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GRADIENT FLOW FOR KOHN-VOGELIUS FUNCTIONAL

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Abstract. The identification problem of an inclusion is considered in the paper. The inclusion is unknown subdomain of a given physical region. The available information on the inclusion is governed by measurements on the boundary of this region. In particular, the single measurement problem of impedance electrotomography and similar inverse problems are included in our approach. The shape identification problem can be solved by the minimization of an objective function taking into account the measurement data. The best choice of such objective function is the Kohn-Vogelius energy functional. The standard regularization of the Kohn-Vogelius functional include the perimeter and Willmore curvature functional evaluated for an admissible inclusion boundary. In the two-dimensional case, a nonlocal existence theorem of strong solutions is proved for the gradient flow dynamical system generated for such a regularization of the Kohn-Vogelius functional. Bibliography: 24 titles.

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1. Introduction

This paper is devoted to applications of the theory of geometric flows to shape optimization problems. The beginning of the modern mathematical theory of shape optimization was laid in monographs [21], and [7]. In the monographs, it was first singled out as an independent scientific discipline. At present, the theory of shape optimization includes a large number of various applied problems.

In this paper we deal with basic $2D$ shape optimization problem which admits the following formulation. Fix an arbitrary bounded simply connected domain $\Omega \subset \mathbb{R}^2$. It

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is supposed to contain the inclusion Ω_i such that $\overline{\Omega_i} \subset \Omega$. The shape of the inclusion is unknown and must be determined together with the solution of the boundary value problem. Let a Jordan curve Γ be the boundary of Ω_i. In this setting the interface Γ split Ω into the inclusion Ω_i and the curvilinear annulus $\Omega_e = \Omega \setminus \overline{\Omega_i}$. Finally, fix an arbitrary constant $a_0 > 0$ and set

$$a(x) = a_0 \text{ in } \Omega_i, \quad a = 1 \text{ in } \Omega_e. \tag{1.1}$$

As a basic example, we consider the single measurement identification problem arising in the electrical impedance tomography, [3]. Electrical impedance tomography is used in medical imaging to reconstruct the electric conductivity of a part of the body from measurements of currents and voltages at the surface. The problem can be formulated as follows:

For given $g, h : \partial\Omega \to \mathbb{R}$ satisfying the condition

$$g \in L^2(\partial\Omega), \quad h \in W^{1/2,2}(\partial\Omega), \quad \int_{\partial\Omega} g\, ds = 0 \tag{1.2}$$

it is necessary to find Γ and an electric potential $u : \Omega \to \mathbb{R}$ satisfying the equations

$$\text{div } (a\nabla u) = 0 \text{ in } \Omega, \quad a\nabla u \cdot \boldsymbol{\nu} = g, \quad u = h \text{ on } \partial\Omega,$$

where $\boldsymbol{\nu}$ is the outward normal vector to $\partial\Omega$. More generally, the problem of identification is to determine the shape of an inclusion by the additional boundary condition. This inverse problem is ill-posed and in general case has no solution. Its approximate solution can be found by using the shape optimization approach. Thus we come to the following variational problem. Denote by $v, w : \Omega \to \mathbb{R}$ the solutions to boundary value problems

$$\text{div } (a\nabla v) = 0 \qquad \text{div } (a\nabla w) = 0 \qquad \text{in } \Omega, \tag{1.3}$$

$$a\nabla v \cdot \boldsymbol{\nu} = g \qquad w = h \qquad \text{on } \partial\Omega, \tag{1.4}$$

$$\int_{\partial\Omega} v\, dx = 0. \tag{1.5}$$

Next, define a positive objective function that vanishes if and only if $v = w =: u$. The most successful choice of the objective functional is the Kohn-Vogelius energy functional, which is given by the formula, [10],

$$J(\Gamma) = \int_\Omega a\nabla(v - w) \cdot \nabla(v - w)\, dx. \tag{1.6}$$

Note that for fixed h and g, it depends only on Γ.

Unfortunately, shape optimization problems as stated with no additional geometric constrains are ill-posed, see [17], [22] for examples. The reason is that microstructures tend to form, which are associated with a weak convergence of the characteristic functions along a minimizing sequence Ω_i^m, $m \geq 1$. Indeed, in the absence of strong compactness of the minimizing sequences of designs, the optimal state should be attained by a fine mixture of different phases.

The widely used method to cope with these difficulties is to penalize the shape perimeter by adding a regularizing term to the objective functional:

$$\epsilon_p \mathcal{L} + J. \tag{1.7}$$

Here $\mathcal{L}$ is the perimeter of Ω_i, $\epsilon_p > 0$ is the regularization parameter. If $\Gamma = \partial\Omega_i$ is a regular curve, then $\mathcal{L}$ is the length of Γ. This penalization was proposed in [5] by analogy with the Mumford-Shah functional, [15], in the theory of image segmentation processes. The stronger regularization may be obtained if we impose constrains on the curvatures of Γ. This approach also was motivated by the theory of image processing, [16]. The only possible geometrically invariant penalization functional depending on curvatures is the 1-dimensional Willmore functional (Euler elastica) defined by the equality

$$\mathcal{E}_e(\Gamma) = \frac{1}{2}\int_\Gamma |k|^2\,ds, \tag{1.8}$$

where k is the curvature vector of Γ. Therefore, we can define the strong regularization of an objective function as follows

$$\mathcal{E} + J, \quad \text{where} \quad \mathcal{E} = \epsilon_e\,\mathcal{E}_e + \epsilon_p\,\mathcal{L}. \tag{1.9}$$

Here ϵ_j, $j = e, p$, are some positive constants. Note that the penalization term can be interpreted as the cost of structure manufacturing. Hence ϵ_j, $j = e, p$, are not supposed to be small. Without loss of generality we will assume that $\epsilon_j = 1$, which leads to the following expression for $\mathcal{E}$

$$\mathcal{E} = \int_\Gamma \big(\frac{1}{2}|k|^2 + 1\big)\,ds = \mathcal{E}_e + \mathcal{L}. \tag{1.10}$$

The most important question of the theory is the construction of a robust algorithm for the numerical study of shape optimization problems. The standard approach is to use the steepest descent method based on the shape calculus developed by Sokolowski and Zolesio (1992), [21]. See also Delfour and Zolesio (2001), [7], and references therein. The shape calculus works for inclusions Ω_i with the regular boundary $\Gamma = \partial\Omega_i$. In this setting, the objective function J is considered as a functional defined on the totality of smooth curves Γ. This assumption is natural from the practical point of view. Without loss of generality we may restrict our considerations by the class of twice differentiable immersions (parametrized curves) $f : \mathbb{S}^1 \to \mathbb{R}^2$ with $\Gamma = f(\mathbb{S}^1)$ diffreomorphic to the circle $\mathbb{S}^1$. In this framework, we will use the denotation $J(f)$ along with the denotation $J(\Gamma)$. The main goal of the shape calculus is to develop the method of differentiation of objective functions with respect to shapes of geometrical objects.

The shape derivative of an objective function is defined as follows. Choose an arbitrary vector field $X : \mathbb{S}^1 \to \mathbb{R}^2$ and consider the immersion

$$f^t(\theta) = f(\theta) + tX(\theta), \quad t \in (-1, 1), \quad \theta \in \mathbb{S}^1.$$

The curves $\Gamma^t = f^t(\mathbb{S}^1)$, $t \in (-1, 1)$, determine 1-parametric family of perturbations of Γ. The shape derivative $\dot{J}$ of J in the direction X is defined by the equality

$$\dot{J}(\Gamma)\,[X] = \frac{d}{dt}\,J(\Gamma^t)\Big|_{t=0}. \tag{1.11}$$

If it admits the Hadamard representation

$$\dot{J}(\Gamma)\,[X] = \int_\Gamma \phi\, n\cdot X\,ds,\ \phi \in L^1(\Gamma), \tag{1.12}$$

where n is the inward normal to $\Gamma = \partial\Omega_i$, then the vector field

$$dJ(\theta) := \phi(\theta)n(\theta), \quad \theta \in \mathbb{S}^1, \tag{1.13}$$

is said to be the gradient of J at the point f.

For the transmission single measurement identification problem, the gradient dJ of the Kohn-Vogelius objective function (1.6) is defined by the equality, see [3],

$$dJ = 2\big(a\partial_n v\big[\partial_n v\big] - a\partial_n w\big[\partial_n w\big]\big)n - \big[a\nabla v\cdot\nabla v - a\nabla w\cdot\nabla w\big]n, \tag{1.14}$$

where v, w are solutions to problem (1.4), $[\,\cdot\,]$ denotes the jump across Γ.

The similar definition holds for the geometric energy functional $\mathcal{E}$ (see Lemma 3.3 in Section 3). Note that that the shape gradient can be regarded as a normal vector field on Γ.

If f is sufficiently smooth, for example $f \in C^4$, then the mapping $f - \delta\, d(\mathcal{E} + J)\,(f)$ defines an immersion of $\mathbb{S}^1$ into $\mathbb{R}^2$ for all sufficiently small $\delta > 0$. In the steepest descent method, the optimal immersion f and the corresponding shape $\Gamma = f(\mathbb{S}^1)$ are determined as a limit of the sequence of immersions

$$f_{n+1} = f_n - \delta\,\big(d\mathcal{E}(f_n) + dJ(f_n)\big), \quad n \geq 0, \tag{1.15}$$

and the corresponding sequence of curves $\Gamma_n = f_n(\mathbb{S}^1)$. Here the energy $\mathcal{E}$ is defined by (1.10), δ is a fixed positive number, usually small, f_0 is an arbitrary admissible initial shape. Relation (1.15) can be considered as the time discretization of the Cauchy problem

$$\partial_t f(t) = -\big(d\mathcal{E}(f(t)) + dJ(f(t))\big), \quad f(0) = f_0. \tag{1.16}$$

Note that here t is an artificial quasi-time related to the steepest descent method.

Since $\mathcal{E}(f(t)) + J(f(t))$ is a decreasing function of t, a solution to problem (1.16) can be considered as an approximate solution to the penalized variational problem

$$\min\,\big(\mathcal{E} + J\big).$$

Hence the existence of a solution to Cauchy problem (1.16) guarantees the well-posedness of the steepest descent method. In its turn, the existence of the limit $\lim_{t\to\infty} f(t)$ guarantees the convergence of the method. Hence the task is to investigate the well posedness of Cauchy problem (1.16). The main goal of this paper is the proof of the existence of global smooth solution to the Cauchy problem (1.16) for an arbitrary smooth initial data, see Section 3. The paper is organized as follows.

In Section 2, we give basic definitions and characterize the elementary properties of curves with finite elastic energy. We also collect the basic facts from the theory of Sobolev spaces, which will be used throughout the paper.

In Section 3, we formulate the main result on existence of global solution to Cauchy problem (1.16). We give the outline of the proof and formulate the main a priori estimates of solutions. Furthermore, we consider in details the dependence of smoothness of the gradient dJ on the curvature of the interface Γ. In Section 5, we give the proof of main estimates for the gradient of Kohn-Vogelius functional. In Sections 6 and 7, we give the proof of a priori estimates for the gradient flow of penalized Kohn-Vogelius functional. In Sections 8–12, we collect auxiliary results which are used throughout of the paper.

2. Preliminaries

In this section we collect the basic facts on the theory of planar curves and the theory of Sobolev spaces on the real axis.

2.1. Geometric lemmata.

Further we will consider special class of immersions $f : \mathbb{S}^1 \to \mathbb{R}^2$ satisfying the conditions

$$\int_\Gamma \big(\frac{1}{2}|k|^2 + 1\big)\, ds \leq E_0, \quad \Gamma = f(\mathbb{S}^1). \tag{2.1}$$

Our considerations are based on the following elementary lemmas on the properties of such immersions. The first gives the double-side estimates for the length $\mathcal{L}$ in terms of the energy bound E_0.

Lemma 2.1. *The estimate*

$$\frac{2}{E_0} \leq \mathcal{L} \leq E_0 \tag{2.2}$$

holds true for every curve Γ satisfying condition (2.1).

Proof. The proof is given in Section 8. □

The second lemma provides the local graph representation of planar curves with square integrable curvature. Let us consider the following construction.

Choose an arbitrary immersion satisfying condition (2.1). Let $z = f(\theta_z) \in \Gamma$ be an arbitrary point. Fix arc-length coordinate s such that

$$s(z) = 0 \text{ and } -\mathcal{L}/2 \le s < \mathcal{L}/2.$$

For every $0 < \kappa < \mathcal{L}/2$, denote by Γ_κ the arc

$$x = f(s), \quad -\kappa < s < \kappa.$$

Next, introduce the Cartesian coordinates (x_1, x_2) with origin at z such that the axis of abscissa is directed along the tangent vector $\tau(\theta_z)$ and the axis of ordinate is directed along the normal vector $n(\theta_z)$. The consequent results are independent of choice of z. Now, our task is to show that the curve Γ is locally represented as a graph of $C^{1+\alpha}$ function in a neighborhood of the origin z.

Lemma 2.2. *Under the above assumptions, there exist positive numbers κ, α, β, and c, depending only on the constant E_0 in* (2.1), *and the function $\eta \in C^1(-\alpha, \beta)$, $\eta(0) = 0$, with the following properties*

$$\begin{gathered} 0 < c^{-1} \le \kappa, \alpha, \beta \le c < \infty, \\ \|\eta'\|_{C(-\alpha,\beta)} \le 1/6, \quad \|\eta''\|_{L^2(-\alpha,\beta)} \le c\|k\|_{L^2(\Gamma_{3\varkappa})}. \end{gathered} \tag{2.3}$$

Here $\eta'(x_1) = \partial_{x_1}\eta(x_1)$. Moreover, the mapping $x_1 \to (x_1, \eta(x_1))$ defines C^1-parametrization of the arc $\Gamma_{3\kappa}$ and takes diffeomorphically the interval $(-\alpha, \beta)$ onto this arc.

Proof. The proof is given in Section 8. □

Lemma 2.2 gives the simple criterium on the absence of self intersections of curves Γ satisfying the energy condition (2.1).

Corollary 2.3. *Let an immersion $f : \mathbb{S}^1 \to \mathbb{R}^2$ meets all requirements of Lemma 2.2. Furthermore assume that there is $\nu > 0$ with the property*

$$\operatorname{dist}\,(\Gamma \setminus \Gamma_{3\kappa},\, \Gamma_{2\kappa}) \ge \nu. \tag{2.4}$$

Then Γ has no self-intersections. Conversely, if Γ has no self-intersections, then inequality (2.4) *holds for some $\nu > 0$.*

Proof. The corollary is an obvious consequence of Lemma 2.2. □

The second corollary extends the previous results to the case of families of immersions with finite elastic energy. Let us consider a family of immersions $f(t, \cdot) : \mathbb{S}^1 \to \mathbb{R}^2$, $t \in [0, T]$. Every immersion $f(t, \cdot)$, satisfying condition (2.1), defines $\mathcal{L}(t)$- periodic function of the arc-length variable s,

$$\overline{f}(t, s) = f(t, \theta(s)).$$

Note that the periods $\mathcal{L}(t)$ are uniformly bounded from below and above by the constants $2/E_0$ and E_0. Moreover, the functions $\partial_s^2 \overline{f}(s, t)$ are uniformly bounded in $L^2(-\mathcal{L}(t)/2, \mathcal{L}(t)/2)$. It follows that the set of the mappings $\overline{f}(t, \cdot)$, $t \in [0, T]$, satisfying (2.1), is relatively compact in $C^1(\mathbb{R})$.

Assume that a family of immersions $f(t)$, $t \in [0, T]$, satisfies the following conditions

- **G.1** The curves $\Gamma(t) = f(t, \mathbb{S}^1)$ have no self-intersections.
- **G.2** The immersions $f(t)$ satisfy energy condition (2.1) with the constant E_0 independent of t.
- **G.3** The set of the mappings $f(t, \cdot)$, $t \in [0, T]$ is compact in the space $C(\mathbb{S}^1, \mathbb{R}^2)$.

It follows from Lemma 2.2 that for every $f(t, \theta)$, $t \in (0, T)$, there is $\kappa \in (0, 2/E_0)$ which meets all requirements of this lemma and is independent of t.

Corollary 2.4. *Let a family of immersions $f(t,\cdot):\mathbb{S}^1\to\mathbb{R}^2$ satisfies conditions* **G.1-G.3**. *Then there is $\nu>0$ such that*

$$\text{dist}\,(\Gamma(t)\setminus\Gamma_{3\kappa}(t),\,\Gamma_{2\kappa}(t))\geq\nu \tag{2.5}$$

for all $t\in[0,T]$ and for all arcs $\Gamma_{3\kappa}(t)$ given by Lemma 2.2.

Proof. The proof is given in Section 8. □

2.2. **Function spaces.**
Sobolev spaces of periodic functions. For every integer $r\geq 0$, denote by $H^r_\sharp$, the Sobolev space of all $\mathcal{L}$ -periodic mappings with the finite norm

$$\|f\|^2_{H^r_\sharp}=\int_0^{\mathcal{L}}(|f|^2+|\partial_s^r f|^2)\,ds. \tag{2.6}$$

For real $r\geq 0$, the space $H^r_\sharp$ is defined by the interpolation. Note that the equivalent norm in $H^r_\sharp$ may be defined by the equality

$$\|f\|^2_{H^r_\sharp}=\sum_{m\in\mathbb{Z}}(1+|m|^2)^r|f_m|^2,$$

where the Fourier coefficients

$$f_m=\frac{1}{\sqrt{\mathcal{L}}}\int_0^{\mathcal{L}}e^{-\frac{2\pi}{\mathcal{L}}m\,s\,i}\,f(s)\,ds.$$

If Γ is a rectifiable Jordan curve of the length $\mathcal{L}$, then the curvature of Γ, the gradient of Kohn-Vogelius functional, tangent and normal vectors of Γ can be regarded as $\mathcal{L}$-periodic functions of the arc-length variable s. By this reason, we will use the parallel denotations for $H^r_\sharp$:

$$H^r_\sharp=W^{r,2}_\sharp=H^r(\Gamma)=W^{r,2}(\Gamma). \tag{2.7}$$

Remark 2.5. *In Sections 3, 6, and 7, we will consider one-parameter families of curves $\Gamma(t)$, $t\in(0,T)$, with the lengths uniformly bounded from above and uniformly separated from 0. In this case the Sobolev spaces of periodic functions depend on the temporal variable t and should be denoted by $H^r_\sharp(t)$. By abuse of notation, further we omit the symbol t and will write $H^r_\sharp$ instead of $H^r_\sharp(t)$.*

Inequalities. Further, we will use the simplest one-dimensional versions of the Sobolev, interpolation, and Gagliardo-Nirenberg inequalities. The first is the embedding inequality

$$\|f\|_{L^\infty(0,\mathcal{L})}\leq c\|f\|_{H^\sigma_\sharp}\quad\text{for all}\quad\sigma>1/2; \tag{2.8}$$

the second is the standard interpolation inequality

$$\|f\|_{H^\varrho_\sharp}\leq c\,\|f\|^{1-\frac{\varrho}{r}}_{L^2(0,\mathcal{L})}\,\|f\|^{\frac{\varrho}{r}}_{H^r_\sharp}\quad\text{for all}\quad 0\leq\varrho\leq r; \tag{2.9}$$

and the third is the Gagliardo-Nirenberg inequality, [18],

$$\|\partial_s^\varrho f\|_{L^{\frac{2r}{\varrho}}(0,\mathcal{L})}\leq c\,\|f\|^{1-\frac{\varrho}{r}}_{L^\infty(0,\mathcal{L})}\,\|f\|^{\frac{\varrho}{r}}_{H^r_\sharp}\quad\text{for all}\quad 0<\varrho\leq r. \tag{2.10}$$

We also will use the Moser inequality

$$\|u\,v\|_{H^r_\sharp}\leq c\|u\|_{L^\infty(0,\mathcal{L})}\,\|v\|_{H^r_\sharp}+c\|v\|_{L^\infty(0,\mathcal{L})}\,\|u\|_{H^r_\sharp},\quad 0\leq r<\infty. \tag{2.11}$$

Here the constant c depends only on $\mathcal{L}$ and the exponents σ, ϱ, r.

Sobolev spaces on real line. For every integer $r \geq 0$, denote by $H^r(\mathbb{R})$, the Sobolev space of mappings $f : \mathbb{R} \to \mathbb{R}$ with the finite norm

$$\|f\|^2_{H^r(\mathbb{R})} = \int_{\mathbb{R}} (|f|^2 + |\partial_s^r f|^2)\, ds. \tag{2.12}$$

For real $r \geq 0$, the norm in $H^r(\mathbb{R})$ may be defined by the equality

$$\|f\|^2_{H^r(\mathbb{R})} = \int_{\mathbb{R}} (1 + |\xi|^2)^r |\hat{f}(\xi)|^2\, d\xi,$$

where the Fourier transform

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-s\xi\, i}\ f(s)\, ds.$$

We also will use the denotation

$$H^r(\mathbb{R}) = W^{r,2}(\mathbb{R}).$$

3. Results

3.1. **Estimates of Kohn-Vogelius functional.** The existence of smooth solutions to the gradient flows equations for shape optimization problems guarantees that the steepest descent method is well defined and gives the robust algorithm for numeric calculations of an optimal shape. In this section we give outline of main ideas of the proofs of existence and smoothness results for the gradient flows in the shape optimization theory. In order to be clear, we restrict our considerations to the single measurement identification problem for the Kohn-Vogelius functional. Recall the formulation of this problem given in Section 1. Let us consider a simply connected bounded *hold all* domain $\Omega \subset \mathbb{R}^2$ with an inclusion $\Omega_i \subset \Omega$ bounded by a Jordan curve Γ. The interface Γ splits Ω into the simply connected inclusion Ω_i and two-connected curvilinear annulus $\Omega_e = \Omega \setminus \overline{\Omega}_i$. Define the conductivity coefficient a by the relations

$$a = 1 \ \text{ in } \ \Omega_e, \quad a = a_0 = \text{const.} > 0 \ \text{ in } \ \Omega_i. \tag{3.1}$$

Finally, choose arbitrary functions $g, h : \partial\Omega \to \mathbb{R}$ satisfying the conditions

$$h \in W^{1/2,2}(\partial\Omega), \quad g \in L^2(\Omega), \quad \int_{\partial\Omega} g\, ds = 0. \tag{3.2}$$

The Kohn-Vogelius energy functional is defined as follows, [10],

$$J(\Omega_i) = \int_{\Omega} a\nabla(v - w) \cdot \nabla(v - w)\, dx. \tag{3.3}$$

Here $v, w : \Omega \to \mathbb{R}$ satisfy the equations and boundary conditions

$$\text{div } a\nabla v = 0 \qquad \text{div } a\nabla w = 0 \qquad \text{in } \ \Omega, \tag{3.4}$$

$$a\nabla v \cdot n = g \qquad w = h \qquad \text{on } \ \partial\Omega, \tag{3.5}$$

$$\int_{\partial\Omega} v\, dx = 0. \tag{3.6}$$

Under the above assumptions, boundary value problems (3.4)-(3.5) admit the only weak solutions $v, w \in W^{1,2}(\Omega)$ satisfying the orthogonality condition (3.6), such that

$$\|v\|_{W^{1,2}(\Omega)} \leq c\|g\|_{L^2(\partial\Omega)}, \quad \|w\|_{W^{1,2}(\Omega)} \leq c\|h\|_{W^{1/2,2}(\partial\Omega)}. \tag{3.7}$$

Here c depends only on Ω and the constant a_0 in the definition (3.1) of a. Hence the Kohn-Vogelius functional is well defined as a function of Ω_i or equivalently of Γ.

Assume, in addition, that the data have additional smoothness properties

$$\partial\Omega, \Gamma \in C^{2+\alpha}, \quad h \in C^{2+\alpha}(\partial\Omega), \quad g \in C^{1+\alpha}(\partial\Omega)), \quad \alpha \in (0,1). \tag{3.8}$$

Denote by v^-, w^- the restrictions of v, w on Ω_e and by v^+, w^+ the restrictions of v, w on Ω_i. It follows from the Schauder estimates for solutions to elliptic equations that

$v^-, w^- \in C^{2+\alpha}(\overline{\Omega}_e)$ and $v^+, w^+ \in C^{2+\alpha}(\overline{\Omega}_i)$. For every function Φ with Φ^- and Φ^+ continuous in $\overline{\Omega}_e$ and $\overline{\Omega}_i$, the denotation $\big[\Phi\big]$, stands for the jump of Φ across Γ,

$$\big[\Phi\big](x) = \lim_{\Omega_e \ni y \to x} \Phi^-(y) - \lim_{\Omega_i \ni y \to x} \Phi^+(y) \quad \text{for all } x \in \Gamma.$$

For strong solutions to transmission problem (3.5) we have

$$\big[a\partial_n v\big] \equiv \big[a\nabla v\big] \cdot n = 0, \quad \big[a\partial_n w\big] \equiv \big[a\nabla w\big] \cdot n = 0, \quad \big[v\big] = \big[w\big] = 0. \tag{3.9}$$

With this notation the gradient dJ of the Kohn-Vogelius objective function (1.6) is defined as follows, see [3],

$$dJ = 2\big(a\partial_n v\, \big[\partial_n v\big] - a\partial_n w\, \big[\partial_n w\big]\big)\, n - \big[a\nabla v \cdot \nabla v - a\nabla w \cdot \nabla w\big]\, n, \tag{3.10}$$

3.1.1. *Estimates of dJ.* In this section we consider in details the gradient dJ of the Kohn-Vogelius functional. Our goal is to derive the estimates of dJ in the Sobolev spaces $H^r_\sharp$ in terms of the geometric characteristics of the interface Γ. By virtue of representation (3.10), the normal vector field $dJ : \Gamma \to \mathbb{R}^2$ is the quadratic form of the derivatives of solutions v, w to boundary value problems (3.4)-(3.6). First we derive the estimates for a general transmission problem. Assume that the interface Γ satisfies the following conditions

H.1 The Jordan curve $\Gamma \subset \Omega$ satisfies the energy condition

$$\frac{1}{2}\int_\Gamma k^2 ds + \mathcal{L} \le E_0.$$

H.2 There is $\nu > 0$ with the property

$$\text{dist}\,(\Gamma \setminus \Gamma_{3\kappa},\, \Gamma_{2\kappa}) \ge \nu,$$

for every arc $\Gamma_{3\kappa}$ with κ, defined by Lemma 2.2.

H.3 There is $\varrho > 0$ such that dist $(\Gamma, \partial\Omega) > \varrho$.

By virtue of Corollary 2.4, every curve Γ satisfying Conditions **H.1**- **H.3** is a Jordan curve of the class $C^{1+\alpha}$, $0 < \alpha < 1/2$. It splits the domain Ω into two parts. The first $\Omega_i \Subset \Omega$ (inclusion) is a simply connected domain with boundary Γ. The second is the curvilinear annulus $\Omega_e = \Omega \setminus \overline{\Omega_i}$ bounded by Γ and $\partial\Omega$. For simplicity, we will assume that $\partial\Omega$ is a Jordan curve of the class C^∞. We adopt the convention that Γ has the positive orientation. This means that the point $z(s)$ moves along Γ in the counter-clockwise direction while s increases. In its turn, the tangent vector τ and the normal vector n form the moving orthonormal frame with the positive orientation. This means that n is inward normal vector to $\partial\Omega_i = \Gamma$.

Next, introduce the piece-wise constant function $a : \Omega \to \mathbb{R}^+$ (conductivity) defined by the equalities (3.1).

Model transmission problem. Let $w \in W^{1,2}(\Omega)$ be a weak solution to the equation

$$\text{div}\,(a\,\nabla w) = 0 \quad \text{in} \quad \Omega.$$

We do not impose boundary conditions on w. Denote by w^- and w^+ the restrictions of w onto subdomains Ω_e and Ω_i,

$$w^- := w \quad \text{in} \quad \Omega_e, \qquad w^+ := w \quad \text{in} \quad \Omega_i.$$

If Γ is sufficiently smooth, then w is continuous on Γ. In other words, $w^- = w^+$ on Γ. However, the normal derivative of w has a jump across Γ. Next set

$$\partial_n w^- = \nabla w^- \cdot n, \quad \partial_n w^+ = \nabla w^+ \cdot n \quad \text{on} \quad \Gamma.$$

Our task is to estimate $\partial_n w^\pm$ via the curvature of Γ. The following theorem on the estimates of $\partial_n w^\pm$ is the first main result of this section. Recall definition (2.7) of the Sobolev spaces $H^r_\sharp = H^r(\Gamma)$ of periodic functions.

Theorem 3.1. *Under the above assumptions, the estimate*

$$\|\partial_n w^{\pm}\|_{H^{m+1/2}_\sharp} \le c\,(\,1 + \|\partial_s^m k\|_{L^2(\Gamma)}\,)\,\|w\|_{W^{1,2}(\Omega)} \tag{3.11}$$

holds for every integer $m \ge 0$. Here c depends only on m and on the constants E_0, ν, ρ in Conditions **H.1-H.3**.

Proof. The proof is given in Section 4. □

Estimates of dJ. Note that the solutions v, w to problems (3.4)-(3.6) meet al requirements of Theorem 3.1 and admit the estimates

$$\|v\|_{W^{1,2}(\Omega)} + \|w\|_{W^{1,2}(\Omega)} \le c(g,h).$$

This result along with representation (3.10) and the multiplicative estimates in Sobolev spaces leads to the following theorem, which is the second main result of this section.

Theorem 3.2. *Assume that a curve Γ satisfies conditions* **H.1-H.3** *and $k \in H^r_\sharp$, for some integer $r \ge 1$. Then for every $\beta \in [0,1/2)$, there is a constant c, depending on r, β, and constants E_0, ν, ρ in conditions* **H.1-H.3**, *such that the gradient $dJ(s)$ of the Kohn-Vogelius functional admits the estimate*

$$\|dJ\|_{H^{r+\beta}} \le c(1 + \|k\|_{H^r_\sharp}). \tag{3.12}$$

In particular, we have

$$\|\partial_s^r dJ\|_{L^q(0,\mathcal{L})} \le c(1 + \|k\|_{H^r_\sharp}). \tag{3.13}$$

for every $q \in [1,\infty)$. In this case the constant c depends in addition on q.

3.2. **Gradient flow. Existence theory.**

3.2.1. *Problem formulation.* The standard formulation of the geometric flow equations deals with immersions (parametrized curves). Further we will assume that the interface Γ admits the representation $\Gamma = f(\mathbb{S}^1)$, where the immersion $f : \mathbb{S}^1 \to \mathbb{R}^2$ is unknown and should be defined along with the solution to the gradient flow problem (1.16). Note that f is a 2π periodic function of the angle variable $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. The element of the length of Γ equals

$$ds = \sqrt{g(\theta)}\,d\theta, \quad g = |\partial_\theta f|^2,$$

where g is the only nontrivial coefficient of the first fundamental form of the curve Γ. In this setting, the derivative with respect to the arc-length variable s

$$\partial_s = \frac{1}{\sqrt{g(\theta)}}\partial_\theta$$

becomes the nonlinear differential operator depending on f.

The tangent vector

$$\tau(\theta) = \partial_s f(\theta) := |\partial_\theta f|^{-1}\,\partial_\theta f(\theta),$$

and the normal vector

$$n(\theta) = \tau^\perp(\theta) = (-\tau_2,\,\tau_1),$$

form the positive oriented moving frame on Γ. Notice that n is the unit inward normal vector to $\partial\Omega_i = \Gamma$. The curvature vector k is defined by the equalities

$$k(\theta) = \partial_s\tau(\theta) = \partial_s^2 f(\theta). \tag{3.14}$$

Notice that k is orthogonal to τ and is directed along the normal vector n.

The Euler elastic energy $\mathcal{E}_e$ and the perimeter $\mathcal{L}$ are defined by the equalities

$$\mathcal{E}_e = \int_\Gamma \frac{k^2}{2}\,ds, \quad \mathcal{L} = \int_\Gamma ds = \int_0^{2\pi}\sqrt{g}\,d\theta. \tag{3.15}$$

We take the penalization energy in the form

$$\mathcal{E} = \mathcal{E}_e + \mathcal{L} = \int_\Gamma \big(\frac{k^2}{2} + 1\big)\,ds, \tag{3.16}$$

The gradient of $\mathcal{E}$ is given by the following lemma.

Lemma 3.3. *Under the above assumptions, we have*

$$d\mathcal{E}_e(f) = \nabla_s \nabla_s k + \frac{1}{2}|k|^2 k, \quad d\mathcal{L} = -k, \tag{3.17}$$

$$d\mathcal{E}(f) = \nabla_s \nabla_s k + \frac{1}{2}|k|^2 k - k. \tag{3.18}$$

Here the normal connection ∇_s for every vector field $\Phi : \Gamma \to \mathbb{R}^2$, is defined by the equality

$$\nabla_s \Phi = \partial_s \Phi - (\partial_s \Phi \cdot \tau)\, \tau. \tag{3.19}$$

Identities (3.18) are classic (see for instance [8]). They are very particular case of the $3D$ Willmore variation formula.

We are now in a position to specify the gradient flow equation

$$\partial_t f + d\mathcal{E} + dJ = 0, \quad f(0) = f_0.$$

for the penalized Kohn-Vogelius functional. Applying Lemma 3.3 we can rewrite this equation in the form

$$\partial_t f + \nabla_s \nabla_s k + \frac{1}{2}|k|^2 k - k + dJ = 0 \text{ for } t > 0, \;\; f(0) = f_0. \tag{3.20}$$

The gradient dJ is defined by relation (3.10) and can be regarded as nonlinear nonlocal operator acting on Γ. Hence (3.20) is a nonlinear operator equation. It may be considered as a nonlocal perturbation of the Euler elastic flow.

3.2.2. *Existence theorem.* In this subsection we prove the main theorem on the existence of global smooth solution to problem (3.20). Assume that the initial data satisfy the following conditions:

I.1 The even integer number $m \geq 10$

I.2 The initial curve $\Gamma_0 = f_0(\mathbb{S}^1)$ satisfies conditions **H.1**-**H.3** of Theorem 3.1.

I.3 There is a constant E_m such that

$$\int_{\Gamma_0} |\partial_s^r k_0|^2 \, ds \leq E_m \;\; \text{for all} \;\; 0 \leq r \leq m. \tag{3.21}$$

I.4 The length element $\sqrt{g_0} = |\partial_\theta f_0|$ satisfies the condition

$$\|\sqrt{g_0}\|_{C^{m-5}(\mathbb{S}^1)} \leq c_g < \infty. \tag{3.22}$$

Theorem 3.4. *Assume that the initial data satisfy Conditions* **I.1**-**I.4**. *Then there is a maximal $T \in (0, \infty]$ with the following properties. Problem* (3.20) *has a solution such that*

$$f \in C(0, T'; C^{m-5}(\mathbb{S}^1)), \; \partial_t f \in C(0, T'; C^{m-9}(\mathbb{S}^1)) \text{ for all } 0 < T' < T. \tag{3.23}$$

Moreover, the Jordan curves $\Gamma(t) = f(t, \mathbb{S}^1)$, $t \in [0, T)$, are separated from $\partial\Omega$ and have no self-intersections. If $T < \infty$, then there is a sequence $f(t_j)$, $t_j \to T$ as $j \to \infty$, such that $\operatorname{dist}\,(\Gamma(t_j), \partial\Omega) \to 0$, *or (and) $f(t_j)$ converge in $C^1(\mathbb{S}^1)$ as $j \to \infty$ to some immersion f_∞ such that the curve $f_\infty(\mathbb{S}^1)$ has a self-intersection.*

The proof is standard and consists of three steps. The first is the proof of the local solvability of problem (3.20) on the small time intervals. The second most important step is the proof of the global a priori estimates for smooth solutions to problem (3.20) in Sobolev and Hölder classes. These estimates and the extension method entail the existence of smooth solution which meets all requirements of Theorem 3.4.

A detailed proof of short-time existence is outside of the scope of this paper. Note that equation (3.20) is a degenerate parabolic equation with added lower order operator dJ. In our case the local existence result can be obtained by writing the flow as a graph over the initial date. In particular, the problem can be reduced to a scalar parabolic equation for the distance function, [6]. See also [8] and [11] for useful arguments.

Hence out main task is to derive global a priori estimates for solutions to problem (3.20). This part of the proof is technical and lengthy. Our approach is based on the estimates for Kohn-Vogelius functional given by Theorem 3.2 and methods developed in [1], [8], and [12]. The results are given by the following two theorems. The first constitutes the Sobolev a priori estimates for the curvature k as a function of the arc-length variable s.

Theorem 3.5. *Let* $f : [0,T]\times\mathbb{S}^1 \to \mathbb{R}^2$ *be a smooth solution to problem* (3.20) *with initial data satisfying condition*

$$\int_{\Gamma(0)} |\partial_s^m k_0|^2 ds \leq E_m < \infty, \quad \mathcal{E}(0) \leq E_0 < \infty, \tag{3.24}$$

where $m \geq 6$ *is an even integer. Furthermore assume that there are two positive constants* ν *and* ρ *with the following properties. For every* $t \in [0,T]$*, the curve* $\Gamma(t)$ *satisfies Conditions* **H.1-H.3** *with fixed constant* ν *and* dist $(\Gamma(t), \partial\Omega) > \varrho$ *independent of* t*. Then there is a constant* c*, depending only on* E_0*,* ν*,* ρ*,* T*, and* m*, such that*

$$\sup_{t\in[0,T]} \|k(t)\|^2_{H^{m-2}(t)} + \int_0^T \|k(t)\|^2_{H^m(t)}\, dt \leq cE_m + c(1+T). \tag{3.25}$$

Proof. The proof is given in Section 6. □

The second theorem gives the a priori estimates for solutions to problem (3.20) in the Hölder classes.

Theorem 3.6. *Let a smooth solution to problem* (3.20) *meets all requirements of Theorem 3.5 with even integer* $m \geq 10$*. Furthermore assume that the initial data satisfies conditions* **I.1-I.4** *of Theorem 3.4. Then there is a constant* c*, depending only on* T*,* ν*,* ϱ*,* m *and the constants* E_m*,* c_g *in conditions* **I.1-I.4***, such that*

$$\|f\|_{C(0,T;C^{m-5}(\mathbb{S}^1))} + \|f\|_{C^1(0,T;C^{m-9}(\mathbb{S}^1))} \leq c. \tag{3.26}$$

Proof. The proof is given in Section 7. □

In order to complete the proof of Theorem 3.4 we use the extension method. Without loss of generality we may assume that $f_0 \in C^\infty(\mathbb{S}^1)$. Hence the problem has a C^∞- solution f defined on some small interval $(0,T)$. By virtue of Theorem 3.6, this solution meets all requirements of Theorem 3.4 for every even $m < \infty$. Moreover, every immersion $f(t)$, $t \in [0,T)$, satisfies conditions **H.1-H.3** of Theorem (3.2) with some constants $\nu(t) > 0$ and $\varrho(t) > 0$. Let $(0,T)$ be the maximal interval of existence of such a solution. There are two possibilities

$$\text{either} \quad \liminf_{t\to T} \nu(t)\varrho(t) > 0 \quad \text{or} \quad \liminf_{t\to T} \nu(t)\varrho(t) = 0.$$

Let us prove that $T = \infty$ in the first case. Assume in contrary to our claim that $T < \infty$. There is $\delta > 0$ such that quantities $\nu(t)$ and $\varrho(t)$ are uniformly separated from zero on the interval $[T-\delta, T)$, i.e.,

$$\nu(t) > \nu > 0 \quad \varrho(t) > \rho > 0$$

for some ν and ρ. Hence $f(t)$ meet all requirements of Theorem 3.6 on the interval $[T-\delta, T)$ with the initial data $f(T-\delta)$. It follows from this theorem that

$$\|f(t)\|_{C^{m-5}(\mathbb{S}^1)} + \|\partial_t f(t)\|_{C^{m-9}(\mathbb{S}^1)} \leq c(m) \quad \text{for all} \quad t \in [T-\Delta, T].$$

Recall that here $m \geq 10$ is an arbitrary even integer. Hence the immersions $f(t)$ converges in every space $C^m(\mathbb{S}^1)$ to some immersion $f_\infty \in C^\infty(\mathbb{S}^1)$ which obviously satisfies conditions **I.1-I.4**. The local existence theory implies the existence of smooth solution to equation (3.20) with initial data $f(T) = f_\infty$ on some interval $[T, T+\delta)$. This contradicts the maximality of T.

It remains to consider the case when $T < \infty$ and hence $\liminf \nu(t)\varrho(t) = 0$. Obviously there exist a sequence t_j such that

$$\nu(t_j)\rho(t_j) \to 0, \quad t_j \to T \ \text{ as } \ j \to \infty.$$

If $\varrho(t_j) \to 0$ as $t_j \to T$, then dist $(\Gamma(t_j), \partial\Omega) \to 0$ as $t_j \to T$ and the assertion follows. Let us consider the case

$$\nu(t_j) \to 0, \quad t_j \to T \ \text{ as } \ j \to \infty. \tag{3.27}$$

Recall that the immersions $x = f(t,s)$, $s \in [0, \mathcal{L}(t)]$, are uniformly bounded in $C^{1+\alpha}[0, \mathcal{L}(t)]$. Moreover, the bound depends only on the constant E_0. Furthermore, by virtue of Lemma 2.1, the perimeters $\mathcal{L}(t)$ are uniformly bounded from above and uniformly separated from 0. After passing to a subsequence we may assume that the sequence $\mathcal{L}(t_j)$ converges to some positive $\mathcal{L}_\infty$ as $t_j \to T$. The sequence of immersions $f(t_j, s)$ converges in C^1 norm to an immersion $f_\infty(s)$ on every compact subset of $[0, \mathcal{L}_\infty)$. It is clear that the energy of the correspondent curve Γ_∞ does not exceed E_0. It remains to prove that the limiting curve Γ_∞ has a self-intersection. To this end, note that the set of curves $\{\Gamma(t_j)\} \cup \Gamma\infty$ is compact in the uniform metric. If the limiting curve has no self- intersections, then every curve from this set has no-self-intersections. From this and Corollary 2.5 we conclude that $\nu(t_j) \geq \nu > 0$ for some ν independent of j, which contradicts to relation (3.27). This completes the proof of Theorem 3.4.

Since the energy $\mathcal{E}(t_j)$ of the curve $\Gamma(t_j)$ is bounded by the constant E_0, it follows from Lemma 2.2 that the functions $f_j(s) = f(t_j, s)$ are uniformly bounded in $C^{1+\alpha}$ norm for $0 \leq \alpha < 1/2$. Hence after passing to a subsequence we may assume that $\Gamma(t_j)$ convege uniformly to C^1 curve Γ_∞. Obviously either Γ_∞ has a self-intersection or (and) it touches $\partial\Omega$. This completes the proof of Theorem 3.4.

4. Model transmission problem

4.1. **Transmission problem. Notation. Results.** Let us consider the following construction. Fix an arbitrary positive κ an ρ and introduce the rectangles

$$Q_0 = (-2\kappa, 2\kappa) \times (-2\rho, 2\rho), \quad Q = (-\kappa, \kappa) \times (-\rho, \rho) \tag{4.1}$$

in the plane of variable $y = (y_1, y_2)$. Next, fix an arbitrary integer $r \geq 1$ and introduce the systems of numbers

$$\kappa_m = \kappa\big(2 - \frac{m}{r}\big), \quad \rho_m = \rho(2 - \frac{m}{r}), \quad 1 \leq m \leq r,$$

and the corresponding domains

$$Q_m = (-\kappa_m, \kappa_m) \times (-\rho_m, \rho_m), \quad Q_r = Q. \tag{4.2}$$

Choose an arbitrary function φ with the properties

$$\varphi \in C_0^\infty(Q), \quad 0 \leq \varphi \leq 1, \quad \varphi = 1 \ \text{ in } \ Q/2, \quad \partial_{y_2}\varphi = 0 \text{ for } y_2 = 0. \tag{4.3}$$

and a system of functions φ_m, $1 \leq m \leq r$, such that

$$\begin{gathered} \varphi_m \in C_0^\infty(Q_m), \quad 0 \leq \varphi_m \leq r - 1, \\ \varphi_m = 1 \ \text{ in } \ Q_{m+1} \ \text{ for } \ 1 \leq m \leq r-1, \quad \varphi_r = \varphi. \end{gathered} \tag{4.4}$$

Next, introduce (2×2)-matrix $N(y_1)$ with the properties

$$N = N^\top, \quad C_N^{-1}\mathbb{I} \leq N \leq C_N\mathbb{I}, \quad \|N\|_{W^{1,2}(-2\kappa, 2\kappa)} \leq C_N, \tag{4.5}$$

where C_N is some fixed constant. Finally set

$$a(y_2) = 1 \ \text{ for } \ y_2 < 0, \quad a(y_2) = a_0 = \ \text{const.} > 0 \ \text{ for } \ y_2 \geq 0.$$

Let $u : Q_0 \to \mathbb{R}$ be a solution to the elliptic equation

$$\text{div } (aN\nabla u) = 0 \ \text{ in } \ Q_0. \tag{4.6}$$

We do not impose boundary conditions for u. Instead of this we assume that it admits the estimate

$$\|u\|_{L^2(Q_0)} + \|\nabla u\|_{L^2(Q_0)} \le C_u < \infty \tag{4.7}$$

It follows from L^∞ interior estimates for solutions of divergent elliptic equations that

$$\|u\|_{C(Q_m)} \le c < \infty \quad \text{for} \quad m \ge 1, \tag{4.8}$$

where the constant c depends only on C_N, C_u, a, and κ, ρ.

Now set

$$v = \varphi\, u, \tag{4.9}$$

where φ is defined by (4.3).

The main goal of this subsection is to estimate the one-sided conormal derivatives of v *on the interface* $\{x_2 = 0\}$. To this end we introduce the system of functions

$$v_m = \varphi_m\, u, \quad 1 \le m \le r, \tag{4.10}$$

where φ_m are given by (4.4).

Proposition 4.1. *Under the above assumptions, the functions* $v_m = \varphi_m u$, $1 \le m \le r$, *admit the estimates*

$$\|\partial_1^m \nabla v_m\|_{L^2(Q_m)} \le c\,(\,1 + \|\partial_1^m N\|_{L^2(-2\kappa,2\kappa)}\,). \tag{4.11}$$

Here the constant c *depends only on* r, φ, *and the constants* C_N, C_u *in* (4.5) *and* (4.7).

The second main result of this section is the following proposition, which provides the estimates for the conormal derivatives of u on the interface $\{y_2 = 0\}$. Split the rectangle Q into two parts

$$Q^- = (-\kappa, \kappa) \times (-\rho, 0), \quad Q^+ = (-\kappa, \kappa) \times (0, \rho),$$

separated by the interface segment

$$\ell = (-\kappa, \kappa) \times \{0\}.$$

Denote by $u^\pm$ and $v^\pm$ the restrictions of the functions u and $v = \varphi u$ to $Q^\pm$. We also denote by $\partial_N u^\pm$ and $\partial_N v^\pm$ the conormal derivatives

$$\partial_N u^\pm = (N_{21}\partial_1 + N_{22}\partial_2)\, u^\pm, \quad \partial_N v^\pm = (N_{21}\partial_1 + N_{22}\partial_2) v^\pm \quad \text{on} \quad \ell. \tag{4.12}$$

We will consider the conormal derivatives as functions defined on the interval $(-\kappa, \kappa)$. Since the function φ is compactly supported in this interval, we may assume that $\varphi\partial_N u^\pm$ and $\partial_N v^\pm$ are extended by zero to $\mathbb{R}$. Now recall definition of Sobolev spaces $H^r(\mathbb{R})$ in subsection 2.2.

Proposition 4.2. *Assume that all assumptions of Proposition 4.1 are satisfied. Then the estimate*

$$\|\varphi\partial_N u^\pm\|_{W^{r-1/2,2}(\mathbb{R})} \le c(1 + \|\partial_1^r N\|_{L^2(-2\kappa,2\kappa)}) \tag{4.13}$$

holds true for all integers $r \ge 1$.

The rest of the subsection is devoted to the proof of Propositions 4.1 and 4.2.

4.2. **Proof of Proposition 4.1. Part I.** We proceed with the induction principle estimating step by step the function v_m. Our first step is the derivation of recurrent system of elliptic equations for these v_m and their derivatives.

4.2.1. *Extended system of equations.* It is easily seen that the functions v_m, $1 \le m \le r$ satisfy the following recurrent system of differential equations

$$\operatorname{div}\,\big(aN\nabla v_m\big) = \operatorname{div}\,\mathcal{B}_{m-1} + \mathcal{C}_{m-1} \quad \text{in} \quad Q_0, \quad 1 \le m \le r, \tag{4.14}$$

which is understood in the sense of distributions. Here $v_0 = u$ and

$$\mathcal{B}_{m-1} = v_{m-1}\, a\, N\nabla\varphi_m, \quad \mathcal{C}_{m-1} = a\nabla\varphi_m \cdot N\,\nabla v_{m-1}. \tag{4.15}$$

Indeed, we have

$$\int_{Q_0} aN\nabla v_m \cdot \nabla\zeta\, dy = \int_{Q_0} aN\nabla(\varphi_m u)\cdot\nabla\zeta\, dy =$$
$$\int_{Q_0} aNv_{m-1}\nabla\varphi_m\cdot\nabla\zeta\, dy + \int_{Q_0} aN\varphi_m\nabla u\cdot\nabla\zeta\, dy.$$

On the other hand, we have

$$\int_{Q_0} aN\varphi_m\nabla u\cdot\nabla\zeta\, dy = \int_{Q_0} aN\nabla u\cdot\nabla(\zeta\,\varphi_m)dy - \int_{Q_0} aN\nabla u\cdot\nabla\varphi_m\,\zeta\, dy$$
$$= -\int_{Q_0} aN\nabla u\cdot\varphi_m\,\zeta\, dy = -\int_{Q_0} aN\nabla v_{m-1}\cdot\nabla\varphi_m\,\zeta\, dy.$$

Thus we get

$$\int_{Q_0} aN\nabla v_m\cdot\nabla\zeta\, dy = \int_{Q_0} av_{m-1}N\nabla\varphi_m\cdot\nabla\zeta\, dy - \int_{Q_0} aN\nabla\varphi_m\nabla v_{m-1}\zeta\, dy,$$

which obviously yields (4.14). We are interested in the smoothness properties of solutions to system (4.14)-(4.15) with respect to the variable y_1. To this end, notice that

$$\partial_1^m \operatorname{div}\,\big(aN\nabla v_m\big) = \operatorname{div}\,\partial_1^m\mathcal{B}_{m-1} + \partial_1^m\mathcal{C}_{m-1} \quad \text{in} \quad Q_0, \quad 1 \le m \le r, \tag{4.16}$$

and

$$\partial_1^m \operatorname{div}\,\big(aN\nabla v_m\big) = \operatorname{div}\,\big(aN\nabla\partial_1^m v_m\big) + \operatorname{div}\,\mathcal{A}_m. \tag{4.17}$$

Here

$$\mathcal{A}_m = \sum_{i+j=m,\, j\ge 1} \mathcal{A}_{ij}^m, \quad \mathcal{A}_{ij}^m = C_j^m\, a\partial_1^j N\,\nabla\partial_1^i v_m. \tag{4.18}$$

Thus we get

$$\operatorname{div}\,\big(aN\nabla\partial_1^m v_m\big) = -\operatorname{div}\,\mathcal{A}_m + \operatorname{div}\,\partial_1^m\mathcal{B}_{m-1} + \partial_1^m\mathcal{C}_{m-1} \quad \text{in} \quad Q_m \tag{4.19}$$

for every $1 \le m \le r$. Now our task is to estimate the quantities $\mathcal{A}_m$, $\mathcal{B}_{m-1}$, $\mathcal{C}_{m-1}$, and $\mathcal{D}_{m-1}$.

4.2.2. *Basis of induction. Auxiliary Lemma.* In this subsection we prove the following lemma, which gives the basis of the induction process. In what follows, we will denote by c various constants depending on the rectangle Q_0, and r, φ_m, as well as the constants C_N and C_u in (4.5) and (4.7).

Lemma 4.3. *Under the above assumptions,*

$$\|\partial_1\nabla v_1\|_{L^2(Q_1)} \le c. \tag{4.20}$$

Proof. The proof is given in Section 10. □

Remark 4.4. *Notice that*

$$\|\partial_1\mathcal{B}_1\|_{L^2(Q_1)} + \|\mathcal{C}_1\|_{L^2(Q_1)} \le c. \tag{4.21}$$

Auxiliary lemma. Now we have to organize the induction process. To this end, we have to derive the recursion system of estimates for the quantities $\mathcal{A}_m$, $\mathcal{B}_{m_1}$ and $\mathcal{C}_{m-1}$. Notice that for $m=1$ the desired estimates follows from Remark 4.4. Hence it suffices to consider the case $m \geq 2$.

Lemma 4.5. *For every integer $m \in [2, r]$ and $\sigma \in (1/2, 1)$, there is a constant c such that*

$$\|\mathcal{A}_m\|_{L^2(Q_m)} \leq c(1 + \|\partial_1^m N\|_{L^2(-\kappa_m, \kappa_m)} + \|\partial_1^m \nabla v_m\|^{\lambda}_{L^2(Q_m)}), \tag{4.22}$$

where $\lambda = (m-2+\sigma)/(m-1) < 1$,

$$\begin{aligned} &\|\partial_1^{m-1} \mathcal{C}_{m-1}\|_{L^2(Q_m)} \leq \\ &c(\, 1 + \|\partial_1^{m-1} N\|_{L^2(-\kappa_m, \kappa_m)} + \|\partial_1^{m-1} \nabla v_{m-1}\|_{L^2(Q_m)}\,), \end{aligned} \tag{4.23}$$

$$\begin{aligned} &\|\partial_1^m \mathcal{B}_{m-1}\|_{L^2(Q_m)} \leq \\ &c\big(1 + \|\partial_1^m N\|_{L^2(-\kappa_m, \kappa_m)} + \|\partial_1^{m-1} \nabla v_{m-1}\|_{L^2(Q_{m-1})}\big). \end{aligned} \tag{4.24}$$

Proof. The proof is given in Section 11 □

4.3. **Proof of Proposition 4.1. Part II.** We are now in a position to complete the proof of Proposition 4.1. Recall the denotations for the rectangles Q and Q_0,

$$Q = (-\kappa, \kappa) \times (-\rho, \rho), \quad Q_0 = (-2\kappa, 2\kappa) \times (-2\rho, 2\rho).$$

Let us consider the sequences of domains $Q = Q_r \subset Q_{r-1} ... \subset Q_0$ and functions φ_m, $0 \leq m \leq r$, defined by relations (4.2)-(4.4). Recall that $\varphi_r = \varphi$. Let us also consider the sequence of the functions $v_m = \varphi_m u$. It is necessary to prove that the estimate

$$\|\partial_1^m \nabla v_m\|_{L^2(Q)} \leq c(1 + \|\partial_1^m N\|_{L^2(-\kappa, \kappa)}) \tag{4.25}$$

holds true for all $1 \leq m \leq r$. Notice that for $m = 1$, this estimate obviously follows from estimate in Lemma 4.3. Now we proceed with the induction principle. Assume that the inequality

$$\|\partial_1^{m-1} \nabla v_{m-1}\|_{L^2(Q_{m-1})} \leq c(1 + \|\partial_1^{m-1} N\|_{L^2(-\kappa_{m-1}, \kappa_{m-1})}) \tag{4.26}$$

holds for some $m \geq 2$. Notice that the function v_m satisfies equation (4.19) that reads

$$\operatorname{div}\big(aN \nabla \partial_1^m v_m\big) = -\operatorname{div} \mathcal{A}_m + \operatorname{div} \partial_1^m \mathcal{B}_{m-1} + \partial_1^m \mathcal{C}_{m-1} \quad \text{in} \quad Q_m, \tag{4.27}$$

where $\mathcal{A}_m$, $\mathcal{B}_{m-1}$, and $\mathcal{C}_{m-1}$ are given by (4.15) and (4.18). Recall that the positive matrix aN is uniformly bounded from below and above. Multiplying both sides of (4.27) by $\partial_1^m v_m$, integrating the result by parts over Q_m, and applying the Cauchy inequality we arrive at the estimate

$$\|\partial_1^m \nabla v_m\|_{L^2(Q_m)} \leq \|\mathcal{A}_m\|_{L^2(Q_m)} + \|\partial_1^m \mathcal{B}_{m-1}\|_{L^2(Q_m)} + \|\partial_1^{m-1} \mathcal{C}_{m-1}\|_{L^2(Q_m)}.$$

It follows from estimates in Lemma 4.5 that

$$\begin{aligned} &\|\partial_1^m \nabla v_m\|_{L^2(Q_m)} \leq c + \|\partial_1^m \nabla v_m\|^{\lambda}_{L^2(Q_m)} + \\ &c\|\partial_1^m N\|_{L^2(-\kappa_m, \kappa_m)} + c\|\partial_1^{m-1} \nabla v_{m-1}\|_{L^2(Q_{m-1})}), \end{aligned}$$

where $0 < \lambda < 1$. From this and the induction hypothesis we conclude that

$$\|\partial_1^m \nabla v_m\|_{L^2(Q_m)} \leq c\,\big(\,1 + \|\partial_1^m N\|_{L^2(-\kappa_m, \kappa_m)}\,\big).$$

This completes the proof of the induction step. Applying the induction principle we obtain desired estimate (4.11) for all $m \in [1, r]$. This completes the proof of Proposition 4.1.

4.4. **Conormal derivative. Proof of Proposition 4.2.** Split the rectangle Q into two parts

$$Q^- = (-\kappa, \kappa) \times (-\rho, 0), \quad Q^+ = (-\kappa, \kappa) \times (0, \rho),$$

separated by the interface segment

$$\ell = (-\kappa, \kappa) \times \{0\}.$$

Denote by $u^\pm$ and $v^\pm$ the restrictions of the functions u and $v = \varphi u$ to $Q^\pm$. We also denote by $\partial_N u^\pm$ and $\partial_N v^\pm$ the conormal derivatives

$$\partial_N u^\pm = (N_{21}\partial_1 + N_{22}\partial_2)\, u^\pm, \quad \partial_N v^\pm = (N_{21}\partial_1 + N_{22}\partial_2) v^\pm \quad \text{on} \quad \ell. \tag{4.28}$$

We will consider the conormal derivatives as a function defined on the interval $(-\kappa, \kappa)$. Since the function φ vanishes for $|x_1| \geq \kappa$, we may assume that $\varphi\partial_N u^\pm$ and $\partial_N v^\pm$ are extended by zero to $\mathbb{R}$. Recall that for every $s \geq 0$, the Sobolev space $W^{s,2}(\mathbb{R})$ is defined as the completion of the space $C_0^\infty(\mathbb{R})$ with respect to the norm

$$\|u\|^2_{W^{s,2}(\mathbb{R})} = \int_{\mathbb{R}} (1 + \xi^2)^s\, |\hat{u}(\xi)|^2\, d\xi, \tag{4.29}$$

where $\hat{u}$ is the Fourier transform of u.

Now, we estimate $\varphi\partial_N u^-$. The proof of inequality (4.13) for $\varphi\partial_N u^+$ is similar. First we show that such an inequality holds true for $\partial_N v^-$. It suffices to prove that the estimate

$$\Big| \int_{\mathbb{R}} \partial_1^r(\partial_N v^-)\, \zeta\, dy_1 \Big| \leq c\,(1 + \|\partial_1^r N\|_{L^2(-2\kappa, 2\kappa)}\,)\|\zeta\|_{W^{1/2,2}(\mathbb{R})} \tag{4.30}$$

holds for all compactly supported $\zeta \in W^{1/2,2}(\mathbb{R})$. Recall that v is extended by zero to $\mathbb{R}^2$. Choose an arbitrary function $\zeta \in W^{1/2,2}(\mathbb{R})$. By virtue of the extension theorem for Sobolev functions, the function ζ can be extended to the strip $Q_\infty = \mathbb{R} \times (-\rho, 0)$ such that the extension ζ^* admits the estimate

$$c^{-1}\|\zeta\|_{W^{1/2,2}(\mathbb{R})} \leq \|\zeta^*\|_{W^{1,2}(Q_\infty)} \leq c\|\zeta\|_{W^{1/2,2}(\mathbb{R})}. \tag{4.31}$$

Multiplying both the sides of (4.16) by ζ and integrating the result by parts over Q_0^- we arrive at the integral identity

$$\int_{\mathbb{R}} \partial_1^r \partial_N v^-\, \zeta\, dy_1 = \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 + \mathbf{I}_4, \tag{4.32}$$

where

$$\begin{aligned} \mathbf{I}_1 &= \int_{\mathbb{R}} \partial_1^r(v_{r-1}\partial_N\varphi)\zeta\, dy_1, \\ \mathbf{I}_2 &= \int_{Q_0^-} \partial_1^r(N\nabla v) \cdot \nabla\zeta^*\, dy, \\ \mathbf{I}_3 &= -\int_{Q_0^-} \partial_1^r(v_{r-1}N\nabla\varphi) \cdot \nabla\zeta^*\, dy, \\ \mathbf{I}_4 &= -\int_{Q_0^-} \partial_1^{r-1}(\nabla\varphi \cdot N\nabla v_{r-1})\partial_1\zeta^*\, dy. \end{aligned} \tag{4.33}$$

Let us estimate step by step the quantities $\mathbf{I}_i$. We have

$$|\mathbf{I}_1| \leq c\|\partial_1^{r-1}(v_{r-1}N\nabla\varphi)\|_{W^{1/2,2}(\mathbb{R})}\, \|\zeta\|_{W^{1/2,2}(\mathbb{R})}. \tag{4.34}$$

Recall that the function $\partial_1^{r-1}(v_{r-1}N\nabla\varphi)$ is compactly supported in Q. From this and the Poincare inequality we obtain

$$\|\partial_1^{r-1}(v_{r-1}N\nabla\varphi)\|_{W^{1/2,2}(\mathbb{R})} \leq c\|\partial_1^{r-1}(v_{r-1}N\nabla\varphi)\|_{W^{1,2}(Q_0)} \leq \\ c\|\nabla\partial_1^{r-1}(v_{r-1}N\nabla\varphi)\|_{L^2(Q_0)}.$$

Noting that N is independent of y_2, we get

$$\begin{aligned}&\|\partial_1^{r-1}(v_{r-1}N\nabla\varphi)\|_{W^{1/2,2}(\mathbb{R})}\leq\\ &c\|\partial_1^{r}(v_{r-1}N\nabla\varphi)\|_{L^2(Q_0)}+c\|\partial_1^{r-1}(\partial_2 v_{r-1}N\nabla\varphi)\|_{L^2(Q_0)}+\boldsymbol{\mu},\end{aligned}\tag{4.35}$$

where

$$\boldsymbol{\mu}=\|\partial_1^{r-1}(v_{r-1}N\partial_2\nabla\varphi)\|_{L^2(Q_0)}.\tag{4.36}$$

Note that the quantities $\partial_1^r(v_{r-1}N\nabla\varphi)$ and $\partial_1^{r-1}(\partial_2 v_{r-1}N\nabla\varphi)$ are compactly supported in $Q=Q_r$. Since $a=1$ in Q_0^-, it follows from formulae (4.15) for $\mathcal{B}_{r-1}$, $\mathcal{C}_{r-1}$ and estimates (4.23), (4.24) for $\mathcal{B}_{r-1}$, $\mathcal{C}_{r-1}$ in Lemma 4.5 that

$$\begin{aligned}&\|\partial_1^{r}(v_{r-1}N\nabla\varphi)\|_{L^2(Q_0)}+\|\partial_1^{r-1}(\partial_2 v_{r-1}N\nabla\varphi)\|_{L^2(Q_0)}\leq\\ &c\|\partial_1^{r}\mathcal{B}_{r-1}\|_{L^2(Q_0)}+\|\partial_1^{r-1}\mathcal{C}_{r-1}\|_{L^2(Q_0)}\leq\\ &c\big(1+\|\partial_1^{r}N\|_{L^2(-2\kappa,2\kappa)}\big).\end{aligned}\tag{4.37}$$

Let us estimate $\boldsymbol{\mu}$. We have

$$\partial_1^{r-1}(v_{r-1}N\partial_2\nabla\varphi)=\sum c_{j,\alpha,\beta}\ \partial_1^{j}N\ \partial_1^{\alpha}v_{r-1}\ \partial_1^{\beta}\partial_2\nabla\varphi,$$

where the sum is taken over the set of indexes

$$j,\alpha,\beta\geq 0,\quad j+\alpha+\beta=r-1.$$

Since N is uniformly bounded, we have

$$\|\partial_1^{j}N\|_{L^\infty(-2\kappa,2\kappa)}\leq c\big(1+\|\partial_1^{j+1}N\|_{L^2(-2\kappa,2\kappa)}\big).$$

It follows that

$$\boldsymbol{\mu}\leq\sum_{j=0}^{j=r-1}\big(1+\|\partial_1^{j+1}N\|_{L^2(-\kappa,\kappa)}\big)\sum_{\alpha+\beta=r-1-j}\|\partial_1^{\alpha}v_{r-1}\ \partial_1^{\beta}\partial_2\nabla\varphi\|_{L^2(Q_0)}.$$

Note that all derivatives of φ are bounded. Thus we get

$$\sum_{\alpha+\beta=r-1-j}\|\partial_1^{\alpha}v_{r-1}\ \partial_1^{\beta}\partial_2\nabla\varphi\|_{L^2(Q_0)}\leq c\big(1+\|\partial_1^{r-1-j}v_{r-1}\|_{L^2(Q_0)}\big).$$

From this and estimate (4.11) in Proposition 4.1 we conclude that

$$\sum_{\alpha+\beta=r-1-j}\|\partial_1^{\alpha}v_{r-1}\ \partial_1^{\beta}\partial_2\nabla\varphi\|_{L^2(Q)}\leq c\big(1+\|\partial_1^{r-1-j}N\|_{L^2(-2\kappa,2\kappa)}\big),$$

which yields

$$\boldsymbol{\mu}\leq\sum_{j=0}^{r-1}\big(1+\|\partial_1^{j+1}N\|_{L^2(-2\kappa,2\kappa)}\big)\ \big(1+\|\partial_1^{r-1-j}N\|_{L^2(-2\kappa,2\kappa)}\big).$$

Since N is uniformly bounded, we may apply the interpolation inequality to obtain

$$\begin{aligned}\boldsymbol{\mu}\leq\sum_{j=0}^{r-1}\big(1+\|\partial_1^{r}N\|^{\frac{j+1}{r}}_{L^2(-2\kappa,2\kappa)}\big)\ \big(1+\|\partial_1^{r}N\|^{\frac{r-1-j}{r}}_{L^2(-2\kappa,2\kappa)}\big)\leq&\\ c\big(1+\|\partial_1^{r}N\|_{L^2(-2\kappa,2\kappa)}\big).&\end{aligned}$$

Combining this estimate with (4.37) and (4.35) we obtain

$$\|\partial_1^{r-1}(v_{r-1}N\nabla\varphi)\|_{W^{1/2,2}(\mathbb{R})}\leq c\big(1+\|\partial_1^{r}N\|_{L^2(-2\kappa,2\kappa)}\big).$$

From this and expression (4.32) for $\mathbf{I}_1$ we conclude that

$$|\mathbf{I}_1|\leq c\left(1+\|\partial_1^{r}N\|_{L^2(-2\kappa,2\kappa)}\right)\ \|\zeta\|_{W^{1/2,2}(\mathbb{R})}.\tag{4.38}$$

Let us estimate the integral $\mathbf{I}_2$. Note that

$$\partial_1^{r}(N\nabla v)=N\nabla\partial_1^{r}v+\mathcal{A}_r\ \text{ in }\ Q_0^-,\tag{4.39}$$

where $\mathcal{A}_r$ is given by equality (4.18). It follows from estimate (4.14) in Lemma 4.5 and estimate (4.11) in Proposition 4.1 that

$$\|\mathcal{A}_r\|_{L^2(Q)} \le c\big(1+\|\partial_1^r N\|_{L^2(-\kappa,\kappa)}\big) + c\|\partial_1^r \nabla v\|^{\lambda}_{L^2(Q)} \le c\big(1+\|\partial_1^r N\|_{L^2(-\kappa,\kappa)}\big), \tag{4.40}$$

where $0 \le \lambda < 1$. On the other hand, estimate (4.11) in Proposition 4.1 yields

$$\|N\nabla \partial_1^r v\|_{L^2(Q)} \le c\,\big(1+\|\partial_1^r N\|_{L^2(-2\kappa,2\kappa)}\big).$$

Combining this estimate with inequality (4.40) and identity (4.39) we obtain

$$\|\partial_1^r(N\nabla v)\|_{L^2(Q^-)} \le \|N\nabla\partial_1^r v\|_{L^2(Q^-)} + \|\mathcal{A}_r\|_{L^2(Q^-)} \le c\big(1+\|\partial_1^r N\|_{L^2(-\kappa,\kappa)}\big).$$

Recall that the function ζ^* is extended by zero from Q^- to the strip Q_∞. It follows from this and the Cauchy inequality that

$$\Big|\int_{Q_0} \partial_1^r(N\nabla v^-)\cdot\nabla\zeta^*\,dy\Big| \le c\big(1+\|\partial_1^r N\|_{L^2(-2\kappa,2\kappa)}\big)\,\|\nabla\zeta^*\|_{L^2(Q_\infty)} \le c\big(1+\|\partial_1^r N\|_{L^2(-2\kappa,2\kappa)}\big)\,\|\zeta\|_{W^{1/2,2}(\mathbb{R})}.$$

This result along with expression (4.33) for $\mathbf{I}_2$ implies the desired estimate

$$|\mathbf{I}_2| \le c\,\big(1+\|\partial_1^r N\|_{L^2(-2\kappa,2\kappa)}\big)\ \|\zeta\|_{W^{1/2,2}(\mathbb{R})}. \tag{4.41}$$

In order to estimate the integrals $\mathbf{I}_3$ and $\mathbf{I}_4$, note that

$$\partial_1^r(v_{r-1}N\nabla\varphi) = \partial_1^r\mathcal{B}_{r-1}, \quad \partial_1^{r-1}(\nabla\varphi\cdot N\nabla v_{r-1}) = \partial_1^{r-1}\mathcal{C}_{r-1} \ \text{ in } \ Q_0^-.$$

From this, estimates (4.23)-(4.24) and expression (4.33) for $\mathbf{I}_i$ we obtain

$$\begin{aligned}|\mathbf{I}_3| + |\mathbf{I}_4| \le\\ c(\|\partial_1^r\mathcal{B}_{r-1}\|_{L^2(Q)} + \|\partial_1^{r-1}\mathcal{C}_{r-1}\|_{L^2(Q)})\,\|\nabla\zeta^*\|_{L^2(Q)} \le\\ c(1+\|\partial_1^r N\|_{L^2(-2\kappa,2\kappa)})\,\|\nabla\zeta^*\|_{L^2(Q)} \le\\ c\,\big(1+\|\partial_1^r N\|_{L^2(-2\kappa,2\kappa)}\big)\ \|\zeta\|_{W^{1/2,2}(\mathbb{R})}.\end{aligned}$$

Combining this estimate with estimates (4.38) and (4.41) we obtain the estimate

$$|\mathbf{I}_i| \le c\,\big(1+\|\partial_1^r N\|_{L^2(-2\kappa,2\kappa)}\big)\ \|\zeta\|_{W^{1/2,2}(\mathbb{R})}, \quad i=1,\dots 4.$$

Finally notice that this estimate along with equality (4.32) obviously yields desired estimate (4.30).

In order to complete the proof of Proposition 4.2, it remains to obtain the estimate, similar to (4.30), for the function $\varphi\partial_N u^-$. Recall that $v=\varphi u$, where φ is an arbitrary function of the class $C_0^\infty(Q)$. Now we specify φ. To this end, choose an arbitrary function $\phi\in C_0^\infty(-\kappa,\kappa)$ and fix the function $\iota(x_2)$, $x_2\in\mathbb{R}$, such that

$$\iota\in C_0^\infty(\mathbb{R}), \quad \iota(x_2)=1 \ \text{ for } \ |x_2|\le\rho/3, \quad \iota(x_2)=0 \ \text{ for } \ |x_2|\ge 2\rho/3.$$

Now set

$$\varphi\equiv\varphi_r = \phi(x_1)\iota(x_2).$$

With this notation we have

$$\partial_N v^- = \phi\partial_N u^- + \partial_1\phi N_{21}u^- = \phi\partial_N u^- + N_{21}\partial_1\phi\, v_{r-1}^- \ \text{ on } \ \ell,$$

since u coincides with v_{r-1} on the support of $\nabla\varphi$. Note that

$$N_{21}\partial_1\varphi\, v_{r-1}^- = v_{r-1}N\nabla\varphi\cdot\mathbf{e}_2 \ \text{ on } \ \ell.$$

Thus we get

$$\varphi\partial_N u^- = \partial_N v^- - (v_{r-1}\partial_N\varphi) \ \text{ on } \ \ell.$$

In order to prove desired estimate (4.13) in Proposition 4.13 for $\partial_N u^-$, it suffices to show that the inequality

$$\Big| \int_{\mathbb{R}} \partial_1^r \, (\varphi \partial_N u^-) \, \zeta \, dy_1 \Big| \le c \, (1 + \|\partial_1^r N\|_{L^2(-2\kappa, 2\kappa)}) \, \|\zeta\|_{W^{1/2,2}(\mathbb{R})} \tag{4.42}$$

holds for all compactly supported functions $\zeta \in W^{1/2,2}(\mathbb{R})$. In its turn, estimate (4.42) is a straightforward consequence of estimates

$$\Big| \int_{\mathbb{R}} \partial_1^r \, (\partial_N v^-) \, \zeta \, dy_1 \Big| \le c \, (1 + \|\partial_1^r N\|_{L^2(-2\kappa, 2\kappa)}) \, \|\zeta\|_{W^{1/2,2}(\mathbb{R})}, \tag{4.43}$$

$$\Big| \int_{\mathbb{R}} \partial_1^r \, (v_{r-1} \partial_N \varphi) \, \zeta \, dy_1 \Big| \le c \, (1 + \|\partial_1^r N\|_{L^2(-2\kappa, 2\kappa)}) \, \|\zeta\|_{W^{1/2,2}(\mathbb{R})}. \tag{4.44}$$

It is easy to see that estimate (4.43) coincides with estimate (4.30). On the other hand, the left hand side of (4.44) obviously equals the quantity $|\mathbf{I}_1|$ defined by (4.33). Therefore, estimate (4.44) obviously follows from estimate (4.34) for $\mathbf{I}_1$. It remains to note that (4.43) and (4.44) imply desired estimate (4.42). This completes the proof of Proposition 4.2.

5. Kohn-Vogelius functional. Proofs of Theorems 3.1 and 3.2

Recall the formulation of the problem. Our goal is to derive estimates of the gradient dJ of the Kohn-Vogelius functional J in terms of the geometric characteristics of the interface Γ. The results are based on the normal derivatives estimate for solutions to the transmission problem given in the previous section. These estimates establish the dependence of the smoothness properties of solutions to a transmission problem with respect to the smoothness properties of the interface Γ. Recall the conditions **H.1**-**H.3** imposed on Γ.

H.1 The Jordan curve $\Gamma \subset \Omega$ satisfies the energy inequality

$$\frac{1}{2} \int_{\Gamma} |k^2| ds + \mathcal{L} \le E_0.$$

H.2 There is $\nu > 0$ with the property

$$\text{dist } (\Gamma \setminus \Gamma_{3\kappa}, \, \Gamma_{2\kappa}) \ge \nu$$

for every arc $\Gamma_{3\kappa}$ with κ defined by Lemma 2.2.

H.3 There is $\varrho > 0$ such that

$$\text{dist } (\Gamma, \partial\Omega) > \varrho.$$

By virtue of Corollary 2.4, every curve Γ satisfying conditions **H.1**-**H.3** is a Jordan curve of the class $C^{1+\alpha}$, $0 < \alpha < 1/2$. Γ splits the domain Ω into two parts. The first $\Omega_i \Subset \Omega$ (inclusion) is a simply connected domain with boundary Γ. The second is the curvilinear annulus $\Omega_e = \Omega \setminus \overline{\Omega_i}$ bounded by Γ and $\partial\Omega$. For simplicity, we will assume that $\partial\Omega$ is a Jordan curve of the class C^∞.

Next, recall that the piecewise constant function $a : \Omega \to \mathbb{R}^+$ (conductivity) is defined by the equalities

$$a = 1 \ \text{ in } \ \Omega_e, \quad a = a_0 \ \text{ in } \ \Omega_i.$$

5.1. **Proof of Theorem 3.1.** Let $w \in W^{1,2}(\Omega)$ be a weak solution to the equation

$$\text{div } (a \, \nabla w) = 0 \ \text{ in } \ \Omega. \tag{5.1}$$

Denote by w^- and w^+ the restrictions of w onto subdomains Ω_e and Ω_i,

$$w^- := w \ \text{ in } \ \Omega_e, \quad w^+ := w \ \text{ in } \ \Omega_i. \tag{5.2}$$

If Γ is sufficiently smooth, then w is continuous on Γ. In other words, $w^- = w^+$ on Γ. However, the normal derivative of w has a jump across Γ. The following remark is important for the further analysis. Set

$$\partial_n w^- = \nabla w^- \cdot n, \quad \partial_n w^+ = \nabla w^+ \cdot n \text{ on } \Gamma. \tag{5.3}$$

Our task is to to prove the estimate (3.11):

$$\|\partial_n w^\pm\|_{H^{m+1/2}(\Gamma)} \le c\,(\,1 + \|\partial_s^m k\|_{L^2(\Gamma)}\,)\,\|w\|_{W^{1,2}(\Omega)},$$

where $m \ge 0$ is an arbitrary integer, c depends only on m and on the constants E_0, ν, ρ in Conditions **H.1**-**H.3**. We split the proof of estimate (3.11) into three steps. First, we define a standard neighborhood of an arbitrary point $z \in \Gamma$ and the special mapping which takes diffeomorphically the standard neighborhood onto a rectangle. Next we employ Proposition 4.2 in Section 4 in order to obtain the local version of estimate (3.11). Finally, we use the local estimate and a partition of unity to complete the proof of (3.11).

5.1.1. *Standard neighborhood and standard mapping.* Note that the immersion f and the curve $\Gamma = f(\mathbb{S}^1)$ satisfy all conditions of Lemma 2.2. Choose an arbitrary $z \in \Gamma$ and consider the subarc $\Gamma_{2\kappa}$ of the arc $\Gamma_{3\kappa}$ defined by Lemma 2.2. It follows from this lemma that $\Gamma_{2\kappa}$ admits the representation

$$\Gamma_{2\kappa} : x_2 = \eta(x_1), \quad x_1 \in (-\gamma, \delta)$$

with positive γ, δ, depending only on κ. Next, assume that Γ has no self intersections and is compactly embedded into a bounded domain $\Omega \subset \mathbb{R}^2$. Set

$$2\rho = \min\big\{\nu,\ \text{dist}\,(\Gamma, \partial\Omega)\big\} > 0, \tag{5.4}$$

where $\nu > 0$ is given by Condition **H.2**.

Definition 5.1. *Under the above assumptions, the standard neighborhood $\Sigma_{2\kappa}$ of the point z is the curvilinear quadrangle defined in the Cartesian coordinates system associated with z by the equalities*

$$\Sigma_{2\kappa} = \big\{x = (x_1, x_2) : -\gamma < x_1 < \delta, \quad -2\rho + \eta(x_1) < x_2 < 2\rho + \eta(x_1)\big\}. \tag{5.5}$$

Our next task is to define the special mapping, which takes the standard neighborhood onto the rectangle. To this end introduce new variables

$$y_1 = s(x_1), \quad y_2 = x_2 - \eta(x_1). \tag{5.6}$$

It is easy to see that the mapping $y = y(x)$ takes diffeomorphically the standard neighborhood $\Sigma_{2\kappa}$ onto the rectangle

$$Q_0 = (-2\kappa, 2\kappa) \times (-2\rho, 2\rho). \tag{5.7}$$

Introduce the matrices M and N defined by the equalities

$$M := y'(x) = \begin{pmatrix} \sqrt{1+\eta'^2} & 0 \\ -\eta' & 1 \end{pmatrix}, \quad N = (\det M)^{-1}\, M\, M^\top. \tag{5.8}$$

Notice that these matrices depend only on x_1 and hence only on $s \in (-2\kappa, 2\kappa)$. Introduce the function $\Theta : (-2\kappa, 2\kappa) \to \mathbb{R}$ defined by the equalities

$$\Theta(s) = \arctan \eta'(x_1(s)), \quad \Theta(0) = 0, \tag{5.9}$$

in the Cartesian coordinates system associated with the origin z. Recall that $x = 0$ and $s = 0$ at the chosen point z.

Lemma 5.2. *Under the above assumptions, we have*

$$\Theta \in W^{1,2}(-2\kappa, 2\kappa), \quad |\Theta| \le \frac{\pi}{18}, \tag{5.10}$$

$$\tau(s) = \big(\cos\Theta,\ \sin\Theta\big), \quad n(s) = (-\sin\Theta,\ \cos\Theta), \quad k(s) = \Theta'(s)\, n(s). \tag{5.11}$$

Moreover, the matrices M and N admit the representation

$$M = \begin{pmatrix} \dfrac{1}{\cos\Theta} & 0 \\ -\tan\Theta & 1 \end{pmatrix}, \quad N = \frac{1}{\cos\Theta}\begin{pmatrix} 1 & -\sin\Theta \\ -\sin\Theta & 1 \end{pmatrix}. \tag{5.12}$$

Proof. Introduce the function

$$\lambda(s(x_1)) = \eta(x_1), \quad s \in (-2\kappa, 2\kappa).$$

We have

$$\lambda'(s)\sqrt{1+\eta'(x_1)^2} = \eta'(x_1) \text{ and hence } \lambda'(s)^2 = \eta'(x_1)^2 - \lambda'(s)^2\,\eta'(x_1)^2,$$

which gives

$$\eta' = \frac{\lambda'}{\sqrt{1-\lambda'^2}}, \quad 1+\eta'^2 = \frac{1}{1-\lambda'^2}. \tag{5.13}$$

In particular, we have

$$M = \begin{pmatrix} \dfrac{1}{\sqrt{1-\lambda'^2}} & 0 \\ -\dfrac{\lambda'}{\sqrt{1-\lambda'^2}} & 1 \end{pmatrix}. \tag{5.14}$$

Next notice that

$$\tau = \frac{1}{\sqrt{1+\eta'^2}}\,(1,\eta') = (\sqrt{1-\lambda'^2},\, \lambda') = (\cos\Theta,\, \sin\Theta) \tag{5.15}$$

with

$$\Theta = \arctan\eta' = \arcsin\lambda'.$$

We thus get the following formulae for the curvature vector k and the normal n

$$k = \partial_s\tau = \Theta'(s)n, \quad n = (-\sin\Theta, \cos\Theta).$$

Identity (5.15) implies that the matrices M and N admit representation (5.13). It remains to note that $|\tan\Theta| = |\eta'| \le 1/6$, which yields the estimate $|\Theta| \le \pi/18$. □

Corollary 5.3. *Under the assumptions of Lemma 5.2, the estimate*

$$\|\partial_s^m N\|_{L^2(-2\kappa,2\kappa)} \le c(m)(1 + \|\partial_s^{m-1}k\|_{L^2(-2\kappa,2\kappa)}) \tag{5.16}$$

holds for every integer $m \ge 1$.

Proof. Since $|\Theta| \le \pi/18$, it follows from the estimates of composite functions in Sobolev spaces that

$$\|\partial_s^m N\|_{L^2(-2\kappa,2\kappa)} \le c(C_N)\big(1 + \|\partial_s^m\Theta\|_{L^2(-2\kappa,2\kappa)}\big),$$

where

$$C_N = \sum_{k=0}^{m} \sup_{|\Theta|\le\pi/18} |\partial_\Theta^k N(\Theta)|.$$

It remains to note that $\partial_s^m\Theta = \partial_s^{m-1}k$. □

5.2. **Local estimates.** In this paragraph we prove the local estimates of the normal derivative of a weak solution to equation (5.1). The result is given by the following proposition. Fix an arbitrary point $z \in \Gamma$. Without loss of generality we may assume that the arc-length variable s equals zero at z. Let κ, depending only on the constant E_0 in Condition **H.1**, be given by Lemma 2.2. Choose an arbitrary function $\phi \in C_0^\infty(-\kappa, \kappa)$. Furthermore, assume that the functions $\phi(s)$ and $\phi(s)\partial_n w^\pm(s)$ are extended by zero to the real axis $\mathbb{R}$.

Proposition 5.4. *Under the above assumptions, the estimate*

$$\|\phi\, \partial_n w^\pm\|_{W^{m+1/2,2}(\mathbb{R})} \le c\big(1 + \|\partial_s^m k\|_{L^2(-2\kappa,2\kappa)}\big)\, \|w\|_{W^{1,2}(\Omega)} \tag{5.17}$$

holds for every integer $m \ge 0$. Here c depends only on m and on the constants E_0,ν, ρ in Conditions **H.1**-**H.3** *of Theorem 3.2.*

Proof. Notice that estimate (5.17) is invariant with respect to dilatation w. Hence without loss of generality we may assume that $\|w\|_{W^{1,2}(\Omega)} = 1$. For an arbitrary fixed $z \in \Gamma$, denote by $\Sigma_{2\kappa}$ the standard neighborhood determined by Definition 5.1. Split $\Sigma_{2\kappa}$ into two disjoint parts $\Sigma_{2\kappa}^-$ and $\Sigma_{2\kappa}^+$ defined by the equalities

$$\begin{aligned}\Sigma_{2\kappa}^- &= \big\{x = (x_1, x_2) : \, -\gamma < x_1 < \delta, \quad \eta(x_1) - 2\rho < x_2 < \eta(x_1)\,\big\},\\ \Sigma_{2\kappa}^+ &= \big\{x = (x_1, x_2) : \, -\gamma < x_1 < \delta, \quad \eta(x_1) < x_2 < 2\rho + \eta(x_1)\,\big\}.\end{aligned} \tag{5.18}$$

Here (x_1, x_2) is the local Cartesian coordinates system with the origin at z, defined in Lemma 2.2. Notice that the ordinate axis x_2 is directed inside Ω_i, which yields

$$\Sigma_{2\kappa}^- = \Sigma_{2\kappa} \cap \Omega_e, \quad \Sigma_{2\kappa}^+ = \Sigma_{2\kappa} \cap \Omega_i, \quad \Sigma_{2\kappa} = \Sigma_{2\kappa}^- \cup \Sigma_{2\kappa}^+ \cup \Gamma_{2\kappa}.$$

In particular, the coefficient a equals 1 in $\Sigma_{2\kappa}^-$ and equals a_0 in $\Sigma_{2\kappa}^+$. The function w serves as a solution to equation (5.1) and the integral identity

$$\int_{\Sigma_{2\kappa}} a\nabla w \cdot \nabla\zeta \, dx = 0 \tag{5.19}$$

holds for all $\zeta \in W_0^{1,2}(\Sigma_{2\kappa})$. The standard change of variables

$$y_1 = s(x_1), \quad y_2 = x_2 - \eta(x_1).$$

takes diffeomorphically the standard neighborhood $\Sigma_{2\kappa}$ onto the rectangle

$$Q_0 = (-2\kappa, 2\kappa) \times (-2\rho, 2\rho).$$

In its turn, the standard change of the variables (5.6) takes diffeomorphically curvilinear quadrangles $\Sigma_{2\kappa}^\pm$ onto the rectangles

$$Q_0^- = (-2\kappa, 2\kappa) \times (-2\rho, 0), \quad Q_{2\kappa}^+ = (-2\kappa, 2\kappa) \times (0, 2\rho).$$

We have

$$Q_0 = Q_0^- \cup Q_0^+ \cup \ell, \text{ where } \quad \ell = (-2\kappa, 2\kappa) \times \{0\}.$$

Now set

$$u(y) = w(x(y)), \quad u^\pm(y) = w^\pm(x(y)) = u(y)\Big|_{Q_0^\pm}. \tag{5.20}$$

Recall that w serves as a weak solution to equation (5.1) in the standard neighborhood $\Sigma_{2\kappa}$. In particular, it satisfies integral identity (5.19). Notice that

$$\nabla_x w(x(y)) = M^\top \nabla_y u(y), \quad dx = (\det M)^{-1} dy.$$

The change of variables $x \to y$ in (5.19) leads to the following integral identity for the function $u : Q_0 \to \mathbb{R}$,

$$\int_{\Sigma_{2\kappa}} a\, N\nabla u \cdot \nabla\zeta \, dy = 0 \text{ for all } \zeta \in W_0^{1,2}(Q_0). \tag{5.21}$$

Here the matrix N is given by Lemma 5.2. The conormal derivative $\partial_N u^-$ on the segment ℓ is defined by the equality

$$\partial_N u^- = N_{21}\partial_{y_1} u^- + N_{22}\partial_{y_2} u^- = \frac{1}{\cos\Theta}\big(- \partial_{y_1} u^- \sin\Theta + \partial_{y_2} u^-\big), \tag{5.22}$$

where $\Theta(s)$ is given by Lemma 5.2. It is easily seen that N and u meet all requirements of Proposition 4.2. Now choose an arbitrary $\phi \in C_0^\infty(-\kappa,\kappa)$ and assume that the function $\phi\partial_N u^-(s)$ extended by 0 to the real axis. Applying estimate (4.13) in Proposition 4.2 and estimate (5.16) in Corollary 5.3 we conclude that the inequality

$$\|\phi\partial_N u^-\|_{W^{r-1/2}(\mathbb{R})} \le c\big(1+\partial_s^r N\|_{L^2(-2\kappa,2\kappa)}\big) \le c\big(1+\|\partial_s^{r-1}k\|_{L^2(-2\kappa,2\kappa)}\big) \tag{5.23}$$

holds for every integer $r \ge 1$.

Recall that $s \equiv y_1$. Next, formulae (5.11) and (5.12) in Lemma 5.2 along with formula (5.22) imply

$$\begin{gathered} n(s)\cdot\nabla_x w^-(x(y)) \equiv n(s)M^\top\cdot\nabla v^- = \\ (-\sin\Theta,\cos\Theta)\cdot\begin{pmatrix} \dfrac{1}{\cos\Theta} & -\tan\Theta \\ 0 & 1\end{pmatrix}\nabla u = \\ \frac{1}{cos\Theta}(-\sin\Theta\,\partial_{y_1}u + \partial_{y_2}u) = \partial_N u^- \quad\text{on}\quad \ell. \end{gathered} \tag{5.24}$$

It follows that for every $\phi \in C_0^\infty(-\kappa,\kappa)$, we have

$$\phi(s)\partial_n w^-(s) = \phi(s)\partial_N u^-(s) \quad\text{for}\quad s\in(-2\kappa,2\kappa).$$

Hence desired estimate (5.17) is the straightforward consequence of (5.23). This completes the proof of Proposition 5.4. □

5.3. **Globalization.** Now we employ Proposition 5.4 in order to complete the proof of Theorem 3.1. To this end, we use the method of partition of unity.

The partition of unity. Let Γ satisfies all conditions of Theorem 3.1 and $\kappa \in (0,\mathcal{L}/2)$ be given by Lemma 2.2. Recall that $\mathcal{L}$ and κ depends only on the constant E_0 in the energy constraint **H.1**. Choose a finite collection of points $z_k \in \Gamma$ with the arc-length coordinates s_k such that

$$s_k = \frac{k}{N}\mathcal{L}, \quad 0\le k\le N-1, \tag{5.25}$$

where N is an arbitrary integer satisfying the condition

$$\frac{1}{N} < \frac{\kappa}{4}.$$

Now choose an arbitrary function $\psi \in C_0^\infty(-\mathcal{L}/2,\mathcal{L}/2)$ with the properties

$$\psi(s)\ge 0, \quad \psi(s)=0 \;\text{ for }\; |s|\ge\kappa, \quad \psi(s)=1 \;\text{ for }\; |s|\le\kappa/2. \tag{5.26}$$

We will consider $\psi(s)$ as a function defined on Γ. In other words, it can be regarded as $\mathcal{L}$-periodic function defined on $\mathbb{R}$. For every integer $k\in[0,N-1]$, define the function

$$\psi_k(s) = \psi(s-s_k).$$

The function $\psi_k : \Gamma \to \mathbb{R}$ is compactly supported on the arc of the length 2κ centered at z_k. Moreover, $\psi_k = 1$ on the arc of length κ centered at z_k. These arcs cover the curve Γ. It is easily seen that for every $z\in\Gamma$, at least one of the functions ψ_k equals 1 in a neighborhood of z.

Now set

$$\phi_k = \frac{\psi_k}{\sum\limits_{j=0}^{N-1}\psi_j}. \tag{5.27}$$

It is clear that every nonnegative function $\phi_k \in C^\infty(\Gamma)$ is compactly supported in the arc of length 2κ centered at z_k and

$$\sum_{k=0}^{N-1} \phi_k = 1. \tag{5.28}$$

Introduce the functions ω_k with the properties

$$\omega_k = \phi_k \partial_n w^-, \quad \partial_n w^- = \sum_{k=0}^{N-1} \omega_k. \tag{5.29}$$

It is clear that for every $m \ge 0$, we have

$$\|\partial_n w^-\|_{H^{m+1/2}(\Gamma)} \le \sum_{k=0}^{N-1} \|\omega_k\|_{H^{m+1/2}(\Gamma)}. \tag{5.30}$$

See Subsection 2.2 for the definition of spaces $H^r(\Gamma) = H^r_\sharp$.

Global estimates of the normal derivatives. It follows from (5.30) that it suffices to estimate ω_k in the space $H^{m+1/2}(\Gamma)$. To this end we use the following construction. Choose an arbitrary compactly supported function $F : (-\kappa, \kappa) \to \mathbb{R}$. There are two ways to extend F to the real line. The first way is to extend F by zero to $\mathbb{R}$. We denote this extension by $\overline{F}$,

$$\overline{F}(s) = F(s) \quad \text{for} \quad |s| < \kappa, \quad \overline{F} = 0 \quad \text{otherwise.}$$

The second way is to extend F $\mathcal{L}$-periodically to $\mathbb{R}$. We denote this extension by $F_\sharp$. These extension are connected by the relation

$$F_\sharp(s) = \sum_{k=-\infty}^{\infty} \overline{F}(s + k\,\mathcal{L}). \tag{5.31}$$

The following lemma establishes estimates of Sobolev norms of $\overline{F}$ and $F_\sharp$.

Lemma 5.5. *There is the constant c depending only on $\mathcal{L}$ and m such that*

$$\|F_\sharp\|_{H^{m+1/2}_\sharp} \le c\|\overline{F}\|_{H^{m+1/2}(\mathbb{R})} \quad \textit{for all} \quad m \ge 0. \tag{5.32}$$

The norms in the spaces $H^{m+1/2}_\sharp$ and $H^{m+1/2}(\mathbb{R})$ are defined in Subsection 2.2.

Proof. The proof is given in Subsection 9.2. □

We are now in a position to complete the proof of Theorem 3.1. To this end, it suffices to prove that every function ω_k defined by (5.29) admits the estimate

$$\|\omega_k\|_{H^{m+1/2}(\Gamma)} \le c\left(1 + \|\partial_s^m\, k\|_{L^2(\Gamma)}\right) \|w\|_{W^{1,2}(\Omega)}. \tag{5.33}$$

Fix an arbitrary integer $k \in [0, N-1]$ and the corresponding point $z_k = f(s_k)$. After the shift of the coordinate s we may assume that $s_k = 0$. Let the arc Γ_k centered at z_k is defined by Lemma 2.2 with z replaced by z_k. Introduce the function $F : \Gamma_\kappa \to \mathbb{R}$ given by the equality

$$F = \phi_k(s)\, \partial_n w^-(s), \quad s \in (-\kappa, \kappa). \tag{5.34}$$

Notice that $\phi_k \in C_0^\infty(-\kappa, \kappa)$. Let $\overline{F}$ be the extension by 0 to the real line. It follows from estimate (5.17) in Lemma 5.5 that

$$\|\overline{F}\|_{H^{m+1/2}(\mathbb{R})} \le c\left(1 + \|\partial_s^m k\|_{L^2(\Gamma)}\right) \|w\|_{W^{1,2}(\Omega)}. \tag{5.35}$$

On the other hand, relations (5.29) and (5.31) imply the equality

$$F_\sharp = \omega_k \quad \text{on} \quad \Gamma.$$

From this, estimate (5.35), and estimate (5.32) in Lemma 5.5 we finally obtain

$$\|\omega_k\|_{H^{m+1/2}(\Gamma)} = \|F_\sharp\|_{H^{m+1/2}(\Gamma)} \le c\|\overline{F}\|_{H^{m+1/2}(\mathbb{R})} \le \\ c\left(1 + \|\partial_s^m k\|_{L^2(\Gamma)}\right)\, \|w\|_{W^{1,2}(\Omega)},$$

which yields desired estimate (5.33). This completes the proof of Theorem 3.1.

5.4. **Proof of Theorem 3.2.** Recall that the Kohn-Vogelius functional is defined by the formula

$$J(\Gamma) = \int_\Omega a\nabla(v-w)\cdot\nabla(v-w)\,dx. \tag{5.36}$$

Here $v, w : \Omega \to \mathbb{R}$ satisfy the equations and boundary conditions

$$\begin{array}{lll} \operatorname{div}\, a\nabla v = 0 & \operatorname{div}\, a\nabla w = 0 & \text{in } \Omega, \\ a\nabla v\cdot n = g & w = h & \text{on } \partial\Omega. \end{array} \tag{5.37}$$

The gradient dJ is defined by equality (1.14),

$$dJ = 2\big(a\nabla v\cdot n\big[\partial_n v\big] - a\nabla w\cdot n\big[\partial_n w\big]\big)n - \big[a\nabla v\cdot\nabla v - a\nabla w\cdot\nabla w\big]n. \tag{5.38}$$

Let a curve Γ satisfies conditions **H.1**-**H.3** and the curvature $k \in H^r(\Gamma)$ with an integer $r \ge 0$. It is necessary to prove that for every $\beta \in [0, 1/2)$, the gradient dJ admits the estimate

$$\|dJ\|_{H^{r+\beta}(\Gamma} \le c(1 + \|\partial_s^r k\|_{L^2(\Gamma)}), \tag{5.39}$$

where the constant c depends on r, β, and constants E_0, ν, ϱ in conditions **H.1**-**H.3**. Note that estimate (5.39) with a suitable β and the embedding theorems imply the inequality

$$\|\partial_s^r dJ\|_{L^q(\Gamma)} \le c(1 + \|\partial_s^r k\|_{L^2(\Gamma)}), \tag{5.40}$$

which holds for every $q \in [1, \infty)$. In this case the constant c depends in addition on q. The rest of the section is devoted to the proof of estimate (5.39). The key observation is that this estimate is straightforward consequence of Theorem 3.1. It follows from inequalities (3.7) that the solutions v and w to problems (5.37) admit the estimate

$$\|v\|_{W^{1,2}(\Omega)} + \|w\|_{W^{1,2}(\Omega)} \le c(g, h).$$

Hence they meet all requirements of Theorem 3.1. Applying this theorem we conclude that estimates

$$\|\partial_n v^\pm\|_{H^{m+1/2}(\Gamma)} + \|\partial_n w^\pm\|_{H^{m+1/2}(\Gamma)} \le c\left(1 + \|\partial^m k\|_{L^2(\Gamma)}\right) \tag{5.41}$$

holds true for all integers $m \ge 0$. Next, it follows from (5.38) that dJ is a quadratic form of the normal derivatives $\partial_n v^\pm$ and $\partial_n w^\pm$. Hence it suffices to estimates the products $\partial_n v^\pm\,\partial_n w^\pm$ and $\partial_n v^\pm\,\partial_n w^\mp$. Let us estimate $\partial_n v^-\partial_n w^-$. The proofs of estimates of other terms are similar. It is necessary to show that

$$\|\partial_n v^-\partial_n w^-\|_{H^{m+\beta}(\Gamma)} \le c\left(1 + \|\partial^m k\|_{L^2(\Gamma)}\right) \tag{5.42}$$

for all integers $m \ge 1$ and for all $\beta \in [0, 1/2)$. It follows from the Moser inequality that

$$\begin{aligned}\|\partial_n v^-\partial_n w^-\|_{H^{m+\beta}(\Gamma)} \le{}& c\,\|\partial_n v^-\|_{L^\infty(\Gamma)}\,\|\partial_n w^-\|_{H^{m+\beta}(\Gamma)} \\ &+c\,\|\partial_n w^-\|_{L^\infty(\Gamma)}\,\|\partial_n v^-\|_{H^{m+\beta}(\Gamma)}.\end{aligned} \tag{5.43}$$

By virtue of the embedding theorem, the estimates

$$\|\partial_n v^-\|_{L^\infty(\Gamma)} \le c(\sigma)\,\|\partial_n v^-\|_{H^{1/2+\sigma}(\Gamma)}, \quad \|\partial_n w^-\|_{L^\infty(\Gamma)} \le c(\sigma)\,\|\partial_n w^-\|_{H^{1/2+\sigma}(\Gamma)}$$

hold for every $\sigma > 0$. Applying the interpolation inequality to the right hand sides of these estimates we obtain

$$\|\partial_n v^-\|_{H^{1/2+\sigma}(\Gamma)} \le c\,\|\partial_n v^-\|^{1-\frac{\sigma}{m}}_{H^{1/2}(\Gamma)}\,\|\partial_n v^-\|^{\frac{\sigma}{m}}_{H^{m+1/2}(\Gamma)},$$

$$\|\partial_n w^-\|_{H^{1/2+\sigma}(\Gamma)} \le c\,\|\partial_n w^-\|^{1-\frac{\sigma}{m}}_{H^{1/2}(\Gamma)}\,\|\partial_n w^-\|^{\frac{\sigma}{m}}_{H^{m+1/2}(\Gamma)}.$$

Inequality (5.41) with $m = 0$ implies

$$\|\partial_n v^-\|_{H^{1/2}(\Gamma)} + \|\partial_n w^-\|_{H^{1/2}(\Gamma)} \leq c.$$

Hence for every $m \geq 1$, we have

$$\begin{aligned} &\|\partial_n v^-\|_{L^\infty(\Gamma)} \leq c\, \|\partial_n v^-\|^{\frac{\sigma}{m}}_{H^{m+1/2}(\Gamma)}, \\ &\|\partial_n w^-\|_{L^\infty(\Gamma)} \leq c\, \|\partial_n w^-\|^{\frac{\sigma}{m}}_{H^{m+1/2}(\Gamma)}. \end{aligned} \tag{5.44}$$

Next, the interpolation inequality implies

$$\|\partial_n v^-\|_{H^{m+\beta}(\Gamma)} \leq c\, \|\partial_n v^-\|^{\frac{1}{m}(\frac{1}{2}-\beta)}_{H^{1/2}(\Gamma)} \ \|\partial_n v^-\|^{1-\frac{1}{m}(\frac{1}{2}-\beta)}_{H^{m+1/2}(\Gamma)},$$

$$\|\partial_n w^-\|_{H^{m+\beta}(\Gamma)} \leq c\, \|\partial_n w^-\|^{\frac{1}{m}(\frac{1}{2}-\beta)}_{H^{1/2}(\Gamma)} \ \|\partial_n w^-\|^{1-\frac{1}{m}(\frac{1}{2}-\beta)}_{H^{m+1/2}(\Gamma)}.$$

It follows that

$$\|\partial_n v^-\|_{H^{m+\beta}(\Gamma)} \leq c\, \|\partial_n v^-\|^{1-\frac{1}{m}(\frac{1}{2}-\beta)}_{H^{m+1/2}(\Gamma)},$$

$$\|\partial_n w^-\|_{H^{m+\beta}(\Gamma)} \leq c\, \|\partial_n w^-\|^{1-\frac{1}{m}(\frac{1}{2}-\beta)}_{H^{m+1/2}(\Gamma)}.$$

Substituting these inequalities along with inequalities (5.44) into (5.43) we obtain

$$\begin{aligned} \|\partial_n v^-\partial_n w^-\|_{H^{m+\beta}(\Gamma)} \leq\ & c\, \|\partial_n v^-\|^{\frac{\sigma}{m}}_{H^{m+1/2}(\Gamma)} \ \|\partial_n w^-\|^{1-\frac{1}{m}(\frac{1}{2}-\beta)}_{H^{m+1/2}(\Gamma)} \\ & + \|\partial_n w^-\|^{\frac{\sigma}{m}}_{H^{m+1/2}(\Gamma)} \ \|\partial_n v^-\|^{1-\frac{1}{m}(\frac{1}{2}-\beta)}_{H^{m+1/2}(\Gamma)}. \end{aligned} \tag{5.45}$$

Now set

$$\sigma = \frac{1}{2} - \beta, \quad \lambda = \frac{\sigma}{m}.$$

We have

$$\frac{1}{m}\big(\frac{1}{2} - \beta\big) = \lambda, \quad 1 - \frac{1}{m}\big(\frac{1}{2} - \beta\big) = 1 - \lambda.$$

From this and (5.45) we conclude that

$$\begin{aligned} \|\partial_n v^-\partial_n w^-\|_{H^{m+\beta}(\Gamma)} \leq\ & c\, \|\partial_n v^-\|^{\lambda}_{H^{m+1/2}(\Gamma)} \ \|\partial_n w^-\|^{1-\lambda}_{H^{m+1/2}(\Gamma)} \\ & + \|\partial_n w^-\|^{\lambda}_{H^{m+1/2}(\Gamma)} \ \|\partial_n v^-\|^{1-\lambda}_{H^{m+1/2}(\Gamma)}. \end{aligned}$$

Applying the Young inequality we arrive at the estimate

$$\|\partial_n v^-\partial_n w^-\|_{H^{m+\beta}(\Gamma)} \leq c\, \|\partial_n v^-\|_{H^{m+1/2}(\Gamma)} + c\, \|\partial_n w^-\|_{H^{m+1/2}(\Gamma)}.$$

It remains to note this inequality along with inequality (5.41) obviously imply desired estimate (5.42). This completes the proof of Theorem 3.2.

6. A priori estimates. Proof of Theorem 3.5

In this section we prove Theorem 3.5. To this end, we have to derive the Sobolev a priori estimates of solutions to the Cauchy problem

$$\begin{aligned} &\partial_t f + \nabla_s^2 k + \frac{|k|^2}{2} k - k + dJ = 0 \ \text{ in } \ (0,T) \times \mathbb{S}^1, \\ &f(0,\theta) = f_0(\theta), \quad \theta \in \mathbb{S}^1 \end{aligned} \tag{6.1}$$

for an immersion $f(t,\theta)$, $t > 0$, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. Here θ is the angle variable on $\mathbb{S}^1$, and f is 2π-periodic function of θ. Recall that s is arc-length variable on the curve $\Gamma(t) = f(t, \mathbb{S}^1)$ associated with f. We have

$$\begin{aligned} &\partial_s = \frac{1}{|\partial_\theta f|}\,\partial_\theta, \quad \nabla_s = \partial_s \cdot -(\tau\cdot\partial_s\cdot)\,\tau, \\ &\tau = \partial_s f, \quad k = \partial_s^2 f = \partial_s \tau. \end{aligned} \tag{6.2}$$

Note that the time derivative ∂_t is evaluated for fixed angle variable θ. With this notation ∂_s and ∇_s become nonlinear differential operators.

Furthermore, we denote by $\mathcal{L}(t)$ the length

$$\mathcal{L}(t) = \int_{\Gamma(t)} ds := \int_0^{2\pi} \sqrt{g(t,\theta)}\, d\theta, \quad \sqrt{g} = |\partial_\theta f|, \tag{6.3}$$

of $\Gamma(t) = f(t, \mathbb{S}^1)$.

Now fix an arbitrary even integer $m \geq 6$. Throughout of the section we assume that all conditions of Theorem 3.5 are satisfied. In particular, there are two positive constants E_0 and E_m such that the initial curvature $k_0 = \partial_s^2 f_0(s)$ satisfies the inequalities

$$\int_{\Gamma(0)} (\frac{1}{2}|k_0|^2 + 1) ds \leq E_0, \quad \int_{\Gamma(0)} |\partial_s^{m-2} k_0|^2\, ds \leq E_m. \tag{6.4}$$

The proof of Theorem 3.5 falls into six steps. Our first task is introduction of notation and presentation of auxiliary material.

6.1. **Commutators and connections.**

6.1.1. *Commutators.* In the proofs of a priori estimates in Theorem 3.5 the multiplication of both sides of (6.1) by the higher order derivatives of k followed by integration by parts is performed. This procedure requires the calculation of the commutator of spatial and time derivatives. In order to do this, we introduce additional notation. Let us consider a one-parametric family of immersions $f : [0,T] \times \mathbb{S}^1 \to \mathbb{R}^2$. Set

$$V = \partial_t f, \quad \pi_2 = 2V \cdot k, \quad \pi_1 = \frac{1}{2} \partial_s \pi_2. \tag{6.5}$$

Denote by Π a projection and by ∇_t a differential operator:

$$\Pi\, \Phi = \Phi - (\tau \cdot \Phi)\, \tau, \quad \nabla_t \Phi = \Pi\, \partial_t \Phi, \tag{6.6}$$

where $\Phi : [0,T] \times \mathbb{S}^1 :\to \mathbb{R}^2$ is an arbitrary smooth vector field. In particular, we have

$$\nabla_s \Phi = \partial_s \Phi - (\tau \cdot \partial_s \Phi)\, \tau = \Pi\, \partial_s \Phi, \quad \nabla_t \Phi = \partial_t \Phi - (\tau \cdot \partial_t \Phi)\, \tau = \Pi\, \partial_t \Phi.$$

The operator $\partial_s^2\, \partial_t$ takes the following form.

Lemma 6.1. *Under the above assumptions, we have*

$$\Pi \partial_s^2 \partial_t\, f = \nabla_t k - \pi_2\, k. \tag{6.7}$$

Proof. Let $\Phi : [0,T] \times \mathbb{S}^1 :\to \mathbb{R}^2$ be an arbitrary smooth vector field. We have

$$(\partial_s \partial_t - \partial_t \partial_s)\Phi = \frac{1}{|\partial_\theta f|} \partial_\theta\, \partial_t \Phi - \partial_t \Big(\frac{1}{|\partial_\theta f|}\, \partial_\theta \Phi \Big) =$$

$$-\partial_t \Big(\frac{1}{|\partial_\theta f|} \Big)\, \partial_\theta\, \Phi = \frac{1}{|\partial_\theta f|^3}\, (\partial_\theta f \cdot \partial_\theta \partial_t\, f)\, \partial_\theta \Phi =$$

$$(\partial_s\, f \cdot \partial_s \partial_t\, f)\, \partial_s \Phi = -(\partial_s^2 f \cdot \partial_t f)\, \partial_s \Phi = -\frac{1}{2} \pi_2\, \partial_s \Phi.$$

Here we use the relation $\partial_s f \cdot \partial_t f = \tau \cdot \partial_t f = 0$, which follows from equation (6.1). Thus we get

$$\partial_s \partial_t \Phi = \partial_t \partial_s \Phi - \frac{1}{2} \pi_2\, \partial_s \Phi. \tag{6.8}$$

Next, we have

$$\partial_s^2 \partial_t \Phi = \partial_s(\partial_s \partial_t \Phi - \partial_t \partial_s \Phi) + (\partial_s \partial_t (\partial_s \Phi) - \partial_t \partial_s (\partial_s \Phi) + \partial_t \partial_s^2 \Phi.$$

From this and (6.8) we obtain

$$\partial_s^2 \partial_t \Phi = \partial_t \partial_s^2 \Phi - \frac{1}{2} \partial_s (\pi_2 \partial_s \Phi) - \frac{1}{2} \pi_2 \partial_s^2 \Phi.$$

Substituting $\Phi = f$ in this relation and recalling the equalities $\tau = \partial_s f$, $k = \partial_s^2 f$ we arrive at the identity

$$\partial_s^2 \partial_t f = \partial_t k - \frac{1}{2}\partial_s(\pi_2)\tau - \pi_2 k.$$

Applying to both sides of this identity the projection Π and noting that

$$\Pi\,\partial_t = \nabla_t, \quad \Pi\,\tau = 0, \quad \Pi\,k = k$$

we obtain the desired equality (6.7).

$\square$

Now we evaluate the commutator of the differential operators ∇_s^2 and ∇_t. The result is given by the following lemma.

Lemma 6.2. *Let $\Phi : [0,T] \times \mathbb{S}^1$ be a smooth normal field, i.e., Φ is orthogonal to the tangent vector τ. Then we have*

$$\nabla_s^2 \nabla_t \Phi = \nabla_t \nabla_s^2 \Phi - \pi_2 \nabla_s^2 \Phi - \pi_1 \nabla_s \Phi. \tag{6.9}$$

Proof. Following Lemma 2.1 in [1], Lemma 2.1 in [8], and Lemma 1 in [12] we have the identity

$$\nabla_s^2 \nabla_t \Phi = \nabla_t \nabla_s^2 \Phi - \mathcal{C}_2 - \mathcal{C}_1 - \mathcal{C}_0, \tag{6.10}$$

where $\mathcal{C}_2 = \pi_2 \nabla_s^2 \Phi$ and

$$\mathcal{C}_1 = (k \cdot \nabla_s V + \nabla_s k \cdot V)\nabla_s \Phi + 2\big((k \cdot \nabla_s \Phi)\nabla_s V - (\nabla_s V \cdot \nabla_s \Phi)k\big),$$
$$\mathcal{C}_0 = \big((\nabla_s k \cdot \Phi)\nabla_s V - (\nabla_s V \cdot \Phi)\nabla_s k\big) + \big((k \cdot \Phi)\nabla_s^2 V - (\nabla_s^2 V \cdot \Phi)k\big).$$

Note that $\mathcal{C}_i$ are orthogonal to τ since ∇_s and k are parallel to n. It is easy to see that

$$\big(\nabla_s k \cdot \Phi)\nabla_s V - (\nabla_s V \cdot \Phi)\nabla_s k\big) \cdot \Phi = 0.$$

Note that the multipliers in this equality are parallel to n. Hence, either $(\nabla_s k \cdot \Phi)\nabla_s V - (\nabla_s V \cdot \Phi)\nabla_s k = 0$ or Φ is orthogonal to the normal vector n. In the latter case Φ is orthogonal to n and τ and hence $\Phi = 0$. From this we conclude that

$$(\nabla_s k \cdot \Phi)\nabla_s V - (\nabla_s V \cdot \Phi)\nabla_s k = 0.$$

Next we have

$$\big((k \cdot \Phi)\nabla_s^2 V - (\nabla_s^2 V \cdot \Phi)k\big) \cdot \Phi = 0.$$

Arguing as before we conclude that either $(k \cdot \Phi)\nabla_s^2 V - (\nabla_s^2 V \cdot \Phi)k = 0$ or $\Phi = 0$. We thus get

$$(k \cdot \Phi)\nabla_s^2 V - (\nabla_s^2 V \cdot \Phi)k = 0.$$

Combining the obtained results we conclude that $\mathcal{C}_0 = 0$. Now consider the quantity $\mathcal{C}_1$. We have

$$\big((k \cdot \nabla_s \Phi)\nabla_s V - (\nabla_s V \cdot \nabla_s \Phi)k\big) \cdot \nabla_s \Phi = 0.$$

Repeating the previous arguments we conclude that either $(k \cdot \nabla_s \Phi)\nabla_s V - (\nabla_s V \cdot \nabla_s \Phi)k = 0$ or $\nabla_s \Phi = 0$ is orthogonal to n and τ. Hence

$$(k \cdot \nabla_s \Phi)\nabla_s V - (\nabla_s V \cdot \nabla_s \Phi)k = 0$$

and

$$\mathcal{C}_1 = (k \cdot \nabla_s V + \nabla_s k \cdot V)\nabla_s \Phi.$$

Since V and k are orthogonal to τ, we have

$$k \cdot \nabla_s V + \nabla_s k \cdot V = k \cdot \partial_s V + \partial_s k \cdot V = \partial_s(k \cdot V) = \pi_1,$$

which yields the equality $\mathcal{C}_1 = \pi_1 \nabla_s \Phi$. Combining this result with the equalities $\mathcal{C}_2 = \pi_2 \nabla_s^2 \Phi$, $\mathcal{C}_0 = 0$, and relation (6.10) we obtain the desired identity (6.9). $\square$

6.1.2. *Relation between ∇_s and ∂_s. Estimates of π_i.* The important ingredient of the theory are estimates $H^r_\sharp$- Sobolev's norm of the curvature k via L^2-norm of the connection $\nabla^r_s k$. In this section we consider this problem in many details. We begin with the observation that every smooth normal vector field $\Phi : \mathbb{S}^1 \to \mathbb{R}^2$ admits the representation

$$\Phi = \varphi n, \quad \varphi = \Phi \cdot n. \tag{6.11}$$

It is easily seen that for every integer $r > 0$ we have

$$\nabla^r_s \Phi = \partial^r_s \varphi\, n. \tag{6.12}$$

The following lemma is the main result of this section.

Lemma 6.3. *Let $\Phi : \mathbb{S}^1 \to \mathbb{R}^2$ be an arbitrary smooth vector field and $\sigma \in (1/2, 1)$. Then for every integer $r > 0$, there is a constant c, depending only on σ, r, and E_0, such that*

$$\|\partial^r_s \Phi - \nabla^r_s \Phi\|_{H^0_\sharp} \leq c\big(\|\Phi\|_{H^0_\sharp} + \|k\|_{H^0_\sharp}\big)^{1+\frac{1-\sigma}{r}} \big(\|\Phi\|_{H^r_\sharp} + \|k\|_{H^r_\sharp}\big)^{1-\frac{1-\sigma}{r}}. \tag{6.13}$$

Proof. The proof is given in Section 12. □

If we assume that L^2-norms ($H^0_\sharp$ -norms) of Φ and k are uniformly bounded, then Lemma 6.3 leads to the efficient estimate of the deviation $\partial_s \Phi - \nabla_s \Phi$. The corresponding result is given by the following

Lemma 6.4. *Let under the assumptions of Lemma 6.3,*

$$\|\Phi\|_{H^0_\sharp} \leq C_\Phi, \quad \|k\|_{H^0_\sharp} \leq E_0.$$

Then for every $\epsilon \in (0, 1)$, there is a constant C, depending on E_0, C_Φ, integer $r \geq 1$, and ϵ, such that

$$(1-\epsilon)\|\Phi\|_{H^r_\sharp} - \epsilon\|k\|_{H^r_\sharp} - C \leq \|\nabla^r_s \Phi\|_{L^2(0,\mathcal{L})} \leq (1+\epsilon)\|\Phi\|_{H^r_\sharp} + \epsilon\|k\|_{H^r_\sharp} + C. \tag{6.14}$$

Proof. Estimate (6.13) and the conditions of lemma imply the inequality

$$\|\partial^r_s \Phi - \nabla^r_s \Phi\|_{H^0_\sharp} \leq C\big(\|\Phi\|_{H^r_\sharp} + \|k\|_{H^r_\sharp}\big)^{1-\frac{1-\sigma}{r}}.$$

Since $1/2 < \sigma < 1$ it follows from this and the Young inequality that

$$\|\partial^r_s \Phi - \nabla^r_s \Phi\|_{H^0_\sharp} \leq C + \epsilon\big(\|\Phi\|_{H^r_\sharp} + \|k\|_{H^r_\sharp}\big).$$

We thus get the inequality

$$\big|\, \|\partial^r_s \Phi\|_{H^0_\sharp} - \|\nabla^r_s \Phi\|_{H^0_\sharp} \,\big| \leq C + \epsilon\big(\|\Phi\|_{H^r_\sharp} + \|k\|_{H^r_\sharp}\big).$$

which yields

$$\begin{aligned} \|\partial^r_s \Phi\|_{H^0_\sharp} - \epsilon\big(\|\Phi\|_{H^r_\sharp} + \|k\|_{H^r_\sharp}\big) - C &\leq \|\nabla^r_s \Phi\|_{H^0_\sharp} \\ &\leq \|\partial^r_s \Phi\|_{H^0_\sharp} + \epsilon\big(\|\Phi\|_{H^r_\sharp} + \|k\|_{H^r_\sharp}\big) + C. \end{aligned}$$

Noting that

$$\|\Phi\|_{H^r_\sharp} - C_\Phi \leq \|\partial^r_s \Phi\|_{L^2(0,\mathcal{L})} \leq \|\Phi\|_{H^r_\sharp} + C_\Phi,$$

we arrive at desired estimate (6.14). □

Corollary 6.5. *Let $\|k\|_{L^2(0,\mathcal{L})} \leq \sqrt{2E_0}$ and an integer $r \geq 1$. Then for every $\epsilon \in (0, 1)$ there exists a constant C, depending only on E_0, r, ϵ, such that*

$$(1-2\epsilon)\|k\|_{H^r_\sharp} - C \leq \|\nabla^r_s k\|_{H^0_\sharp} \leq (1+2\epsilon)\|k\|_{H^r_\sharp} + C. \tag{6.15}$$

Proof. It suffices to note that $\Phi = k$ and $C_\Phi = \sqrt{2E_0}$ meet all requirements of Lemma 6.4. □

Estimates of π_i. In this paragraph we give estimates for the coefficients π_i defined by (6.5). The results are given by the following Lemmas.

Lemma 6.6. *Under the assumptions of Theorem 3.5 for every integer $r \geq 0$, there is a constant c depending on r, such that*

$$\|V(t)\|_{H^r_\sharp} \leq c(1 + \|k(t)\|_{H^{r+2}_\sharp}). \tag{6.16}$$

Proof. Notice that

$$V \equiv \partial_t f = -\nabla_s^2 k - \frac{1}{2}|k|^2 k + k - dJ. \tag{6.17}$$

By virtue of Corollary 6.5

$$\|\nabla_s^2 k\|_{H^r_\sharp} \leq c + c\|\nabla_s^{r+2} k\|_{H^0_\sharp} \leq c + c\|k\|_{H^{r+2}_\sharp}. \tag{6.18}$$

Next, the Moser inequality (2.11) implies the estimate

$$\||k|^2 k\|_{H^r_\sharp} \leq c\|k\|^2_{L^\infty(0,\mathcal{L})}\, \|k\|_{H^r_\sharp}.$$

Now choose an arbitrary $\sigma \in (1/2, 1)$. Recall that the embedding $H^\sigma_\sharp \hookrightarrow L^\infty(0,\mathcal{L})$ is continuous. From this, the interpolation inequality, and the estimate $\|k\|_{L^2(0,\mathcal{L})} \leq c$ we obtain

$$\begin{aligned}\||k|^2 k\|_{H^r_\sharp} &\leq c\, \|k\|^2_{H^\sigma_\sharp}\, \|k\|_{H^r_\sharp} \leq c\, \|k\|^{\frac{2\sigma}{r+2}}_{H^{r+2}_\sharp}\, \|k\|^{\frac{r}{r+2}}_{H^{r+2}_\sharp} = \\ &c\, \|k\|^{\frac{r+2\sigma}{r+2}}_{H^{r+2}_\sharp} \leq c\, (1 + \|k\|_{H^{r+2}_\sharp}).\end{aligned} \tag{6.19}$$

Finally, Theorem 3.2 on the estimates of the Kohn-Vogelius functional and estimate (6.15) in Corollary 6.5 imply

$$\|dJ\|_{H^r_\sharp} \leq c + c\|k\|_{H^r_\sharp}. \tag{6.20}$$

Combining estimates (6.18) - (6.20) and recalling relation (6.17) we arrive at desired estimate (6.16). □

We are now in a position to estimate the quantities π_i. The result is given by the following lemma.

Lemma 6.7. *Assume that all assumptions of Theorem 3.5 are satisfied. Let an integer $r \geq 0$ and $\sigma \in (1/2, 1)$ be given. Then there is c, depending on r, σ and the constants ν, ϱ in Theorem 3.5, such that*

$$\|\pi_2(t)\|_{H^r_\sharp} \leq c(1 + \|k(t)\|^{1+\frac{\sigma}{r+2}}_{H^{r+2}_\sharp}), \quad \|\pi_1(t)\|_{H^r_\sharp} \leq c(1 + \|k(t)\|^{1+\frac{\sigma}{r+3}}_{H^{r+3}_\sharp}) \tag{6.21}$$

for all $t \in (0, T)$.

Proof. Since $\pi_1 = -\partial_s \pi_2/2$, it suffices to estimate π_2. To shorten notation, we omit the symbol t. Notice that $\pi_2 = k \cdot V$. From this, the Moser inequality, and continuity of the embedding $H^\sigma_\sharp \hookrightarrow L^\infty(0,\mathcal{L})$ we conclude that

$$\|\pi_2\|_{H^r_\sharp} \leq c\|k\|_{H^\sigma_\sharp}\|V\|_{H^r_\sharp} + \|k\|_{H^r_\sharp}\|V\|_{H^\sigma_\sharp}. \tag{6.22}$$

By the interpolation inequality and estimate (6.16) in Lemma 6.6, we have

$$\|V\|_{H^\sigma} \leq \|V\|^{1-\frac{\sigma}{r}}_{H^0_\sharp}\, \|V\|^{\frac{\sigma}{r}}_{H^r_\sharp} \leq c(1 + \|k\|_{H^2_\sharp})^{1-\frac{\sigma}{r}}\, (1 + \|k\|_{H^{r+2}_\sharp})^{\frac{\sigma}{r}}.$$

Since

$$\|k\|_{H^2_\sharp} \leq \|k\|^{1-\frac{2}{r+2}}_{H^0_\sharp}\, \|k\|^{\frac{2}{r+2}}_{H^{r+2}_\sharp} \leq c\, \|k\|^{\frac{2}{r+2}}_{H^{r+2}_\sharp},$$

we have

$$\|V\|_{H^\sigma_\sharp} \leq c\big((1 + \|k\|^{\frac{2}{r+2}}_{H^{r+2}_\sharp}\big)^{1-\frac{\sigma}{r}}\, (1 + \|k\|_{H^{r+2}_\sharp})^{\frac{\sigma}{r}} \leq c(1 + \|k\|_{H^{r+2}_\sharp})^{\mu},$$

where

$$\mu = \frac{2}{r+2}(1-\frac{\sigma}{r}) + \frac{\sigma}{r} = \frac{2+\sigma}{r+2}.$$

From this and the inequality

$$\|k\|_{H^r_\sharp} \le c\,\|k\|^{1-\frac{r}{r+2}}_{H^0_\sharp}\ \|k\|^{\frac{r}{r+2}}_{H^{r+2}_\sharp} \le c\|k\|^{\frac{r}{r+2}}_{H^{r+2}_\sharp}$$

we obtain

$$\|k\|_{H^r_\sharp}\|V\|_{H^\sigma_\sharp} \le c\,\|k\|^{\frac{r}{r+2}}_{H^{r+2}_\sharp}\ (1+\|k\|_{H^{r+2}_\sharp})^{\mu} \le c + c\|k\|^{1+\frac{\sigma}{r+2}}_{H^{r+2}_\sharp}. \tag{6.23}$$

Next, the interpolation inequality implies

$$\|k\|_{H^\sigma_\sharp} \le c\ \|k\|^{1-\frac{\sigma}{r+2}}_{H^0_\sharp}\ \|k\|^{\frac{\sigma}{r+2}}_{H^{r+2}_\sharp} \le c\,\|k\|^{\frac{\sigma}{r+2}}_{H^{r+2}_\sharp},$$

which along with estimate (6.16) in Lemma 6.6 gives

$$\|k\|_{H^\sigma_\sharp}\|V\|_{H^r_\sharp} \le c\,\|k\|^{\frac{\sigma}{r+2}}_{H^{r+2}_\sharp}\ (1+\|k\|_{H^{r+2}_\sharp}) \le c + c\,\|k\|^{1+\frac{\sigma}{r+2}}_{H^{r+2}_\sharp}.$$

Substituting this estimate and estimate (6.23) into (6.22) we arrive at desired estimate (6.21). □

6.2. **Main integral identity.** Now we use formulae (6.7) and (6.9) in order to derive the main integral identity for solutions to problem (6.1). The result is given by the following proposition.

Proposition 6.8. *The following integral identity holds for every smooth solution f : $[0,T]\times\mathbb{S}^1$ to problem* (6.1)*, for every $t_0\in(0,T]$, and for every even integer $m\ge 6$.*

$$\begin{aligned}\frac{1}{2}\int_{\Gamma(t_0)} |\nabla_s^{m-2}k(t_0)|^2\,ds + \int_0^{t_0}\int_{\Gamma(t)} |\nabla_s^m k(t)|^2\,dsdt = \\ \sum_1^4 \mathcal{N}_i + \frac{1}{2}\int_{\Gamma(0)} |\nabla_s^{m-2}k(0)|^2\,ds.\end{aligned} \tag{6.24}$$

Here the quantity $\mathcal{N}_i$ are defined by the equalities

$$\begin{aligned}\mathcal{N}_1 = \sum_{i=0}^{\frac{m-2}{2}}\int_0^{t_0} I_{2i}(t)\,dt + \frac{1}{2}I_{m-2},\quad \mathcal{N}_2 = \sum_{i=1}^{\frac{m-2}{2}}\int_0^{t_0} I_{2i-1}(t)\,dt,\\ I_{2i}(t) = \int_{\Gamma(t)} \nabla_s^{m-2-2i}\big(\,\pi_2\nabla_s^{2i}k\,\big)\cdot\nabla_s^{m-2}k\,ds,\\ I_{2i-1}(t) = \int_{\Gamma(t)} \nabla_s^{m-2-2i}\big(\,\pi_1\nabla_s^{2i-1}k\,\big)\cdot\nabla_s^{m-2}k\,ds,\end{aligned} \tag{6.25}$$

$$\begin{aligned}\mathcal{N}_3 = \int_0^{t_0}\int_{\Gamma(t)} \nabla_s^{m-4}\big(\,(k\cdot\nabla_s^2 k)\,k\big)\cdot\nabla_s^m k\,dsdt,\\ \mathcal{N}_4 = -\int_0^{t_0}\int_{\Gamma(t)} \nabla_s^{m-2}\Upsilon\cdot\nabla_s^m k\,dsdt + \int_0^{t_0}\int_{\Gamma(t)} \nabla_s^{m-4}\big((k\cdot\Upsilon)\,k\big)\cdot\nabla_s^m k\,dsdt,\end{aligned} \tag{6.26}$$

where

$$\Upsilon = \frac{1}{2}|k|^2\,k\,-\,k\,+\,dJ.$$

Proof. Multiplying both sides of (6.1) by $\partial_s^2 \nabla_s^{2m-4} k$ and integrating the result with respect to s and t we arrive at the equality

$$\begin{aligned}\int_0^{t_0}\int_{\Gamma(t)} \Big(\partial_t f\cdot \partial_s^2\nabla_s^{2m-4}k + \nabla_s^2 k\cdot \partial_s^2\nabla_s^{2m-4}k\Big)\,dsdt + \\ \int_0^{t_0}\int_{\Gamma(t)} \Big(\frac12 |k|^2 k - k + dJ\Big)\cdot \partial_s^2\nabla_s^{2m-4}k\,dsdt = 0.\end{aligned} \tag{6.27}$$

The rest of the proof falls into a sequence of lemmas.

Lemma 6.9. *Under the assumptions of Proposition 6.8, we have*

$$\begin{aligned}\int_0^{t_0}\int_{\Gamma(t)} \partial_t f\cdot \partial_s^2\nabla_s^{2m-4}k\,ds = \frac12\int_{\Gamma(t_0)} |\nabla_s^{m-2}\,k(t_0)|^2\,ds \\ -\frac12\int_{\Gamma(0)} |\nabla_s^{m-2}\,k(0)|^2\,ds - \mathcal{N}_1 - \mathcal{N}_2.\end{aligned} \tag{6.28}$$

Proof. Notice that $\nabla_s^\top = -\partial_s \Pi$ and $\nabla_s^{m-2}k = \Pi\nabla_s^{m-2}k$. Since m is even, it follows that

$$\begin{aligned}\int_{\Gamma(t)} \partial_t f\cdot \partial_s^2\nabla_s^{2m-4}k\,ds = \int_{\Gamma(t)} (\nabla_s^\top)^{m-2}\partial_s^2\partial_t\, f\cdot \Pi\,\nabla_s^{m-2}k\,ds = \\ \int_{\Gamma(t)} \nabla_s^{m-2}\Pi\partial_s^2\partial_t\, f\cdot\nabla_s^{m-2}k\,ds.\end{aligned}$$

Recall that the integer m is even. From this and identity (6.7) in Lemma 6.1 we obtain

$$\int_{\Gamma(t)} \partial_t f\cdot \partial_s^2\nabla_s^{2m-4}k\,ds = \int_{\Gamma(t)} \nabla_s^{m-2}\partial_t\,k\cdot\nabla_s^{m-2}k\,ds - \int_{\Gamma(t)} \nabla_s^{m-2}(\pi_2\,k)\cdot\nabla_s^{m-2}k\,ds.$$

Recalling the expression for $I_{2i}(t)$ we can rewrite this equality in the equivalent form

$$\int_{\Gamma(t)} \partial_t f\cdot \partial_s^2\nabla_s^{2m-4}k\,ds = \int_{\Gamma(t)} \nabla_s^{m-2}\nabla_t\,k\cdot\nabla_s^{m-2}k\,ds - I_0(t). \tag{6.29}$$

Let us consider the integral in the right hand side of this equality. Using identity (6.9) in Lemma 6.2 with Φ replaced by k we obtain

$$\begin{aligned}\int_{\Gamma(t)} \nabla_s^{m-2}\nabla_t\,k\cdot\nabla_s^{m-2}k\,ds = \int_{\Gamma(t)} \nabla_s^{m-4}\nabla_t\nabla_s^2\,k\cdot\nabla_s^{m-2}k\,ds - \\ \int_{\Gamma(t)} \nabla_s^{m-4}(\pi_2\nabla_s^2\,k)\cdot\nabla_s^{m-2}k\,ds - \int_{\Gamma(t)} \nabla_s^{m-4}(\pi_1\nabla_s\,k)\cdot\nabla_s^{m-2}k\,ds\end{aligned}$$

or equivalently

$$\int_{\Gamma(t)} \nabla_s^{m-2}\nabla_t\,k\cdot\nabla_s^{m-2}k\,ds = \int_{\Gamma(t)} \nabla_s^{m-4}\nabla_t\nabla_s^2\,k\cdot\nabla_s^{m-2}k\,ds - I_2(t) - I_1(t).$$

Repeating this process and using again identity (6.9) we finally obtain

$$\int_{\Gamma(t)} \nabla_s^{m-2}\nabla_t\,k\cdot\nabla_s^{m-2}k\,ds = \int_{\Gamma(t)} \nabla_t\nabla_s^{m-2}\,k\cdot\nabla_s^{m-2}k\,ds - \sum_{i=1}^{\frac{m-2}{2}}(I_{2i}(t) - I_{2i-1}(t)).$$

Notice that

$$\nabla_t\nabla_s^{m-2}\,k\cdot\nabla_s^{m-2}k = \partial_t\nabla_s^{m-2}\,k\cdot\nabla_s^{m-2}k$$

since $\nabla_s^{m-2}k$ is orthogonal to τ. Thus we get

$$\int_{\Gamma(t)} \nabla_s^{m-2}\nabla_t\,k\cdot\nabla_s^{m-2}k\,ds = \int_{\Gamma(t)} \partial_t\nabla_s^{m-2}\,k\cdot\nabla_s^{m-2}k\,ds - \sum_{i=1}^{\frac{m-2}{2}}(I_{2i}(t) - I_{2i-1}(t)).$$

Combining this equality with (6.29) we arrive at the identity

$$\begin{aligned}\int_{\Gamma(t)} \partial_t f \cdot \partial_s^2 \nabla_s^{2m-4} k\, ds = \int_{\Gamma(t)} \partial_t \nabla_s^{m-2} k \cdot \nabla_s^{m-2} k\, ds \\ - \sum_{i=0}^{\frac{m-2}{2}} I_{2i}(t) - \sum_{i=1}^{\frac{m-2}{2}} I_{2i-1}(t)).\end{aligned} \tag{6.30}$$

Now consider the integral

$$\int_0^{t_0} \int_{\Gamma(t)} \partial_t \nabla_s^{m-2} k \cdot \nabla_s^{m-2} k\, dsdt.$$

Our task is to integrate by parts with respect to t. The difficulty is that the curve $\Gamma(t)$ and ds depend on the time variable. In order to cope with this difficulty we rewrite this integral in terms of the original independent variable θ. The change of variable $s \to \theta$ leads to the equality

$$\int_0^{t_0} \int_{\Gamma(t)} \partial_t \nabla_s^{m-2} k \cdot \nabla_s^{m-2} k\, dsdt = \int_0^{t_0} \int_0^{2\pi} \partial_t \nabla_s^{m-2} k \cdot \nabla_s^{m-2} k \sqrt{g}\, d\theta dt,$$

where $\sqrt{g} = |\partial_\theta f|$. Integrating by parts gives the equality

$$\begin{aligned}\int_0^{t_0} \int_0^{2\pi} \partial_t \nabla_s^{m-2} k \cdot \nabla_s^{m-2} k \sqrt{g}\, d\theta dt = \\ \frac{1}{2} \int_0^{2\pi} |\nabla_s^{m-2} k(t_0)|^2 \sqrt{g(t_0)}\, d\theta - \frac{1}{2} \int_0^{2\pi} |\nabla_s^{m-2} k(0)|^2 \sqrt{g(0)}\, d\theta \\ -\frac{1}{2} \int_0^{t_0} \int_0^{2\pi} |\nabla_s^{m-2} k(t)|^2 \partial_t \sqrt{g(t)}\, d\theta dt.\end{aligned} \tag{6.31}$$

Next we have

$$\partial_t \sqrt{g(t)} = \frac{1}{|\partial_\theta f|} (\partial_\theta f \cdot \partial_\theta \partial_t f) = (\partial_s f \cdot \partial_s \partial_t f) \sqrt{g(t)}.$$

Notice that

$$\partial_s f \cdot \partial_s \partial_t f = \tau \cdot \partial_s \partial_t f = -\partial_s \tau \cdot \partial_t f = -k \cdot V = -\frac{1}{2} \pi_2,$$

which yields

$$\begin{aligned}\int_0^{2\pi} |\nabla_s^{m-2} k(t)|^2 \partial_t \sqrt{\mathbf{g}(t)}\, d\theta = -\frac{1}{2} \int_0^{2\pi} |\nabla_s^{m-2} k(t)|^2 \pi_2 \sqrt{g(t)}\, d\theta = \\ -\frac{1}{2} \int_{\Gamma(t)} |\nabla_s^{m-2} k(t)|^2 \pi_2\, ds = -\frac{1}{2} I_{m-2}(t).\end{aligned} \tag{6.32}$$

Next, we have

$$\begin{aligned}\frac{1}{2} \int_0^{2\pi} |\nabla_s^{m-2} k(t_0)|^2 \sqrt{\mathbf{g}(t_0)}\, d\theta - \frac{1}{2} \int_0^{2\pi} |\nabla_s^{m-2} k(0)|^2 \sqrt{g(0)}\, d\theta = \\ \frac{1}{2} \int_{\Gamma(t_0)} |\nabla_s^{m-2} k(t_0)|^2\, ds - \frac{1}{2} \int_{\Gamma(0)} |\nabla_s^{m-2} k(0)|^2\, ds.\end{aligned}$$

Substituting this equality along with equality (6.32) into (6.31) we obtain

$$\begin{aligned}\int_0^{t_0} \int_{\Gamma(t)} \partial_t \nabla_s^{m-2} k \cdot \nabla_s^{m-2} k\, dsdt = \int_0^{t_0} \int_0^{2\pi} \partial_t \nabla_s^{m-2} k \cdot \nabla_s^{m-2} k \sqrt{g}\, d\theta dt \\ = \frac{1}{2} \int_{\Gamma(t_0)} |\nabla_s^{m-2} k(t_0)|^2\, ds - \frac{1}{2} \int_{\Gamma(0)} |\nabla_s^{m-2} k(0)|^2\, ds - \frac{1}{2} \int_0^{t_0} I_{m-2}(t)\, dt.\end{aligned} \tag{6.33}$$

Integrating both sides of equality (6.30) and using relation (6.33) we arrive at the desired equality (6.28).

$\square$

Lemma 6.10. *Under the assumptions of Proposition 6.8 we have*

$$\int_0^{t_0}\int_{\Gamma(t)} \nabla_s^2 k\cdot\partial_s^2\nabla_s^{2m-4}k\,ds = \int_0^{t_0}\int_{\Gamma(t)} |\nabla_s^m k(t)|^2\,dsdt - \mathcal{N}_3. \tag{6.34}$$

Proof. Arguing as in the proof of Lemma 6.9 we obtain

$$\begin{gathered}\int_{\Gamma(t)} \nabla_s^2 k\cdot\partial_s^2\nabla_s^{2m-4}k\,ds = \int_{\Gamma(t)} (\nabla_s^\top)^{m-4}\partial_s^2\nabla_s^2 k\cdot\Pi\,\nabla_s^m k\,ds = \\ \int_{\Gamma(t)} \nabla_s^{m-4}\Pi\,\partial_s^2\,\nabla_s^2\,k\cdot\nabla_s^m k\,ds = \int_{\Gamma(t)} \nabla_s^{m-4}\,\nabla_s\partial_s\,\nabla_s^2\,k\cdot\nabla_s^m k\,ds.\end{gathered} \tag{6.35}$$

Note that

$$\partial_s\nabla_s^2 k = \nabla_s^3 k + (\tau\cdot\partial_s\nabla_s^2 k)\,\tau.$$

Since ∇_s^2 is orthogonal to τ, we have

$$\partial_s\nabla_s^2 k = \nabla_s^3 k - (k\cdot\nabla_s^2 k)\,\tau.$$

It is easy to check that

$$\nabla_s\big((k\cdot\nabla_s^2 k)\,\tau\big) = (k\cdot\nabla_s^2 k)\,\nabla_s\tau = (k\cdot\nabla_s^2 k)\,k,$$

which yields

$$\nabla_s\partial_s\nabla_s^2 k = \nabla_s^4 k - (k\cdot\nabla_s^2 k)\,k.$$

Substituting this relation into (6.35) we finally obtain

$$\begin{gathered}\int_{\Gamma(t)} \nabla_s^2 k\cdot\partial_s^2\nabla_s^{2m-4}k\,ds = \\ \int_{\Gamma(t)} |\nabla_s^m k|^2\,ds - \int_{\Gamma(t)} \nabla_s^{m-4}\big((k\cdot\nabla_s^2 k)\,k\big)\cdot\nabla_s^m k\,ds,\end{gathered}$$

which along with the expression for $\mathcal{N}_3$ implies desired equality (6.28). □

Recall the denotation

$$\Upsilon = \frac{1}{2}|k|^2\,k\;-\;k\;+\;dJ.$$

Lemma 6.11. *Under the assumptions of Proposition 6.8 we have*

$$\int_0^{t_0} \Upsilon\cdot\partial_s^2\nabla_s^{2m-4} = -\mathcal{N}_4. \tag{6.36}$$

Proof. The proof imitates the proof of Lemma 6.10. We have

$$\begin{aligned}\int_{\Gamma(t)} \Upsilon\cdot\partial_s^2\nabla_s^{2m-4}k\,ds &= \int_{\Gamma(t)} (\nabla_s^\top)^{m-4}\partial_s^2\Upsilon\cdot\Pi\,\nabla_s^m k\,ds \\ &= \int_{\Gamma(t)} \nabla_s^{m-4}\,\nabla_s\partial_s\,\Upsilon\cdot\nabla_s^m k\,ds.\end{aligned} \tag{6.37}$$

Notice that Υ is orthogonal to τ. It follows that

$$\partial_s\Upsilon = \nabla_s\Upsilon + (\tau\cdot\partial_s\Upsilon)\,\tau = \nabla_s\Upsilon - (k\cdot\Upsilon)\,\tau.$$

Next we have

$$\nabla_s\big((k\cdot\Upsilon)\,\tau\big) = (k\cdot\Upsilon)\,\nabla_s\tau = (k\cdot\Upsilon)\,k,$$

which yields

$$\nabla_s\partial_s\Upsilon = \nabla_s^2\Upsilon - (k\cdot\Upsilon)\,k.$$

Substituting this relation into (6.36) we arrive at the identity

$$\begin{gathered}\int_{\Gamma(t)} \Upsilon\cdot\partial_s^2\nabla_s^{2m-4}k\,ds = \\ \int_{\Gamma(t)} \nabla_s^{m-2}\Upsilon\cdot\nabla_s^m k\,ds - \int_{\Gamma(t)} \nabla_s^{m-4}\big((k\cdot\Upsilon)\,k\big)\cdot\nabla_s^m k\,ds.\end{gathered}$$

which along with the expression for $\mathcal{N}_4$ yields (6.36). □

We are now in a position to complete the proof of Proposition 6.8. To this end, it suffices to substitute equalities (6.28), (6.34), and (6.36) into integral identity (6.27).

□

6.3. **Estimates of reminders $\mathcal{N}_i$.**

6.3.1. *Estimates of $\mathcal{N}_1$ and $\mathcal{N}_2$.* Estimates (6.21) for π_i lead to the basic estimates for the reminders $\mathcal{N}_i$ in the integral identity (6.24). The first result in this direction is given by the following

Lemma 6.12. *Assume that all assumptions of Theorem 3.5 are satisfied. Let an even integer $m \geq 4$ and $\sigma \in (1/2, 1)$ be given. Then, there is c, depending on m and σ, such that*

$$|\mathcal{N}_i| \leq c \int_0^T \big(1 + \|k(t)\|_{H^m_\sharp}^{2-\frac{2(1-\sigma)}{m}}\big) \quad \textit{for} \quad i = 1, 2. \tag{6.38}$$

Proof. The proof is based on Lemma 6.7 and Corollary 6.5. We give the proof only for the quantity $\mathcal{N}_1$. The proof for $\mathcal{N}_2$ is similar. It follows from representation (6.25) for $\mathcal{N}_1$ that

$$\begin{gathered}\mathcal{N}_1 = \sum_{i=0}^{\frac{m-2}{2}} \int_0^{t_0} I_{2i}(t)\, dt + \frac{1}{2} I_{m-2}(t),\\ I_{2i}(t) = \int_{\Gamma(t)} \nabla_s^{m-2-2i}\big(\pi_2 \nabla_s^{2i} k\big) \cdot \nabla_s^{m-2} k\, ds.\end{gathered} \tag{6.39}$$

Hence it suffices to estimate $I_{2i}(t)$. To simplify the notation, we omit the symbol t. By the Cauchy inequality, we have

$$|I_{2i}| \leq \|\nabla_s^{m-2-2i}\big(\pi_2 \nabla_s^{2i} k\big)\|_{H^0_\sharp} \|\nabla_s^{m-2} k\|_{H^0_\sharp}. \tag{6.40}$$

Recall that for every normal vector field $\Phi = \varphi n$, we have

$$\nabla_s^r \Phi = \partial_s^r \varphi\, n.$$

It follows that

$$\pi_2 \nabla_s^{2i} k = \pi_2 \partial_s^{2i} K\, n, \quad K = k \cdot n$$

and

$$\nabla_s^{m-2-2i}\big(\pi_2 \nabla_s^{2i} k\big) = \partial_s^{m-2-2i}(\pi_2 \partial_s^{2i} K)\, n\,.$$

Thus we get

$$\|\nabla_s^{m-2-2i}\big(\pi_2 \nabla_s^{2i} k\big)\|_{H^0_\sharp} \leq \|\pi_2 \partial_s^{2i} K\|_{H^{m-2-2i}_\sharp}. \tag{6.41}$$

The Moser inequality and the continuity of the embedding $H^\sigma_\sharp \hookrightarrow L^\infty(0, \mathcal{L})$ imply the estimate

$$\begin{aligned}\|\pi_2 \partial_s^{2i} K\|_{H^{m-2-2i}_\sharp} &\leq c\, \|\pi_2\|_{H^{m-2-2i}_\sharp} \|\partial_s^{2i} K\|_{H^\sigma_\sharp} + c\, \|\pi_2\|_{H^\sigma_\sharp} \|\partial_s^{2i} K\|_{H^{m-2-2i}_\sharp}\\ &\leq c\, \|\pi_2\|_{H^{m-2-2i}_\sharp} \|K\|_{H^{2i+\sigma}_\sharp} + c\, \|\pi_2\|_{H^\sigma_\sharp} \|K\|_{H^{m-2}_\sharp}.\end{aligned} \tag{6.42}$$

It follows from the interpolation inequality and estimate (6.21) in Lemma 6.7 that

$$\begin{gathered}\|\pi_2\|_{H^\sigma_\sharp} \leq \|\pi_2\|_{H^0_\sharp}^{1-\frac{\sigma}{m-2}} \|\pi_2\|_{H^{m-2}_\sharp}^{\frac{\sigma}{m-2}} \leq\\ c\, \big(1 + \|k\|_{H^2_\sharp}^{1+\frac{\sigma}{2}}\big)^{1-\frac{\sigma}{m-2}} \big(1 + \|k\|_{H^m_\sharp}^{1+\frac{\sigma}{m}}\big)^{\frac{\sigma}{m-2}}.\end{gathered} \tag{6.43}$$

Since $\|k\|_{H^0_\sharp}$ is uniformly bounded, the interpolation inequality implies

$$\|k\|_{H^2_\sharp} \leq c \|k\|_{H^m_\sharp}^{\frac{2}{m}}.$$

Substituting this inequality in (6.43) we arrive at the estimate

$$\|\pi_2\|_{H^\sigma_\sharp} \le c(1+\|k\|^\iota_{H^m_\sharp}), \tag{6.44}$$

where

$$\iota = \frac{2}{m}(1+\frac{\sigma}{2})(1-\frac{\sigma}{m-2}) + (1+\frac{\sigma}{m})\frac{\sigma}{m-2} = \frac{2+2\sigma}{m}.$$

Next notice that $\|K\|_{H^0_\sharp} = \|k\|_{H^0_\sharp}$ is uniformly bounded, which yields

$$\|K\|_{H^{m-2}_\sharp} \le c\|K\|^{\frac{m-2}{m}}_{H^m_\sharp}.$$

From this and (6.44) we obtain

$$\|\pi_2\|_{H^\sigma_\sharp}\ \|K\|_{H^{m-2}_\sharp} \le c\,(1+\|k\|_{H^m_\sharp})^{\frac{2+2\sigma}{m}}\ \|K\|^{\frac{m-2}{m}}_{H^m_\sharp}. \tag{6.45}$$

Next, estimate (6.21) in Lemma 6.7 and the interpolation inequality imply

$$\|\pi_2\|_{H^{m-2-2i}_\sharp}\|K\|_{H^{2i+\sigma}_\sharp} \le c\,\big(1+\|k\|^{1+\frac{\sigma}{m-2i}}_{H^{m-2i}_\sharp}\big)\ \|K\|^{\frac{2i+\sigma}{m}}_{H^m_\sharp} \le \\ c\,\big(1+\|k\|^{\frac{m-2i}{m}(1+\frac{\sigma}{m-2i})}_{H^m_\sharp}\big)\ \|K\|^{\frac{2i+\sigma}{m}}_{H^m_\sharp},$$

which gives

$$\|\pi_2\|_{H^{m-2-2i}_\sharp}\|K\|_{H^{2i+\sigma}_\sharp} \le c(1+\|k\|^{\frac{m-2i+\sigma}{m}}_{H^m_\sharp})\,\|K\|^{\frac{2i+\sigma}{m}}_{H^m_\sharp}. \tag{6.46}$$

Substituting (6.45) and (6.46) into (6.42) and next into (6.41) we arrive at the inequality

$$\|\nabla^{m-2-2i}_s\big(\pi_2\nabla^{2i}_s k\big)\|_{L^2(0,\mathcal{L})} \le c(1+\|k\|_{H^m_\sharp})^{\frac{2+2\sigma}{m}}\ \|K\|^{\frac{m-2}{m}}_{H^m_\sharp} + \\ c(1+\|k\|^{\frac{m-2i+\sigma}{m}}_{H^m_\sharp})\,\|K\|^{\frac{2i+\sigma}{m}}_{H^m_\sharp}. \tag{6.47}$$

Now out task is to estimate $H^r_\sharp$-norm of K by $H^r_\sharp$-norm of k. To this end, notice that the identity $\nabla^r_s k = \partial^r_s K\, n$ and boundedness of L^2-norm of $K = k\cdot n$ yields the estimates

$$\|K\|_{H^r_\sharp} \le c + \|\partial^r_s K\|_{H^0_\sharp} = c + c\|\nabla^r_s k\|_{H^0_\sharp},$$

which holds for every integer $r \ge 0$. Applying estimate (6.15) in Corollary 6.5 we arrive at the inequality

$$\|K\|_{H^r_\sharp} \le c + c\|k\|_{H^r_\sharp}.$$

Substituting this inequality with $r = m$ into (6.47) leads to the estimates

$$\|\nabla^{m-2-2i}_s\big(\pi_2\nabla^{2i}_s k\big)\|_{H^0_\sharp} \le c(1+\|k\|^{\frac{m-2i+\sigma}{m}}_{H^m_\sharp})\,(1+\|k\|^{\frac{2i+\sigma}{m}}_{H^m_\sharp}) \\ +(1+\|k\|^{\frac{2+2\sigma}{m}}_{H^m_\sharp})\,(1+\|k\|_{H^m_\sharp})^{\frac{m-2}{m}} \le c(1+\|k\|_{H^m_\sharp})^{1+\frac{2\sigma}{m}}. \tag{6.48}$$

In order to complete the proof, note that estimate (6.15) and the interpolation inequality imply the estimate

$$\|\nabla^{m-2}_s k\|_{H^0_\sharp} \le c + c\|k\|_{H^{m-2}_\sharp} \le c + c\|k\|^{1-\frac{2}{m}}_{H^m_\sharp}.$$

Combining this estimate with (6.48) and (6.40) we finally obtain the inequality

$$|I_{2i}(t)| \le c(1+\|k(t)\|_{H^m(t)})^{1+\frac{2\sigma}{m}}(1+\|k(t)\|_{H^m_\sharp})^{1-\frac{2}{m}} \le c(1+\|k(t)\|_{H^m_\sharp})^{2-\frac{2(1-\sigma)}{m}},$$

which along with expression (6.39) yields desired estimate (6.38). □

6.3.2. *Estimates of $\mathcal{N}_3$ and $\mathcal{N}_4$.* In order to complete the proof of Theorem 3.5 it remains to estimate the reminders $\mathcal{N}_3$ and $\mathcal{N}_4$ given by (6.26). The result is given by the following lemmas.

Lemma 6.13. *Assume that all assumptions of Theorem 3.5 are satisfied. Let an integer $m \geq 4$ and $\sigma \in (1/2, 1)$ be given. Then there is c, depending on m and σ, such that*

$$|\mathcal{N}_3| \leq c \int_0^{t_0} (1 + \|k(t)\|_{H^m_\sharp}^{2-\frac{2(1-\sigma)}{m}}). \tag{6.49}$$

Proof. Recall that

$$\mathcal{N}_3 = \int_0^{t_0} \int_{\Gamma(t)} \nabla_s^{m-4}\big((k \cdot \nabla_s^2 k)\, k\big) \cdot \nabla_s^m k \, ds dt.$$

Hence out task is to estimate the integrand in the right hand side of this formula. To simplify the notation we omit the symbol t. By the Cauchy inequality, we have

$$\Big| \int_\Gamma \nabla_s^{m-4}((k \cdot \nabla_s^2 k)k) \cdot \nabla_s^m k \, ds \Big| \leq \|\nabla_s^{m-4}((k \cdot \nabla_s^2 k)k)\|_{H^0_\sharp} \, \|\nabla_s^m k\|_{H^0_\sharp}. \tag{6.50}$$

Notice that $k = K\, n$ and $\nabla_s^2 k = \partial_s^2 K\, n$, which yields

$$(k \cdot \nabla_s^2 k)k = K^2 \partial_s^2 K\, n, \quad \nabla_s^{m-4}((k \cdot \nabla_s^2 k)k) = \partial_s^{m-4}(K^2 \partial_s^2 K)\, n.$$

From this we conclude that

$$\|\nabla_s^{m-4}((k \cdot \nabla_s^2 k)k)\|_{H^0_\sharp} \leq \|K^2 \partial_s^2 K\|_{H^{m-4}_\sharp}. \tag{6.51}$$

The Moser inequality, and continuity of the embedding $H^\sigma_\sharp \hookrightarrow L^\infty(0, \mathcal{L})$ imply the estimate

$$\begin{aligned} &\|\nabla_s^{m-4}((k \cdot \nabla_s^2 k)k)\|_{H^0_\sharp} \leq \\ &c\|K\|^2_{L^\infty(0,\mathcal{L})} \|K\|_{H^{m-2}_\sharp} + c\|\partial_s^2 K\|_{L^\infty(0,\mathcal{L})} \|K\|_{L^\infty(0,\mathcal{L})} \|K\|_{H^{m-4}_\sharp} \\ &\leq c\|K\|^2_{H^\sigma_\sharp} \|K\|_{H^{m-2}_\sharp} + c\|K\|_{H^{2+\sigma}_\sharp} \|K\|_{H^\sigma_\sharp} \|K\|_{H^{m-4}}. \end{aligned} \tag{6.52}$$

Since the $\|K\|_{H^0_\sharp}$ is uniformly bounded, the interpolation inequality yields

$$\|K\|_{H^\sigma_\sharp} \leq c\|K\|^{\frac{\sigma}{m}}_{H^m_\sharp}, \quad \|K\|_{H^{m-2}_\sharp} \leq c\|K\|^{1-\frac{2}{m}}_{H^m_\sharp},$$

$$\|K\|_{H^{2+\sigma}_\sharp} \leq c\|K\|^{\frac{2+\sigma}{m}}_{H^m_\sharp}, \quad \|K\|_{H^{m-4}_\sharp} \leq c\|K\|^{1-\frac{4}{m}}_{H^m_\sharp}.$$

Substituting this inequalities into (6.52) we obtain

$$\|\nabla_s^{m-4}((k \cdot \nabla_s^2 k)k)|_{H^0_\sharp} \leq c\|K\|^{1-\frac{2(1-\sigma)}{m}}_{H^m_\sharp}. \tag{6.53}$$

Next, inequality (6.15) in Lemma 6.5 leads to the estimate

$$\|K\|_{H^m_\sharp} \leq c + c\|\partial^m K\|_{H^0_\sharp} = c + c\|\nabla_s^m k\|_{H^0_\sharp} \leq c + c\|k\|_{H^m_\sharp},$$

which along with (6.53) yields the estimate

$$\|\nabla_s^{m-4}((k \cdot \nabla_s^2 k)k)|_{H^0_\sharp} \leq c(1 + \|k\|_{H^m_\sharp})^{1-\frac{2(1-\sigma)}{m}}. \tag{6.54}$$

Applying again inequality (6.15) we obtain

$$\|\nabla_s^m k\|_{H^0_\sharp} \leq c(+\|k\|_{H^m_\sharp}).$$

Substituting this inequality and inequality (6.54) we finally arrive at the estimate

$$\Big| \int_{\Gamma(t)} \nabla_s^{m-4}((k(t) \cdot \nabla_s^2 k(t))k(t)) \cdot \nabla_s^m k(t)\, ds \Big| \leq c\,(1 + \|k(t)\|_{H^m_\sharp})^{2-\frac{2(1-\sigma)}{m}},$$

which obviously yields desired estimate (6.49). □

Now our task is to estimate the reminder

$$\mathcal{N}_4 = \int_0^{t_0} \int_{\Gamma(t)} \nabla_s^{m-4}\big((k \cdot \Upsilon)\, k\big) \cdot \nabla_s^m k \, ds dt - \int_0^{t_0} \int_{\Gamma(t)} \nabla_s^{m-2} \Upsilon \cdot \nabla_s^m k \, ds dt,$$

where

$$\Upsilon = \frac{1}{2} |k|^2 \, k \, - \, k \, + \, dJ.$$

Lemma 6.14. *Assume that all assumptions of Theorem 3.5 are satisfied. Let an integer $m \geq 4$ and $\sigma \in (1/2, 1)$ be given. Then there is c depending on m and σ, such that*

$$|\mathcal{N}_4| \leq c \int_0^{t_0} (1 + \|k(t)\|_{H^m_\sharp}^{2 - \frac{2(1-\sigma)}{m}}). \tag{6.55}$$

Proof. Introduce the functions

$$\begin{aligned} M_1 &= -\frac{1}{2} \nabla_s^{m-2} (|k|^2 k) + \frac{1}{2} \nabla_s^{m-4} (|k|^4 k), \\ M_2 &= -\nabla_s^{m-2} (k) + \nabla_s^{m-4} (|k|^2 k), \\ M_3 &= -\nabla_s^{m-2} (dJ) + \nabla_s^{m-4} ((k \cdot dJ) k). \end{aligned} \tag{6.56}$$

It is easily seen that

$$\mathcal{N}_4 = \sum_{i=1}^{3} \int_0^{t_0} \int_{\Gamma(t)} M_i(t) \cdot \nabla_s^m k \, ds dt.$$

Estimate (6.15) in Corollary 6.5 implies

$$\|\nabla_s^m k\|_{H^0_\sharp} \leq c(1 + \|k\|_{H^m_\sharp}).$$

From this and the Cauchy inequality we obtain the estimate

$$|\mathcal{N}_4| \leq c \sum_{i=1}^{3} \int_0^{t_0} \|M_i(t)\|_{H^0_\sharp(t)} \, \big(1 + \|k(t)\|_{H^m_\sharp(t)} dt \big) \,. \tag{6.57}$$

Now our task is to estimate $H^0_\sharp$-norm of M_i First we derive estimate for the quantity M_1. We begin with the observation that

$$\begin{aligned} |k|^2 k = K^3 \, n, &\quad (k \cdot (|k|^2 k))\, k = K^5 \, n, \\ \nabla_s^r (K^3 n) = \partial_s^r (K^3) \, n, &\quad \nabla_s^r (K^5 n) = \partial_s^r (K^5) \, n, \end{aligned} \tag{6.58}$$

where $K = k \cdot n$ is the scalar curvature and $r \geq 0$ is an arbitrary integer. It follows from this and the Moser inequality that

$$\|\nabla^{m-2} (|k|^2 k)\|_{H^0_\sharp} \leq c \|K^3\|_{H^{m-2}_\sharp} \leq c \|K\|^2_{H^\sigma_\sharp} \|K\|_{H^{m-2}_\sharp}.$$

Since $\|K\|_{H^0_\sharp}$ is uniformly bounded, we may apply the interpolation inequality to obtain

$$\|\nabla^{m-2} (|k|^2 k)\|_{H^0_\sharp} \leq c \|K\|_{H^m_\sharp}^{\frac{2\sigma + m - 2}{m}}.$$

Note that estimate (6.15) in Corollary 6.5 and the identity $\nabla_s^m k = \partial_s^m K \, n$ imply the inequality

$$\|K\|_{H^m_\sharp} \leq c + c \|\partial_s^m K\|_{H^0_\sharp} = c + c \|\nabla_s^m k\|_{H^0_\sharp} \leq c + c \|k\|_{H^m_\sharp}.$$

It follows that

$$\|\nabla^{m-2} (|k|^2 k)\|_{H^0_\sharp} \leq c (1 + \|k\|_{H^m_\sharp})^{1 - \frac{2(1-\sigma)}{m}}. \tag{6.59}$$

Repeating these arguments and using identities (6.58) we obtain

$$\|\nabla^{m-4} (|k|^4 k)\|_{H^0_\sharp} \leq c \, (1 + \|k\|_{H^m_\sharp}^{1 - \frac{4(1-\sigma)}{m}}).$$

Since $\sigma \in (1/2, 1)$, we conclude from this and expression (6.56) for M_1 that

$$\|M_1\|_{H^0} \le c(1 + \|k\|_{H^m})^{1-\frac{2(1-\sigma)}{m}}. \tag{6.60}$$

Arguing as before we obtain the estimate

$$\|M_2\|_{H^0_\sharp} \le c(1 + \|k\|_{H^m_\sharp})^{1-\frac{4(1-\sigma)}{m}}. \tag{6.61}$$

It remains to estimate $H^0_\sharp$-norm of M_3. Recall that dJ is a normal field and set

$$dJ = \psi\, n, \quad \psi = dJ \cdot n, \quad \text{which yields} \;\; (k \cdot dJ)k = K^2 \psi\, n.$$

Thus we get

$$\nabla_s^{m-2} dJ = \partial_s^{m-2} \psi\, n, \quad \nabla_s^{m-4}\big((k \cdot dJ)k\big) = \partial_s^{m-4}(K^2\psi)\, n,$$

which leads to the inequality

$$\|M_3\|_{H^0_\sharp} \le c\|\psi\|_{H^{m-2}_\sharp} + \|K^2\psi\|_{H^{m-4}}.$$

The Moser inequality and the embedding theorem imply the estimate

$$\|M_3\|_{H^0_\sharp} \le c\|\psi\|_{H^{m-2}_\sharp} + c\|K\|^2_{H^\sigma_\sharp}\|\psi\|_{H^{m-4}_\sharp} + c\|K\|_{H^\sigma_\sharp}\|\psi\|_{H^\sigma_\sharp}\|K\|_{H^{m-4}}.$$

Since $\|\psi\|_{H^0_\sharp} = \|dJ\|_{H^0_\sharp}$ and $\|k\|_{H^0_\sharp}$ are uniformly bounded, we may apply the interpolation inequality to obtain the estimate

$$\|M_3\|_{H^0_\sharp} \le c\|\psi\|^{1-\frac{2}{m}}_{H^m_\sharp} + \|K\|^{\frac{2\sigma}{m}}_{H^m_\sharp}\|\psi\|^{1-\frac{4}{m}}_{H^m_\sharp} + \|K\|^{\frac{\sigma}{m}}_{H^m_\sharp}\|\psi\|^{\frac{\sigma}{m}}_{H^m_\sharp}\|K\|^{1-\frac{4}{m}}_{H^m}. \tag{6.62}$$

Next notice that by virtue of estimate (6.15), we have

$$\|K\|_{H^m_\sharp} \le c(1 + \|\partial_s^m K\|_{H^0_\sharp}) = c(1 + \|\nabla_s^m k\|_{H^0_\sharp}) \le c(1 + \|k\|_{H^m_\sharp}).$$

Recall that $\|dJ\|_{H^0}$ and $\|k\|_{H^0}$ are uniformly bounded. From this and estimate (6.14) in Lemma 6.4 with $r = m$ and $\Phi = dJ$ we conclude that

$$\|\psi\|_{H^m_\sharp} \le c(1 + \|\partial_s^m \psi\|_{H^0_\sharp}) = c(1 + \|\nabla_s^m dJ\|_{H^0_\sharp}) \le c(1 + \|k\|_{H^m_\sharp} + \|dJ\|_{H^m_\sharp}).$$

By virtue of Theorem 3.2, we have

$$\|dJ\|_{H^{m+\beta}_\sharp} \le c(1 + \|k\|_{H^m_\sharp}), \quad \beta \in [0, 1/2),$$

which gives

$$\|\psi\|_{H^m_\sharp} \le c(1 + \|k\|_{H^m_\sharp}).$$

Substituting the obtained estimates for $\|K\|_{H^m_\sharp}$ and $\|\psi\|_{H^m_\sharp}$ into (6.62) and noting that $\sigma \in (1/2, 1)$ we finally obtain

$$\|M_3\|_{H^0_\sharp} \le (1\|k\|_{H^m_\sharp})^{1-\frac{2}{m}}. \tag{6.63}$$

Combining estimates (6.58), (6.61) and (6.63) we arrive at the inequality

$$\sum_{i=1}^{3} \|M_i(t)\|_{H^0(t)} \le (1 + \|k\|_{H^m(t)})^{1-\frac{2(1-\sigma)}{m}}.$$

Substituting this inequality into (6.57) we obtain desired estimate (6.55). $\square$

6.4. **Proof of Theorem 3.5.**

Proof. We are now in a position to complete the proof of Theorem 3.5. We begin with the observation that the estimates of $\mathcal{N}_i$ given by Lemmas 6.12, 6.13, and 6.14 imply the inequality

$$\sum_{i=1}^{4} |\mathcal{N}_i| \leq c \int_0^{t_0} (1 + \|k(t)\|_{H_\sharp^m}^{2-\frac{2(1-\sigma)}{m}})\, dt\,.$$

Since $\sigma \in (1/2, 1)$ we may apply the Young inequality to obtain

$$\sum_{i=1}^{4} |\mathcal{N}_i| \leq c_\epsilon t_0 + \epsilon \int_0^{t_0} \|k(t)\|_{H_\sharp^m}^2\, dt, \tag{6.64}$$

where $\epsilon \in (0, 1)$ is an arbitrary number and the constant c_ϵ depends on ϵ. Next, estimate (6.15) in Corollary 6.5 yields the inequalities

$$\|\nabla_s^{m-2} k\|_{H_\sharp^0} \geq (1 - 2\epsilon)\|k\|_{H_\sharp^{m-2}}^2 - c_\epsilon, \quad \|\nabla_s^m k\|_{H_\sharp^0} \geq (1 - 2\epsilon)\|k\|_{H_\sharp^m}^2 - c_\epsilon. \tag{6.65}$$

Substituting these inequalities in the main integral identity (6.24) we arrive at the estimate

$$\frac{1-2\epsilon}{2}\|k(t_0)\|_{H_\sharp^{m-2}}^2 + (1 - 2\epsilon)\int_0^{t_0} \|k(t)\|_{H_\sharp^m}^2\, dt \leq$$
$$c_\epsilon(1 + t_0) + \frac{1}{2}\int_{\Gamma(0)} |\nabla_s^{m-2} k(0)|^2\, ds.$$

Setting $\epsilon = 1/6$ we finally obtain

$$\|k(t_0)\|_{H_\sharp^{m-2}}^2 + \int_0^{t_0} \|k(t)\|_{H_\sharp^m}^2\, dt \leq$$
$$c(1 + t_0) + \frac{3}{2}\int_{\Gamma(0)} |\nabla_s^{m-2} k(0)|^2\, ds.$$

This completes the proof of Theorem 3.5. □

7. Proof of Theorem 3.6

Proof. Since $k = \partial_s^2 f$, it follows from estimate (3.25) in Theorem 3.5 that

$$\sup_{[0,T]} \|f(t)\|_{H_\sharp^m} \leq c(T + 1) + E_m\,. \tag{7.1}$$

Recall that the length of the curve $\Gamma(t)$ is uniformly bounded from below and above by the constants $2/E_0$ and E_0. Hence the embedding theorems holds in the spaces $H_\sharp^r(t)$, $r \geq 0$, $t \in [0, T]$, with embedding constants independent of t. Since the embedding $H^m(t) \hookrightarrow C^{m-1}(0, \mathcal{L}(t))$ is continuous, it follows from this and (7.1) that f and k as a function of variables t and s admits the estimate

$$\sup_{[0,T]} \|f(t)\|_{C^{m-1}(0,\mathcal{L}(t))} + \|k(t)\|_{C^{m-3}(0,\mathcal{L}(t))} \leq c(T + 1) + E_m. \tag{7.2}$$

Notice that s is an auxiliary variable and the basic independent variable is $\theta \in \mathbb{S}^1$. Hence our task is to estimate the derivatives of f as a function of the variable θ. To this end, note that

$$\frac{\partial s(t, \theta)}{\partial\theta} = \sqrt{g(t, \theta)}, \quad \text{where} \quad \sqrt{g(t, \theta)} = |\partial_\theta f(t, \theta)|. \tag{7.3}$$

Let us estimate the length element $\sqrt{g}$. It is easily seen that

$$\partial_t\sqrt{g(t)} = \frac{1}{|\partial_\theta f|}(\partial_\theta f \cdot \partial_\theta\partial_t f) = (\partial_s f \cdot \partial_s\partial_t f)\sqrt{g(t)}\,.$$

Notice that

$$\partial_s f \cdot \partial_s\partial_t f = \tau \cdot \partial_s\partial_t f = -\partial_s\tau \cdot \partial_t f = -k \cdot V = -\frac{1}{2}\pi_2,$$

which yields the ordinary differential equation for $\sqrt{g}$:

$$\partial_t\sqrt{g} = -\frac{1}{2}\pi_2\sqrt{g} \quad \text{or} \quad \sqrt{g(t)} = \sqrt{g_0}\, e^{-\frac{1}{2}\int_0^t \pi_2 dt}. \tag{7.4}$$

It follows from estimate (6.21) in Lemma 6.7 that

$$\|\pi_2\|_{H^{m-4}_\sharp} \le c(1+\|k\|_{H^{m-2}_\sharp})^{1+\frac{\sigma}{m-2}},$$

which gives

$$\sup_{[0,T]}\|\pi_2(t)\|_{C^{m-5}(0,\mathcal{L}(t))} \le c. \tag{7.5}$$

Next we have for every integer $r \ge 1$

$$\partial_\theta^r\pi_2 = \sum_{\rho=1}^r \partial_s^\rho\pi_2 \sum_\alpha c_{\rho\alpha}(\partial_\theta s)^{\alpha_1}\,(\partial_\theta^2 s)^{\alpha_2}\dots(\partial_\theta^r s)^{\alpha_r},$$

where the interior sum is taken over all nonnegative integer vectors α such that

$$\alpha_1+\alpha_2+\dots\alpha_r = \rho, \quad \alpha_1+2\alpha_2+\dots+r\alpha_r = r,$$

and $c_{\rho\alpha}$ are some constants. In particular, we have

$$\partial_\theta^r\pi_2 = \sum_{\rho=1}^r \partial_s^\rho\pi_2 \sum_\alpha c_{\rho\alpha}(\sqrt{g})^{\alpha_1}\,(\partial_\theta\sqrt{g})^{\alpha_2}\dots(\partial_\theta^{r-1}\sqrt{g})^{\alpha_r}\,.$$

Differentiating both sides of equation (7.4) with respect to θ we obtain

$$\partial_t(\partial_\theta^r\sqrt{g}) = -\frac{1}{2}\pi_2\,(\partial_\theta^r\sqrt{g}) + \mathcal{R}_r, \tag{7.6}$$

where

$$\mathcal{R}_r = -c_\sigma\sum_{\sigma=0}^{r-1}\partial_\theta^\sigma\sqrt{g}\Big(\sum_{\rho=1}^{r-\sigma}\partial_s^\rho\pi_2\sum_\alpha c_{\rho\alpha}(\sqrt{g})^{\alpha_1}\,(\partial_\theta\sqrt{g})^{\alpha_2}\dots(\partial_\theta^{r-\sigma-1}\sqrt{g})^{\alpha_r}\Big),$$

and

$$\alpha_1+\alpha_2+\dots\alpha_{r-\sigma} = \rho, \quad \alpha_1+2\alpha_2+\dots+r\alpha_{r-\sigma} = r-\sigma.$$

Notice that $\mathcal{R}_r$ contains only the derivatives $\partial_\theta^j\sqrt{g}$ of order $j \le r-1$. Moreover by virtue of (7.5), we have

$$|\partial_s^r\pi_2(t,\theta)| \le c \;\text{ for all }\; t\in[0,T] \;\text{ and }\; r \le m-5,$$

which yields the estimate

$$|R_r| \le c\big(1+\sum_{i=0}^{r-1}|\partial_\theta^i\sqrt{g}|\big)^N, \tag{7.7}$$

where the integer N depends only on r. It follows from this and (7.6) that

$$\sup_{[0,T]\times\mathbb{S}^1}|\partial_\theta^r\sqrt{g}| \le c\sup_{\mathbb{S}^1}|\partial_\theta^r\sqrt{g_0}| + c\sup_{[0,T]\times\mathbb{S}^1}\big(1+\sum_{i=0}^{r-1}|\partial_\theta^i\sqrt{g}|\big)^N \tag{7.8}$$

for all $1 \le r \le m-5$.
It follows from the conditions of Theorem 3.6 that

$$\sup_{\mathbb{S}^1}|\partial_\theta^r\sqrt{g_0}| \le c \;\text{ for all }\; r \le m-5\,.$$

On the other hand, we have

$$\sqrt{g} = \sqrt{g_0}e^{-\frac{1}{2}\int_0^t\pi_2}, \;\text{ which yields }\; \sqrt{g}^{\pm 1} \le c.$$

From this, (7.8), and the induction principle we conclude that

$$\sup_{[0,T]}\|\sqrt{g(t)}\|_{C^{m-6}(\mathbb{S}^1)} \le c. \tag{7.9}$$

This inequality along with the relation

$$\partial_\theta s(t,\theta) = \sqrt{g(t,\theta)}$$

leads to the estimate

$$\|s(t)\|_{C^{m-5}(\mathbb{S}^1)} \le c.$$

Combining this result with (2.11) we arrive at a priori estimate (3.26) for f,

$$\|f\|_{C(0,T;C^{m-5}(\mathbb{S}^1))} \le c.$$

Employing equation (3.20) and repeating the previous arguments we obtain desired estimate (3.26) for $\partial_t f$. This complete the proof of Theorem 3.6. □

8. Proof of Lemmas 2.1, 2.2, and Corollary 2.4

Proof of Lemma 2.1. It suffices to estimate $\mathcal{L}$ from below. The estimate from above obviously follows from (5.11). Since Γ is a C^1 Jordan curve, the degree of the mapping $\tau : \Gamma \to \mathbb{S}^1$ equals 1. Hence there exists $s^* \in (0,\mathcal{L})$ such that $\tau(s^*) = -\tau(0)$. We have

$$2 = |\tau(s^*) - \tau(0)| \le \int_0^{s^*} |k|\, ds \le \sqrt{\mathcal{L}}\Big(\int_0^{\mathcal{L}} |k|^2\, ds\Big)^{1/2} \le \sqrt{2}\sqrt{\mathcal{L}}\sqrt{E_0},$$

which implies the estimate $\sqrt{\mathcal{L}} \ge \sqrt{2/E_0}$. The proof of Lemma 2.1 is completed.

Proof of Lemma 2.2. We will consider the immersion $f = (f_1, f_2)$ as a function of the arc-length variable $s \in [-\mathcal{L}/2, \mathcal{L}/2]$. Obviously it can be regarded as $\mathcal{L}$-periodic function on $\mathbb{R}$. We have $\partial_s f_1 = \tau_1$, $\partial_s f_2 = \tau_2$. In the Cartesian system of coordinates associated with z, we have $\tau(0) = (1,0)$. Notice that τ is $\mathcal{L}$-periodic and

$$\|\partial_s \tau\|_{L^2(-\mathcal{L}/2,\mathcal{L}/2)} = \|k\|_{L^2(-\mathcal{L}/2,\mathcal{L}/2)} \le \sqrt{2E_0}.$$

It follows from this and embedding theorem that

$$\|\tau\|_{C^\alpha(-\mathcal{L}/2,\mathcal{L}/2)} + \|f\|_{C^{1+\alpha}(-\mathcal{L}/2,\mathcal{L}/2)} \le c(\alpha, E_0) \tag{8.1}$$

for all $\alpha \in [0, 1/2)$. This means that the curve Γ belongs to the class $C^{1+\alpha}$ and its smoothness properties depends only on α and E_0. In particular, there is a positive κ, depending only on E_0, such that

$$|\partial_s f_1(s) - 1| \le \frac{1}{12}, \quad |\partial_s f_2(s)| \le \frac{1}{12} \ \text{ for all } \quad s \in [-3\kappa, 3\kappa]$$

and

$$0 < c(E_0)^{-1} \le \kappa \le c(E_0) < \infty$$

Therefore, the mapping $x_1 = f(s)$, $s \in [-3\kappa, 3\kappa]$ is diffeomorphic. We denote its image by $[-\alpha, \beta]$. The mapping

$$\eta(x_1) = f_2(f_1^{-1}(x_1)), \quad x_1 \in [-\alpha, \beta]$$

is well defined and continuously differentiable. Moreover, the identity $\eta' = \partial_s f_2 (\partial_s f_1)^{-1}$ yields the desired estimate (2.3) for η'. Next, we have

$$\eta''(x_1) = (\partial_s^2 f_2 (\partial_s f_1)^{-2})(s(x_1)) - 2(\partial_s f_2 (\partial_s f_1)^{-3} \partial_s^2 f_1)(s(x_1)).$$

Since the absolute values of the derivatives $(\partial_s f_1)^{\pm 1}$, $\partial_s f_2$ are bounded by 2 on the segment $[-3\kappa, 3\kappa]$, it follows that

$$\int_{-\alpha}^{\beta} |\eta''|^2\, dx_1 \le c \int_{-3\kappa}^{3\kappa} |\partial_s^2 f|^2 ds \le cE_0.$$

This completes the proof of Lemma 2.2.

Proof of Corollary 2.4. By virtue of Corollary 2.3, we have for every $t \in [0,T]$,

$$\nu(t) = \inf_{z \in \Gamma(t)} \ \text{dist}\ (\Gamma(t) \setminus \Gamma_{3\kappa}(t),\, \Gamma_{2\kappa}(t)) > 0\,.$$

Here $\Gamma_{3\kappa}(t)$ and $\Gamma_{2\kappa}(t))$ are the arcs centered at z and defined by Lemma 2.2 with Γ replaced by $\Gamma(t)$.

It is necessary to prove that $\inf_t \nu(t) > 0$. Suppose, in contrary to the our claim, that there are sequences t_n and $z_n \in \Gamma(t_n)$ with the properties

$$\text{dist } (\Gamma(t_n) \setminus \Gamma^n_{3\kappa}(t_n), \Gamma^n_{2\kappa}(t_n)) \to 0 \quad \text{as} \quad n \to \infty.$$

Here $\Gamma^n_{3\kappa}(t_n)$ and $\Gamma^n_{2\kappa}(t_n)$ are the arcs centered at z_n and defined by Lemma 2.2 with Γ replaced by $\Gamma(t_n)$.

After passing to a subsequence we may assume that $t_n \to t_\infty \in [0, T]$ and $z_n \to z^* \in \mathbb{R}^2$ as $n \to \infty$. It follows that there are sequences $z'_n \in \Gamma(t_n) \setminus \Gamma_{3\kappa}(t_n)$ and $z''_n \in \Gamma_{2\kappa}(t_n)$ such that $|z'_n - z''_n| \to 0$ as $n \to \infty$. Choose the arc-length coordinates on $\Gamma(t_n)$ such that the corresponding arc-length coordinates equal zero at z^*. It follows that

$$z'_n = f(t_n, s'_n), \quad z''_n = f(t_n, s''_n),$$

where

$$s'_n \in (-\mathcal{L}(t_n)/2, \mathcal{L}(t_n)/2) \setminus (-3\kappa, 3\kappa), \quad s''_n \in (-2\kappa, 2\kappa).$$

Passing to subsequences we may assume that

$$s'_n \to s'_\infty \in (-\mathcal{L}(t_\infty)/2, \mathcal{L}(t_\infty)/2) \setminus (-3\kappa, 3\kappa), \quad s''_n \to s"_\infty \in (-2\kappa, 2\kappa).$$

It follows from condition **H.3** that

$$z'_n = \overline{f}(t_n, s'_n) \to \overline{f}(t_\infty, s'_\infty) \quad \text{and} \quad z''_n = \overline{f}(t_n, s''_n) \to \overline{f}(t_\infty, s''_\infty).$$

Hence

$$f(t_\infty, s'_\infty) = f(t_\infty, s''_\infty) \quad \text{and} \quad s'_\infty \neq s''_\infty. \tag{8.2}$$

On the other hand, Condition **H.2** implies that the curve $\Gamma(t_\infty)$ has no self-intersections. This contradicts to relations (8.2).

9. Sobolev spaces

9.1. **Anisotropic spaces.** In the proof of basic results in Section 4, the analysis of function classes with specific smoothness properties with respect to specific space variables is required. In this subsection we collect the basis facts about such spaces. The results will be used in Sections 10 and 11.

Let

$$Q_m = (-\kappa_m, \kappa_m) \times (-\rho_m, \rho_m), \quad m \in [1, r]$$

be a rectangle defined by (3.1). Fix an arbitrary smooth function $\phi : Q_m \to \mathbb{R}$ compactly supported in the interval $y_1 \in (-\kappa_m, \kappa_m)$ for every $y_2 \in (-\rho_m, \rho_m)$. Assume that ϕ is extended by 0 to the strip $\mathbb{R} \times (-\rho_m, \rho_m)$. Introduce the norm

$$\begin{aligned} \|\phi\|_{X^\infty} &= \sup_{y_1 \in \mathbb{R}} \|\phi(y_1, \cdot)\|_{L^2(-\rho_m, \rho_m)}, \\ \|\phi\|^2_{Y^j} &= \|\phi\|^2_{L^2(Q_m)} + \|\partial^j_1 \phi\|^2_{L^2(Q_m)}. \end{aligned} \tag{9.1}$$

Since $\phi(\cdot, y_2)$ is smooth and compactly supported in $\mathbb{R}$, we can rewrite this definition in terms of the Fourier transform. The Fourier transform $\hat{\phi}(\xi, y_2)$, $(\xi, y_2) \in \mathbb{R} \times (-\rho_m, \rho_m)$, is defined by the equalities

$$\hat{\phi}(\xi, y_2) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi y_1} \phi(y_1, y_2)\, dy_1, \quad \phi(y_1, y_2) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi y_1} \hat{\phi}(\xi, y_2)\, d\xi.$$

With this notation definition (9.1) for Y^j-norm can be written in the equivalent form

$$\|\phi\|^2_{Y^j} = \int_{-\rho_m}^{\rho_m} \Big(\int_{\mathbb{R}} (1 + \xi^2)^j \, |\hat{\phi}(\xi, y_2)|^2 \, d\xi \Big)\, dy_2. \tag{9.2}$$

Using (9.2) we can define Y^β-norm for any $\beta \in \mathbb{R}$:

$$\|\phi\|^2_{Y^\beta} = \int_{-\rho_m}^{\rho_m} \Big(\int_{\mathbb{R}} (1 + \xi^2)^\beta \, |\hat{\phi}(\xi, y_2)|^2 \, d\xi \Big)\, dy_2. \tag{9.3}$$

Notice two elementary inequalities for the introduced norms.

Lemma 9.1. *Let ϕ be a smooth function compactly supported in Q_m. Assume that ϕ is extended by 0 to $\mathbb{R}\times(-\rho_m,\rho_m)$. Then*

$$\|\phi\|_{X^\infty} \le c(\sigma)\ \|\phi\|_{Y^\sigma} \quad \text{for every} \quad \sigma > 1/2, \tag{9.4}$$

$$\|\phi\|_{Y^\gamma} \le c(\sigma)\ \|\phi\|_{L^2(Q_m)}^{1-\frac{\gamma}{\sigma}}\ \|\phi\|_{Y^\sigma}^{\frac{\gamma}{\sigma}} \quad \text{for every} \quad \sigma > 0 \quad \text{and} \quad 0<\gamma<\sigma. \tag{9.5}$$

Proof. In order to prove the first estimate (9.4) in Lemma 9.1, note that

$$\sup_{y_1\in\mathbb{R}} |\phi(y_1,y_2)| \le \frac{1}{\sqrt{2\pi}}\int_R |\hat\phi(\xi,y_2)|\,d\xi.$$

Thus we get

$$\begin{aligned} \|\phi\|_{X^\infty}^2 = \sup_{y_1\in\mathbb{R}}\int_{-\rho_m}^{\rho_m} |\phi(y_1,y_2)|^2\,dy_2 \le \int_{-\rho_m}^{\rho_m}\Big\{\sup_{y_1\in\mathbb{R}}|\phi(y_1,y_2)|\Big\}^2\,dy_2 \le \\ c\int_{-\rho_m}^{\rho_m}\Big\{\int_{\mathbb{R}}|\hat\phi(\xi,y_2)|\,d\xi\Big\}^2\,dy_2. \end{aligned} \tag{9.6}$$

Since $\sigma>1/2$, the Cauchy inequality implies the estimate

$$\begin{aligned} \Big\{\int_{\mathbb{R}}|\hat\phi(\xi,y_2)|\,d\xi\Big\}^2 \le \int_{\mathbb{R}}(1+\xi^2)^{-\sigma}\,d\xi\ \int_{\mathbb{R}}(1+\xi^2)^{\sigma}\ \|\hat\phi(\xi,y_2)|^2\,d\xi \\ \le c\int_{\mathbb{R}}(1+\xi^2)^{\sigma}\ |\hat\phi(\xi,y_2)|^2\,d\xi. \end{aligned}$$

Substituting this estimate into (9.6) we finally arrive at the inequality

$$\|\phi\|_{X^\infty}^2 \le \int_{-\rho_m}^{\rho_m}\int_{\mathbb{R}}(1+\xi^2)^{\sigma}\ |\hat\phi(\xi,y_2)|^2\,d\xi\,dy_2,$$

which along with (9.3) yields the desired estimate (9.4).

It remains to prove the interpolation inequality (9.5). We begin with the observation that

$$(1+\xi^2)^{\gamma}|\hat\phi(\xi,y_2)|^2 = (|\hat\phi(\xi,y_2)|^2)^{1-\vartheta}\,\big((1+\xi^2)^{\gamma}|\hat\phi(\xi,y_2)|^2)^{\vartheta}\big),\quad \vartheta=\gamma/\sigma.$$

From this and the Hölder inequality we conclude that

$$\int_{\mathbb{R}}(1+\xi^2)^{\gamma}|\hat\phi(\xi,y_2)|^2\,d\xi \le \Big(\int_{\mathbb{R}}|\hat\phi(\xi,y_2)|^2\,d\xi\Big)^{1-\vartheta}\ \Big(\int_{\mathbb{R}}(1+\xi^2)^{\sigma}|\hat\phi(\xi,y_2)|^2\,d\xi\Big)^{\vartheta}.$$

Combining this inequality with (9.2) and applying the Hölder inequality we finally arrive at the estimate

$$\begin{aligned} \|\phi\|_{Y^\gamma}^2 = \int_{-\rho_m}^{\rho_m}\Big\{\int_{\mathbb{R}}(1+\xi^2)^{\gamma}|\hat\phi(\xi,y_2)|^2\,d\xi\Big\}^2\,dy^2 \le \\ \int_{-\rho_m}^{\rho_m}\Big\{\Big(\int_{\mathbb{R}}|\hat\phi(\xi,y_2)|^2\,d\xi\Big)^{1-\vartheta}\ \Big(\int_{\mathbb{R}}(1+\xi^2)^{\sigma}|\hat\phi(\xi,y_2)|^2\,d\xi\Big)^{\vartheta}\Big\}\,dy_2 \le \\ \Big(\int_{-\rho_m}^{\rho_m}\int_{\mathbb{R}}|\hat\phi(\xi,y_2)|^2\,d\xi dy_2\Big)^{1-\vartheta}\ \Big(\int_{-\rho_m}^{\rho_m}\int_{\mathbb{R}}(1+\xi^2)^{\sigma}|\hat\phi(\xi,y_2)|^2\,d\xi dy_2\Big)^{\vartheta} = \\ \|\phi\|_{Y^0}^{2(1-\vartheta)}\,\|\phi\|_{Y^\sigma}^{2\vartheta}. \end{aligned}$$

Recalling that $\vartheta=\gamma/\sigma$ and

$$\|\phi\|_{Y^0}^2 = \int_{-\rho_m}^{\rho_m}\int_{\mathbb{R}}|\hat\phi(\xi,y_2)|^2\,d\xi dy_2 = \int_{-\rho_m}^{\rho_m}\int_{\mathbb{R}}|\phi(y)|^2\,dy_1dy_2 = \|\phi\|_{L^2(Q_m)}^2$$

we obtain the desired interpolation inequality (9.5). □

9.2. **Proof of Lemma 5.5.** Without loss of generality we may assume that $\mathcal{L} = 2\pi$. For the sake of simplicity, introduce the temporary notation:

$$\overline{F} = u, \quad F_\sharp = v = \sum_{k=-\infty}^{\infty} u(s + 2k\pi). \tag{9.7}$$

It is necessary to prove that

$$\|v\|_{H_\sharp^{m+1/2}} \le \|u\|_{H^{m+1/2}(\mathbb{R})}. \tag{9.8}$$

Step 1. New norm in $H^{m+1/2}(\mathbb{R})$. It is convenient to introduce the equivalent norm on the space $H^{m+1/2}(\mathbb{R})$. Recall that

$$\|u\|^2_{H^{m+1/2}(\mathbb{R})} = \int_{\mathbb{R}} (1 + |\xi|^2)^{m+1/2} \, |\hat{u}(\xi)|^2 \, d\xi. \tag{9.9}$$

Introduce the pseudodifferential operators $\mathbf{S}$ and $\mathbf{T}$ defined in terms of the Fourier transform by the equalities

$$\widehat{\mathbf{S}\, u}(\xi) = \frac{1}{(1 + |\xi|^2)^{1/4}} \; \hat{u}(\xi), \quad \widehat{\mathbf{T}\, u}(\xi) = \mathcal{T}(\xi) \, \hat{u}(\xi), \tag{9.10}$$

where

$$\mathcal{T}(\xi) = \frac{(-i\xi)^{m+1}}{(1 + |\xi|^2)^{1/4}} \; \hat{u}(\xi). \tag{9.11}$$

It is clear that

$$\mathbf{T} = (\partial_s)^{m+1} \; \mathbf{S}. \tag{9.12}$$

Introduce the Hilbert norm defined by

$$|u|^2_{H^{m+1/2}(\mathbb{R})} = \|u\|^2_{L^2(\mathbb{R})} + \|\mathbf{T}\, u\|^2_{L^2(\mathbb{R})}, \tag{9.13}$$

or equivalently

$$|u|^2_{H^{m+1/2}(\mathbb{R})} = \int_{\mathbb{R}} (1 + |\mathcal{T}(\xi)|^2) \, |\hat{u}(\xi)|^2 \, d\xi. \tag{9.14}$$

Since

$$c^{-1}(1 + |\xi|^2)^{m+1/2} \le 1 + |\mathcal{T}(\xi)|^2 \le c(1 + |\xi|^2)^{m+1/2},$$

the norms $\|\cdot\|_{H^{m+1/2}(\mathbb{R})}$ and $|\cdot|_{H^{m+1/2}(\mathbb{R})}$ are equivalent, i.e.,

$$c^{-1}\|u\|_{H^{m+1/2}(\mathbb{R})} \le |u|_{H^{m+1/2}(\mathbb{R})} \le c\|u\|_{H^{m+1/2}(\mathbb{R})}. \tag{9.15}$$

Finally introduce the function

$$\Phi(\xi) = \mathcal{T}(\xi) \, \hat{u}(\xi), \quad \xi \in \mathbb{R}. \tag{9.16}$$

It is clear that

$$|u|^2_{H^{m+1/2}(\mathbb{R})} = \|\hat{u}\|^2_{L^2(\mathbb{R})} + \|\Phi\|^2_{L^2(\mathbb{R})}. \tag{9.17}$$

Step 2. New norm in $H_\sharp^{m+1/2}$. It is convenient to introduce the equivalent norm on the space $H_\sharp^{m+1/2}$. Recall that

$$\|v\|^2_{H_\sharp^{m+1/2}} = \sum_{k=-\infty}^{\infty} (1 + |k|^2)^{m+1/2} \, |v_k|^2, \tag{9.18}$$

where the Fourier coefficients are defined by

$$v_k = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-iks} \, v(s) \, ds.$$

Introduce the operators $\mathbf{S}_\sharp$ and $\mathbf{T}_\sharp$ defined in the Fourier basis by the equalities

$$(\mathbf{S}_\sharp \, v)_k = \frac{1}{(1 + |k|^2)^{1/4}} \, v_k, \quad (\mathbf{T}_\sharp \, v)_k = \mathcal{T}(k) \, v_k, \tag{9.19}$$

where $\mathcal{T}$ is defined by (9.11). It is clear that

$$\mathbf{T}_\sharp = (\partial_s)^{m+1}\,\mathbf{S}_\sharp. \tag{9.20}$$

Introduce the Hilbert norm defined by

$$|v|^2_{H_\sharp^{m+1/2}} = \sum_k |v_k|^2 + \sum_k |\mathcal{T}(k)|^2\,|v_k|^2. \tag{9.21}$$

Arguing as before we conclude that the norms $\|\cdot\|_{H^{m+1/2}\sharp}$ and $|\cdot|_{H_\sharp^{m+1/2}}$ are equivalent, i.e.,

$$c^{-1}\|v\|_{H_\sharp^{m+1/2}} \le |v|_{H_\sharp^{m+1/2}} \le c\|v\|_{H_\sharp^{m+1/2}}. \tag{9.22}$$

Finally introduce the sequence

$$\Phi_\sharp(k) = \mathcal{T}(k)\,v_k, \quad -\infty < k < \infty. \tag{9.23}$$

It is clear that

$$|v|^2_{H_\sharp^{m+1/2}} = \sum_k |v_k|^2 + \sum_k |\Phi_\sharp(k)|^2. \tag{9.24}$$

Hence we can rewrite inequality (9.8) in the form

$$|v|_{H_\sharp^{m+1/2}} \le c|u|_{H^{m+1/2}(\mathbb{R})}.$$

From this and representations (9.17), (9.24) for the norms in the spaces $H_\sharp^{m+1/2}$ and $H^{m+1/2}(\mathbb{R})$ we conclude that this equality is equivalent to the following

$$\sum_k |v_k|^2 + \sum_k |\Phi_\sharp(k)|^2 \le c(\|\hat{u}\|^2_{L^2(\mathbb{R})} + \|\Phi\|^2_{L^2(\mathbb{R})}). \tag{9.25}$$

Step 3. Relation between Φ *and* $\Phi_\sharp$. Let a compactly supported function u and a periodic function v are connected by relation (9.7),

$$v = \sum_{k=-\infty}^{\infty} u(s + 2k\pi).$$

Furthermore assume that Φ and $\Phi_\sharp$ are defined by (9.16) and (9.23). Then we have the identity

$$\Phi_\sharp(k) = \Phi(k), \quad -\infty < k < \infty. \tag{9.26}$$

Indeed, we have

$$\Phi_\sharp(k) = \mathcal{T}(k)\,v_k = \mathcal{T}(k)\frac{1}{\sqrt{2\pi}}\int_0^{2\pi} e^{-iks}v(s)\,ds =$$

$$\sum_{j=-\infty}^{\infty}\frac{\mathcal{T}(k)}{\sqrt{2\pi}}\int_0^{2\pi} e^{-iks}u(s+2\pi j)\,ds = \sum_{j=-\infty}^{\infty}\frac{\mathcal{T}(k)}{\sqrt{2\pi}}\int_{2\pi j}^{2\pi(j+1)} e^{-iks}u(s)\,ds$$

$$= \mathcal{T}(k)\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-iks}u(s)\,ds = \Phi(k).$$

Similarly we have

$$v_k = \hat{u}(k), \quad -\infty < k < \infty. \tag{9.27}$$

Substituting (9.26) and (9.27) into (9.25) we conclude that it suffices to prove the inequality

$$\sum_k (|\hat{u}(k)|^2 + |\Phi(k)|^2) \le c\int_{\mathbb{R}} (|\hat{u}(\xi)|^2 + |\Phi(\xi)|^2)\,d\xi. \tag{9.28}$$

Step 4. The proof of inequality (9.28) is based on the following lemma

Lemma 9.2. *Under the above assumptions we have*

$$\|\Phi\|_{H^1(\mathbb{R})} \le c\|\hat{u}\|_{L^2(\mathbb{R}} + c\|\Phi\|_{L^2(\mathbb{R})}, \quad \|\hat{u}\|_{H^1(\mathbb{R})} \le c\|\hat{u}\|_{L^2(\mathbb{R})}. \tag{9.29}$$

Proof. Recall that

$$\mathbf{T}u = \partial_s^{m+1}\mathbf{S}u, \quad \widehat{\mathbf{S}u} = (1+|\xi|^2)^{-1/4}\,\hat{u}.$$

The operator $\mathbf{S}$ is the Bessel potential of order $1/2$. It admits the integral representation

$$\mathbf{S}\,u(s) = \int_{\mathbb{R}} \mathbf{b}(s-t)\,u(t)\,dt, \tag{9.30}$$

where the Bessel kernel $\mathbf{b}$ has the following properties, see [2], Ch. 1. On the interval $(-\kappa,\kappa)$ it has the representation

$$\mathbf{b}(s) = \frac{c(\kappa)}{\sqrt{|s|}} + o(s), \quad o \in C^\infty[-\kappa,\kappa], \tag{9.31}$$

Outside of this interval the kernel $\mathbf{b}(s)$ admits the estimate

$$|\partial_s^r \mathbf{b}(s)| \le c(r)e^{-\beta|s|} \ \text{ for all } \ |s|\ge\kappa, \ \ r\ge 0. \tag{9.32}$$

Here β is some positive exponent. Choose an arbitrary function $\zeta\in C_0^\infty(\mathbb{R})$ such that

$$\zeta(s)=1 \ \text{ for } \ |s|\le 2\kappa, \quad \zeta(s)=0 \ \text{ for } \ |s|\ge 3\kappa.$$

We have

$$\mathbf{T}u = \zeta\mathbf{T}u + (1-\zeta)\mathbf{T}u$$

Obviously we have

$$\int_{\mathbb{R}} |s\,\zeta\,\mathbf{T}u|^2\,ds \le c\kappa^2\int_{\mathbb{R}}|\mathbf{T}u|^2\,ds. \tag{9.33}$$

Recall that u is compactly supported in the interval $(-\kappa,\kappa)$. From this and estimate (9.32) we conclude that the inequalities

$$|\mathbf{T}u(s)| = \Big|\int_{\mathbb{R}} \partial_s^{m+1}\mathbf{b}(s-t)u(t)\,dt\Big| \le$$
$$c\int_{\mathbb{R}} e^{-\beta|s-t|}u(t)\,dt \le ce^{-\beta|s|}\int_{\mathbb{R}}|u|\,dt \le ce^{-\beta|s|}\,\|u\|_{L^2(\mathbb{R})}$$

hold for every s with $|s|\ge 2\kappa$.

Since $(1-\zeta)\,\mathbf{T}u$ equals zero in the segment $[-2\kappa,2\kappa]$, it follows from this that

$$\int_{\mathbb{R}} |s(1-\zeta)\,\mathbf{T}u(s)|^2\,ds \le c\int_{\mathbb{R}} s^2\,e^{-2\beta|s|}\,ds\ \|u\|^2_{L^2(\mathbb{R})}$$
$$\le c\|u\|^2_{L^2(\mathbb{R})}.$$

Combining this estimate with (9.33) we obtain

$$\int_{\mathbb{R}}|s\mathbf{T}u|^2\,ds \le c\big(\,\|u\|^2_{L^2(\mathbb{R})} + \|\mathbf{T}u\|^2_{L^2(\mathbb{R})}\,\big). \tag{9.34}$$

Next notice that

$$\widehat{s\mathbf{T}u} = \frac{1}{i}\partial_\xi\widehat{\mathbf{T}u} = \frac{1}{i}\partial_\xi\Phi, \quad \widehat{\mathbf{T}u} = \Phi.$$

From this, (9.34), and the Plancherel theorem we obtain

$$\int_{\mathbb{R}}|\partial_\xi\Phi|^2\,d\xi \le c\big(\,\|\hat{u}\|^2_{L^2(\mathbb{R})} + \|\Phi\|^2_{L^2(\mathbb{R})}\,\big),$$

which gives the desired estimate (9.29) for Φ. Repeating these arguments with essential simplifications we finally obtain estimate (9.29) for $\hat{u}$, and the lemma follows. □

We are now in a position to complete the proof of Lemma 5.5. To this end, it suffices to prove inequality (9.28). Notice that

$$|\Phi(k)|^2 \le 2|\Phi(\xi)|^2 + 2|\Phi(\xi)-\Phi(k)|^2.$$

Next we have

$$|\Phi(\xi)-\Phi(k)|\leq\Big(\int_k^{k+1}|\partial_\xi\Phi|^2\,d\xi\Big)^{1/2}\quad\text{for}\quad\xi\in[k,k+1].$$

It follows that

$$|\Phi(k)|^2\leq 2|\Phi(\xi)|^2+2\int_k^{k+1}|\partial_\xi\Phi|^2\,d\xi\quad\text{for}\quad\xi\in[k,k+1].$$

Integrating both sides of these inequality over the segment $[k,k+1]$ we arrive at the inequality

$$|\Phi(k)|^{\leq}2\int_k^{k+1}(|\Phi(\xi)|^2+|\partial_\xi\Phi|^2\,d\xi.$$

Summation this inequality with respect to k gives

$$\sum_{-\infty}^{\infty}|\Phi(k)|^2\leq 2\int_{\mathbb{R}}(|\Phi(\xi)|^2+|\partial_\xi\Phi|^2)\,d\xi=\|\Phi\|^2_{H^1(\mathbb{R})}.$$

From this and Lemma 9.2 we conclude that

$$\sum_{-\infty}^{\infty}|\Phi(k)|^2\leq c\,(\|\hat u\|^2_{L^2(\mathbb{R})}+\|\Phi\|^2_{L^2(\mathbb{R})}).\tag{9.35}$$

Repeating these arguments gives

$$\sum_{-\infty}^{\infty}|\hat u(k)|^2\leq c\,\|\hat u\|^2_{L^2(\mathbb{R})}.\tag{9.36}$$

Combining (9.35) and (9.36) we obtain desired inequality (9.28). This completes the proof of Lemma 5.5.

10. Proof of Lemma 4.3

We begin with the observation that the function u satisfies the divergent elliptic equation

$$\text{div }(aN\nabla u)=0\ \text{ in }\ Q_0$$

with the matrix aN bounded from below and above by the constants, depending on C_N and a_0. It follows from this, inequality

$$\|u\|_{L^2(Q_0)}+\|\nabla u\|_{L^2(Q_0)}\leq C_u<\infty$$

and di-Giorgi-Nash-Moser estimate for the Hölder norm of solutions to the divergent elliptic equations that the estimate

$$\|u\|_{C^\alpha(Q')}\leq c(Q')\ \text{ for some }\quad\alpha\in[0,1),$$

holds for every compact set $Q'\Subset Q_0$. We thus get the estimate

$$\|v_1\|_{C(Q_1)}\leq c\,\|u\|_{C(Q_1)}\leq c.\tag{10.1}$$

For $m=1$, equation (4.19) reads

$$\begin{gathered}\text{div }\big(aN\nabla\partial_1 v_1\big)=-\text{ div }(a\partial_1 N\nabla v_1)+\\ \text{div }\big(\partial_1 u\,aN\nabla\varphi_1+u\,a\partial_1(N\nabla\varphi_1)\big)+\partial_1(a\nabla\varphi_1\cdot N\nabla u)\ \text{ in }\ Q_1.\end{gathered}\tag{10.2}$$

Now we estimate every term in the right hand side of this equation. Since a is bounded and N is independent of y_2, we have

$$\begin{gathered}\|a\partial_1 N\nabla v_1\|_{L^2(Q_1)}\leq c\,\|\partial_1 N\nabla v_1\|_{L^2(Q_1)}\leq\\ c\|\partial_1 N\|_{L^2(-\kappa_1,\kappa_1)}\sup_{y_1\in(-\kappa_1,\kappa_1)}\|\nabla v_1\|_{L^2(-\rho_1,\rho_1)}=\\ c\,\|\partial_1 N\|_{L^2(-\kappa_1,\kappa_1)}\|\nabla v_1\|_{X^\infty}\leq c\,\|\nabla v_1\|_{X^\infty}.\end{gathered}$$

Here we use estimate (4.5) for $\partial_1 N$. From this and inequalities (4.4)-(4.5) with $\phi = \nabla v_1$ we conclude that for every fixed $\sigma \in (0,1)$,

$$\begin{aligned} \|a\partial_1 N\nabla v_1\|_{L^2(Q_1)} &\leq c\,\|\nabla v_1\|_{X^\infty} \leq c\,\|\nabla v_1\|_{Y^\sigma} \\ &\leq c\,\|\nabla v_1\|^{1-\sigma}_{L^2(Q_1)}\|\nabla v_1\|^\sigma_{Y^1} \leq c\,\|\nabla v_1\|^\sigma_{Y^1}. \end{aligned} \tag{10.3}$$

Let us estimate the second term in the right hand side of (10.2). Recall that L^2-norm of ∇u and $\partial_1 N$ are uniformly bounded by the constants C_N and C_u. In its turn, L^∞ norms of u and N are bounded by C_N and the constant in inequality (10.1). It follows from this that

$$\begin{aligned} &\|a\partial_1 u\, N\nabla\varphi_1 + u\, a\, \partial_1(N\nabla\varphi_1)\|_{L^2(Q_1)} \leq \\ &c\,\|\nabla u\|_{L^2(Q_1)} + c\,\|\partial_1(N\nabla\varphi_1)\|_{L^2(Q_1)} \leq c. \end{aligned} \tag{10.4}$$

It remains to estimate the third term in the right hand side of (10.2). We have

$$\|a\nabla\varphi_1 \cdot N\nabla u\|_{L^2(Q_1)} \leq c\,\|\nabla u\|_{L^2(Q_1)} \leq c. \tag{10.5}$$

Multiplying both sides of (10.2) by $\partial_1 v_1$ and integrating the result by parts we arrive at the equality

$$\begin{aligned} \int_{Q_1} aN\nabla\partial_1 v_1 \cdot \nabla\partial_1 v_1 dy = -\int_{Q_1} a\,\partial_1 N\nabla v_1 \cdot \nabla\partial_1 v_1\, dy + \\ \int_{Q_1} (\partial_1 u\; a\, N\nabla\varphi_1 + u\, a\, \partial_1(N\nabla\varphi_1)) \cdot \nabla\partial_1 v_1\, dy + \int_{Q_1} (a\nabla\varphi_1 \cdot N\nabla u)\, \partial_1^2 v_1\, dy. \end{aligned}$$

Applying the Cauchy inequality in the right hand side and employing estimates (10.3)-(10.5) we obtain

$$\int_{Q_1} aN\nabla\partial_1 v_1 \cdot \nabla\partial_1 v_1 dy \leq c\,(\|\nabla v_1\|^\sigma_{Y^1} + 1)\;\|\nabla\partial_1 v_1\|_{L^2(Q_1)}. \tag{10.6}$$

Recall that the matrix aN is bounded from below by the constant depending only on C_N and a_0, which gives

$$\int_{Q_1} aN\nabla\partial_1 v_1 \cdot \nabla\partial_1 v_1 dy \geq c^{-1}\|\nabla\partial_1 v_1\|^2_{L^2(Q_1)}.$$

Obviously we have

$$\begin{aligned} \|\nabla v_1\|^\sigma_{Y^1} \leq c(\|\nabla_1 v_1\|_{L^2(Q_1)} + \|\partial_1\nabla_1 v_1\|_{L^2(Q_1)})^\sigma \\ \leq c(1 + \|\partial_1\nabla_1 v_1\|_{L^2(Q_1)})^\sigma \leq c + c\|\partial_1\nabla v_1\|^\sigma_{L^2(Q_1)}. \end{aligned}$$

Substituting the obtained estimates into (10.6) we arrive at the inequality

$$\|\nabla\partial_1 v_1\|_{L^2(Q_1)} \leq c(\|\nabla\partial_1 v_1\|^\sigma_{L^2(Q_1} + c, \quad \sigma \in (0,1),$$

which obviously yields the desired estimate (4.20)

$$\|\partial_1\nabla v_1\|_{L^2(Q_1)} \leq c.$$

11. Proof of Lemma 4.5

11.1. **Proof of estimate (4.22) for $\mathcal{A}_m$.** Introduce the temporary notation

$$H = \partial_1 N, \quad w = \partial_1\nabla v_m.$$

Since $\varphi_1 = 1$ on the support of φ_m, it follows from the definition (4.10) of v_m and estimate (4.20) in Lemma 4.3 that

$$\|H\|_{L^2(-\kappa_m,\kappa_m)} + \|w\|_{L^2(Q_m)} \leq c. \tag{11.1}$$

We have

$$\mathcal{A}_m = \sum_{i+j=m-2,\, i,j\geq 0} a\partial_1^j H\partial_1^i\nabla w + a\partial_1^m N\nabla v_m,$$

which yields

$$\|\mathcal{A}_m\|_{L^2(Q_m)} \le \sum_{i+j=m-2,i,j\ge 0} F_{ij} + \|\partial_1^m N\nabla v_m\|_{L^2(Q_m)}, \tag{11.2}$$

where

$$F_{ij} = \|\partial_1^j H\,\partial_1^i\, w\|_{L^2(Q_m)}.$$

Let us estimate F_{ij}. We have

$$F_{ij}^2 = \int_{Q_m} |\partial_1^j H|^2\,|\partial_1^i w|^2\,dy \le$$

$$\Big\{\int_{-\kappa_m}^{\kappa_m} |\partial_1^j H|^2\,dy_1\Big\}\Big\{\sup_{y_1\in(-\kappa_m,\kappa_m)}\int_{-\rho_m}^{\rho_m} |\partial_1^i w(y_1,y_2)|^2\,dy_2\Big\}.$$

Choose an arbitrary $\sigma \in (1/2, 1)$. Recalling the definition (9.1) of norms X^∞, Y^j and inequality (9.4) we obtain

$$F_{ij} \le c\|H\|_{W^{j,2}(-\kappa_m,\kappa_m)}\,\|\partial_1^i w\|_{X^\infty} \le c\|H\|_{W^{j,2}(-\kappa_m,\kappa_m)}\,\|\partial_1^i w\|_{Y^\sigma}. \tag{11.3}$$

Next, we employ the interpolation inequality in Sobolev space and the special interpolation inequality (9.5) to obtain

$$\|H\|_{W^{j,2}(-\kappa_m,\kappa_m)} \le c\|H\|^{1-\frac{j}{m-1}}_{L^2(-\kappa_m,\kappa_m)}\,\|H\|^{\frac{j}{m-1}}_{W^{m-1,2}(-\kappa_m,\kappa_m)} \le c\|H\|^{\frac{j}{m-1}}_{W^{m-1,2}(-\kappa_m,\kappa_m)}$$

and

$$\|\partial_1^i w\|_{Y^\sigma} \le \|w\|_{Y^{i+\sigma}} \le c\|w\|^{1-\frac{i+\sigma}{m-1}}_{L^2(Q_m)}\,\|w\|^{\frac{i+\sigma}{m-1}}_{Y^{m-1}} \le c\|w\|^{\frac{i+\sigma}{m-1}}_{Y^{m-1}}.$$

Substituting these inequalities in (11.3) we arrive at the estimate

$$F_{ij} \le c\|H\|^{\frac{j}{m-1}}_{W^{m-1,2}(-\kappa_m,\kappa_m)}\,\|w\|^{\frac{i+\sigma}{m-1}}_{Y^{m-1}}.$$

Notice that

$$\frac{j}{m-1} + \frac{i+\sigma}{m-1} = \frac{j+i+\sigma}{m-1} = \frac{m-2+\sigma}{m-1} = \lambda < 1.$$

Thus we get

$$F_{ij} \le c\big(\ \|H\|^{\alpha}_{W^{m-1,2}(-\kappa_m,\kappa_m)}\,\|w\|^{\beta}_{Y^{m-1}}\ \big)^{\lambda},$$

where

$$\alpha + \beta = \frac{1}{\lambda}\Big(\frac{j}{m-1} + \frac{i+\sigma}{m-1}\Big) = 1.$$

Applying the Young inequality we obtain

$$F_{ij} \le c\big(\ \|H\|^{\lambda}_{W^{m-1,2}(-\kappa_m,\kappa_m)} + \|w\|^{\lambda}_{Y^{m-1}}\ \big). \tag{11.4}$$

It remains to estimate the last term in inequality (11.2). By virtue of estimate (4.20) in Lemma 4.3, we have

$$\begin{aligned}\|\partial_1^m N\nabla v_m\|_{L^2(Q_m)} &\le c\ \|\partial_1^m N\|_{L^2(-\kappa_m,\kappa_m)}\ \|\nabla v_m\|_{X^\infty}\\ &\le c\|\partial_1^m N\|_{L^2(-\kappa_m,\kappa_m)}\ \|\nabla v_m\|_{Y^1} \le c\|\partial_1^m N\|_{L^2(-\kappa_m,\kappa_m)}.\end{aligned}$$

Substituting this estimate and estimate (11.4) into (11.2) we arrive at the inequality

$$\|\mathcal{A}_m\|_{L^2(Q_m)} \le c\|H\|^{\lambda}_{W^{m-1,2}(-\kappa_m,\kappa_m)} + c\,\|w\|^{\lambda}_{Y^{m-1}} + c\|\partial_1^m N\|_{L^2(-\kappa_m,\kappa_m)}. \tag{11.5}$$

Since

$$w = \partial_1\nabla v_m \quad\text{and}\quad \|w\|_{Y^{m-1}} \le c(\|w\|_{L^2(Q_m)} + \|\partial_1^{m-1} w\|_{L^2(Q_m)}),$$

it follows from estimate (4.20) in Lemma 4.3 that

$$\|w\|_{Y^{m-1}} \le c\big(1 + \|\partial_1^m \nabla v_m\|_{L^2(Q_m)}\big). \tag{11.6}$$

On the other hand, the equality $H = \partial_1 N$ implies the estimate

$$\|H\|_{Y^{m-1}} \le c\big(1 + \|\partial_1^m N\|_{L^2(-\kappa_m,\kappa_m)}\big). \tag{11.7}$$

Substituting (11.6) and (11.7) into (11.5) and noting that $0 < \lambda < 1$ we obtain desired inequality (4.22).

11.2. **The proof of estimate** (4.23) **for** $\mathcal{C}_{m-1}$. Recall that

$$\mathcal{C}_{m-1} = a\nabla\varphi_m \cdot N\nabla v_{m-1},$$

which yields

$$\begin{aligned}\|\partial_1^{m-1}\mathcal{C}_{m-1}\|_{L^2(Q_m)} &\leq c\sum_{p=0}^{m-1}\|\partial_1^p(N\nabla v_{m-1})\|_{L^2(Q_m)} \\ &\leq c + c\|\partial_1^{m-1}(N\nabla v_{m-1})\|_{L^2(Q_m)}.\end{aligned}\tag{11.8}$$

Notice that for $m=1$ estimate (4.23) is obviously true. Next, it follows from (11.8) that

$$\|\partial_1\mathcal{C}_1\|_{L^2(Q_1)} \leq c + c\|\partial_1(N\nabla v_1)\|_{L^2(Q_m)} \leq$$
$$c + c\|N\partial_1\nabla v_1\|_{L^2(Q_1)} + \|\partial_1 N\nabla v_1\|_{L^2(Q_1)}.$$

From this, estimate (4.20) in Lemma 4.3, and inequality (4.5) we conclude that

$$\|\partial_1\mathcal{C}_1\|_{L^2(Q_1)} \leq c + c\|\partial_1(N\nabla v_1)\|_{L^2(Q_m)} \leq$$
$$c + \|\partial_1 N\|_{L^2(-\kappa_m,\kappa_m)}\|\nabla v_1\|_{X^\infty} \leq c + c\|\nabla v_1\|_{X^\infty}.$$

Employing estimates (9.4) and (4.20) we finally obtain

$$\|\partial_1\mathcal{C}_1\|_{L^2(Q_1)} \leq c + c\|\nabla v_1\|_{Y^1} \leq c + \|\partial_1\nabla v_1\|_{L^2(Q_m)} \leq c.$$

Hence estimate (4.23) holds true for $m = 1, 2$. Let us consider the case $m \geq 3$. By virtue of (11.8), we have

$$\begin{aligned}&\|\partial_1^{m-1}\mathcal{C}_{m-1}\|_{L^2(Q_m)} \leq c + c\|\partial_1^{m-1}(N\nabla v_{m-1})\|_{L^2(Q_m)} \leq \\ &c + c\|\partial_1^{m-1}\nabla v_{m-1}\|_{L^2(Q_m)} + \sum_{i+j=m-1, j\geq 1}\|\partial_1^j N\,\partial_1^i\nabla v_{m-1}\|_{L^2(Q_m)}.\end{aligned}\tag{11.9}$$

Arguing as before we obtain

$$\|\partial_1^j N\,\partial_1^i\nabla v_{m-1}\|_{L^2(Q_m)} \leq c\|N\|_{W^{j,2}(-\kappa_m,\kappa_m)}\ \|\partial_1^i\nabla v_{m-1}\|_{X^\infty}.$$

Fix an arbitrary $\sigma \in (1/2, 1)$. Estimate (9.4) implies

$$\|\partial_1^i\nabla v_{m-1}\|_{X^\infty} \leq c\|\partial_1^i\nabla v_{m-1}\|_{Y^\sigma} \leq c\|\nabla v_{m-1}\|_{Y^{i+\sigma}}.$$

Thus we get

$$\|\partial_1^j N\,\partial_1^i\nabla v_{m-1}\|_{L^2(Q_m)} \leq c\|N\|_{W^{j,2}(-\kappa_m,\kappa_m)}\ \|\nabla v_{m-1}\|_{Y^{i+1}}.\tag{11.10}$$

It follows from (4.5) and the interpolation inequality that

$$\begin{aligned}\|N\|_{W^{j,2}(-\kappa_m,\kappa_m)} &\leq \|N\|_{W^{1,2}(-\kappa_m,\kappa_m)}^{\frac{m-1-j}{m-2}}\|N\|_{W^{m-1,2}(-\kappa_m,\kappa_m)}^{\frac{j-1}{m-2}} \\ &\leq c\|N\|_{W^{m-1,2}(-\kappa_m,\kappa_m)}^{\frac{j-1}{m-2}}.\end{aligned}\tag{11.11}$$

On the other hand, interpolation inequality (9.5) and estimate (4.20) imply

$$\begin{aligned}&\|\nabla v_{m-1}\|_{Y^{i+1}} \leq c\|\nabla v_{m-1}\|_{Y^1}^{1-\frac{i}{m-2}}\|\nabla v_{m-1}\|_{Y^{m-1}}^{\frac{i}{m-2}} \leq \\ &c\|\nabla v_{m-1}\|_{Y^{m-1}}^{\frac{i}{m-2}} \leq c(1 + \|\partial_1^{m-1}\nabla v_{m-1}\|_{L^2(Q_m)})^{\frac{i}{m-2}}.\end{aligned}\tag{11.12}$$

Substituting (11.11)-(11.12) into (11.10) and noting that

$$\begin{aligned}\|N\|_{W^{m-1,2}(-\kappa_m,\kappa_m)} &\leq c(\|N\|_{L^2(-\kappa_m,\kappa_m)} + \|\partial_1^{m-1}N\|_{L^2(-\kappa_m,\kappa_m)}) \\ &\leq c(1 + \|\partial_1^{m-1}N\|_{L^2(-\kappa_m,\kappa_m)})\end{aligned}$$

we arrive at the estimate

$$\|\partial_1^j N\,\partial_1^i \nabla v_{m-1}\|_{L^2(Q_m)} \le c(1+\|\partial_1^{m-1}N\|_{L^2(-\kappa_m,\kappa_m)})^{\frac{j-1}{m-2}}\,(1+\|\partial_1^{m-1}\nabla v_{m-1}\|_{L^2(Q_m)})^{\frac{i}{m-2}}.$$

Since

$$\frac{j-1}{m-2}+\frac{i}{m-2}=\frac{i+j-1}{m-2}=1,$$

we can apply the Young inequality to obtain

$$\|\partial_1^j N\,\partial_1^i \nabla v_{m-1}\|_{L^2(Q_m)} \le c\big(1+\|\partial_1^{m-1}N\|_{L^2(-\kappa_m,\kappa_m)}+\|\partial_1^{m-1}\nabla v_{m-1}\|_{L^2(Q_m)}\big).$$

Substituting this inequality into (11.9) we finally obtain desired estimate (4.23)

11.3. **The proof of estimate (4.24) for $\mathcal{B}_{m-1}$.** Since a and N are independent of y_1, it follows from the expression (4.15) for $\mathcal{B}_1$ that

$$\begin{aligned}\|\partial_1^m \mathcal{B}_{m-1}\|_{L^2(Q_m)} \le{}& c\|\partial_1^m(v_{m-1}\nabla\varphi_m)\|_{L^2(Q_m)}\\ &+c\sum_{i+j=m,\,i\le m-1}\|\partial_1^j N\,\partial_1^i(v_{m-1}\nabla\varphi_m\|_{L^2(Q_m)}.\end{aligned}\tag{11.13}$$

We have

$$\|\partial_1^m(v_{m-1}\nabla\varphi_m)\|_{L^2(Q_m)}\le c\sum_{0\le p\le m}\|\partial_1^p(v_{m-1}\nabla\varphi_m)\|_{L^2(Q_{m-1})}$$

which along with the Poincare inequality yields the estimate

$$\begin{aligned}\|\partial_1^m(v_{m-1}\nabla\varphi_m)\|_{L^2(Q_m)}&\le c(1+\|\partial_1^m v_{m-1}\|_{L^2(Q_{m-1})})\\ &\le c(1+\|\partial_1^{m-1}\nabla v_{m-1}\|_{L^2(Q_{m-1})}).\end{aligned}\tag{11.14}$$

For nonnegative integers i,j, satisfying relations $i+j=m$, $0\le i\le m-1$, we have

$$\|\partial_1^j N\,\partial_1^i(v_{m-1}\nabla\varphi_m)\|_{L^2(Q_m)}\le c\|\partial_1^j N\|_{L^2(-\kappa_m,\kappa_m)}\,\|\partial_1^i(v_{m-1}\nabla\varphi_m)\|_{X^\infty}.\tag{11.15}$$

Recall definitions (9.1) and (9.3) of the Banach spaces X^∞ and Y^s. Now choose an arbitrary $\sigma\in(1/2,1)$. By the embedding inequality (9.5), we have

$$\|\partial_1^i(v_{m-1}\nabla\varphi_m)\|_{X^\infty}\le c\|v_{m-1}\nabla\varphi_m\|_{Y^{i+\sigma}},$$

which along with (11.15) yields

$$\|\partial_1^j N\;\partial_1^i(v_{m-1}\nabla\varphi_m)\|_{L^2(Q_m)}\le c\|\partial_1^j N\|_{L^2(-\kappa_m,\kappa_m)}\;\|v_{m-1}\nabla\varphi_m\|_{Y^{i+\sigma}}.\tag{11.16}$$

Next notice that by virtue of estimates (4.5) and (4.20), we have

$$\|N\|_{W^{1,2}(-\kappa_m,\kappa_m)}+\|v_{m-1}\nabla\varphi_n\|_{Y^1}\le c.$$

From this and the interpolation inequality we obtain

$$\|\partial_1^j N\|_{L^2(-\kappa_m,\kappa_m)}\le c\|N\|^{\frac{j-1}{m-1}}_{W^{m,2}(-\kappa_m,\kappa_m)},\quad \|v_{m-1}\nabla\varphi_m\|_{Y^{i+\sigma}}\le c\|v_{m-1}\nabla\varphi_m\|^{\frac{i+\sigma-1}{m-1}}_{Y^m}.$$

Since

$$\frac{j-1}{m-1}+\frac{i+\sigma-1}{m-1}=1-\frac{1-\sigma}{m-1}\le 1,$$

it follows from (11.16) and the Young inequality that

$$\|\partial_1^j N\;\partial_1^i(v_{m-1}\nabla\varphi_m)\|_{L^2(Q_m)}\le c(1+\|N\|_{W^{m,2}(-\kappa_m,\kappa_m)}+\|v_{m-1}\nabla\varphi_m\|_{Y^m}).\tag{11.17}$$

Notice that

$$\|N\|_{W^{m,2}(-\kappa_m,\kappa_m)}\le c(1+\|\partial_1^m N\|_{L^2(-\kappa_m,\kappa_m)}).$$

Arguing as in the proof of (11.14) we obtain

$$\begin{aligned}\|v_{m-1}\nabla\varphi_m\|_{Y^m} &\le c(1+\|\partial_1^m(v_{m-1}\nabla\varphi_m)\|_{L^2(Q_m)}\\ &\le c(1+\|\partial_1^{m-1}\nabla v_{m-1}\|_{L^2(Q_{m-1})}).\end{aligned}$$

Substituting these inequalities into (11.17) we arrive at the estimate

$$\|\partial_1^j N\ \partial_1^i(v_{m-1}\nabla\varphi_m)\|_{L^2(Q_m)} \le c(1+\|\partial_1^m N\|_{L^2(-\kappa_m,\kappa_m)}+\|\partial_1^{m-1}\nabla v_{m-1}\|_{L^2(Q_{m-1})}).$$

Combining this estimate with estimates (11.14) and (11.13) we arrive at the desired estimate (4.24) for $\mathcal{B}_{m-1}$.

12. Proof of Lemma 6.3

Introduce the denotations

$$\Phi=\varphi\, n,\quad k=K\, n,\quad \varphi=\Phi\cdot n,\quad K=k\cdot n, \tag{12.1}$$

where K is the scalar curvature. We have

$$\partial_s^r\Phi-\nabla_s^r\Phi=\partial_s^r(\varphi\, n)-\partial_s^r\varphi\, n=\sum_{i=0}^{r-1}c_i\partial_s^i\varphi\,\partial_s^{r-i}n.$$

Note that

$$\partial_s n=\partial_s\tau^\perp=k^\perp,\quad k^\perp=(-k_2,k_1)=-K\tau.$$

It follows that

$$\partial_s^r\Phi-\nabla_s^r\Phi=\sum_{i=0}^{r-1}c_i\partial_s^i\varphi\,\partial_s^{r-1-i}k^\perp. \tag{12.2}$$

Let us estimate L^2-norm of every term in the right hand side. First we consider the case when

$$i\ge 1,\quad j=r-1-i\ge 1.$$

The Hölder inequality implies the estimate

$$\|\partial_s^i\varphi\partial_s^j k\|^2_{H^0_\sharp}=\int_\Gamma|\partial_s^i\varphi|^2|\partial_s^j k|^2\,ds\le\Big(\int_\Gamma|\partial_s^i\varphi|^{\frac{2(r-1)}{i}}\,ds\Big)^{\frac{i}{r-1}}\Big(\int_\Gamma|\partial_s^j k|^{\frac{2(r-1)}{j}}\,ds\Big)^{\frac{j}{r-1}},$$

which can be written in the equivalent form

$$\|\partial_s^i\varphi\partial_s^j k\|_{H^0_\sharp}\le\|\partial_s^i\varphi\|_{L^{\frac{2(r-1)}{i}}(0,\mathcal{L})}\|\partial_s^j k\|_{L^{\frac{2(r-1)}{j}}(0,\mathcal{L})}.$$

From this and the the Gagliardo-Nirenberg inequality (2.10) we obtain

$$\|\partial_s^i\varphi\partial_s^j k\|_{L^2(0,\mathcal{L})}\le c(\|\varphi\|^{1-\frac{i}{r-1}}_{L^\infty(0,\mathcal{L})}\|k\|^{1-\frac{j}{r-1}}_{L^\infty(0,\mathcal{L})})\ (\|\varphi\|^{\frac{i}{r-1}}_{H^{r-1}_\sharp}\|k\|^{\frac{j}{r-1}}_{H^{r-1}_\sharp}). \tag{12.3}$$

It follows from the embedding inequality (2.8) that for $\sigma\in(1/2,1)$, we have

$$\|\varphi\|_{L^\infty(0,\mathcal{L})}\le c\|\varphi\|_{H^\sigma_\sharp}.$$

Next, the interpolation inequality gives

$$\|\varphi\|_{H^\sigma_\sharp}\le\|\varphi\|^{1-\frac{\sigma}{r}}_{H^0_\sharp}\ \|\varphi\|^{\frac{\sigma}{r}}_{H^r_\sharp},\quad \|\varphi\|_{H^{r-1}_\sharp}\le\|\varphi\|^{\frac{1}{r}}_{H^0_\sharp}\ \|\varphi\|^{1-\frac{1}{r}}_{H^r_\sharp},$$

which yields

$$\|\varphi\|^{1-\frac{i}{r-1}}_{L^\infty(0,\mathcal{L})}\ \|\varphi\|^{\frac{i}{r-1}}_{H^{r-1}_\sharp}\le c\|\varphi\|^{\alpha_i}_{H^0_\sharp}\ \|\varphi\|^{\beta_i}_{H^r_\sharp},$$

where

$$\alpha_i=(1-\frac{\sigma}{r})(1-\frac{i}{r-1})+\frac{i}{r(r-1)},\quad \beta_i=\frac{\sigma}{r}(1-\frac{i}{r-1})\ +\frac{r-1}{r}\frac{i}{r-1}. \tag{12.4}$$

Repeating these arguments we obtain

$$\|k\|_{L^\infty(0,\mathcal{L})}^{1-\frac{j}{r-1}}\ \|k\|_{H^{r-1}_\sharp}^{\frac{j}{r-1}} \le c\|k\|_{H^0_\sharp}^{\alpha_j}\ \|k\|_{H^r_\sharp}^{\beta_j}.$$

The quantities α_j and β_j are given by the formulae (12.4) with i replaced by j. From this and (12.3) we conclude that

$$\|\partial_s^i\varphi\ \partial_s^j k\|_{H^0_\sharp} \le c\big(\|\varphi\|_{H^0_\sharp}^{\alpha_i}\|k\|_{H^0_\sharp}^{\alpha_j}\big)\ \big(\|\varphi\|_{H^r_\sharp}^{\beta_i}\|k\|_{H^r_\sharp}^{\beta_j}\big). \tag{12.5}$$

Since $i+j=r-1$ we have

$$\alpha_i+\alpha_j = 1+\frac{1-\sigma}{r},\quad \beta_i+\beta_j = 1-\frac{1-\sigma}{r}.$$

In other words, we have

$$\alpha_i=(1+\frac{1-\sigma}{r})\alpha_i^*,\ \ \alpha_j=(1+\frac{1-\sigma}{r})\alpha_j^*,\quad \beta_i=(1-\frac{1-\sigma}{r})\beta_i^*,\ \beta_j=(1-\frac{1-\sigma}{r})\beta_j^*,$$

where $\alpha_i^*+\alpha_j^*=1$ and $\beta_i^*+\beta_j^*=1$. From this and the Young inequality we obtain

$$\|\varphi\|_{H^0_\sharp}^{\alpha_i}\|k\|_{H^0_\sharp}^{\alpha_j} \le (\|\varphi\|_{H60_\sharp}+\|k\|_{H^0_\sharp})^{1+\frac{1-\sigma}{r}},$$

$$\|\varphi\|_{H^r_\sharp}^{\beta_i}\|k\|_{H^r_\sharp}^{\beta_j} \le (\|\varphi\|_{H^r_\sharp}+\|k\|_{H^r_\sharp})^{1-\frac{1-\sigma}{r}}.$$

Combining these estimates with (12.5) we conclude that the inequality

$$\|\partial_s^i\varphi\ \partial_s^j k\|_{H^0_\sharp} \le c(\|\varphi\|_{H^0_\sharp}+\|k\|_{H^0_\sharp})^{1+\frac{1-\sigma}{r}}\ (\|\varphi\|_{H^r_\sharp}+\|k\|_{H^r_\sharp})^{1-\frac{1-\sigma}{r}} \tag{12.6}$$

holds for every $1\le i\le r-2$ and $j=r-1-i$. It remains to consider the cases $i=0$ and $i=r-1$. We have for $\sigma\in(1/2,1)$,

$$\|\varphi\ \partial_s^{r-1}k\|_{H^0_\sharp}+\|\partial_s^{r-1}\varphi\ k\|_{H^0_\sharp} \le c\|\varphi\|_{H^\sigma_\sharp}\ \|k\|_{H^{r-1}\sharp}+c\|k\|_{H^\sigma_\sharp}\ \|\varphi\|_{H^{r-1}_\sharp}.$$

Next, the interpolation inequality implies

$$\|\varphi\|_{H^\sigma_\sharp} \le c\|\varphi\|_{H^0_\sharp}^{1-\frac{\sigma}{r}}\ \|\varphi\|_{H^r_\sharp}^{\frac{\sigma}{r}},\quad \|\varphi\|_{H^{r-1}_\sharp} \le c\|\varphi\|_{H^0_\sharp}^{\frac{1}{r}}\ \|\varphi\|_{H^r_\sharp}^{1-\frac{1}{r}}.$$

The similar estimates holds true for k. Thus we get

$$\begin{gathered}\|\varphi\ \partial_s^{r-1}k\|_{H^0_\sharp}+\|\partial_s^{r-1}\varphi\ k\|_{H^0_\sharp} \le\\ c(\|k\|_{H^0_\sharp}^{1-\frac{\sigma}{r}}\|\varphi\|_{H^0_\sharp}^{\frac{1}{r}})\ (\|k\|_{H^r}^{\frac{\sigma}{r}}\ \|\varphi\|_{H^r}^{1-\frac{1}{r}})+\\ c(\|\varphi\|_{H^0_\sharp}^{1-\frac{\sigma}{r}}\|k\|_{H^0_\sharp}^{\frac{1}{r}})\ (\|\varphi\|_{H^r}^{\frac{\sigma}{r}}\ \|k\|_{H^r}^{1-\frac{1}{r}}).\end{gathered}$$

It follows that

$$\begin{aligned}&\|\varphi\ \partial_s^{r-1}k\|_{H^0_\sharp}+\|\partial_s^{r-1}\varphi\ k\|_{H^0_\sharp} \le\\ &c\left(\|k\|_{H^0_\sharp}^{1-\frac{\sigma}{r}}\ \|\varphi\|_{H^0_\sharp}^{\frac{1}{r}}+\|\varphi\|_{H^0_\sharp}^{1-\frac{\sigma}{r}}\ \|k\|_{H^0_\sharp}^{\frac{1}{r}}\right)\ \left(\|k\|_{H^r_\sharp}^{1-\frac{\sigma}{r}}\ \|\varphi\|_{H^r_\sharp}^{1-\frac{1}{r}}+\|\varphi\|_{H^r_\sharp}^{1-\frac{\sigma}{r}}\ \|k\|_{H^r_\sharp}^{1-\frac{1}{r}}\right).\end{aligned}$$

Using the simple inequality

$$a^\lambda b^\mu+a^\mu b^\lambda \le 2a^{\lambda+\mu}+2b^{\lambda+\mu},\quad a,b,\lambda,\mu\ge 0,$$

we finally obtain the estimate

$$\begin{aligned}&\|\varphi\ \partial_s^{r-1}k\|_{H^0_\sharp}+\|\partial_s^{r-1}\varphi\ k\|_{H^0_\sharp} \le\\ &c\left(\|k\|_{H^0_\sharp}^{1+\frac{1-\sigma}{r}}+\|\varphi\|_{H^0_\sharp}^{1+\frac{1-\sigma}{r}}\right)\left(\|k\|_{H^r_\sharp}^{1-\frac{1-\sigma}{r}}+\|\varphi\|_{H^r_\sharp}^{1-\frac{1-\sigma}{r}}\right).\end{aligned}$$

It follows from this estimate and estimate (12.6) that the inequality

$$\|\partial_s^i\varphi\ \partial_s^j k\|_{H^0_\sharp} \le c\big(\|\varphi\|_{H^0_\sharp}+\|k\|_{H^0_\sharp}\big)^{1+\frac{1-\sigma}{r}}\ \big(\|\varphi\|_{H^r_\sharp}+\|k\|_{H^r_\sharp}\big)^{1-\frac{1-\sigma}{r}}$$

holds for every $0 \le i \le r-1$ and $j = r-1-i$. Combining this result with identity (12.2) we obtain desired estimate (6.13).

References

[1] A. Dall'acqua, P. Pozzi, *A Willmore-Helfrich L^2-flows of curves with natural boundary conditions*, Commun. Anal. Geom., **22**:4 (2014), 617–669. Zbl 1319.53070

[2] D. Adams, L. Hedberg, *Function spaces and potential theory*, Springer-Verlag, Berlin, 1995. Zbl 0834.46021

[3] L. Afraites, M. Dambrine, D. Kateb, *Shape methods for the transmission problem with a single measurment*, Numer. Funct. Anal. Optimization, **28**:5-6 (2007), 519–551. Zbl 1114.49020

[4] G. Allaire, E. Bonnetier, G. Francfort, F. Jouve. *Shape optimization by the homogenization method*. Numer. Math., **76**:1 (1997), 27–68. Zbl 0889.73051

[5] L. Ambrosio, G. Buttazzo, *An optimal design problem with perimeter penalization*, Calc. Var. Partial Differ. Equ., **1**:1 (1993), 55–69. Zbl 0794.49040

[6] K.-S. Chou, X.-P. Zhu, *The curve shortening problem*, Chapman & Hall/CRC Press, Boca Raton, 2001. Zbl 1061.53045

[7] M. Delfour, J. Zolésio, *Shapes and geometries. Analysis, differential calculus, and optimization*, SIAM, Philadelphia, 2001. Zbl 1002.49029

[8] G. Dziuk, E. Kuwert, R. Schatzle, *Evolution of elastic curves in $\mathbb{R}^n$: Existence and computation*, SIAM J. Math. Anal., **33**:5 (2002), 1228–1245. Zbl 1031.53092

[9] K. Eppler H. Harbrecht, *Shape optimization for 3D electrical impedance tomography*, In R. Glowinski (ed.) et al., *Free and moving boundaries. Analysis, simulation and control*, Lecture Notes in Pure and Applied Mathematics, **252**, Chapman & Hall/CRC, Boca Raton, 2007, 165–183. Zbl 1403.49042

[10] R. Kohn, M. Vogelius, *Determining conductivity by boundary measurements*. Comm. Pure Appl. Math., **37** (1984), 289–298. Zbl 0586.35089

[11] N. Koiso, *On the motion of a curve towards elastica*, in Besse, Arthur L. (ed.), *Actes de la table ronde de géométrie différentielle*, (Luminy, 1992), Sémin. Congr. **1**, Soc. Math. France, Paris, 1996, 403–436. Zbl 0884.58089

[12] C.-C. Lin, *L^2-flow of elastic curves with clamped boundary conditions*, J. Differ. Equ., **252**:12 (2012), 6414–6428. Zbl 1243.35089

[13] C. Mantegazza, M. Posetta, *The Lojasievicz-Simon inequality for the elastic flows*, arXiv: 2007.16093v3 [math AP] 18 Dec 2020.

[14] M. Müller, F. Rupp, *A Li-Yau inequality for the 1-dimensional Willmore energy*, arXiv:2011.08509, 21 Jan 2021.

[15] D. Mumford, J. Shah, *Optimal approximations by piecewise smooth functions and associated variaional problems*, Commun. Pure Appl. Math., **42**:5 (1989), 577–685. Zbl 0691.49036

[16] D. Mumford, *Elastica and computer vision*, In Bajaj, Chandrajit L. (ed.), *Algebraic geometry and its applications*, Springer-Verlag, New York, 1993. Zbl 0798.53003

[17] F. Murat. *Contre-exemples pour divers problèmes oú le contrôle intervient dans les coeffcients*, Ann. Mat. Pura Appl., IV Serie, **112** (1977), 49–68. Zbl 0349.49005

[18] L. Nirenberg, *On elliptic partial differential equations*, Ann. Sc. Norm. Super. Pisa, Sci. Fis. Mat., III. Ser., **13** (1959), 115–162. Zbl 0088.07601

[19] J. Roche, J. Sokołowski, *Numerical methods for shape identification problems*, Control Cybern., **25**:5 (1996), 867–894. Zbl 0876.65048

[20] F. Rupp, A. Spener, *Existence and convergence of the length-preserving elastic flows of clamped curves*, arXiv:2009.06991v [math AP] 15 Sep 2020.

[21] J. Sokołowski, J. Zolesio, *Introduction to shape optimization: shape sensitivity analysis*, Springer, Berlin etc., 1992. Zbl 0761.73003

[22] L. Tartar, *Problèmes de contrôle des coeffcients dans des équations aux dérivées partielles*, In A. Bensoussan and J.L. Lions, editors, *Control Theory, numer. Meth., Computer Syst. Mod.*, Lecture Notes Econ. math. Syst., **107**, Springer, Berlin Heidelberg, 1975. Zbl 0318.49010

[23] Y. Wen, *Curve straightening flow deforms closed plane curves with nonzero rotation number to circles*, J. Differ. Equ., **120**:1 (1995), 89–107. Zbl 0913.53003

[24] G. Wheeler, *Global analysis of the generalised Helfrich flow of closed curves immersed in* $\mathbb{R}^n$, Preprint 2012, arXiv:1205.5939

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