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СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ
Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru
ISSN 1813-3304

Vol. 20, No. 2, pp. 1037–1051 (2023)
https://doi.org/10.33048/semi.2023.20.064

УДК 510.67
MSC 03C15,03C64

CONSTANT EXPANSION OF THEORIES AND THE NUMBER OF COUNTABLE MODELS

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Communicated by S.V. Sudoplatov

Abstract: The present paper is dedicated to the method of constant expansion of a complete theory for studying its number of countable models. This paper aims to rehabilitate the method of constant expansion by demonstrating its continued relevance and its potential for use in counting the number of countable models. The main result reveals that the question of reducing the number of countable models from the continuum to a countable number by a constant expansion of a theory remains unanswered, contrary to previous beliefs.

Keywords: small theory, the number of countable models, expansion by constants, non-orthogonality of types, ordered structures.

1 Introduction and preliminaries

Let T be a countable complete theory of a language $\mathcal{L}$. Denote by $S_n(T)$ the set of all complete n-types of T over an empty set. Let $p \in S_n(T)$ be non-principal, and let $\bar{c} = (c_1, c_2, ..., c_n) \notin \mathcal{L}$ be a tuple of new constants. The theory $T^* := T \cup p(\bar{c})$ is called a constant expansion of T. The theory T^* is an $\mathcal{L}^*$-theory, where $\mathcal{L}^* := \mathcal{L} \cup \{c_1, c_2, ..., c_n\}$.

Baizhanov, B., Umbetbayev, O., Constant expansion of theories and the number of countable models.

This research is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP14971869).

Received July, 17, 2023, Published November, 12, 2023.

R.E. Woodrow [1] provided an example of a theory T such that $I(T,\aleph_0) = 4$ and $I(T^*,\aleph_0) = \aleph_0$ for some constant expansion T^* of T. M.G. Peretyatkin in his article [2] constructed an example of a theory T such that $I(T,\aleph_0) = 3$ and $I(T^*,\aleph_0) = \aleph_0$. He formulated the following question: "Does there exist a countable complete theory with a finite number of countable models whose a constant expansion has a fewer number of countable models?" He also asked if it is possible to reduce the number of countable models from $\aleph_0$ to a finite number n. In [3] B. Omarov answered Peretyatkin's question and proved that for every $n < \omega$ there exists a theory T_n such that $I(T_n,\aleph_0) = n+5$ and $I(T_n^*,\aleph_0) = 5$. Moreover, he showed that the case $I(T,\aleph_0) = \aleph_0$, $I(T^*,\aleph_0) < \aleph_0$ is also possible [4].

A.D. Taimanov asked if it is possible that $I(T,\aleph_0) = 2^{\aleph_0}$ and $I(T^*,\aleph_0)$ is finite or countable. B. Omarov tried to construct an example with $I(T,\aleph_0) = 2^{\aleph_0}$ and $I(T^*,\aleph_0) < \aleph_0$. Omarov's construction in [3] does not give an answer (Theorem 2) and in general, Taimanov's question is still open.

The transition from T to T^* preserves different properties of a theory. If T is either ω-stable, $\aleph_1$-categorical, o-minimal, weakly o-minimal, o-stable, (N)IP or (N)SOP, then so is T^*. Note that the definition and properties of o-stable theories one can find in [5, 6].

Let T be a countable complete theory in a countable language $\mathcal{L}$. The well-known Vaught's conjecture postulates if $I(T,\aleph_0) > \aleph_0$ implies $I(T,\aleph_0) = 2^{\aleph_0}$. We are investigating whether this number of countable non-isomorphic models is preserved when a new constant is added to the language [7]. Can the constant expansion method be used when working on Vaught's conjecture? Since non-small theories have the maximal number of countable models, we restrict to small theories.

We denote models of elementary theories by Gothic letters $\mathfrak{M}$, $\mathfrak{N}$, ... and we use corresponding Latin letters M, N, ... to denote their universes.

For a subset $A \subset N$ (which is not necessary definable) we denote:

$$A^+ := \{\gamma \in N \mid \forall a \in A : N \models a < \gamma\};$$
$$A^- := \{\gamma \in N \mid \forall a \in A : N \models \gamma < a\}.$$

We study linearly ordered theories and suppose that $<$ is an $\emptyset$-definable linear order relation. For subsets C and D of a linearly ordered set M we use the notation $C < D$ if $c < d$ for every $c \in C$ and $d \in D$.

Definition 1. *We say a formula $\phi(x)$ splits a linearly ordered set B (not necessarily definable) if $\phi(M) \cap B < \neg\phi(M) \cap B$.*

2 Constant expansion

Let $p(\bar{x}) \rightleftharpoons p$ and $q(\bar{y}) \rightleftharpoons q$ be two complete types over a finite subset A of some model of T. We say that $p(\bar{x})$ is *not almost orthogonal* to $q(\bar{y})$, $p(\bar{x}) \not\perp^a q(\bar{y})$, if there is a $\mathcal{L}_A$-formula $\varphi(\bar{x},\bar{y},\bar{a})$, where $\bar{a} \in A$, such that for a model $\mathfrak{M} \models T$ realizing $p(\bar{x})$, for some (equivalently, for any) realization $\bar{\alpha} \in p(\mathfrak{M})$, $\emptyset \neq \varphi(\bar{\alpha}, M, \bar{a}) \subsetneq q(\mathfrak{M})$. A formula $\varphi(\bar{x},\bar{y})$ with parameters in

A is said to be (p,q)-*preserving*, a $(p \to q)$-*formula*, or a $(q \leftarrow p)$-*formula* if $\varphi(\bar{a},\bar{y}) \vdash q(\bar{y})$ holds for any realization $\bar{a}$ of p [8].

Note that in general the relation of not almost orthogonality is transitive, but not symmetric. An equivalence relation can be defined by setting $p \sim_{\not\perp^a} q$ if and only if $p \not\perp^a q$ and $q \not\perp^a p$.

Definition 2. [9] *Two types $p(\bar{x})$ and $q(\bar{y})$ from $S(A)$ are called weakly orthogonal, if $p(\bar{x}) \cup q(\bar{y})$ has a unique extension to a complete $(l(\bar{x})+l(\bar{y}))$-type over A.*

The relation of weak orthogonality of two types $p(\bar{x})$ and $q(\bar{y})$ is denoted by $p(\bar{x}) \perp^{\omega} q(\bar{y})$.

It follows from the definition that $p(\bar{x})$ is not weakly orthogonal to $q(\bar{y})$ ($p(\bar{x}) \not\perp^{\omega} q(\bar{y})$) if there are an $\mathcal{L}_A$-formula $H(\bar{x},\bar{y})$, $\bar{a} \in p(\mathfrak{M})$, and $\bar{b}_1, \bar{b}_2 \in q(\mathfrak{M})$ such that $\bar{b}_1 \in H(M,\bar{a})$ and $\bar{b}_2 \notin H(M,\bar{a})$.

We describe Omarov's constructions from [3] of theories T_0, T_1, and T_2 such that $T_0 \subset T_1 \subset T_2$ and show that T_2 constructed in his work ([3], Theorem 3) does not have continuum of countable models, but has the countable number of countable models and a constant expansion T_2^* with a finite number of countable models (Theorem 2). That is, $I(T_2,\aleph_0) = \aleph_0$ and $I(T_2^*,\aleph_0) < \omega$.

The language of T_0 is $\mathcal{L}_0 = \{<, f^1, H^1\}$, where f is a unary functional symbol and H is a unary predicate symbol.

The axioms of T_0 are:

1. $<$ is a dense linear ordering without endpoints.
2. $\forall x(f(f(x)) = x)$.
3. $\forall x(H(x) \Leftrightarrow (f(x) = x))$.
4. $\neg(\exists x \exists y)(x < y < f(x) < f(y))$.
5. $\forall x \forall y \exists z(x < y \to (x < z < y \wedge f(z) = z))$.
6. $\forall x \forall y \exists z(x < y \to z < x < y < f(z))$.
7. $\forall x \forall y \forall z \exists t(x \leq f(x) < y \leq f(y) < z \leq f(z) \to (t < x < f(y) < f(t) < z \vee f(x) < t < y < f(z) < f(t)))$.
8. $\forall x \forall y \exists z \exists t(x \leq f(x) < y \leq f(y) \to z < x \leq f(x) < f(z) < t < y \leq f(y) < f(t))$.
9. $\forall x \forall y \forall z \exists t(x < y \leq f(y) < z < f(z) < f(x) \to x < t < y < f(z) < f(t) < f(x))$.

In [3] Omarov proved that T_0 is an $\aleph_0$-categorical theory and therefore T_0 is complete.

We extend $\mathcal{L}_0$ with a countable number of constants: $\mathcal{L}_1 = \mathcal{L}_0 \cup \{a_i\}_{i<\omega}$. The axioms of T_1 are:

1. The axioms of T_0.
2. For every $i < \omega$, $a_i < a_{i+1} < f(a_{i+1}) < f(a_i)$.

By analogy with Ehrenfeucht's example, Omarov considered the following partial non-principal 1-type of T_1:

$$p(x) = \{a_i < x < f(a_i) \mid i < \omega\}.$$

We introduce the following notation:

$$A_l(x) := f(x) > x,\ A_r(x) := f(x) < x,\ A(x) := A_l(x) \vee A_r(x).$$

The type $p(x)$ can be extended to the following complete types:

$$p^1(x) = p(x) \cup \{H(x)\};$$
$$p^2(x) = p(x) \cup \{A_l(x)\};$$
$$p^3(x) = p(x) \cup \{A_r(x)\};$$
$$p(\mathfrak{M}) = p^1(\mathfrak{M}) \cup p^2(\mathfrak{M}) \cup p^3(\mathfrak{M}).$$

According to [3], T_1 has five countable non-isomorphic models obtained by considering all possible five cases for $p(\mathfrak{M})$.

Lemma 1. [3] *The theory T_1 has five countable non-isomorphic models which are defined by the set of realizations of $p(x)$:*

I. $p(\mathfrak{M}) = \emptyset$ then $\mathfrak{M}$ is a prime model of T_1.

II. $p(\mathfrak{M}) = p^1(\mathfrak{M}) = \{a\}$ and $\mathfrak{M} \models H(a)$.

III. $p(\mathfrak{M}) = [a, f(a)]$, $a \in p^2(\mathfrak{M})$.

IV. $p(\mathfrak{M}) = C \cup D$ such that $C < D$, and for every $c \in C$, $f(c) \in C$ and for every $d \in D$, $f(d) \in D$, thus it is a non-homogeneous model of T_1.

V. $p(\mathfrak{M}) \models T_0$. In this case the model $\mathfrak{M}$ is an $\aleph_0$-saturated model of T_1.

By Axiom 4 and Axiom 8 for case IV we have $\mathfrak{M} = \mathfrak{M}(c, d)$ for $c \in C$ and $d \in D$ (Claim 4). Note that in case V, $\mathfrak{M}$ is *limit over the type $p^2 \in S(T)$* according to Sudoplatov's classification [10, 11].

We provide an explanation when $|p(\mathfrak{M})| = \aleph_0$ as to why the case $p(\mathfrak{M}) = [\alpha, \beta]$ is impossible for some α and β satisfying $\models H(\alpha) \wedge H(\beta)$.

Proposition 1. *Let $\mathfrak{M}$ be a countable model of T_1, $\alpha, \beta \in p^1(\mathfrak{M})$ and $l(x, \alpha)$, $r(z, \beta)$ be the sets of formulas:*

$$l(x, \alpha) = \{H(x)\} \cup \{a_i < x < \alpha \mid i < \omega\} \cup \{\phi(x, \alpha)\},$$

and

$$r(z, \beta) = \{H(z)\} \cup \{\beta < z < f(a_i) \mid i < \omega\} \cup \{\psi(z, \beta)\},$$

where $\phi(x, \alpha) = \exists y(y < x < f(y) < \alpha)$ and $\psi(z, \beta) = \exists y(\beta < y < z < f(y))$. Then $l(x, \alpha)$ is consistent, and for every realization $l(x, \alpha)$ in a countable saturated model $\mathfrak{N}$ of T_1 it is ensured that the smallest element of $p(\mathfrak{N})$ cannot be an element satisfying $H(x)$. The same is true for the largest element of $p(\mathfrak{N})$ for $r(z, \beta)$.

Proof. Let $\mathfrak{M} \prec \mathfrak{N}$ and $\alpha_1 \in N \setminus M$ and such that $\mathfrak{N} \models l(\alpha_1, \alpha)$ and, consequently, $\mathfrak{N} \models \phi(\alpha_1, \alpha)$. Suppose there exists $\gamma \notin p(\mathfrak{N})$ such that $a_i < \gamma < a_{i+1}$. Then $f(\gamma) < a_{i+1}$ which contradicts Axiom 4. Therefore $\gamma \in p(\mathfrak{N})$ and the following is true for γ:

$$\mathfrak{N} \models \gamma < \alpha_1 < f(\gamma) < c.$$

□

Let $\mathcal{L}_2 = \mathcal{L}_1 \cup \{h_i^1, b_i, c_i\}_{i<\omega}$, where b_i, c_i are new constants and h_i is a functional symbol for each $i < \omega$.

The axioms of T_2 are:

1. The axioms of T_1.
2. $a_i < c_i < f(c_i) < a_{i+1}, i < \omega$.
3. $c_0 < b_i < b_{i+1} < f(b_{i+1}) < f(b_i) < f(c_0), i < \omega$.

The structure with b_i that lies between c_0 and $f(c_0)$ is the model of T_1.

4. h_i is a bijective mapping preserving $<$ and f between two different sets definable by the following formulas:

$$A(x) \wedge (c_i < x < f(c_i)) \text{ and } A(x) \wedge (c_{i+1} < x < f(c_{i+1})), i < \omega.$$

That is $h_i : A(M) \cap (c_i, f(c_i)) \to A(M) \cap (c_{i+1}, f(c_{i+1}))$.

5. $\forall x(c_i < x < c_{i+1} \to (H(x) \leftrightarrow h_i(x) = x)), i < \omega$.

Notice that for $\alpha \in H(M)$ such that $c_i < \alpha < f(c_i)$ the formula

$$K(x, \alpha) := H(\alpha) \wedge \exists y(A(y) \wedge c_i < y < \alpha \wedge h_i(y) = x)$$

defines an irrational cut in $(c_{i+1}, f(c_{i+1}))$ and provides that the following formula is satisfiable:

$$\neg\forall x[(H(x) \wedge c_i < x < f(c_i)) \to \exists y(H(y) \wedge \forall z(A_l(z) \wedge (c_i < z < x < f(z) \to h_i(z) < y < h_i(f(z)))))].$$

The proof of the completness of T_2 follows from $\aleph_0$ categoricity of T_0. Fix a finite language $\mathcal{L}$ with no functional symbols, and let $\mathfrak{M}_1 = \langle M_1, \Sigma\rangle$ and $\mathfrak{M}_2 = \langle M_2, \Sigma\rangle$ be $\mathcal{L}$-structures.

We use Ehrenfeucht-Fraisse game $G_n(\mathfrak{M}_1, \mathfrak{M}_2)$ for $n = 1, 2, \ldots$. The game will have n rounds. On the ith round player I plays first and either plays $a_i \in M_1$ or $b_i \in M_2$. On player II's turn, if player I played $a_i \in M_1$, then player II must play $b_i \in M_2$, and if player I plays $b_i \in M_2$, then player II must play $a_i \in M_1$. The game stops after the nth round. Player II wins if $\{(a_i, b_i) : i = 1, \ldots, n\}$ is the graph of a partial embedding from $\mathfrak{M}_1$ into $\mathfrak{M}_2$.

Theorem 1. [12] *Let $\mathcal{L}$ be a finite language without function symbols and let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be $\mathcal{L}$-structures. Then $\mathfrak{M}_1 \equiv \mathfrak{M}_2$ if and only if the second player has a winning strategy in $G_n(\mathfrak{M}_1, \mathfrak{M}_2)$ for all n.*

Theorem 1 implies the following Corollary 1.

Corollary 1. *Let $\mathcal{L}$ be an infinite language without function symbols and let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be $\mathcal{L}$-structures. Then $\mathfrak{M}_1 \equiv \mathfrak{M}_2$ if and only if for every finite $\mathcal{L}_0 \subset \mathcal{L}$*

$$\langle M_1, \mathcal{L}_0\rangle \equiv \langle M_2, \mathcal{L}_0\rangle.$$

Let $\Sigma = \langle =, <, P^2_f, A^1_l, A^1_r, H^1, P^2_{h_0}, \ldots P^2_{h_i}, a_i, c_j, b_k, \rangle_{i,j,k<\omega}$. And fix $\Sigma_1 \subset \Sigma_0 \subset \Sigma$ by the following:

$$\Sigma_0 = \langle =, <, P^2_f, A^1_l, A^1_r, H^1, P^2_{h_0}, \ldots P^2_{h_{i-1}}, a_0, \ldots a_m, c_0, \ldots c_m, b_0, \ldots, b_k\rangle,$$
$$\Sigma_1 = \langle =, <, P^2_f, A^1_l, A^1_r, H^1\rangle,$$

where $m, k < \omega$.

$\mathfrak{M} = \langle M, \Sigma\rangle$ and $\mathfrak{N} = \langle N, \Sigma\rangle$ be two countable models of T_2.

We describe the winning strategy of player II for a Σ_0.

Before starting the game, the player II establishes an isomorphism g between structures defined on intervals:

$$\langle(-\infty, a_0); \Sigma_1\rangle \cong \langle(-\infty, a'_0); \Sigma_1\rangle;$$
$$\langle(a_0, c_0); \Sigma_1\rangle \cong \langle(a'_0, c'_0); \Sigma_1\rangle;$$
$$\langle(f(c_0), a_1); \Sigma_1\rangle \cong \langle(f(c'_0), a'_1); \Sigma_1\rangle;$$
$$\langle(a_1, c_1); \Sigma_1\rangle \cong \langle(a'_1, c'_1); \Sigma_1\rangle;$$
$$\langle(f(c_1), a_2); \Sigma_1\rangle \cong \langle(f(c'_1), a'_2); \Sigma_1\rangle;$$
$$\dots$$
$$\langle(a_m, c_m); \Sigma_1\rangle \cong \langle(a'_m, c'_m); \Sigma_1\rangle;$$
$$\langle(f(c_m), a_{m+1}); \Sigma_1\rangle \cong \langle(f(c'_m), a'_{m+1}); \Sigma_1\rangle.$$

For $0 \le i \le k$

$$\langle(f(a_{i+1}), f(a_i)); \Sigma_1\rangle \cong \langle(f(a'_{i+1}), f(a'_i)); \Sigma_1\rangle;$$
$$\langle(a_{m+1}, f(a_{m+1})); \Sigma_1\rangle \cong \langle(a'_{m+1}, f(a'_{m+1})); \Sigma_1\rangle;$$
$$\langle(f(a_0), \infty); \Sigma_1\rangle \cong \langle(f(a'_0), \infty); \Sigma_1\rangle.$$

The isomorphism of these structures follows from the fact that they are models of T_0.

Note that it is possible to establish an order relation $<_r$ on all elements of $\bigcup\limits_{i<m+1} (c_i, f(c_i))$. Below we define this relation.

Let $\alpha \in (c_i, f(c_i))$ and $\beta \in (c_j, f(c_j))$, then if $i = j$, $\alpha < \beta$ then $\alpha <_r \beta$.

Let $\models H(\alpha)$, then

$$\alpha <_r \beta \Leftrightarrow \exists d \in A'_r(M) \cap (c_i, f(c_i)) \text{ such that } \alpha < d \text{ and}$$

if $j < i$ then

$$(h_j^{-1}(...(h_{i-2}^{-1}(h_{i-1}^{-1}(d)))...)) < \beta,$$

or if $i < j$ then

$$(h_{j-1}(...(h_{i+1}(h_i(d)))...)) < \beta.$$

$j < i$, $\models A(\alpha)$
if $(h_j^{-1}(...(h_{i-2}^{-1}(h_{i-1}^{-1}(\alpha)))...)) = \beta$, then $\alpha =_r \beta$,
if $(h_j^{-1}(...(h_{i-2}^{-1}(h_{i-1}^{-1}(\alpha)))...)) < \beta$, then $\alpha <_r \beta$,
if $(h_j^{-1}(...(h_{i-2}^{-1}(h_{i-1}^{-1}(\alpha)))...)) > \beta$, then $\alpha >_r \beta$,
$i < j$, $\models A(\alpha)$
if $(h_{j-1}(...(h_{i+1}(h_i(\alpha)))...)) = \beta$, then $\alpha =_r \beta$,
if $(h_{j-1}(...(h_{i+1}(h_i(\alpha)))...)) < \beta$, then $\alpha <_r \beta$,
if $(h_{j-1}(...(h_{i+1}(h_i(\alpha)))...)) > \beta$, then $\alpha >_r \beta$.

And similarly it is possible to establish an order relation $<_r$ on all elements of $\bigcup\limits_{i<m+1} (c'_i, f(c'_i))$.

At step 0 we have defined the definable closure of constants on $M_0 \subset M$ and $N_0 \subset N$ such that

$$M_0 = dcl_{\Sigma_1}(M'_0) \text{ and } N_0 = dcl_{\Sigma_1}(N'_0),$$

and

$$\mathfrak{M}_0 = \langle M_0, \Sigma_0\rangle \cong \mathfrak{N}_0 = \langle N_0, \Sigma_0\rangle,$$

where

$$M'_0 = \{a_0, \dots, a_m, c_0, \dots, c_m, b_0, \dots, b_k, b^i_j = (h_{i-1}(...(h_0(b_j))...)), f(b^i_j) \mid j < k, 0 \le i \le m\},$$
$$N'_0 = \{a'_0, \dots, a'_m, c'_0, \dots, c'_m, b'_0, \dots, b'_k, b'^i_j = (h_{i-1}(...(h_0(b_j))...)), f(b'^i_j) \mid j < k, 0 \le i \le m\}.$$

At step $k+1$ we have an isomorphism of the chosen finite structures $M_0 \subset M_1 \subset \cdots \subset M_k$ and $N_0 \subset N_1 \subset \cdots \subset N_k$ such that $\mathfrak{M}_i = \langle M_i, \Sigma_0\rangle \cong \mathfrak{N}_i = \langle N_i, \Sigma_0\rangle$.

Let player I takes $d \in M \setminus M_k$, then two cases are possible:

Case 1.

$$d \in (-\infty, a_0) \cup (a_0, c_0) \cup (f(c_0), a_1) \cup \cdots \cup (f(c_m), a_{m+1}) \cup (f(a_{m+1}), f(a_m)) \cup \cdots \cup (f(a_1), f(a_0)) \cup (f(a_0), \infty),$$

Case 2. there is some i, $0 \le i \le m$, $d \in (c_i, f(c_i))$. This case has multiple subcases.

Case 1

The player II take $d' \in N$ such that $d' = g(d)$ and

$$\langle M_k \cup \{d\}; \Sigma_0\rangle \cong \langle N_k \cup \{d'\}; \Sigma_0\rangle.$$

Let $M_{k+1} := dcl_{\Sigma_1}(M_k \cup \{d\})$ and $N_{k+1} := dcl_{\Sigma_1}(N_k \cup \{d'\})$.

Case 2

Let $e_1, e_2 \in M_k$ be such that $e_1 <_r d <_r e_2$ and for every $e \in M_k$, $e \le_r e_1$ or $e_2 \le_r e$.

Case 2.1

Let $\mathfrak{M} \models H(d)$. There are two possible cases: there is $\gamma \in (c_i, f(c_i))$ such that

2.1.1 Suppose that $\mathfrak{M} \models e_1 <_r \gamma < d <_r e_2 <_r f(\gamma)$;

2.1.2 Suppose that $\mathfrak{M} \models f(\gamma) <_r e_1 <_r d < \gamma <_r e_2$.

Then player II take $e'_1 <_r d' <_r e'_2$ such that there is $\gamma' \in (c'_i, f(c'_i))$ such that

$\mathfrak{N} \models e'_1 <_r \gamma' < d' <_r e'_2 <_r f(\gamma')$ if and only if $\mathfrak{M} \models e_2 <_r f(\gamma)$ and

$\mathfrak{N} \models f(\gamma') <_r e'_1 <_r d' < \gamma' <_r e'_2$ if and only if $\mathfrak{M} \models f(\gamma) <_r e_1$.

Let $M_{k+1} := M_k \cup \{d\}$.

Case 2.2

Let $\mathfrak{M} \models A_l(d)$. There are two possible cases:

2.2.1 Suppose that $\mathfrak{M} \models e_1 <_r d < f(d) <_r e_2$

There are two subcases of 2.2.1, there is $\gamma \in (c_i, f(c_i))$ such that

2.2.1.1 Suppose that $\mathfrak{M} \models e_1 <_r \gamma < d < f(d) <_r e_2 <_r f(\gamma)$;

2.2.1.2 Suppose that $\mathfrak{M} \models f(\gamma) < e_1 <_r d < f(d) < \gamma <_r e_2$.

Then player II takes $d' \in N$, $e'_1 <_r d' < f(d') <_r e'_2$ such that there is $\gamma' \in (c'_i, f(c'_i))$:

$\mathfrak{N} \models e'_1 <_r \gamma' < d' < f(d') <_r e'_2 <_r f(\gamma)$ if and only if $\mathfrak{M} \models e_2 <_r f(\gamma)$ and

$\mathfrak{N} \models f(\gamma') < e'_1 <_r d' < f(d') < \gamma' <_r e'_2$ if and only if $\mathfrak{M} \models f(\gamma) <_r e_1$.

2.2.2 Suppose that $\mathfrak{M} \models e_1 <_r d <_r e_2 <_r f(d)$

Then player II takes $d' \in (c'_i, f(c'_i))$ such that $\mathfrak{N} \models e'_1 <_r d' <_r e'_2 <_r f(d')$ with the following condition:

for any $e \in M_k$ and corresponding $e' \in N_k$ the following holds:
$\mathfrak{N} \models f(d') <_r e'$ if and only if $\mathfrak{M} \models f(d) <_r e$ and
$\mathfrak{N} \models e' <_r f(d')$ if and only if $\mathfrak{M} \models e <_r f(d)$.
The last provides the isomorphism

$$\langle M_k \cup \{d\}; \Sigma_0\rangle \cong \langle N_k \cup \{d'\}; \Sigma_0\rangle.$$

Case 2.3 Let $\mathfrak{M} \models A_r(d)$. This case is analogical to the Case 2.2.
In Case 2 we define structures

$$M_{k+1} = dcl_{\Sigma_0}(M_k \cup \{d\}) \text{ and } N_{k+1} = dcl_{\Sigma_0}(N_k \cup \{d'\}).$$

In accordance with this, in the general case we get

$$\mathfrak{M}_{k+1} = \langle M_{k+1}, \Sigma_0\rangle \cong \mathfrak{N}_{k+1} = \langle N_{k+1}, \Sigma_0\rangle.$$

Let us introduce the following notation:

$$b_i^j := h_j(h_{j-1}(\dots h_0(b_i)\dots)).$$

We define the partial non-principal 1-types of the theory T_1:

$$p(x) = \{a_i < x < f(a_i) \mid i < \omega\};$$
$$q(x) = \{b_i < x < f(b_i) \mid i < \omega\};$$
$$q_j(x) = \{b_i^j < x < f(b_i^j) \mid i, j < \omega\}.$$

3 Number of countable models

Next, we analyze Omarov's example in terms of (non-)orthogonality (as well as weak and almost orthogonality). Consider the properties of the following complete 1-types that were not considered in [3] in terms of weak orthogonality.

For $j < \omega$ consider extensions of $q_j(x)$ to complete types. Every element from $q(\mathfrak{M})$ must satisfy exactly one of the following 1-formulas: $H(x)$, $A_l(x)$, $A_r(x)$. Therefore $q_j(x)$ has exactly three completions:

$$q_j^1(x) = q_j(x) \cup \{H(x)\};$$
$$q_j^2(x) = q_j(x) \cup \{A_l(x)\};$$
$$q_j^3(x) = q_j(x) \cup \{A_r(x)\};$$
$$q_j(\mathfrak{M}) = q_j^1(\mathfrak{M}) \cup q_j^2(\mathfrak{M}) \cup q_j^3(\mathfrak{M}).$$

Since h_j is a mapping from the set of realizations of q_j^2 to the set of realizations of q_{j+1}^2, that is $h_j(q_j^2(\mathfrak{M})) = q_{j+1}^2(\mathfrak{M})$, and for any $\alpha \in q_j^2(\mathfrak{M})$, $h_j(\alpha) \in q_{j+1}^2(\mathfrak{M})$, this means $q_j^2 \not\perp^a q_{j+1}^2$. Since h_j is a bijection, we have $q_{j+1}^2 \not\perp^a q_j^2$. Thus for any $n, j < \omega$, $q_n^2 \sim_{\not\perp^a} q_j^2$.

The following formula is q_j^2-preserving:

$$\varphi(x, \alpha) = A_l(x) \wedge (\alpha < x < f(\alpha)).$$

Denote by $F(x, \alpha) := H(x) \wedge \exists y_1 \exists y_2((y_1 < x < y_2) \wedge \varphi(y_1, \alpha) \wedge \varphi(y_2, \alpha))$. Then we have $F(M, \alpha) \subset q_j^1(M)$. Thus,

$$q_j^2(x) \not\perp^a q_j^1(x). \quad (1)$$

Note that the converse is not true.

We show that $q_j^1(x) \not\perp^w q_i^1(x)$, $i \neq j$. Let $\alpha \in q_j^1(\mathfrak{M})$. First, we show that the following holds: $q_j^1(x) \not\perp^w q_{j+k}^1(x)$. For $k \geq 1$ denote

$$S^{j+k}(x,\alpha) := \exists y_1 \exists y_2 \big[(h_{j-1}(h_{j-2}(...(h_1(h_0(b_n)))...))) < y_1 < y_2 < \alpha < \\ < (h_{j-1}(h_{j-2}(...(h_1(h_0(f(b_n))))...))) \wedge A_l(y_1) \wedge A_l(y_2) \wedge \\ \wedge(h_{j+k-1}(...(h_{j+1}(h_j(y_1)))...) < x) \wedge (x < h_{j+k-1}(...(h_{j+1}(h_j(y_2)))...))\big],$$

and for $i < j$ denote

$$S^i(x,\alpha) := \exists y_1 \exists y_2 [(h_{j-1}(h_{j-2}(...h_0(b_n)...))) < y_1 < y_2 < \alpha < \\ < (h_{j-1}(h_{j-2}(...(h_0(f(b_n)))...))) \wedge A_l(y_1) \wedge A_l(y_2) \wedge \\ \wedge(h_i^{-1}(...(h_{j-2}^{-1}(h_{j-1}^{-1}(\bar{y}_1)))...) < x) \wedge (x < h_i^{-1}(...(h_{j-2}^{-1}(h_{j-1}^{-1}(y_2)))...))].$$

And, similarly to the remark to Axiom 5, $S^{j+k}(x,\alpha)$ splits q_{j+k}^s, for every $s \in \{1,2,3\}$. In particular $S^{j+k}(x,\alpha)$ splits $q_{j+k}(M)$, if

$$S^{j+k}(M,\alpha) \cup (\neg S^{j+k}(M,\alpha) \cap q_{j+k}(M)) = q_{j+k}(M)$$

and,

$$S^{j+k}(M,\alpha) < (\neg S^{j+k}(M,\alpha) \cap q_{j+k}(M)).$$

The same is true for the formula $S^i(M,\alpha)$.

Lemma 2. *For every $s \in \{1,2,3\}$, for every $j \neq l < \omega$, $q_j^1 \not\perp^w q_l^s$, $q_j^1 \perp^a q_l^s$, and, because h_j is a bijection preserving the ordering on $A_l(x) \vee A_r(x)$ and by (1) we have $q_j^2 \not\perp^a q_l^s$ and $q_j^3 \not\perp^a q_l^s$.*

Proof. Let $\alpha \in q_j^1(\mathfrak{M})$, then for every $j,k < \omega, k \neq 0$, $S^{j+k}(x,\alpha)$ splits q_{j+k}^1, q_{j+k}^2, and q_{j+k}^3, also for every $i < j$, $S^i(x,\alpha)$ splits q_i^1, q_i^2, and q_i^3. Thus, $S^{j+k}(x,\alpha) \wedge H(x)$ splits q_{j+k}^1, $S^{j+k}(x,\alpha) \wedge A_l(x)$ and $S^{j+k}(x,\alpha) \wedge A_r(x)$ splits respectively q_{j+k}^2 and q_{j+k}^3. The same holds for $S^i(x,\alpha)$.

From II of Lemma 1, it follows that $p^1 \perp^a p^2$, $p^1 \perp^a p^3$, and, consequently, $q_j^1 \perp^a q_j^2$, $q_j^1 \perp^a q_j^3$. Since h_j, h_j^{-1} are bijections, $q_j^1 \perp^a q_l^s$ for every $s \in \{1,2,3\}$. □

Theorem 2. *The theory T_2 has countably many countable models, the expansions of T_2 by the complete 1-type $q_j^1(x)$ or by the 1-type $q_j^2(x)$ have a finite number of countable models.*

Proof. The proof is done in two stages. In the first step we show that $I(T_2,\aleph_0) = \aleph_0$. Further, we prove that $I(T_2 \cup q_j^2(c),\aleph_0) = 25$.

I. Let $\mathfrak{M}$ be a countable saturated model of T_2 and $\{\alpha_0,\alpha_1,...,\alpha_j,...\}_{j<\omega} \subset M$ be a set of some realizations of types $q_j^1(\mathfrak{M})$ such that $\alpha_0 \models q^1(x)$ and $\alpha_j \models q_j^1(x)$.

We denote by $\mathfrak{M}_j$ a prime model of T_2 over α_j. Let $\mathfrak{M}_{j,k} = \mathfrak{M}(\alpha_j,\alpha_k)$ be a prime model over α_j and α_k.

Claim 1. *The theory T_2 has at least countably many non-isomorphic countable models.*

Proof. Since $q_j^1 \perp^a q_j^2$, $q_j^1 \perp^a q_j^3$ and $q_j^1 \perp^a q_k^s$, $k \neq j < \omega$, for every $s \in \{1,2,3\}$, then the 1-types q_k^1, q_k^2, q_k^3, q_j^2, and q_j^3 are not realized in $\mathfrak{M}_j$, that is $q_k(\mathfrak{M}) = \emptyset$, $q_j^1(\mathfrak{M}) = q_j(\mathfrak{M})$, and $|q_j(\mathfrak{M})| = 1$. So we have at least countably many non-isomorphic countable models, because $\mathfrak{M}_j \not\cong \mathfrak{M}_k$, $k \neq j$. That is, for every distinct α_j and α_k, we have either $q_j(\mathfrak{M}_j) = \{\alpha_j\}$ and $q_j(\mathfrak{M}_k) = \emptyset$, or $q_k(\mathfrak{M}_j) = \emptyset$ and $q_k(\mathfrak{M}_k) = \{\alpha_k\}$. □

Claim 2. *For every countable model $\mathfrak{N}$ of T_2, if $q^2(\mathfrak{N}) \neq \emptyset$, then $|q_j^2(\mathfrak{N})| = |q_j^3(\mathfrak{N})| = |q_j^1(\mathfrak{N})| = \aleph_0$.*

Proof. If $q^2(\mathfrak{N}) \neq \emptyset$, then there exists $\beta \in q^2(\mathfrak{N})$ such that $\beta < f(\beta)$, and $f(\beta) \in q^3(\mathfrak{N})$, since the function f is q-preserving. Consequently, within the interval $(\beta, f(\beta))$, there exists a countable number of elements from $q^s(\mathfrak{N})$, where $s \in \{1,2,3\}$. Furthermore, due to the existence of a bijection between $q^2(\mathfrak{N}) \cup q^3(\mathfrak{N})$ and $q_j^2(\mathfrak{N}) \cup q_j^3(\mathfrak{N})$, it follows that $q_j^2(\mathfrak{N})$ and $q_j^3(\mathfrak{N})$ are countable. Moreover, since $H(\mathfrak{N})$ is mutually dense with $q_j^2(\mathfrak{N}) \cup q_j^3(\mathfrak{N})$, it can be concluded that $q_j^1(\mathfrak{N})$ is also countable. □

Claim 3. *For every $j \neq k < \omega$, $\mathfrak{M}_{j,k}$ contains an infinite number of realizations of q_i^1, q_i^2 and q_i^3 for $i < \omega$. And consequently, for every $i < \omega$,*

$$\langle q_j^2(\mathfrak{M}_{j,k}); =, <, f\rangle \cong \langle q_i^2(\mathfrak{M}_{j,k}); =, <, f\rangle.$$

Proof. As follows from the remark to Axiom 5, α_j using formulas $S^k(M, \alpha_j)$ defines an irrational cut in q_k.

Assuming $j < k$, $k - j = m$. There are two cases: $\alpha_k \in S^{j+m}(M, \alpha_j) \cap H(M)$ and $\alpha_k \notin S^{j+m}(M, \alpha_j) \cap H(M)$. If $\alpha_k \in S^{j+m}(M, \alpha_j) \cap H(M)$, then there exists β such that $(\alpha_k < \beta) \wedge S^k(\beta, \alpha_j) \wedge H(\beta)$, since, $S^k(M, \alpha_j) \cap H(M)$ does not have a maximal element. Consequently, the formula $S^k(x, \alpha_j) \wedge (\alpha_k < x < \beta) \wedge A_l(x)$ has infinitely many solutions, indicating that the definable set of this formula is infinite. Since h_i and h_i^{-1} are bijections for any $i < \omega$, $\langle q_j^2(\mathfrak{M}_{j,k}); =, <, f\rangle \cong \langle q_i^2(\mathfrak{M}_{j,k}); =, <, f\rangle$.

If $\alpha_k \notin S^{j+m}(M, \alpha_j) \cap H(M)$ then there exists β such that $(\beta < \alpha_k) \wedge \neg S^k(\beta, \alpha_j) \wedge H(\beta)$, since, $\neg S^k(M, \alpha_j) \cap H(M)$ does not have a minimal element.

Consequently, the formula $\neg S^k(x, \alpha_j) \wedge (\beta < x < \alpha_k) \wedge A_l(x)$ has infinitely many solutions, indicating that the definable set of this formula is infinite. Since h_i and h_i^{-1} are bijections for any $i < \omega$,

$$\langle q_j^2(\mathfrak{M}_{j,k}); =, <, f\rangle \cong \langle q_i^2(\mathfrak{M}_{j,k}); =, <, f\rangle.$$

□

Claim 4. *For every $j \neq k$, in the case of a prime model $\mathfrak{M}_{j,k}$, we have $q^1(\mathfrak{M}_{j,k}) \cup q^2(\mathfrak{M}_{j,k}) \cup q^3(\mathfrak{M}_{j,k}) = C \cup D$ such that $C < D$, and for every $c \in C$, $f(c) \in C$ and for every $d \in D$, $f(d) \in D$.*

Proof. For every countable model $\mathfrak{N}$ of T_2 if $|q_j^1(\mathfrak{N})| = 1$, then the prime model $\mathfrak{M}_j$ is isomorphic to $\mathfrak{N}$ in case $p(N) = \emptyset$, because for every $s \in \{1, 2, 3\}$, $p \perp^w q_j^s$. Note that $tp(\alpha_k/\alpha_j)$ is non-principal, since non-principal 1-type $q_k^1(x)$ has exactly two non-principal extensions over α_j, that is $q_k^1(x) \cup \{S^k(x, \alpha_j)\}$ and $q_k^1(x) \cup \{\neg S^k(x, \alpha_j)\}$ $(q^1(\mathfrak{M}) = (q_k^1(\mathfrak{M}) \cap S^k(M, \alpha_j)) \cup (q_k^1(\mathfrak{M}) \cap \neg S^k(M, \alpha_j)))$. As well as the type $tp(\alpha_j/\alpha_k)$ is non-principal. Let $\alpha_k \in S^k(M, \alpha_j) \wedge H(M)$, this means α_k lies to the left of $S^k(x, \alpha_j)$ and, consequently, $\alpha_j \in \neg S^j(M, \alpha_k) \wedge H(M)$. If $\alpha_k \in \neg S^k(x, \alpha_j) \cap H(M)$, then $\alpha_j \in S^j(x, \alpha_k) \cap H(M)$. Note that $S^k(x, \alpha_j)$ is determined by composition of monotone bijections of kind h_i which preserve the ordering. Then by Proposition 1 for $j \neq k < \omega$ we obtain that:

$$r_1(x, \alpha_j) = \{H(x)\} \cup \{b_n^k < x < S^k(x, \alpha_j) \mid n < \omega\} \cup \{\phi(x, \alpha_j)\} \subset tp(\alpha_k/\alpha_j),$$

where $\phi(x, \alpha_j) = \exists y(y < x < f(y) \wedge S^k(f(y), \alpha_j))$. There exists $\beta \in q_k(\mathfrak{N})$ such that $\mathfrak{N} \models (\beta < \alpha_k < f(\beta)) \wedge S^k(f(\beta), \alpha_j) \wedge H(\alpha_k)$. Likewise, we obtain that:

$$r_2(x, \alpha_k) = \{H(x)\} \cup \{\neg S^j(x, \alpha_k) \wedge x < f(b_n^j) \mid n < \omega\} \cup \{\psi(x, \alpha_k)\} \subset \\ \subset tp(\alpha_j/\alpha_k),$$

where $\psi(x, \alpha_k) = \exists y(y < x < f(y) \wedge \neg S^j(y, \alpha_k))$. There exists $\gamma \in q_j(\mathfrak{N})$ such that $\mathfrak{N} \models (\gamma < \alpha_j < f(\gamma)) \wedge \neg S^j(\gamma, \alpha_k) \wedge H(\alpha_j)$.

Consider $S^0(M_{j,k}, \alpha_k)$ and $S^0(M_{j,k}, \alpha_j)$. Denote by

$$O(x, y, \alpha_k, \alpha_j) := O_1(x, \alpha_k, \alpha_j) \vee O_2(y, \alpha_k, \alpha_j)$$

a formula that is the disjunction of the following formulas:

$$O_1(x) := O_1(x, \alpha_k, \alpha_j) = \exists y[S^0(x, \alpha_k) \wedge \neg S^0(f(x), \alpha_k) \wedge S^0(f(x), \alpha_j) \wedge \\ \wedge (f(x) < y) \wedge S^0(y, \alpha_j) \wedge \neg S^0(f(y), \alpha_j)],$$

$$O_2(y) := O_2(y, \alpha_k, \alpha_j) = \exists x[S^0(x, \alpha_k) \wedge \neg S^0(f(x), \alpha_k) \wedge S^0(f(x), \alpha_j) \wedge \\ \wedge (f(x) < y) \wedge S^0(y, \alpha_j) \wedge \neg S^0(f(y), \alpha_j)].$$

The following properties hold:

$$O_1(M_{j,k}, \alpha_k, \alpha_j) < O_2(M_{j,k}, \alpha_k, \alpha_j)$$

and

$$O_1(M_{j,k}, \alpha_k, \alpha_j) \cup O_2(M_{j,k}, \alpha_k, \alpha_j) \subset q(\mathfrak{M}_{j,k}).$$

Consider the following formulas:

$$O_{1,1}(z) := \exists x(O_1(x) \wedge f(x) = z),\ O_{2,1}(t) := \exists y(O_2(y) \wedge f(y) = t).$$

So by definition of a definable subset we have:

$$O_1(M_{j,k}) < O_{1,1}(M_{j,k}) < O_2(M_{j,k}) < O_{2,1}(M_{j,k}).$$

By Axiom 5 and Axiom 6 of T_0 the sets $[A_l(M_{j,k}) \cap q(\mathfrak{M}_{j,k})]$ and $[H(M_{j,k}) \cap q(\mathfrak{M}_{j,k})]$ are mutually dense, and

$$[A_l(M_{j,k}) \cap q(\mathfrak{M}_{j,k})]^- = [H(M_{j,k}) \cap q(\mathfrak{M}_{j,k})]^- = q(\mathfrak{M}_{j,k})^-.$$

Similarly,

$$[A_r(M_{j,k}) \cap q(\mathfrak{M}_{j,k})]^+ = [H(M_{j,k}) \cap q(\mathfrak{M}_{j,k})]^+ = q(\mathfrak{M}_{j,k})^+.$$

Notice that

$$O_1(M_{j,k}) \cup O_2(M_{j,k}) \subset A_l(M_{j,k})$$

and

$$O_{1,1}(M_{j,k}) \cup O_{2,1}(M_{j,k}) \subset A_r(M_{j,k}).$$

For the previous formulas the following is true:

$$O_1(M_{j,k})^- = q(\mathfrak{M}_{j,k})^-, O_{2,1}(M_{j,k})^+ = q(\mathfrak{M}_{j,k})^+. \tag{2}$$

By applying Axiom 8 of T_0, we obtain the following mutually disjoint sets of realizations:

$$C = \{a \in \mathfrak{M}_{j,k} \mid \exists b \in O_1(M_{j,k}), \mathfrak{M}_{j,k} \models b < a < f(b)\};$$
$$D = \{a \in \mathfrak{M}_{j,k} \mid \exists b \in O_2(M_{j,k}), \mathfrak{M}_{j,k} \models b < a < f(b)\}.$$

Then $C < D$ and $C, D \subset q(\mathfrak{M}_{j,k})$ since f is q-preserving. Thus, let us prove that $C \cup D = q(\mathfrak{M}_{j,k})$. From (2) it follows that:

$$C^- = q(\mathfrak{M}_{j,k})^-,\ D^+ = q(\mathfrak{M}_{j,k})^+.$$

Suppose there exists a set $E \subset q(\mathfrak{M}_{j,k})$ such that $C < E < D$. Then for every elements $\alpha \in C$, $\beta \in E$ and $\gamma \in D$ according to Axiom 7, there is some c such that $c < \alpha < f(\alpha) < \beta < f(\beta) < f(c)$ or $c < \beta < f(\beta) < \gamma < f(\gamma) < f(c)$. Then $\beta \in C$ or $\beta \in D$.

□

Claim 5. *Let $\mathfrak{N}$ be a countable model of T_2 such that $q^2(\mathfrak{N}) \neq \emptyset$. Then the type of isomorphism of $q(\mathfrak{N}) \neq \emptyset$ can be one of the following:*

1. $q(\mathfrak{N}) \models T_0$. In this case the realization set $q(\mathfrak{N})$ is a model of the theory T_0.

2. $q(\mathfrak{N}) = [\alpha, f(\alpha)]$.

3. $q(\mathfrak{N}) = C \cup D$ such that $C < D$, and for every $c \in C$, $f(c) \in C$ and for every $d \in D$, $f(d) \in D$.

Proof. By Claim 2, it is stated that the set $q(\mathfrak{N})$ is infinite, and it is not necessary for it to be a model of T_0.

Let us consider two cases for the structure $\langle q(\mathfrak{N}); =, <, f^1\rangle$:

Case 1. $\langle q(\mathfrak{N}); =, <, f^1\rangle \models \forall x \forall y (x < y \to \exists z (z < x < y < f(z)))$.

Case 2. $\langle q(\mathfrak{N}); =, <, f^1\rangle \models \neg\forall x \forall y (x < y \to \exists z (z < x < y < f(z)))$.

In Case 1, $q(\mathfrak{N})$ is a convex subset of N, and f acts on this set since f is q-preserving. Consequently, $\langle q(\mathfrak{N}); =, <, f^1\rangle$ satisfies all axioms of T_0.

In Case 2, $\langle q(\mathfrak{N}); =, <, f^1\rangle \models \exists x \exists y (x < y \wedge \forall z \neg (z < x < y < f(z)))$. This means that for some $\alpha, \beta \in q(\mathfrak{N})$, the following holds:

$$\langle q(\mathfrak{N}); =, <, f^1\rangle \models \alpha < \beta \wedge \forall z \neg (z < \alpha < \beta < f(z)).$$

Subcase 2.1: Suppose $\beta \leq f(\alpha)$. Since there is no element $\gamma \in q(\mathfrak{N})$ such that $\langle q(\mathfrak{N}); =, <, f^1\rangle \models \gamma < \alpha < f(\alpha) < f(\gamma)$, we can conclude that for a countable saturated model $\mathfrak{M}$ of T_2, which is an elementary extension of $\mathfrak{N}$, for any $\gamma \in q(\mathfrak{M})$ with $\mathfrak{M} \models \gamma < \alpha$, the type $tp(\gamma/\alpha)$ is not isolated. Then, for the prime model over α, $\mathfrak{M}(\alpha)$, we have $\mathfrak{M}(\alpha) \prec \mathfrak{N}$ and $q(\mathfrak{N}) = q(\mathfrak{M}(\alpha)) = [\alpha, f(\alpha)]$.

Subcase 2.2: By Axiom 8 of T_0, there exist $\gamma, \delta \in q(\mathfrak{M})$ such that $\mathfrak{N} \models \gamma < \alpha < f(\gamma) \wedge \delta < \beta < f(\delta)$. By applying Axiom 8 an infinite number of times, we obtain that $q(\mathfrak{N}) = q(\mathfrak{M}(\alpha, \beta))$, where $\mathfrak{M}(\alpha, \beta)$ is the prime model over α and β, and $q(\mathfrak{M}(\alpha, \beta)) = C \cup D$. □

For an arbitrary countable model $\mathfrak{N}$, there are the following possibilities:

1_i. If $|q^1(\mathfrak{N})| = \emptyset$, $q_j(\mathfrak{N}) = \emptyset$, $i < \omega$.

2. If $q^2(\mathfrak{N}) \neq \emptyset$, then $q_j^2(\mathfrak{N}) \cong \langle q^2(\mathfrak{N}); =, <, f^1\rangle$.

Thus, the analysis of $q(\mathfrak{N})$ using Claim 5 gives us $\aleph_0 + 3 = \aleph_0$. For a fixed type of isomorphism of $q(\mathfrak{N})$, by Lemma 1 of T_1, we have 5 possibilities for $p(\mathfrak{N})$. Since $p^2(x) \perp^\omega q^2(y)$, $q(\mathfrak{N})$ and $p(\mathfrak{N})$ determine the type of isomorphism of $\mathfrak{N}$, we have $I(T_2, \aleph_0) = \aleph_0 \times 5 = \aleph_0$.

II. Let $T_2^* = T_2 \cup q_j^1(c)$ be a non-principal constant expansion of the theory T_2. Then $\mathfrak{M}^*$ is a model of T_2^*, denoted as $\mathfrak{M}^* = (\mathfrak{M}, c)$, where $\mathfrak{M} \models T_2$ and $\models q_j^1(c)$. According to Claim 1 and Claim 5 we have the following cases for the model $\mathfrak{M}$:

1. Let $q_j^1(\mathfrak{M}) = \{c\}$, then $q_j^2(\mathfrak{M}) = \emptyset$, $q_j^3(\mathfrak{M}) = \emptyset$ and $q_k^s(\mathfrak{M}) = \emptyset$ for $k \neq j$, $s \in \{1, 2, 3\}$. If $p(\mathfrak{M}) = \emptyset$, then $\mathfrak{M}^*$ is a prime model of T_2^*.

2. Let $q_j(\mathfrak{M}) = [a, f(a)]$ for some $a \in q_j^2(\mathfrak{M})$. In this case, $\mathfrak{M}^* = \langle \mathfrak{M}, c\rangle$ with $c \in [a, f(a)]$, and

$$\langle q_k(\mathfrak{M}); =, <, f\rangle \cong \langle q_j(\mathfrak{M}); =, <, f\rangle.$$

3. Let $q_j(\mathfrak{M}) = C \cup D$, then $\langle q_k(\mathfrak{M}); =, <, f\rangle \cong \langle q_j(\mathfrak{M}); =, <, f\rangle$ and $c \in C$.

4. Let $q_j(\mathfrak{M}) = C \cup D$, then $\langle q_k(\mathfrak{M}); =, <, f\rangle \cong \langle q_j(\mathfrak{M}); =, <, f\rangle$ and $c \in D$.

5. Let $\langle q_k(\mathfrak{M}); =, <, f\rangle \models T_0$, $c \in q_j(\mathfrak{M})$, and

$$\langle q_k(\mathfrak{M}); =, <, f\rangle \cong \langle q_j(\mathfrak{M});\ =, <, f\rangle.$$

This is an $\aleph_0$-categorical model of the theory T_0. If $\langle p(\mathfrak{M}); =, <, f\rangle \models T_0$, then $\mathfrak{M}^*$ is a countable saturated model of T_2^*.

Considering that the type of isomorphism of the countable model is determined by the type of isomorphisms of $q_j(\mathfrak{M})$ and $p(\mathfrak{M})$, we have $I(T_2 \cup q_j^1(c), \aleph_0) = 5 \times 5 = 25$.

Let $T_2^* = T_2 \cup q_j^2(c)$ be a non-principal constant expansion of the theory T_2, then $\mathfrak{M}^*$ is a model of T_2^* and $\mathfrak{M}^* = (\mathfrak{M}, c)$, where $\mathfrak{M} \models T_2$ and $\models q_j^2(c)$. Then by Claim 5 we have the following cases:

1. $q_j(\mathfrak{M}) = [a, f(a)]$ for some $a \in q_j(\mathfrak{M})$, $\mathfrak{M} \models a < c < f(c) < f(a)$, and
$$\langle q_k(\mathfrak{M}); =, <, f\rangle \cong \langle q_j(\mathfrak{M}); =, <, f\rangle.$$
2. $q_j(\mathfrak{M}) = [a, f(a)]$ for some $a \in q_j(\mathfrak{M})$, $\mathfrak{M}^* \models c = a$, and
$$\langle q_k(\mathfrak{M}); =, <, f\rangle \cong \langle q_j(\mathfrak{M}); =, <, f\rangle.$$
3. $q_j(\mathfrak{M}) = C \cup D$, then $\langle q_k(\mathfrak{M}); =, <, f\rangle \cong \langle q_j(\mathfrak{M}); =, <, f\rangle$ and $c \in C$.
4. $q_j(\mathfrak{M}) = C \cup D$, then $\langle q_k(\mathfrak{M}); =, <, f\rangle \cong \langle q_j(\mathfrak{M}); =, <, f\rangle$ and $c \in D$.
5. $\langle q_k(\mathfrak{M}), =, <, f\rangle \models T_0$, $c \in q_j(\mathfrak{M})$ and
$$\langle q_k(\mathfrak{M}); =, <, f\rangle \cong \langle q_j(\mathfrak{M}); =, <, f\rangle.$$

If $\langle p(\mathfrak{M}), =, <, f\rangle \models T_0$, then $\mathfrak{M}^*$ is a countable saturated model of T_2^*.

Considering that the type of isomorphism of the countable model is determined by the types of isomorphism of $q_j(\mathfrak{M})$ and $p(\mathfrak{M})$, we have $I(T_2 \cup q_j^2(c), \aleph_0) = 5 \times 5 = 25$. □

Taking into account B. Omarov's successful attempt to reduce the number of countable models from $\aleph_0$ to a finite number, we propose the following conditions for constructing an example of a complete theory that reduces the number of countable models from the continuum by means of a constant expansion.

Conjecture 1. *Let T be a small ordered complete theory of a countable language $\mathcal{L}$. Let $I(T, \aleph_0) = 2^{\aleph_0}$. Then there exists a non-principal type $p \in S_1(T)$ such that $I(T^*, \aleph_0) \leq \omega$, where $T^* := T \cup p(c)$ and c is a new constant, if and only if the following holds:*

1. There exists a family of non-principal 1-types $p_1, p_2, \ldots, p_n, \ldots \in S_1(T)$ $(n < \omega)$ such that $tp(\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k}) \perp^a p_j$ for all $k < \omega$, and all pairwise distinct $i_1, i_2, \ldots, i_k, j < \omega$ such that $\alpha_{i_1} \in p_{i_1}(\mathfrak{M})$, $\alpha_{i_2} \in p_{i_2}(\mathfrak{M}), \ldots, \alpha_{i_k} \in p_{i_k}(\mathfrak{M})$, where $\mathfrak{M}$ is a countable saturated model of T.

2. For every $i < \omega$, we have $p \not\perp^a p_i$.

3. There is a finite number of non-principal 1-types, $q_1, q_2, \ldots, q_m \in S_1(T)$, such that for every $j \neq k$ $(1 \leq j \leq m$, $1 \leq k \leq m)$ we have $q_k \perp^a q_j$, $p \perp^a q_k$, and $q_k \perp^a p$.

4. For every non-principal $q \in S(T)$, we have either $q \sim_{\not\perp^a} p$, or for some $i < \omega$, $q \sim_{\not\perp^a} p_i$, or for some k, $1 \leq k \leq m$, $q \sim_{\not\perp^a} q_k$.

The complete theory T_2, presented in [3] (Theorem 3), by Claim 3, does not satisfy the condition 1 of Conjecture 1. Condition 1 of Conjecture 1 ensures the maximal number of countable models, condition 2 of Conjecture 1 reduces the number of countable models by a constant expansion, while conditions 3 and 4 of Conjecture 1 ensure that the number of countable models cannot be increased due to other types.

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