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ON BINOMIAL COEFFICIENTS OF REAL ARGUMENTS

T.I. FEDORYAEVA

Abstract. As is well-known, a generalization of the classical concept of the factorial $n!$ for a real number $x \in \mathbb{R}$ is the value of Euler's gamma function $\Gamma(1+x)$. In this connection, the notion of a binomial coefficient naturally arose for admissible values of the real arguments.

We prove by elementary means a number of properties of binomial coefficients $\binom{r}{\alpha}$ of real arguments r, $\alpha \in \mathbb{R}$ such as analogs of unimodality, symmetry, Pascal's triangle, etc. for classical binomial coefficients. The asymptotic behavior of such generalized binomial coefficients of a special form is established.

Keywords: factorial, binomial coefficient, gamma function, real binomial coefficient.

Introduction

We study binomial coefficients of real arguments. The aim of the investigation is to obtain by elementary methods analogs of the basic properties well known for the classic binomial coefficients. Such properties are of independent interest and, in addition, can simplify the work with binomial coefficients of the form $\binom{n}{m}$ with integer non-negative arguments n and m, given essentially by real values with rounding to an integer (when, for example, floor and ceiling functions for a real number are used, etc.). So, for example, the properties of unimodality and symmetry allow passing from such binomial coefficients $\binom{n}{m}$, $0 \leq m \leq n$ to "close" real binomial coefficients of the form $\binom{r}{\alpha}$, $\alpha \in (-1, r+1)$ and vice versa (see, for example, Proposition 1 and subsequent comments). This approach simplifies the evaluation

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of expressions with discrete binomial coefficients with integer arguments of the specified form.

Note that the binomial coefficients of the form $\binom{r}{n}$, where $r \in \mathbb{R}$ and $n \in \mathbb{N}$, can be defined in the standard way as

$$\binom{r}{n} = \frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}.$$

This approach was discussed in [5], where numerous identities for such binomial coefficients are given. In [4], D. Fowler studied the graph of the function $\binom{r}{\alpha}$ of two real variables r and α, various slices of this graph were constructed using a computer and their analysis was carried out. It is also indicated there an explicit expression for the binomial coefficient $\binom{n}{\alpha}$, where n is a non-negative integer, through elementary functions (see Proposition 3 in Section 2). On the basis of this representation, St.T. Smith investigated the binomial coefficients of the form $\binom{n}{z}$ with complex variable $z \in \mathbb{C}$ and fixed natural number $n \in \mathbb{N}$, a number of properties of such a function of complex argument z is established in [7]. In particular, the derivatives of the first and second orders are calculated, and for the real argument z, increasing and decreasing intervals, zeros of the function, etc are found. It is also noted there the nontriviality of the function investigation $\binom{n}{\alpha}$ of real variable α exactly on the interval $\alpha \in (-1, n+1)$, in contrast to the domain outside this interval. In particular, the increasing and decreasing of this function was established rather difficult.

In this paper we prove by elementary means a number of properties of the binomial coefficients $\binom{r}{\alpha}$ of real arguments r, $\alpha \in \mathbb{R}$, $\alpha \in (-1, r+1)$ (analogs of the properties of unimodality, symmetry, Pascal's triangle, etc. for discrete binomial coefficients), which may be useful in further research (see, for example, [2]).

1. Preliminary information

The article uses the generally accepted concepts and notation of real analysis [3], as well as the standard concepts of combinatorial analysis [5]. Denote by (a, b) the *open real interval* between the numbers $a, b \in \mathbb{R}$, $o(1)$ is an *infinitesimal function* in a neighborhood of ∞, $n!$ is the *factorial* of non-negative integer n, i.e. $n! = n(n-1)\cdots 2\cdot 1$, and wherein we define $0! = 1$, $\binom{n}{m}$, where $0 \le m \le n$, is the (standard) binomial coefficient (with non-negative integer arguments n, m), i.e.

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}.$$

To denote *asymptotic equality* of real-valued functions $f(x)$ and $g(x)$ as $x \to \infty$, we use the notation $f(x) \sim g(x)$, which by definition means that $f(x) = g(x)(1 + r(x))$ in some neighborhood of ∞, where $r(x) = o(1)$, or, equivalently (for functions positive in some neighborhood of ∞)

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

The standard approach is considered, according to which the concept of factorial for non-negative integers extends to real (and even complex) numbers by the *gamma function* $\Gamma(\alpha)$. We will use its definition in the following *Euler-Gauss form* [3, p. 393–394, 812]

$$\Gamma(\alpha)=\lim_{n\to\infty}\frac{(n-1)!\,n^{\alpha}}{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)},\ \alpha\in\mathbb{R}\setminus\{0,-1,-2,\ldots\},\qquad(1)$$

such a limit exists for any specified value of α (see, for example, [6] or [3, p. 393]). In view of the problem statement, we do not consider extensions of the gamma function outside its standard domain of definition. Note that when defining the gamma function in the form of the *Euler integral of the second kind*

$$\Gamma(\alpha)=\int_0^{\infty}x^{\alpha-1}e^{-x}dx,$$

converging for $\alpha>0$, we obtain an equivalent definition on the interval $(0,\infty)$ [3, p. 811]. The gamma function $\Gamma(\alpha)$ is continuous and has continuous derivatives of all orders on $(0,\infty)$, has no real roots, and it is positive on $(0,\infty)$. For any non-negative integer n the following equality holds

$$\Gamma(1+n)=n!\,,\qquad(2)$$

moreover, the next *reduction formula* is valid [3, p. 394]

$$\Gamma(1+\alpha)=\alpha\Gamma(\alpha),\ \alpha\in\mathbb{R}\setminus\{0,-1,-2,\ldots\}.\qquad(3)$$

In addition, the series expansion of $\sin\pi\alpha$ and (1) imply the following *reflection formula* (see, for example, [3,6])

$$\Gamma(\alpha)\,\Gamma(1-\alpha)=\frac{\pi}{\sin\pi\alpha}\,,\ \alpha\in\mathbb{R}\setminus\{0,\pm1,\pm2,\ldots\}.\qquad(4)$$

As a generalization of the discrete binomial coefficient $\binom{n}{m}$, the binomial coefficient of real arguments is defined as follows (see, for example, [1] or [4])

$$\binom{r}{\alpha}=\frac{\Gamma(1+r)}{\Gamma(1+\alpha)\Gamma(1+r-\alpha)}\,.\qquad(5)$$

Note that if $r\in(-1,+\infty)$ and $\alpha\in(-1,r+1)$, the binomial coefficient $\binom{r}{\alpha}$ is defined correctly by the equality (5).

2. Binomial coefficients $\binom{r}{\alpha}$ for $r,\,\alpha\in\mathbb{R}$

Theorem 1 (properties of the binomial coefficient of real arguments)**.** *Let $r\in(-1,+\infty)$ and $\alpha\in(-1,r+1)$. Then*

(i) $\binom{r}{\alpha}>0$, $\binom{r}{0}=1$ *and* $\binom{r}{r}=1$;

(ii) $\binom{0}{\alpha}=\begin{cases}1 & \text{if } \alpha=0,\\ \frac{\sin\pi\alpha}{\pi\alpha} & \text{if } \alpha\neq0 \text{ and } \alpha\in(-1,1);\end{cases}$

(iii) $\binom{r}{r-\alpha}=\binom{r}{\alpha}$;

(iv) $\binom{r}{\alpha}=\binom{r-1}{\alpha-1}+\binom{r-1}{\alpha}$ *if* $r\in(0,+\infty)$ *and* $\alpha\in(0,r)$;

(v) $\binom{r}{\alpha}=\binom{r}{\alpha+1}\frac{\alpha+1}{r-\alpha}$ *if* $r\in(-1,+\infty)$ *and* $\alpha\in(-1,r)$;

(vi) $\binom{r}{\alpha}=\binom{r-1}{\alpha}\frac{r}{r-\alpha}$ *if* $r\in(0,+\infty)$ *and* $\alpha\in(-1,r)$;

(vii) *binomial coefficient* $\phi(\alpha)=\binom{r}{\alpha}$ *is strictly increasing on the interval* $(-1,\frac{r}{2}]$ *and strictly decreasing on the interval* $[\frac{r}{2},r+1)$;

(viii) *binomial coefficient* $\Xi(r) = \binom{r}{\alpha}$ *is strictly increasing for* $\alpha > 0$, *strictly decreasing for* $-1 < \alpha < 0$ *and* $\Xi(r) \equiv 1$ *if* $\alpha = 0$.

Proof. Statement (i) follows from the relations (2), (5).

Prove of (ii). If $\alpha = 0$, the required equality follows from (i). Further, we assume that $\alpha \neq 0$. Using the relations (2)–(5), we obtain

$$\binom{0}{\alpha} = \frac{\Gamma(1)}{\Gamma(1+\alpha)\Gamma(1-\alpha)} = \frac{1}{\alpha\Gamma(\alpha)\Gamma(1-\alpha)} = \frac{\sin \pi\alpha}{\pi\alpha}.$$

Note that if $\alpha \in (-1, r+1)$, then $r - \alpha \in (-1, r+1)$. Therefore, the binomial coefficient $\binom{r}{r-\alpha}$ is defined and the required equality from (iii) is satisfied due to (5). It is also easy to prove combinatorial identities (iv), (v), (vi) from (3) and (5).

Prove of statement (vii). Let α, $\beta \in (-1, r+1)$. From (1) we obtain

$$\Gamma(1+\alpha)\,\Gamma(1+r-\alpha) = \lim_{n\to\infty} \frac{(n-1)!\,(n-1)!\,n^{2+r}}{\prod_{i=1}^{n}(\alpha+i)(r-\alpha+i)}.$$

Hence,

$$\frac{\phi(\alpha)}{\phi(\beta)} = \frac{\Gamma(1+\beta)\,\Gamma(1+r-\beta)}{\Gamma(1+\alpha)\,\Gamma(1+r-\alpha)} = \lim_{n\to\infty} \prod_{i=1}^{n} \delta_i(\alpha,\beta), \text{ where} \tag{6}$$

$$\delta_i(\alpha,\beta) = \frac{(\alpha+i)(r-\alpha+i)}{(\beta+i)(r-\beta+i)}.$$

Note that $\delta_i(\alpha,\beta) > 0$ for every α, $\beta \in (-1, r+1)$ and $i = 1, \ldots, n$. It is also easy to prove that

$$\delta_i(\alpha,\beta) \geq 1 \Leftrightarrow f(\alpha) \geq f(\beta), \tag{7}$$

where $f(x) = -x^2 + xr$ and parabola $f(x)$ is strictly increasing on $(-\infty, \frac{r}{2}]$ as well as strictly decreasing on $[\frac{r}{2}, +\infty)$. Moreover, it is directly established that

$$\delta_1(\alpha,\beta) = 1 + \varepsilon(\alpha,\beta), \text{ where } \varepsilon(\alpha,\beta) = \frac{(r-\alpha-\beta)(\alpha-\beta)}{(\beta+1)(r-\beta+1)}. \tag{8}$$

Let $-1 < \beta < \alpha \leq \frac{r}{2}$. Then $f(\alpha) > f(\beta)$ and $\varepsilon(\alpha,\beta) > 0$. By virtue of (7), we have $\delta_i(\alpha,\beta) \geq 1$ for every $i = 1, \ldots, n$. Hence, from (6) and (8) we obtain

$$\frac{\phi(\alpha)}{\phi(\beta)} \geq \delta_1(\alpha,\beta) = 1 + \varepsilon(\alpha,\beta) > 1.$$

Similarly, if $\frac{r}{2} \leq \beta < \alpha < r+1$, then $f(\alpha) < f(\beta)$ and $\varepsilon(\alpha,\beta) < 0$. Therefore, $0 < \delta_i(\alpha,\beta) < 1$, $i = 1, \ldots, n$ and

$$\frac{\phi(\alpha)}{\phi(\beta)} \leq \delta_1(\alpha,\beta) = 1 + \varepsilon(\alpha,\beta) < 1.$$

Prove of statement (viii). In view of statement (i), we can assume that $\alpha \neq 0$. Let $r < r'$. Note that $1+r+i > 0$, $1+r'+i > 0$ and $\alpha/(1+r+i) < 1$, $\alpha/(1+r'+i) < 1$ for every $i \geq 0$. From (1) we obtain

$$\frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} = \Big(1 - \frac{\alpha}{1+r}\Big) \lim_{n\to\infty} n^{\alpha} \prod_{i=1}^{n-1} \Big(1 - \frac{\alpha}{1+r+i}\Big).$$

Hence, $\Gamma(1+r)/\Gamma(1+r-\alpha) < \Gamma(1+r')/\Gamma(1+r'-\alpha)$ for $\alpha > 0$ (and the reverse strict inequality holds for $\alpha < 0$). In view of (5), we conclude $\Xi(r) < \Xi(r')$ (respectively $\Xi(r) > \Xi(r')$ for $\alpha < 0$). □

Note that the monotonicity properties (vii) and (viii) of Theorem 1 can also be justified by an alternative method based on the broader apparatus of the Theory of the gamma function, when properties of digamma and trigamma functions are applied (the author is grateful to the referee who suggested this approach). Let us give the necessary definitions and properties of these functions (see, for example, handbook [1]).

The digamma function $\Psi(\alpha)$ is defined as the logarithmic derivative of the gamma function $\Gamma(\alpha)$, i.e.

$$\Psi(\alpha) = (\ln \Gamma(\alpha))' = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}\,,\ \alpha > 0. \tag{9}$$

The digamma function $\Psi(\alpha)$ is defined on $(0, \infty)$ and is differentiable everywhere. Its derivative is called the *trigamma function* and denoted by $\Psi_1(\alpha)$. Thus,

$$\Psi_1(\alpha) = \Psi'(\alpha) = (\ln \Gamma(\alpha))^{(2)},\ \alpha > 0. \tag{10}$$

The trigamma function has a known representation as the sum of the following series:

$$\Psi_1(\alpha) = \sum_{n=0}^{\infty} \frac{1}{(\alpha+n)^2}\,,\ \alpha > 0.$$

Hence, $\Psi_1(\alpha)$ is positive and the digamma function $\Psi(\alpha)$ is strictly increasing on $(0, \infty)$ due to (10).

Using the definition of the binomial coefficient (5) for $r \in (-1, \infty)$, $\alpha \in (-1, r+1)$ and equality (9), calculate the derivative of the function $\Phi(\alpha)$:

$$\begin{aligned}
\Phi'(\alpha) &= \Gamma(1+r)\Big(\frac{1}{\Gamma(1+r-\alpha)} \cdot \frac{1}{\Gamma(1+\alpha)}\Big)' \\
&= \Gamma(1+r)\Big(\frac{\Gamma'(1+r-\alpha)}{(\Gamma(1+r-\alpha))^2\Gamma(1+\alpha)} - \frac{\Gamma'(1+\alpha)}{\Gamma(1+r-\alpha)(\Gamma(1+\alpha))^2}\Big) \\
&= \binom{\alpha}{r}\Big(\Psi(1+r-\alpha) - \Psi(1+\alpha)\Big).
\end{aligned} \tag{11}$$

Note that

$$1 + r - \alpha \geq 1 + \alpha \Leftrightarrow \alpha \leq r/2$$

and equality is achieved only if $\alpha = r/2$. Now property (vii) of Theorem 1 follows from Theorem 1(i), the relation (11) and the increase of the digamma function $\Psi(\alpha)$.

Similarly, we calculate

$$\begin{aligned}
\Xi'(r) &= \frac{1}{\Gamma(1+\alpha)} \cdot \Big(\frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)}\Big)' \\
&= \frac{1}{\Gamma(1+\alpha)\Gamma(1+r-\alpha)}\Big(\Gamma'(1+r) - \frac{\Gamma(1+r)\Gamma'(1+r-\alpha)}{\Gamma(1+r-\alpha)}\Big) \\
&= \binom{r}{\alpha}\Big(\Psi(1+r) - \Psi(1+r-\alpha)\Big).
\end{aligned}$$

Note that

$$1 + r \geq 1 + r - \alpha \Leftrightarrow \alpha \geq 0$$

and equality is achieved only if $\alpha = 0$. Now property (viii) of Theorem 1 obviously holds.

The following proposition gives an example of applications of the unimodality and symmetry properties from Theorem 1, when in the asymptotic evaluation of binomial coefficients with non-negative integer arguments, given using rounding functions, we can pass to "close" binomial coefficients of real arguments, and vice versa.

Proposition 1. *Let n be a positive integer and $0 < \alpha < 1$. Then*
(i) *if $0 < \alpha < 1/2$, then for every $n \geq \frac{2}{1-2\alpha}$ the following inequalities hold*

$$\frac{\alpha}{1-\alpha+n^{-1}}\binom{n}{n\alpha} \leq \binom{n}{\lfloor n\alpha \rfloor} \leq \binom{n}{n\alpha};$$

(ii) *if $\alpha = 1/2$, then*

$$\frac{2}{1+n^{-1}}\binom{n-1}{\frac{n-1}{2}} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} \leq \binom{n}{\frac{n}{2}};$$

(iii) *if $1/2 < \alpha < 1$, then for every $n \geq \frac{2}{2\alpha-1}$ the following inequalities hold*

$$\binom{n}{n\alpha} \leq \binom{n}{\lfloor n\alpha \rfloor} \leq \frac{\alpha+n^{-1}}{1-\alpha}\binom{n}{n\alpha}.$$

Proof. Let $0 < \alpha < 1/2$ and $n \geq \frac{2}{1-2\alpha}$. Note that $0 \leq \lfloor n\alpha \rfloor \leq n\alpha \leq \lfloor n\alpha \rfloor + 1 \leq \frac{n}{2} < n$. Now, using statements (v) and (vii) of Theorem 1, we obtain

$$\binom{n}{\lfloor n\alpha \rfloor} \leq \binom{n}{n\alpha},$$

$$\binom{n}{\lfloor n\alpha \rfloor} = \binom{n}{\lfloor n\alpha \rfloor + 1}\frac{\lfloor n\alpha \rfloor + 1}{n - \lfloor n\alpha \rfloor} \geq \binom{n}{n\alpha}\frac{n\alpha}{n-n\alpha+1} = \binom{n}{n\alpha}\frac{\alpha}{1-\alpha+n^{-1}}.$$

Using the inequalities $\frac{n-1}{2} \leq \lfloor \frac{n}{2} \rfloor \leq \frac{n}{2}$, statements (vi) and (vii) of Theorem 1, it is also easy to prove (ii).

Let $1/2 < \alpha < 1$ and $n \geq \frac{2}{2\alpha-1}$. Note that $0 \leq n - \lfloor n\alpha \rfloor - 1 \leq n(1-\alpha) \leq n - \lfloor n\alpha \rfloor \leq \frac{n}{2}$. Now, using statements (iii), (v) and (vii) of Theorem 1, we obtain

$$\binom{n}{\lfloor n\alpha \rfloor} = \binom{n}{n - \lfloor n\alpha \rfloor} \geq \binom{n}{n(1-\alpha)} = \binom{n}{n\alpha},$$

$$\begin{aligned}\binom{n}{\lfloor n\alpha \rfloor} &= \binom{n}{n - \lfloor n\alpha \rfloor} = \binom{n}{n - \lfloor n\alpha \rfloor - 1}\frac{n - (n - \lfloor n\alpha \rfloor) + 1}{n - \lfloor n\alpha \rfloor} \\ &\leq \binom{n}{n(1-\alpha)}\frac{n\alpha+1}{n-n\alpha} = \binom{n}{n\alpha}\frac{\alpha+n^{-1}}{1-\alpha}.\end{aligned}$$

□

Corollary 1. *Let n be a positive integer, $\alpha \in \mathbb{R}$ does not depend on n and $0 < \alpha < 1$. Then the following equality is valid as n tends to infinity*

$$\binom{n}{\lfloor n\alpha \rfloor} = \binom{n}{n\alpha}\Theta(1).$$

Proof. The case $\alpha \neq 1/2$ follows directly from statements (i) and (iii) of Proposition 1. Let $\alpha = 1/2$. From Proposition 2, which has been proved below independently, as $n \to \infty$ we obtain

$$\binom{n}{\frac{n}{2}} \sim \frac{2^n}{\sqrt{\pi n/2}}\,.$$

Therefore,

$$\frac{2}{1+n^{-1}}\binom{n-1}{\frac{n-1}{2}} \sim \frac{2^n}{\sqrt{\pi n/2}}\,.$$

Now the required asymptotic relation follows from Proposition 1(ii). □

The approach outlined in the proof of Proposition 1 is similarly applicable to estimates of classical binomial coefficients of the form

$$\binom{\mu(f(x))}{\nu(g(x))}$$

and the like, where $\mu(x)$ and $\nu(x)$ are the rounding functions $\lfloor x \rfloor$, $\lceil x \rceil$, and $f(x)$, $g(x)$ are real-valued functions of a real argument x. Moreover, for the binomial coefficients of real arguments arising here, it is often possible to easily find an asymptotics in a fairly simple form.

Proposition 2. *Let r take real values, $\alpha \in \mathbb{R}$ does not depend on r and $0 < \alpha < 1$. Then the following asymptotic equality is valid as r tends to infinity*

$$\binom{r}{r\alpha} \sim \sqrt{\frac{1}{2\pi\alpha(1-\alpha)r}}\left(\frac{1}{\alpha}\right)^{\alpha r}\left(\frac{1}{1-\alpha}\right)^{(1-\alpha)r}. \tag{12}$$

Proof. For $r > 0$ the functions $\Gamma(1+r)$, $\Gamma(1+r\alpha)$, $\Gamma(1+r-r\alpha)$ are defined correctly and positive. By virtue of (5), we have

$$\binom{r}{r\alpha} = \frac{\Gamma(1+r)}{\Gamma(1+r\alpha)\,\Gamma(1+r-r\alpha)}\,. \tag{13}$$

For the gamma function, the following generalized Stirling formula is valid (see, for example, [3]):

$$\Gamma(1+x) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \text{ as } x \to \infty.$$

In view of the condition $0 < \alpha < 1$, we have $r\alpha \to \infty$ and $r - r\alpha \to \infty$ as r tends to infinity. Now, using the generalized Stirling formula, we obtain by equivalent transformations the asymptotic equality (12) from (13) . □

Corollary 2. *Let r take non-negative integer values, $\alpha \in \mathbb{R}$ does not depend on r and $0 < \alpha < 1$. Then the asymptotic equality* (12) *is valid as r tends to infinity.*

Proof. Let $g(r)$ be the function on the right side of the asymptotic equality (12) and $f(r) = \binom{r}{r\alpha}/g(r)$. Then $\lim_{r\to\infty} f(r) = 1$ by Proposition 2. Non-negative integers n form an infinitesimal subsequence of the values of real variable r. Therefore, the existing limit value of function $f(r)$ of the real argument as $r \to \infty$ is preserved for function $f(n)$ of the non-negative integer argument as $n \to \infty$. □

Corollary 3. *Let $q \geq 1$, $0 < \Delta < q+1$ and q, Δ do not depend on n. Then the following asymptotic equality is valid as $n \to \infty$*

$$\binom{n}{\frac{n}{q+1}\Delta} \sim \frac{q+1}{\sqrt{2\pi\Delta(q+1-\Delta)n}}(q+1)^n\Delta^{-\frac{\Delta}{q+1}n}(q+1-\Delta)^{-\frac{q+1-\Delta}{q+1}n}\,.$$

Proof. It follows directly from Corollary 2. □

Using Proposition 2, in the next lemma the asymptotic behavior of the following function is established

$$h(n) = \Big(\frac{q}{q+1}\Big)^n \binom{n}{\frac{n}{q+1}\Delta} q^{-\frac{n}{q+1}\Delta}, \text{ where } q \geq 1 \text{ and } 0 \leq \Delta \leq q+1.$$

Lemma 1. *Let $q \geq 1$, $0 \leq \Delta \leq q+1$ and q, Δ do not depend on n. Then the following properties are fulfilled*

(i) *if $\Delta = 0$, then $h(n) = \Big(\frac{q}{q+1}\Big)^n$;*

(ii) *if $0 < \Delta < q+1$, then the following asymptotic equality is valid as $n \to \infty$*

$$h(n) \sim \frac{q+1}{\sqrt{2\pi\Delta(q+1-\Delta)}} \frac{\varepsilon^n_{\Delta,\, q}}{\sqrt{n}}, \tag{14}$$

$$\textit{where } \varepsilon_{\Delta,\, q} = q\, \Delta^{-\frac{\Delta}{q+1}} (q+1-\Delta)^{-\frac{q+1-\Delta}{q+1}} q^{-\frac{\Delta}{q+1}};$$

moreover, $\varepsilon_{\Delta,\, q}$ does not depend on n, $0 < \varepsilon_{\Delta,\, q} < 1$ if $\Delta \neq 1$ and $\varepsilon_{\Delta,\, q} = 1$ if $\Delta = 1$;

(iii) *if $\Delta = q+1$, then $h(n) = \Big(\frac{1}{q+1}\Big)^n$.*

Proof. Statements (i), (iii) follow directly from the definition of the function h(n). Therefore, we assume that $0 < \Delta < q+1$. From the definition of the function h(n) and Corollary 3, we obtain the asymptotic equality (14).

Since $0 < \Delta < q+1$ and $q \geq 1$, we have $\varepsilon_{\Delta,\, q} > 0$. Consider the functions

$$f(x) = x^{\frac{x}{q+1}} (q+1-x)^{\frac{q+1-x}{q+1}} q^{\frac{x}{q+1}} \text{ and } g(x) = \ln f(x)$$

on the interval $(0, q+1)$. Note that

$$\varepsilon_{\Delta,\, q} = \frac{q}{f(\Delta)} = \frac{f(1)}{f(\Delta)}. \tag{15}$$

Hence, $\varepsilon_{\Delta,\, q} = 1$ if $\Delta = 1$. Therefore, we further assume that $\Delta \neq 1$. Let us investigate monotonicity intervals of the function $f(x)$. It is directly calculated

$$g'(x) = \frac{1}{q+1} \ln \frac{qx}{q+1-x}.$$

Note that $g'(x) < 0$ on $(0,1)$, $g'(1) = 0$ and $g'(x) > 0$ on $(1, q+1)$. Therefore, function $g(x)$ strictly decreases on $(0,1]$ and strictly increases on $[1, q+1)$. Function $f(x)$ behaves similarly on the interval $(0, q+1)$. Therefore, if $0 < \Delta < 1$ or $1 < \Delta < q+1$, then $f(\Delta) > f(1)$. Hence, $\varepsilon_{\Delta,\, q} < 1$ due to the equality (15). □

Corollary 4. *Let n be a positive integer, $q \geq 1$, $0 < \Delta < q+1$ and q, Δ do not depend on n. Then the following equality is valid as n tends to infinity*

$$\Big(\frac{q}{q+1}\Big)^n \binom{n}{\lfloor \frac{n}{q+1}\Delta \rfloor} q^{-\frac{n}{q+1}\Delta} = \frac{\varepsilon^n_{\Delta,\, q}}{\sqrt{n}}\, \Theta(1).$$

Proof. It follows directly from Corollary 1 and Lemma 1. □

The outlined approaches (Theorem 1, Corollary 1, Proposition 2, Lemma 1, and Corollary 4) also made it possible to fairly easily investigate the asymptotic behavior of expressions of the form

$$\left(\frac{q}{q+1}\right)^n \sum_{s=0}^{\lfloor \frac{n}{q+1}\Delta \rfloor} \binom{n}{s} q^{-s}$$

as n tends to infinity, where $q \geq 1$ and $0 < \Delta < 1$ do not depend on integer n. Such bounded sums of weighted classical binomial coefficients arose in the asymptotic study of the number of central vertices of almost all n-vertex graphs of given diameter (see [2] for more details).

Further, as noted in [4], in the case of binomial coefficients of the form $\binom{n}{\alpha}$ when n is a non-negative integer, the binomial coefficient is explicitly expressed in terms of elementary functions. The following proposition formalizes this statement and its justification is based on the properties of the gamma function and binomial coefficients of real arguments.

Proposition 3 [4]**.** *Let n be a non-negative integer and real number $\alpha \in (-1, n+1)$. Then the following equality is valid*

$$\binom{n}{\alpha} = \begin{cases} \frac{\sin \pi\alpha}{\pi\alpha} & \textit{if } n = 0 \textit{ and } \alpha \neq 0, \\ \frac{n!}{(n-\alpha)(n-1-\alpha)\cdots(1-\alpha)} \frac{\sin \pi\alpha}{\pi\alpha} & \textit{if } n \geq 1 \textit{ and } \alpha \notin \{0, 1, \ldots, n\}, \\ \frac{n!}{\alpha!(n-\alpha)!} & \textit{if } \alpha \in \{0, 1, \ldots, n\}. \end{cases}$$

Proof. The required equality is proved in Theorem 1 for $n = 0$, and for $\alpha \in \{0, 1, \ldots, n\}$ it follows from the property of the gamma function (2). Let now $n \geq 1$ and $\alpha \notin \{0, 1, \ldots, n\}$. Suppose that $n - i - \alpha \in \{0, -1, -2, \ldots\}$ for some $i \in \mathbb{N}$ and $0 \leq i \leq n - 1$. Then $\alpha - (n - i) \in \{0, 1, 2, \ldots\}$. Hence, $\alpha \in \mathbb{N}$ and therefore $\alpha \in \{0, 1, \ldots, n\}$, got a contradiction. Thus, $n - i - \alpha \notin \{0, -1, -2, \ldots\}$ for every $i = 0, 1, \ldots, n - 1$. By virtue of the reduction formula (3), we have

$$\Gamma(1 + n - \alpha) = (n - \alpha)\Gamma(n - \alpha) = \ldots = \prod_{i=0}^{n-1} (n - i - \alpha)\Gamma(1 - \alpha).$$

Since $\alpha \in \mathbb{R} \setminus \{0, \pm 1, \pm 2, \ldots\}$, from (3)–(5) we obtain the required expression for the binomial coefficient $\binom{n}{\alpha}$. $\square$

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Tatiana Ivanovna Fedoryaeva

Sobolev Institute of Mathematics,
pr. Koptyuga, 4,
630090, Novosibirsk, Russia
Email address: fti@math.nsc.ru