ whence $|B_n| \leqslant |A_F|$ in this case.

Since $\mathcal{A}_F$ is a finite structure and $|B_n| < |B_{n+1}|$ for all $n < \omega$ by Lemma 2(ii), there is $s < \omega$ such that $|A_F| < |B_n|$ for all $n \geqslant s$. Therefore, $f_n(\mathcal{A}_F) \in \mathbf{V}$ for all $n \geqslant s$. Let $J_s = \{n < \omega \mid n \geqslant s\}$; we have then

$$\mathcal{B} \cong \varprojlim \langle J_s, \mathcal{B}_j, \pi_{ij} \rangle \leq_s \prod_{n \in J_s} \mathcal{B}_n;$$

$$f(\mathcal{A}_F) \leq_s \prod_{n \in J_s} f_n(\mathcal{B}) \in \mathbf{V}.$$

This yields that the substructure $f(\mathcal{A}_F)$ of $\mathcal{B}$ generated by the set $\{\gamma(x) \mid x \in F\}$ belongs to $\mathbf{V}$. Using our assumption that $F \cap I \neq \varnothing$ and applying (B_Θ^*), we obtain that there is a homomorphism $g \colon \mathcal{A}_{F \cap I} \to \mathcal{B}$ such that $g\big(\gamma_{F \cap I}^F(x)\big) = f\big(\gamma^F(x)\big) = \gamma(x)$ for all $x \in X_F$. As $\mathcal{A}_{F \cap I} \models \Delta_{F \cap I}^F[\gamma_{F \cap I}^F]$, we conclude that $\mathcal{B} \models \Delta_{F \cap I}^F[\gamma]$ and $\mathcal{B} \models \varphi_F^I$ therefore.

Finally, if $|B| < \omega$ then by Lemma 8, there is $n < \omega$ such that $|B| \leqslant |B_n|$. Hence, $|B| \leqslant |B_n| \leqslant |B_m|$ for all $m \geqslant n$ by Lemma 2(ii). As π_n is a homomorphism from $\mathcal{B}$ onto $\mathcal{B}$, we have that $|B_m| \leqslant |B| \leqslant |B_m|$ and $|B_m| = |B|$ for all $m \geqslant n$ which is impossible as $|\sigma^P| < \omega$. □

According to Claim 1, we have $\mathcal{A} \notin \mathbf{K}_I$, $\mathcal{A}_n \in \mathbf{K}_I$ and $\mathcal{B}_n \notin \mathbf{K}_I$ for all $n < \omega$. By (B_1), $\mathcal{B}_n \in \mathbf{Q}(\mathcal{A}, \mathcal{A}_n)$. As $\mathcal{B}_n \notin \mathbf{V}$ and $\mathcal{A}$ and $\mathcal{A}_n$ are finite structures of finite type, we conclude by (B_2^*) that $\mathcal{B}_n$ is a subdirect product of structures isomorphic to $\mathcal{A}$ or to $\mathcal{A}_n$. Lemma 4(iv) yields that $\mathcal{B}_n \leq_s \mathcal{A} \times \mathcal{A}_n$. For each $n < \omega$, let

$$\alpha_n \colon \mathcal{B}_n \to \mathcal{A} \quad \text{and} \quad \beta_n \colon \mathcal{B}_n \to \mathcal{A}_n$$

$$\begin{tikzcd}
\mathcal{A}_0 & \mathcal{A}_1 \arrow[l, "\varphi_{01}"'] & \mathcal{A}_2 \arrow[l, "\varphi_{12}"'] & \bullet\bullet\bullet \arrow[l, "\varphi_{23}"'] & \mathcal{A}_n \arrow[l, "\varphi_{n-1,n}"'] & \bullet\bullet\bullet \arrow[l, "\varphi_{n,n+1}"'] \\
 & & & \bullet\bullet\bullet & & \bullet\bullet\bullet \\
\mathcal{B}_0 \arrow[uu, "\beta_0"'] \arrow[dd, "\alpha_0"] & \mathcal{B}_1 \arrow[l, "\pi_{01}"'] \arrow[uu, "\beta_1"'] \arrow[dd, "\alpha_1"] & \mathcal{B}_2 \arrow[l, "\pi_{12}"'] \arrow[uu, "\beta_2"'] \arrow[dd, "\alpha_2"] & \bullet\bullet\bullet \arrow[l, "\pi_{23}"'] & \mathcal{B}_n \arrow[l, "\pi_{n-1,n}"'] \arrow[uu, "\beta_n"'] \arrow[dd, "\alpha_n"] & \bullet\bullet\bullet \arrow[l, "\pi_{n,n+1}"'] \\
 & & & \bullet\bullet\bullet & & \bullet\bullet\bullet \\
\mathcal{A} \arrow[r, "\psi_{01}"] & \mathcal{A} \arrow[r, "\psi_{12}"] & \mathcal{A} \arrow[r, "\psi_{23}"] & \bullet\bullet\bullet \arrow[r, "\psi_{n-1,n}"] & \mathcal{A} \arrow[r, "\psi_{n,n+1}"] & \bullet\bullet\bullet
\end{tikzcd}$$

FIGURE 1.

denote the canonical surjective homomorphisms corresponding to these subdirect decompositions. Since $\mathcal{B}_n \notin \mathbf{K}_I$ and $\mathcal{A}_n \in \mathbf{K}_I$, we obtain that $\ker \beta_n \neq 0_{\mathcal{B}_n}$ for each $n < \omega$. This means that, for each $n < \omega$, there exists a relation symbol $p_n \in \sigma^P \cup \{=\}$ of arity m_n and elements $b_1^n, \dots, b_{m_n}^n \in B_n$ such that

$$\mathcal{B}_n \not\models p_n(b_1^n, \dots, b_{m_n}^n) \quad \text{and} \quad \mathcal{A}_n \models p_n\big(\beta_n(b_1^n), \dots, \beta_n(b_{m_n}^n)\big).$$

This yields that $\mathcal{A} \not\models p_n\big(\alpha_n(b_1^n), \dots, \alpha_n(b_{m_n}^n)\big)$. Since $\sigma^P \cup \{=\}$ and A are finite sets, there is an infinite set $J \subseteq \omega$ and a relation symbol $p \in \sigma^P \cup \{=\}$ of arity m such that $p_n = p$ and $m_n = m$ for all $n \in J$.

According to Lemma 4(ii), for each $n < \omega$, there is a surjective homomorphism $\varphi_{n,n+1} \colon \mathcal{A}_{n+1} \to \mathcal{A}_n$ such that $\beta_n \pi_{n,n+1} = \varphi_{n,n+1}\beta_{n+1}$. According to Lemma 4(iii), for each $n < \omega$, there is an isomorphism $\psi_{n,n+1} \colon \mathcal{A} \to \mathcal{A}$ such that $\psi_{n,n+1}\alpha_n\pi_{n,n+1} = \alpha_{n+1}$, see Figure 1. We put for all $i \leqslant n < \omega$:

$$\varphi_{in} = \begin{cases} \mathrm{id}_{\mathcal{A}_i}, & \text{if } i = n; \\ \varphi_{i,i+1} \dots \varphi_{n-1,n}, & \text{if } i < n. \end{cases} \qquad \psi_{in} = \begin{cases} \mathrm{id}_{\mathcal{A}}, & \text{if } i = n; \\ \psi_{n-1,n} \dots \psi_{i,i+1}, & \text{if } i < n. \end{cases}$$

The for all $i \leqslant n < \omega$, we have

$$\beta_i \pi_{in} = \varphi_{in}\beta_n, \quad \psi_{in}\alpha_i\pi_{in} = \alpha_n;$$

see Figure 1. The proof of the following claim repeats the proof of Claim 4 from the proof of [12, Theorem 4].

Claim 4. If $(b_1, \dots, b_m) \in \ker \beta_n(p) \setminus \ker \alpha_n(p)$ for some $n < \omega$ then

$$\big(\pi_{tn}(b_1), \dots, \pi_{tn}(b_m)\big) \in \ker \beta_t(p) \setminus \ker \alpha_t(p)$$

for all $t \leqslant n$.

Since the set $J \subseteq \omega$ is infinite, it follows from Claim 4 that for each $n < \omega$, there are elements $b_1^n, \dots, b_m^n \in B_n$ such that

$$\begin{aligned}
&\mathcal{B}_n \not\models p(b_1^n, \dots, b_m^n); \\
&\mathcal{A}_n \models p\big(\beta_n(b_1^n), \dots, \beta_n(b_m^n)\big) \quad \text{and thus, } (b_1^n, \dots, b_m^n) \in \ker \beta_n(p); \\
&\mathcal{A} \not\models p\big(\alpha_n(b_1^n), \dots, \alpha_n(b_m^n)\big) \quad \text{and thus, } (b_1^n, \dots, b_m^n) \notin \ker \alpha_n(p).
\end{aligned}$$

Claim 5. There are elements $c_1, \dots, c_m \in B$ such that $\mathcal{B} \not\models p(c_1, \dots, c_m)$ and, for each homomorphism $f\colon \mathcal{B} \to \mathcal{D}$ where $\mathcal{D} \in \mathbf{K}_I$ is a finite structure, $\mathcal{D} \models p\big(f(c_1), \dots, f(c_m)\big)$.

Proof of Claim. Exactly as in the proof of Claim 5 from the proof of [12, Theorem 4], we can construct elements $c_1, \dots, c_m \in B$ such that

$$\big(\pi_n(c_1), \dots, \pi_n(c_m)\big) \in \ker \beta_n(p) \setminus \ker \alpha_n(p) \quad \text{for all } n < \omega.$$

As $\big(\pi_n(c_1), \dots, \pi_n(c_m)\big) \notin \ker \alpha_n(p)$ for all $n < \omega$, we have $\mathcal{B} \not\models p(c_1, \dots, c_m)$.

Consider an arbitrary homomorphism $f\colon \mathcal{B} \to \mathcal{D}$ with $\mathcal{D} \in \mathbf{K}_I$ being a finite structure. By Lemma 8, there is $n < \omega$ and a homomorphism $g\colon \mathcal{B}_n \to \mathcal{D}$ such that $f = g\pi_n$. Two cases are possible.

Case 1: $g(\mathcal{B}_n) \in \mathbf{V}$. As $\mathbf{A}$ satisfies (B^*_Θ), there is a homomorphism $h\colon \mathcal{A}_n \to \mathcal{D}$ such that $h\big(\gamma_{F_n}^{\{k\}\cup F_n}(x)\big) = f\big(\gamma^{\{k\}\cup F_n}(x)\big)$ for all $x \in X_{\{k\}\cup F_n}$. As the set $X_{\{k\}\cup F_n}$ generates $\mathcal{B}_n$, we conclude that $g = hf_{F_n}^{\{k\}\cup F_n} = h\beta_n$ whence $f = g\pi_n = h\beta_n\pi_n$.

Case 2: $g(\mathcal{B}_n) \notin \mathbf{V}$. By Lemma 2(iii), we conclude that $g(\mathcal{B}_n) \cong \mathcal{A}_G \leq \mathcal{D} \in \mathbf{K}_I$ for some $G \subseteq \{k\} \cup F_n$. Thus, $\mathcal{A}_G \in \mathbf{K}_I$ whence $G \subseteq (\{k\} \cup F_n) \cap I = F_n$ by Claim 1. By Lemma 4(ii), there is a homomorphism $h\colon \mathcal{A}_n \to \mathcal{D}$ with $g = h\beta_n$; therefore, $f = h\beta_n\pi_n$.

In both cases, $f = h\beta_n\pi_n$ for some homomorphism $h\colon \mathcal{A}_n \to \mathcal{D}$. This implies, in particular, that $\ker \beta_n\pi_n \leq \ker f$. Since $\big(\pi_n(c_1), \dots, \pi_n(c_m)\big) \in \ker \beta_n(p)$, we conclude that $(c_1, \dots, c_m) \in \ker \beta_n\pi_n(p) \subseteq \ker f(p)$. Therefore, $\mathcal{D} \models p\big(f(c_1), \dots, f(c_m)\big)$. □

By Claims 3 and 5, the profinite structure $\mathcal{B} \in \mathbf{K}_I$ does not embed into a Cartesian product of finite structures from $\mathbf{K}_I$, whence it is not profinite with respect to $\mathbf{K}_I$. Hence, the quasivariety $\mathbf{K}_I$ is not profinite.

If $I, J \subseteq \omega$ are such that $I \nsubseteq J$ then there is $i \in I \setminus J$. According to Claim 1, $\mathcal{A}_i \in \mathbf{K}_I \setminus \mathbf{K}_J$, whence $\mathbf{K}_I \nsubseteq \mathbf{K}_J$. It remains to recall that there are continuum many infinite proper subsets $I \subset \omega$. □

In the proof of the following theorem, we use ideas from the proof of [12, Theorem 5] as well as certain statements from the proof of [31, Theorem 2.1].

Theorem 10. *Let a class* $\mathbf{A} = \{\mathcal{A}_F \mid F \in \mathcal{P}_{fin}(\omega)\}$ *of structures of a finite similarity type* σ *be a finite* B^**-class with respect to a quasivariety* $\mathbf{M} \subseteq \mathbf{K}(\sigma)$ *and a variety* $\mathbf{V} \subseteq \mathbf{K}(\sigma)$ *which satisfies* (B^*). *Then there are continuum many Q-universal subquasivarieties in* $\mathbf{M}$ *which are not profinite and have a finitely partitionable* ω*-independent quasi-equational basis relative to* $\mathbf{M}$.

Proof. Again, we fix Θ and an infinite set $I \subseteq \omega$ and assume that $I = \{i_n \mid n < \omega\}$, where $i_n \leqslant i_m$ if and only if $n \leqslant m$ for all $m, n < \omega$. As in the proof of [12, Theorem 5], we also put

$$F_\perp = \{i_1\} \quad \text{and} \quad \mathcal{B}_\perp = \mathcal{A}_{F_\perp} \quad F_n = \{i_0, \dots, i_n\} \quad \text{and} \quad \mathcal{B}_n = \mathcal{A}_{F_n} \quad \text{for all } n < \omega.$$

As in the proof of [30, Theorem 2.1], see also Theorem 7, we consider for each $m < \omega$ the following sentence which is equivalent to a finite set of quasi-identities:

$$\forall \overline{x} \quad \&\Delta_{F_{m+1}}(\overline{x}) \;\longrightarrow\; \&\Delta_{F_m}^{F_{m+1}}(\overline{x}).$$

We denote this sentence by ξ_m.

We have $F_\perp = \{i_0\} \subseteq \{i_0, i_1\} = F_1$. By $\xi_\perp$, we denote the following sentence which is also equivalent to a finite set of quasi-identities:

$$\forall \overline{x} \quad \&\Delta_{F_1}(\overline{x}) \longrightarrow \&\Delta_{F_\perp}^{F_1}(\overline{x}).$$

We put

$$\Xi_I = \{\xi_m \mid m < \omega\} \cup \{\xi_\perp\} \quad \text{and} \quad \mathbf{M}_I = \operatorname{Mod} \Xi_I \cap \mathbf{M}.$$

By Claims 2.5 and 2.7 from the proof of [30, Theorem 2.1], see also Theorem 7, we obtain the following

Claim 1. $\mathbf{M}_I$ is a Q-universal quasivariety which consists of all structures $\mathcal{A} \in \mathbf{M}$ with the following property:

$$\text{if } \mathcal{A}_F \in \mathbf{S}(\mathcal{A}) \text{ for some nonempty } F \in \mathcal{P}_{fin}(\omega) \text{ then } F \nsubseteq I.$$

Moreover, Ξ_I is a finitely partitionable ω-independent quasi-equational basis of $\mathbf{M}_I$ relative to $\mathbf{M}$.

Let $\omega' = \omega \cup \{\perp\}$ where $\perp \leqslant n$ for all n such that $0 < n < \omega$. It is clear that ω' is an up-directed set with respect to $\leqslant$. We construct an inverse spectrum $\Lambda = \langle \omega', \mathcal{B}_j, \pi_{ij} \rangle$ as follows. According to Lemma 2(ii), for each $n < \omega$, there is a surjective homomorphism $\pi_{n,n+1} \colon \mathcal{B}_{n+1} \to \mathcal{B}_n$. We put for all $i \leqslant n < \omega$:

$$\pi_{in} = \begin{cases} \mathrm{id}_{\mathcal{B}_i}, & \text{if } i = n; \\ \pi_{i,i+1} \dots \pi_{n-1,n}, & \text{if } i < n. \end{cases}$$

By Lemma 2(ii), there is a surjective homomorphism $\pi_{\perp,1} \colon \mathcal{B}_1 \to \mathcal{B}_\perp$. We put $\pi_{\perp\perp} = \mathrm{id}_{\mathcal{B}_\perp}$ and $\pi_{\perp n} = \pi_{\perp,1} \dots \pi_{n-1,n}$ for all n such that $0 < n < \omega$:

The next statement follows immediately.

Claim 2. $\Lambda = \langle \omega', \mathcal{B}_j, \pi_{ij} \rangle$ is a surjective inverse limit.

We put $\mathcal{B} = \varprojlim \Lambda$. Since B_n is a finite set for all $n \in \omega'$, we conclude that $B \neq \varnothing$. Moreover, $\mathcal{B} \in \mathbf{SP}(\mathbf{A}) \subseteq \mathbf{Q}(\mathbf{A}) \subseteq \mathbf{M}$.

Claim 3. $\mathcal{B}$ is an infinite structure and $\mathcal{B} \in \mathbf{M}_I$.

Proof of Claim. Exactly as in the proof of Claim 3 from the proof of Theorem 9, one establishes that $\mathcal{B}$ is an infinite structure. We show now that $\mathcal{B} \models \Xi_I$. Assume that $m < \omega$ and that an interpretation $\gamma \colon X_{F_{m+1}} \to B$ is such that $\mathcal{B} \models \Delta_{F_{m+1}}[\gamma]$. This means that there is a homomorphism $f \colon \mathcal{B}_{m+1} \to \mathcal{B}$ such that $f\big(\gamma_{F_{m+1}}(x)\big) = \gamma(x)$ for each $x \in X_{F_{m+1}}$. For each $n \in \omega'$, let $f_n = \pi_n f$ where $\pi_n \colon \mathcal{B} \to \mathcal{B}_n$ is a canonical projection. Two cases are possible.

Case 1: $f_n(\mathcal{B}_{m+1}) \in \mathbf{V}$.

Case 2: $f_n(\mathcal{B}_{m+1}) \notin \mathbf{V}$. In this case, we have by Lemma 2(iii) that $f_n(\mathcal{B}_{m+1}) \cong \mathcal{A}_G \leq \mathcal{B}_n$ for some nonempty set $G \subseteq F_m$. Therefore, we get by (B_2^*) that $G = F_n$. This implies that $f_n(\mathcal{B}_{m+1}) \cong \mathcal{B}_n$ whence $|B_n| \leqslant |B_{m+1}|$ in this case.

Since $\mathcal{B}_{m+1}$ is a finite structure and $|B_n| < |B_{n+1}|$ for all $n < \omega$ by Lemma 2(ii), there is $s < \omega$ such that $|B_{m+1}| < |B_n|$ for all $n \geqslant s$. Therefore, $f_n(\mathcal{B}_{m+1}) \in \mathbf{V}$ for all $n \geqslant s$. Let $J_s = \{n < \omega \mid n \geqslant s\}$; we have then

$$\mathcal{B} \cong \varprojlim \langle J_s, \mathcal{B}_j, \pi_{ij} \rangle \leq_s \prod_{n \in J_s} \mathcal{B}_n;$$

$$f(\mathcal{B}_{m+1}) \leq_s \prod_{n \in J_s} f_n(\mathcal{B}) \in \mathbf{V}.$$

This yields that the substructure $f(\mathcal{B}_{m+1})$ of $\mathcal{B}$ generated by the set $\{\gamma(x) \mid x \in F_{m+1}\}$ belongs to $\mathbf{V}$. As $F_m \neq \varnothing$ and $F_\perp \neq \varnothing$, we apply (B^*_Θ) and obtain that there is a homomorphism $g\colon \mathcal{B}_m \to \mathcal{B}$ such that $g\big(\gamma^{F_{m+1}}_{F_m}(x)\big) = f\big(\gamma^{F_{m+1}}(x)\big) = \gamma(x)$ for all $x \in X_{F_{m+1}}$, as well as there is a homomorphism $g\colon \mathcal{B}_\perp \to \mathcal{B}$ such that $g\big(\gamma^{F_1}_{F_\perp}(x)\big) = f\big(\gamma^{F_1}(x)\big) = \gamma(x)$ for all $x \in X_{F_1}$ whenever $m = 0$. As $\mathcal{B}_m \models \Delta^{F_{m+1}}_{F_m}[\gamma^{F_{m+1}}_{F_m}]$ and $\mathcal{B}_\perp \models \Delta^{F_1}_{F_\perp}[\gamma^{F_1}_{F_\perp}]$, we conclude that $\mathcal{B} \models \Delta^{F_{m+1}}_{F_m}[\gamma]$ and $\mathcal{B} \models \Delta^{F_1}_{F_\perp}[\gamma]$ whenever $m = 0$. This implies that $\mathcal{B} \models \xi_m$ for all $m < \omega$ and $\mathcal{B} \models \xi_\perp$. ☐

Claim 4. $\mathcal{B}$ is not profinite with respect to $\mathbf{M}_I$.

Proof of Claim. Consider an arbitrary homomorphism $f\colon \mathcal{B} \to \mathcal{D}$, where $\mathcal{D} \in \mathbf{M}_I$ is a finite structure. By Lemma 8, there is $n \in \omega'$ and a homomorphism $g\colon \mathcal{B}_n \to \mathcal{D}$ such that $f = g\pi_n$. We have by Lemma 2(iii) and (B^*_2) that either $g(\mathcal{B}_n) \in \mathbf{V}$ or $g(\mathcal{B}_n) \cong \mathcal{A}_G \leq \mathcal{D} \in \mathbf{M}_I$ for some nonempty finite set G such that $G \subseteq F_n \subseteq I$. In the second case, we obtain a contradiction with Claim 1. This contradiction shows that $f(\mathcal{B}) \cong g(\mathcal{B}_n) \in \mathbf{V}$.

If $\mathcal{B}$ were profinite with respect to $\mathbf{M}_I$ then $\mathcal{B} \leq_s \prod_{t \in T} \mathcal{D}_t$, where $\mathcal{D}_t \in \mathcal{M}_I$ is a finite structure for each $t \in T$. According to the above argument $\mathcal{D}_t \in \mathbf{V}$ for each $t \in T$ whence $\mathcal{B} \in \mathbf{V}$. As $\mathbf{V}$ is a variety and $\mathcal{B}_n \in \mathbf{H}(\mathcal{B})$ for all $n \in \omega'$, we conclude that $\mathcal{B}_n \in \mathbf{V}$ for all $n \in \omega'$ which contradicts Lemma 2(i). ☐

By Claims 3 and 4, the profinite structure $\mathcal{B} \in \mathbf{M}_I$ is not profinite with respect to $\mathbf{M}_I$, whence the quasivariety $\mathbf{M}_I$ is not profinite.

If $I, J \subseteq \omega$ are such that $I \nsubseteq J$ then there is $i \in I \backslash J$. According to Claim 1, $\mathcal{A}_i \in \mathbf{M}_J \backslash \mathbf{M}_I$, whence $\mathbf{M}_J \nsubseteq \mathbf{M}_I$. The fact that there are continuum many infinite proper subsets in ω completes the proof. ☐

4. Applications

We present in this section some applications of Theorems 9 and 10 in a compact way. For more details, we refer to [30, 31].

Remark 2. As each B-class is a B*-class, [12, Theorem 4] is a corollary of our Theorem 9 while [12, Theorem 5] is a corollary of Theorem 10. In particular, all the statements of [12, Corollary 1] follow from Theorems 9–10 by Corollary 1.

For the definition of relatively full embedding and almost ff-universal quasivariety, we refer to A. Pultr and V. Trnková [26] and to V. Koubek and J. Sichler [22, 23].

Corollary 11. *If $\mathbf{K} \subseteq \mathbf{K}(\sigma)$ is an almost ff-universal quasivariety of a finite type σ, then there is a quasivariety $\mathbf{K}' \subseteq \mathbf{K}$ which contains continuum many Q-universal non-profinite quasivarieties $\mathbf{M} \subseteq \mathbf{K}'$ with one of the following properties:*

(i) *$\mathbf{M}$ has a finitely partitionable ω-independent quasi-equational basis relative to some subquasivariety $\mathbf{K}'$;*

(ii) **M** *has no finitely partitionable* ω*-independent quasi-equational basis relative to some subquasivariety* $\mathbf{K}'$.

Proof. By Theorem 6.2 and Corollary 6.11 of [30], **K** contains a finite B*-class relative to some subquasivariety $\mathbf{K}' \subseteq \mathbf{K}$ and some variety which satisfies the condition (B*). The desired statement follows from Theorems 9–10. □

The statement of the next corollary was established in [17]. For the definition of the variety **Dm** of differential groupoids, we refer to A. B. Romanowska and J. D. H. Smith [27], see also [15]–[17], [30, Section 4] and [2].

Theorem 12. [31, Theorem 2.2] *Let a class* $\mathbf{A} = \{\mathcal{A}_F \mid F \in \mathcal{P}_{fin}(\omega)\}$ *of structures of a finite similarity type* σ *be a* B*-*class satisfying* (B^*_5) *and* (B*) *with respect to a quasivariety* $\mathbf{M} \subseteq \mathbf{K}(\sigma)$ *and a variety* $\mathbf{V} \subseteq \mathbf{K}(\sigma)$. *Assume that*

$$\Delta_F^{F\cup\{n\}} \text{ is equivalent within } \mathbf{M} \text{ to } \Delta_F \cup \{\varphi_n^F\} \tag{4.1}$$

where φ_n^F *is an atomic formula for all finite nonempty sets* $F \in \mathcal{P}_{fin}(\omega)$ *and all* $n < \omega$ *such that* $n \notin F$. *Then there are continuum many quasivarieties* $\mathbf{K} \subseteq \mathbf{M}$ *with the following properties:*

(i) **K** *has an independent quasi-equational basis relative to* **M**;
(ii) *there is a set* $\mathfrak{I}$ *and quasivarieties* $\mathbf{K}_I \subseteq \mathbf{M}$, $I \in \mathfrak{I}$, *such that* $\mathbf{K} = \bigcap_{I\in\mathfrak{I}} \mathbf{K}_I$ *and the quasivariety* $\mathbf{K}_I$ *has no finitely partitionable* ω*-independent quasi-equational basis relative to* **M**, *but has an* ω*-independent quasi-equational basis relative to* **M** *for all* $I \in \mathfrak{I}$.

If **A** *is a finite* B*-*class then* **K** *above can be chosen Q-universal.*

Corollary 13. *The variety* **Dm** *contains continuum many Q-universal non-profinite subquasivarieties* $\mathbf{M} \subseteq \mathbf{Dm}$ *with one the following properties:*

(i) **M** *has an independent quasi-equational basis relative to* **Dm**;
(ii) **M** *has no independent quasi-equational basis relative to* **Dm**.

Proof. By [30, Proposition 7.1], the variety **Dm** contains a finite B*-class satisfying (B*) with respect to **Dm** and a finitely generated variety $\mathbf{V} \subseteq \mathbf{Dm}$ for which the assumptions of Theorem 12 are satisfied; in particular, the condition (4.1) is fulfilled. The desired statement follows from Theorems 9–10 and Theorem 12. □

Corollary 14. *Let* $\mathbf{K}'$ *be one of the following classes:*

(a) *the variety of lattices generated by the modular lattice* $M_{3,3}$;
(b) *one of the infinitely many pairwise incomparable lattice varieties generated by a single finite simple lattice and not containing the lattice* $M_{3,3}$ *constructed in* [18];
(c) *any variety of* 0*-lattices containing a finite lattice with more than two elements and no prime ideal.*

Then there is a quasivariety $\mathbf{M} \subseteq \mathbf{K}'$ *which contains continuum many Q-universal non-profinite subquasivarieties* $\mathbf{K} \subseteq \mathbf{M}$ *with one the following properties:*

(i) **K** *has a finitely partitionable* ω*-independent quasi-equational basis relative to* **M**;
(ii) **K** *has no finitely partitionable* ω*-independent quasi-equational basis relative to* **M**.

Proof. As it was established in the papers of V. Koubek and J. Sichler [18]–[21], in each of the cases (a)–(c), $\mathbf{K}'$ is an almost ff-universal quasivariety. The desired statement follows from Corollary 11. □

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