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MULTIVALUED QUASIMÖBIUS PROPERTY AND BOUNDED TURNING

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Abstract: The class of multivalued mappings with bounded angular distortion (BAD) property in metric spaces can be considered as a multivalued analogry for quasimöbius mappings. We study the connections between quasimeromorphic self-mappings of $X = \bar{R}^n$ and multivalued mappings $F : X \to 2^X$ with BAD property. The main result of the paper concerns the multivalued mappings $F : D \to 2^{\bar{\mathbf{C}}}$ with BAD property of a domain $D \subset \bar{\mathbf{C}}$. If the image $F(x)$ of each point $x \in D$ is either a point or a continuum with bounded turning then F is proved to be a single-valued quasimöbius mapping. The crucial point in the proof of this result is the local connectedness of the set $F(X)$ for the multivalued continuous mapping $F : X \to 2^Y$ with BAD property. We obtain sufficient conditions providing $F(X)$ to have local connectedness or bounded turning property in the most general case.

Keywords: multivalued quasimöbius mapping, multivalued hyperinjective mapping, Ptolemaic characteristic of tetrad, generalized angle, bounded angular distortion, local connectedness.

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1 Introduction

The definition of quasisymmetric and quasimöbius mappings was given by P. Tukia and J. Väisälä [1], [2], [3] in order to extend the notion of quasiconformality to the case of arbitrary (and in particular, discrete) metric spaces. The quasimöbius property was defined as the existence of some estimate for the distortion of absolute cross-ratio of a quadruple of points (i.e. tetrad) under the mapping. It is also equivalent to the existence of two-sided bounds for the distortion of Ptolemaic characteristics of tetrads (see [4], proposition 3.4). Moreover, it was shown in [5] under some assumptions on metric spaces that the lower estimate alone (for the distortion of Ptolemaic characteristic) is sufficient for the quasimöbius property of the mapping (see [5], theorem 6.2).

A multivalued mapping is called *hyperinjective* if the images of distinct points do not intersect each other. Thus, the multivalued hyperinjective mapping $F : X \to 2^Y$ transforms each tetrad from X into a quadruple of pairwise disjoint sets in Y (so-called *generalized angle*, or *generalized tetrad*). The value of generalized angle was introduced in [4] (see definition 4.1) as a counterpart for the Ptolemaic characteristic of tetrads. Then the condition of lower boundedness of the distortion of this characteristic distinguishes the class BAD (Bounded Angular Distortion) of multivalued mappings in metric spaces which may be considered as a multivalued analogue for quasimöbius mappings. Some topological and metric properties of such mappings were investigated in [5], [6], [7] and [8]. In one special case, the metric spaces X and Y being unit circles in $\bar{\mathbf{C}}$, the multivalued quasimöbius mappings $F : X \to 2^Y$ with BAD property have been thoroughly studied in [9]. Namely, it was proved that the left inverse to F mapping $f = F^{-1} : F(X) \to X$ is of the form $f(z) = [\varphi(z)]^N$ where φ is a quasimöbius homeomorphism of unit circle onto itself.

The purpose of our further research is to study the connection between quasimeromorphic self-mappings of $X = \bar{R}^n$ and multivalued mappings $F : X \to 2^X$ with BAD property. The main result presented below as Theorem 4.1. of section 4 concerns the multivalued mappings $F : D \to 2^{\bar{\mathbf{C}}}$ with BAD property of a domain $D \subset \bar{\mathbf{C}}$. If the image $F(x)$ of each point $x \in D$ is either a point or a continuum with bounded turning then F is proved to be a single-valued quasimöbius mapping. The crucial point in the proof of this result is the local connectedness of the set $F(X)$ for the multivalued continuous mapping $F : X \to 2^Y$ with BAD property. The sufficient conditions providing $F(X)$ to have local connectedness or bounded turning property in the most general case are obtained in Theorem 3.1. of section 3 which will be also useful in our further studies.

2 Definitions and notations

Definition (see [1], Definition 2.7, p.100 or [10], §1, p. 559). A metric space (X, ρ) has *bounded turning* property c-BT with a constant $c \geq 1$ if any two points x' and x'' of X can be joined by a continuum $\tau \subset X$ such that $\mathrm{diam}_\rho(\tau) \leq c \cdot \rho(x', x'')$.

In this definition, the space X is supposed to be connected. In more general case, we use the following

2.1. Definition. A metric space (X, ρ) has *bounded turning* property (c, δ)-BT with constants $c \geq 1$ and $\delta > 0$ if any two points $x', x'' \in X$ with the distance $\rho(x', x'') < \delta$ can be joined by a continuum $\tau \subset X$ such that $\mathrm{diam}_\rho(\tau) \leq c \cdot \rho(x', x'')$.

If $\delta \geq \mathrm{diam}_\rho(X)$, then the property (c, δ)-BT is equivalent to c-BT and provides the connectedness of X. If X is a continuum of bounded diameter $\mathrm{diam}_\rho(X) = D < \infty$, then the property (c, δ)-BT implies c'-BT with the constant $c' = \max\{c, D/\delta\}$.

Definition (see [11], §49.I). A topological space X is *locally connected* if for every point $x \in X$ and it's open neighbourhood U there exists a connected neighbourhood V of x such that $V \subset U$.

2.2. Lemma. *A metric space with bounded turning (c, δ)-BT is locally connected.*

See paragraph 5.1. for the proof.

Definitions. Throughout this paper 2^Y denotes the set of all nonempty subsets of Y. The multivalued mapping $F : X \to 2^Y$ is said to be *hyperinjective* if $x_1 \neq x_2$ implies $F(x_1) \cap F(x_2) = \emptyset$. The images $F(x) \subset Y$ of points $x \in X$ will be called *thick points*. In metric spaces (X, ρ) and (Y, σ) the multivalued mapping $F : X \to 2^Y$ with closed thick points is said to be *continuous* if the convergence $x_n \to x$ in X implies the existence of the topological limit of the sequence $\{F(x_n)\}$ of closed sets in Y and the equality $\mathrm{Lim}_{n\to\infty} F(x_n) = F(x_0)$. For more detailed definition see paragraph 5.2.

2.3. Proposition. *Let X, Y be topological spaces and $F : X \to 2^Y$ be a multivalued hyperinjective continuous mapping.*

(i) If a set $K \subset X$ is compact and each of thick points $F(x)$ for $x \in K$ is compact then the set $F(K) := \cup\{F(x) : x \in K\}$ is compact as well. Moreover, since Y is a Hausdorff space, the left inverse to F mapping $f = F^{-1} : F(K) \to K$ is continuous.

(ii) If $\gamma \subset X$ is a connected set and $F(x_0)$ is connected for some point $x_0 \in \gamma$, then $F(\gamma)$ is connected as well. In particular, if $\gamma \subset X$ is a continuum and all thick points $F(x)$ for $x \in \gamma$ are continua, then $F(\gamma)$ is a continuum in Y.

The proof of Proposition 2.3. will be given in paragraph 5.3.

Definition (see [12], section 32.1). A metric (or semimetric) space (X, ρ) is said to be *Ptolemaic* if the Ptolemy's inequality

$$\rho(x_1, x_2) \cdot \rho(x_3, x_4) + \rho(x_1, x_4) \cdot \rho(x_2, x_3) \geq \rho(x_1, x_3) \cdot \rho(x_2, x_4)$$

holds for every four of distinct points $x_1, x_2, x_3, x_4 \in X$.

In particular, the space $\bar{R}^n$ with chordal metric and the space R^n with Euclidean or chordal metric both are Ptolemaic spaces. Any subspace of a Ptolemaic space remains to be Ptolemaic.

Definition (see [4], definition 4.1). A *generalized angle* in a metric (or semimetric) space (Y, σ) is a quadruple of non-empty pairwise disjoint sets $\Psi = (A_1, A_2; B_1, B_2)$, and it's *value* $\alpha(\Psi)$ is defined as

$$\alpha(\Psi) := \inf_{u_1 \in A_1; u_2 \in A_2} \left(\sup_{v_1 \in B_1; v_2 \in B_2} \frac{\sigma(u_1, u_2) \cdot \sigma(v_1, v_2)}{\sigma(u_1, v_1) \cdot \sigma(u_2, v_2) + \sigma(u_1, v_2) \cdot \sigma(u_2, v_1)} \right).$$

It is clear, that $\alpha(\Psi) \leq 1$ for every generalized angle Ψ in Ptolemaic metric space.

2.4. Definition (see [6], section 4.1). Let (X, ρ) and (Y, σ) be Ptolemaic metric (or semimetric) spaces. Let a real non-decreasing function $\omega : [0,1] \to [0,1]$ be positive on $(0,1]$, and $\omega(0) = 0 = \lim_{t \searrow 0} \omega(t)$. Then a multivalued hyperinjective mapping $F : X \to 2^Y$ is said to have ω-BAD property (Bounded Angular Distortion) with *control function* ω if the estimate $\alpha(F(\Psi)) \geq \omega(\alpha(\Psi))$ is valid for every generalized angle $\Psi = (A_1, A_2; B_1, B_2)$ in X and it's image $F(\Psi) = (F(A_1), F(A_2); F(B_1), F(B_2))$ in Y.

For every control function ω in ω-BAD property the real function $\omega^{(-1)} : [0,1] \to [0,1]$ is defined by formula $\omega^{(-1)}(s) := \sup\{t \in [0,1] : \omega(t) \leq s\}$. In general, the function $\omega^{(-1)}$ is not inverse to ω in the strict sense, but it has all the properties of control function, and $\omega^{(-1)}(\omega(t)) \geq t$ for all $t \in [0,1]$.

It was shown in [6] (section 4.2) that it is sufficient to check the inequality $\alpha(F(\Psi)) \geq \omega(\alpha(\Psi))$ in ω-BAD property only for generalized angles Ψ of the form $\Psi = (\{a_1\}, \{a_2\}; \{b_1\}, \{b_2\})$. In this case, the value

$$\alpha(\Psi) = \alpha(a_1, a_2; b_1, b_2) = \frac{\rho(a_1, a_2) \cdot \rho(b_1, b_2)}{\rho(a_1, b_1) \cdot \rho(a_2, b_2) + \rho(a_1, b_2) \cdot \rho(a_2, b_1)}$$

is called the *Ptolemaic characteristic of tetrad* $(a_1, a_2; b_1, b_2)$.

2.5. Definition. Let (X, ρ) and (Y, σ) be metric spaces, and $F : X \to 2^Y$ be a multivalued hyperinjective mapping. A pair of positive numbers (δ, Δ) is called the *normalizer* of F if there exists a set $\{a_1, a_2, a_3\}$ of points in X such that $\rho(a_i, a_j) \geq \delta$ and $\operatorname{dist}_\sigma(F(a_i), F(a_j)) \geq \Delta$ for all $i, j \in \{1, 2, 3\}$ with $i \neq j$.

It should be noted that a multivalued hyperinjective mapping $F : X \to 2^Y$ with compact thick points has many normalizers. It suffices to take an arbitrary triple of distinct points (a_1, a_2, a_3) in X and put $\delta := \min\{\rho(a_i, a_j) : i \neq j\}$ and $\Delta := \min\{\operatorname{dist}_\sigma(F(a_i), F(a_j)) : i \neq j\}$.

3 Local connectedness and bounded turning

In this section, we prove the following theorem.

3.1. Theorem. *Let (X, ρ) and (Y, σ) be Ptolemaic metric spaces and a multivalued hyperinjective continuous mapping $F : X \to 2^Y$ has ω-BAD property. If all thick points $F(x)$ with $x \in X$ are compact sets with bounded turning property (C_0, δ_0)-BT, then the following statements hold.*

(i) If X is compact and locally connected, then $F(X)$ is also compact and locally connected.

(ii) If X has a finite diameter $\mathrm{diam}_\rho(X) = d < \infty$, normalizer (δ, Δ), and bounded turning property (C, δ')-BT, then $F(X)$ has bounded turning property (C^*, δ^*)-BT. Here, the constants C^* and δ^* depend only on the normalizer (δ, Δ), constants (C_0, δ_0), (C, δ') in BT properties, the diameter d of X, and the control function ω in BAD property.

The proof of the theorem is divided into several consecutive steps.

3.2. Proposition. *Under the assumptions in Theorem 3.1., if the mapping F has a normalizer (δ, Δ), then for every $x_0 \in X$ and $y_0 \in F(x_0)$, there exists a point $a \in X$ such that*

$$\rho(x_0, a) > \delta/2\,, \quad \mathrm{dist}_\sigma(y_0, F(a)) \geq \Delta/2\,. \tag{3.2.1}$$

Proof. Let $A = \{a_1, a_2, a_3\}$ be the triple of points in X which realizes the normalizer (δ, Δ). Then there is a pair of distinct points $\{a_i, a_j\} \subset A$ such that $\rho(x_0, a_i) > \delta/2$ and $\rho(x_0, a_j) > \delta/2$. Indeed, if $\rho(x_0, a') > \delta/2$ for all $a' \in A$, then we can take an arbitrary pair of distinct points in A as $\{a_i, a_j\}$. If $\rho(x_0, a') \leq \delta/2$, then $\rho(a'', x_0) \geq \rho(a'', a') - \rho(a', x_0) > \delta - \delta/2 = \delta/2$ for every $a'' \in A \setminus \{a'\}$. Then $\{a_i, a_j\} = A \setminus \{a'\}$ is the required pair of points.

Next, there exists a pair of distinct points $\{a_k, a_s\}$ such that

$$\min\{\mathrm{dist}_\sigma(y_0, F(a_k)) \geq \Delta/2\,, \quad \mathrm{dist}_\sigma(y_0, F(a_s))\} \geq \Delta/2\,.$$

Indeed, if $\mathrm{dist}_\sigma(y_0, F(a')) \geq \Delta/2$ for all $a' \in A$, then we can take an arbitrary pair of distinct points in A as the pair $\{a_k, a_s\}$. If $\mathrm{dist}_\sigma(y_0, F(a')) < \Delta/2$ for a point $a' \in A$, then there exists a point $y' \in F(a')$ such that $\sigma(y_0, y') < \Delta/2$. Therefore for any point $a'' \in A$ and every $z \in F(a'')$, the inequality $\sigma(y_0, z) \geq \sigma(z, y') - \sigma(y_0, y') > \Delta - \Delta/2 = \Delta/2$ holds. Thus $\{a_k, a_s\} = A \setminus \{a'\}$ is the required pair of points.

Since $\{i, j\} \cap \{k, s\} \neq \emptyset$, there exists a point $a \in \{a_i, a_j\} \cap \{a_k, a_s\}$ for which (3.2.1) is valid.

Proposition 3.2. is proved.

3.3. Proposition. *Let (X, ρ) be a metric space with a pair of distinct points x_0 and a. If $C \geq 1$ and points x_1, x in X are such that*

$$\rho(x_0, x') \leq C \cdot \rho(x_0, x_1) < \rho(x_0, a)/2\,, \tag{3.3.1}$$

then the generalized angle $\Psi = (x_0, x_1; x', a)$ has the value $\alpha(\Psi) > (3{+}4C)^{-1}$.

Proof. Since $\rho(x_0,x') \le C\rho(x_0,x_1)$ and $\rho(x_1,x') \le (1+C)\rho(x_0,x_1)$, then

$$\alpha(\Psi) = \frac{\rho(x_0,x_1)\rho(x',a)}{\rho(x_0,x')\rho(x_1,a)+\rho(x_0,a)\rho(x_1,x')} \ge \frac{\rho(x',a)}{C\rho(x_1,a)+(1+C)\rho(x_0,a)}\,.$$

Then in the numerator, we use the inequalities $\rho(x',a) \ge \rho(x_0,a)-\rho(x_0,x') > \rho(x_0,a)/2$, and in the denominator, we note that $\rho(x_1,a) \le \rho(x_0,a) + \rho(x_0,x_1) < (1+1/(2C))\rho(x_0,a)$. Continuing the inequality, we have

$$\alpha(\Psi) > \frac{1/2}{(1_2C)/2+(1+C)} = \frac{1}{3+4C}\,.$$

The Proposition 3.3. is proved.

3.4. Proposition. *Under the assumptions in Theorem 3.1., let the mapping F have the normalizer (δ,Δ). Let $C \ge 1$ and the points $y_0 \in F(x_0)$ and $y_1 \in F(x_1)$ be such that $x_0 \ne x_1$ and*

$$\rho(x_0,x_1) \le \delta/(4C)\,, \quad \sigma(y_0,y_1) \le (\Delta/4)\omega_C\,, \tag{3.4.1}$$

where $\omega_C := \omega(1/(3+4C))$. Then for every point $x' \in X$ the condition $\rho(x_0,x') \le C\cdot\rho(x_0,x_1)$ implies that

$$F(x') \cap B(y_0, \sigma(y_0,y_1)\cdot 2/\omega_C) \ne \emptyset\,. \tag{3.4.2}$$

Proof. By Proposition 3.2., the estimates (3.2.1) are valid for some point $a \in X$. Let $x' \in X\setminus\{x_0,x_1,a\}$ be such that $\rho(x_0,x') \le C\cdot\rho(x_0,x_1)$. Estimate (3.3.1) in Proposition 3.3. follows from inequalities (3.2.1) and (3.4.1), indeed $\rho(x_0,x') \le C\cdot\rho(x_0,x_1) \le \delta/4 < \rho(x_0,a)/2$. Thus we have the estimate $\alpha(\Psi) > (3+4C)^{-1}$ for the value of the generalized angle $\Psi = (x_0,x_1;x',a)$. For the value of the generalized angle $F(\Psi) = (F(x_0,F(x_1);F(x'),F(a))$, we have the following inequalities

$$\alpha(F(\Psi)) \le \sup_{v'\in F(x');\ v\in F(a)} \frac{\sigma(y_0,y_1)\cdot\sigma(v',v)}{\sigma(y_0,v')\cdot\sigma(y_1,v)+\sigma(y_0,v)\cdot\sigma(y_1,v')}\,.$$

Since $\sigma(y_0,v') \ge \mathrm{dist}_\sigma(y_0,F(x'))$ and $\sigma(y_0,v) \ge \mathrm{dist}_\sigma(y_0,F(a)) \ge \Delta/2)$ then we have

$$\alpha(F(\Psi)) \le$$

$$\frac{\sigma(y_0,y_1)}{\min\{\Delta/2,\ \mathrm{dist}_\sigma(y_0,F(x'))\}}\left(\sup_{v'\in F(x');\ v\in F(a)} \frac{\sigma(v',v)}{\sigma(v',y_0)+\sigma(v,y_0)}\right) \le$$

$$\frac{\sigma(y_0,y_1)}{\min\{\Delta/2,\ \mathrm{dist}_\sigma(y_0,F(x')\}}\,.$$

From the ω-BAD property and the estimate $\alpha(\Psi) > (3+4C)^{-1}$ we get the inequality

$$\frac{\sigma(y_0,y_1)}{\min\{\Delta/2,\ \mathrm{dist}_\sigma(y_0,F(x'))\}} \ge \alpha(F(\Psi)) \ge \omega((3+4C)^{-1}) = \omega_C\,.$$

Hence, $\min\{\Delta/2,\ \mathrm{dist}_\sigma(y_0,F(x'))\} \le \sigma(y_0,y_1)/\omega_C$.

Since $\Delta/2 > \Delta/4 \geq \sigma(y_0, y_1)/\omega_C$ by (3.4.1), then

$$\operatorname{dist}_\sigma(y_0, F(x')) \leq \sigma(y_0, y_1)/\omega_C\,.$$

This means that the thick point $F(x')$ intersects the open ball $B(y_0, R)$ with radius $R = \sigma(y_0, y_1) \cdot 2/\omega_C$.

If $x' = x_0$, then (3.4.2) is obvious. If $x' = x_1$, then $y_1 \in B(y_0, \sigma(y_0, y_1) \cdot 2/\omega_C)$, and (3.4.2) is true. Under the condition $\rho(x_0, x') \leq C \cdot \rho(x_0, x_1)$, the point x' cannot coincide with a since $\rho(x_0, a) > \delta/2 = C \cdot \delta/(4C) \geq C \cdot \rho(x_0, x_1)$.

Thus (3.4.2) holds for every point x' such that $\rho(x_0, x') \leq C \cdot \rho(x_0, x_1)$.

Proposition 3.4. is proved.

3.5. Proposition. *Let the mapping $F : X \to 2^Y$ satisfy the conditions in Theorem 3.1. Let $y_0 \in F(x_0)$ and $\varepsilon \in (0, \delta_0/2)$ be given. Then for every thick point $F(x')$ the set $F(x') \cap B(y_0, \varepsilon)$ (it may be empty) is contained in a component of the set $F(x') \cap B(y_0, (1 + 2C_0)\varepsilon)$.*

Proof. Clearly, it suffices to consider the case of non-empty set $F(x') \cap B(y_0, \varepsilon)$. Let $y', y'' \in F(x') \cap B(y_0, \varepsilon)$. Because of bounded turning property (C_0, δ_0)-BT and the inequality $\sigma(y', y'') \leq \sigma(y', y_0) + \sigma(y'', y_0) < \delta_0$, there exists a continuum $\tau \subset F(x')$ such that $y', y'' \in \tau$ and $\operatorname{diam}_\sigma(\tau) \leq C_0 \cdot \sigma(y', y'') < 2C_0\varepsilon$. Since $\sigma(y_0, y') +\ \operatorname{diam}_\sigma(\tau) < \varepsilon + 2C_0\varepsilon$, $\tau \subset B(y_0.(1 + 2C_0)\varepsilon)$. It means that y' and y'' enter into the same component U of the set $F(x') \cap B(y_0, \varepsilon)$. Since y', y'' have been arbitrarily chosen points in $F(x') \cap B(y_0, \varepsilon)$, the desired result $F(x') \cap B(y_0, \varepsilon) \subset U$ follows.

Proposition 3.5 is proved.

3.6. Proposition. *Let the mapping $F : X \to 2^Y$ satisfy the conditions in Theorem 3.1. and have the normalizer (δ, Δ). Let $C \geq 1$ and the points $y_0 \in F(x_0), y_1 \in F(x_1)$ be such that*

$$0 < \rho(x_0, x_1) \leq \delta/(4C)\,; \quad r = \sigma(y_0, y_1) \leq \min\{\Delta/4, \delta_0/8\} \cdot \omega_C\,. \quad (3.6.1)$$

Let the connected set $\gamma \subset X$ be such that $x_0, x_1 \in \gamma \subset B(x_0, C \cdot \rho(x_0, x_1))$. Then the set

$$F(\gamma) \cap B(y_0, 2r(1 + 2C_0)/\omega_C)$$

has a component W such that $y_0, y_1 \in W$ and

$$F(\gamma) \cap B(y_0, 2r/\omega_C) \subset W.$$

Proof. Let $\mathcal{U}$ be the family of all those components of the set $F(\gamma) \cap B(y_0, 2r(1+2C_0)/\omega_C)$, that intersect the ball $B(y_0, 2r/\omega_C)$. Then $\rho(x_0, x_1) \leq \delta/(4C)$, $\sigma(y_0, y_1) \leq (\Delta/4)\omega_C$ by (3.6.1) and $\rho(x_0, x') < C \cdot \rho(x_0, x_1)$ for all $x' \in \gamma$. Hence, the set $F(x') \cap B(y_0, 2r/\omega_C)$ is not empty (by Proposition 3.4.). It means that for every point $x' \in \gamma$ there exists an element $U(x')$ in the family $\mathcal{U}$.

Since $\varepsilon := 2r/\omega_C < (\delta_0/4) \cdot 2/\omega_C = \delta_0/2$, by Proposition 3.5. the set $F(x') \cap B(y_0, 2r/\omega_C)$ is contained in a component of the set $F(x') \cap B(y_0, 2r(1+$

$2C_0)/\omega_C)$. It follows that the element $U(x')$ corresponding to the point $x' \in \gamma$ contains non-empty set $F(x') \cap B(y_0, 2r/\omega_C)$.

Let us show that the set $W := \cup\{U : U \in \mathcal{U}\}$ has required properties. It is contained in the ball $B(y_0, 2r(1+2C_0)/\omega_C)$ (because every element of $\mathcal{U}$ is contained in that ball) and contains the set $F(\gamma) \cap B(y_0, 2r/\omega_C)$. In particular, $y_0, y_1 \in W$. Therefore, we have to prove that W is connected and then $\mathcal{U} = \{W\}$.

Suppose W to be non-connected. Then there exist open sets $S_1, S_2 \subset Y$ such that $W \subset S_1 \cup S_2$, $S_1 \cap S_2 = \emptyset$ and $W \cap S_1 \neq \emptyset \neq W \cap S_2$. Replacing S_j $(j = 1, 2)$ by open sets $S_j \cap B(y_0, 2r(1+2C_0)/\omega_C)$ we may assume that $S_1 \cup S_2 \subset B(y_0, 2r(1+2C_0)/\omega_C)$. Since every $U \in \mathcal{U}$ is a connected set, for every $U \in \mathcal{U}$ either $U \subset S_1$ or $U \subset S_2$ holds. The restriction $F|\gamma$ is a lower semicontinuous mapping, so the sets $V_j = \{x \in \gamma : F(x) \cap U_j \neq \emptyset\}$ $(j = 1, 2)$ are open in γ. For every point $x \in \gamma$ the set $F(x) \cap B(y_0, 2r/\omega_C)$ is non-empty and is contained in an element $U(x) \in \mathcal{U}$ which is contained either in S_1 or in S_2. It means that for every one point $x \in \gamma$ we have either $x \in V_1$ or $x \in V_2$ but not both $x \in V_1$ and $x \in V_2$. Thus $V_1 \neq \emptyset \neq V_2$, $\gamma = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$ for open sets V_1, V_2, and this contradicts the connectedness of γ.

Proposition 3.6. is proved.

3.7. Proposition. *Let the mapping $F : X \to 2^Y$ satisfy the conditions in Theorem 3.1. and have the normalizer (δ, Δ). Let $d = \operatorname{diam}_\rho(X) < \infty$. Then for each pair of points x', x'' in X the following inequality holds*

$$\operatorname{dist}_\sigma(F(x'), F(x'')) \geq \Delta \cdot \omega\left(\frac{\delta^2}{4d^2}\right) \cdot \omega\left(\rho(x', x'')\frac{\delta}{4d^2}\right). \qquad (3.7.1)$$

Furthermore, $\rho(f(y'), f(y'')) \leq \zeta(\sigma(y', y''))$ for all $y', y'' \in F(X)$, where

$$\zeta(t) := \frac{4d^2}{\delta} \cdot \omega^{(-1)}\left(\frac{t}{\Delta \cdot \omega(\delta^2/(4d^2))}\right). \qquad (3.7.2)$$

Proof. Let the normalizer (δ, Δ) be realized by 3-point set $A = \{a_0, a_1, a_2\}$. For a point $x \in X \setminus A$ let us consider the generalized angle $\Psi = (x, a_j; a_i, a_k)$ where $\{j, i, k\} = \{0, 1, 2\}$. From $\rho(a_i, a_k) \geq \delta$ and $\rho(\,.\,,\,.\,) \leq d$ it follows that

$$\alpha(\Psi) = \frac{\rho(x, a_j)\rho(a_i, a_k)}{\rho(x, a_i)\rho(a_j, a_k) + \rho(x, a_k)\rho(a_j, a_i)} \geq \frac{\delta}{2d^2}\rho(x, a_j). \qquad (3.7.3)$$

The value of generalized angle $F(\Psi) = (F(x), A_j; A_i, A_k)$ (here $A_j := F(a_j)$, $A_i := F(a_i)$, $A_k := F(a_k)$) is

$$\alpha(F(\Psi)) = \inf_{u_1 \in F(x);\ u_2 \in A_j}\left(\sup_{v_1 \in A_i;\ v_2 \in A_k} \frac{\sigma(u_1, u_2)\sigma(v_1, v_2)}{\sigma(u_1, v_1)\sigma(u_2, v_2) + \sigma(u_1, v_2)\sigma(u_2, v_1)}\right).$$

For an arbitrary pair of points $u_1^0 \in F(x)$, $u_2^0 \in A_j$ it follows from estimates $\sigma(u_2^0, v_2) \geq \Delta$ and $\sigma(u_2^0, v_1) \geq \Delta$ that

$$\alpha(F(\Psi)) \leq \frac{\sigma(u_1^0, u_2^0)}{\Delta} \sup_{v_1 \in A_i;\ v_2 \in A_k} \frac{\sigma(v_1, v_2)}{\sigma(v_1, u_1^0) + \sigma(u_1^0, v_2)} \leq \frac{\sigma(u_1^0, u_2^0)}{\Delta}.$$

It means that $\alpha(F(\Psi)) \leq \mathrm{dist}_\sigma(F(x), A_j)/\Delta$. Then ω-BAD property and (3.7.3) give us the inequality

$$\mathrm{dist}_\sigma(F(x), A_j) \geq \Delta \cdot \omega\left(\rho(x, a_j)\frac{\delta}{2d^2}\right) \tag{3.7.4}$$

which holds for every point $x \in X \setminus A$ and every $j \in \{0, 1, 2\}$. If $i \neq j$ we have $\mathrm{dist}_\sigma(A_i, A_j) \geq \Delta \geq \Delta \cdot \omega(\rho(a_i, a_j))\delta/(2d^2)$ and if $x \in A_i$ where $i = j$ the inequality (3.7.4) turns to the equality $0 = 0$. Thus (3.7.4) is true for every $x \in X$ and for each $j \in 0, 1, 2$.

For an arbitrary point $x' \in X$ there exists a pair of points $\{b_1, b_2\}$ such that $\rho(x', b_1) \geq \delta'$ and $\rho(x', b_2) \geq \delta'$ where $\delta' = \delta/2$. Then for $B_1 = F(b_1)$ and $B_2 = F(b_2)$ the estimate (3.7.4) gives us the inequality

$$\mathrm{dist}_\sigma(F(x'), B_j) \geq \Delta \cdot \omega\left(\rho(x', b_j)\frac{\delta}{2d^2}\right) \geq \Delta \cdot \omega\left(\frac{\delta^2}{4d^2}\right)$$

where $j \in \{1, 2\}$. The set $B = \{x', b_1, b_2\}$ defines for the mapping F the normalizer (δ', Δ') with $\Delta' := \Delta \cdot \omega(\delta^2/(4d^2)) \leq \Delta$. The estimate (3.7.4) being applied to an arbitrary point $x'' \in X$ and the normalizer (δ', Δ') gives us the desired inequality (3.7.1):

$$\mathrm{dist}_\sigma(F(x''), F(x')) \geq \Delta' \cdot \omega\left(\rho(x'', x')\frac{\delta'}{2d^2}\right) = \Delta \cdot \omega\left(\frac{\delta^2}{4d^2}\right) \cdot \omega\left(\rho(x', x'')\frac{\delta}{4d^2}\right).$$

Now, for every pair of points $y', y'' \in F(X)$ and the corresponding points $f(y'), f(y'') \in X$ we get the inequality

$$\frac{\sigma(y', y'')}{\Delta \cdot \omega(\delta^2/(4d^2))} \geq \omega\left(\rho(f(y'), f(y''))\frac{\delta}{4d^2}\right).$$

The increasing function $\omega^{(-1)}$ with $\omega^{(-1)}(\omega(p)) \geq p$ (see section 2.4.) being applied to both sides of this inequality gives the desired estimate

$$\rho(f(y'), f(y'')) \leq \frac{4d^2}{\delta} \cdot \omega^{(-1)}\left(\frac{\sigma(y', y'')}{\Delta \cdot \omega(\delta^2/(4d^2))}\right) = \zeta(\sigma(y', y'')).$$

Proposition 3.7 is proved.

3.8. Proof of Theorem 3.1(i). We may assume without loss of generality that $Y = F(X)$. The compact metric space X has a finite diameter $\mathrm{diam}_\rho(X) = d < \infty$. Let (δ, Δ) be a normalizer for F and the function $\zeta(t)$ be defined by (3.7.2).

Let y_0 be an arbitrary point in $F(X)$ and $y_0 \in F(x_0)$ for $x_0 \in X$. We are to show that every open ball $B(y_0, R)$ contains a connected neighbourhood of the point y_0. Let $C := 2$ and $\omega_0 := \Omega_C = \omega(1/11)$.

Since the set $\{t > 0 : \zeta(t) < \delta/(4C)\}$ is not empty and $\zeta(t) \to 0$ as $t \to 0$, there exists $t_0 > 0$ such that $\zeta(t) < \delta/(4C)$ for all $t \leq t_0$. Let

$$R_0 := \min\left\{ \frac{R \cdot \omega_0}{2(1 + C_0)} \ ; \ \frac{\Delta}{4}\omega_0 \ ; \ \frac{\delta_0}{8}\omega_0 \ ; \ t_0 \right\}.$$

The mapping F being lower semicontinuous, the set $V_0 := \{x \in X : F(x) \cap B(y_0, R_0) \neq \emptyset\}$ is an open neighbourhood of the point x_0. The space X is locally connected, so there exists a connected open neighbourhood $V_0' \subset V_0$ of the point x_0. Consider a point $x_1 \in V_0'$ such that $V_0' \subset B(x_0, C\rho(x_0, x_1))$. Since $x_1 \in V_0$, the set $F(x_1) \cap B(y_0, R_0)$ is not empty, so there exists a point y_1 in this set. Then Proposition 3.7. gives the inequalities $\rho(x_0, x_1) \leq \zeta(\sigma(y_0, y_1)) \leq \zeta(R_0) \leq \zeta(t_0) < \delta/(4C)$.

Thus the points x_1 and $y_1 \in F(x_1)$ satisfy the conditions (3.6.1) in Proposition 3.6.:

$$\rho(x_0, x_1) < \delta/(4C) \ ; \ \sigma(y_0, y_1) < R_0 \leq \min\left\{\frac{\Delta}{4}, \frac{\delta_0}{8}\right\}\omega_0 \,.$$

The connected set $\gamma = V_0'$ contains the points x_0, x_1 and is contained in the open ball $B(x_0, C \cdot \rho(x_0, x_1))$. Then by Proposition 3.6. the nonempty set $F(V_0') \cap B(y_0, \sigma(y_0, y_1) \cdot 2(1 + 2C_0)/\omega_0)$ has a component W such that $y_0, y_1 \in W$ and $W_0 \subset W$ where $W_0 := F(V_0') \cap B(y_0, \sigma(y_0, y_1) \cdot 2/\omega_0)$. Besides $W \subset B(y_0, \sigma(y_0, y_1) \cdot 2(1 + 2C_0)/\omega_0) \subset B(y_0, R_0)$.

The set V_0' is open, the set $X \setminus V_0'$ is compact, all thick points of the mapping F are compact. Then by Proposition 2.3(i) the set $F(X) \setminus V_0'$ is compact. Hence the set $F(V_0') = F(X) \setminus F(X \setminus V_0')$ is open. Then W_0 is open and the connected set W is a neighbourhood of the point y_0 which is contained in the given open neighbourhood $B(y_0, R)$ of the point y_0.

It means that $F(X)$ is locally connected at every point $y_0 \in F(X)$.

The proposition (i) in Theorem 3.1. is proved.

3.9. Proof of Theorem 3.1(ii). For the given normalizer (δ, Δ) and the constant $d = \operatorname{diam}_\rho(X)$ let the function $\zeta(t)$ be defined by (3.7.2). Since $\zeta(t) \searrow 0$ as $t \to 0$, there exists $t_0 > 0$ such that $\zeta(t) < \min\{\delta', \delta/(4C)\}$ for all $t \leq t_0$. Let $\omega_C := \omega(1/(3 + 4C))$ and $\delta^* := \min\{(\Delta/4)\omega_C, (\delta_0/8)\omega_C, t_0\}$.

Let points $y_0 \in F(x_0)$ and $y_1 \in F(x_1)$ be such that $\sigma(y_0, y_1) \leq \delta^*$.

It follows from $\delta^* \leq t_0$ that $\rho(x_0, x_1) \leq \zeta(\delta^*) \leq \zeta(t_0) \leq \delta'$. The bounded turning property (C, δ')-BT of the space X gives the existence of a continuum $\gamma \subset X$ such that $x_0, x_1 \in \gamma$ and $\operatorname{diam}_\rho(\gamma) \leq C \cdot \rho(x_0, x_1)$. It means that $\gamma \subset \bar{B}(x_0, C \cdot \rho(x_0, x_1))$. Since $\rho(x_0, x_1) < \delta/(4C)$ and $\sigma(y_0, y_1) \leq \min\{\Delta/4, \delta_0/8\} \cdot \omega_C$, we may apply Proposition 3.6. to γ and obtain a connected set $W \subset F(\gamma)$ such that $y_0, y_1 \in W$ and $W \subset \bar{B}(y_0, \sigma(y_0, y_1) \cdot 2(1+2C_0)/\omega_C))$. The set $F(\gamma)$ being compact (by Proposition 2.3.), the set $\bar{W} \subset F(\gamma)$ is a continuum such that $y_0, y_1 \in \bar{W}$ and $\operatorname{diam}_\sigma(\bar{W}) \leq \sigma(y_0, y_1) \cdot C^*$ where $C^* := 4(1 + 2C_0)/\omega_C$.

Due to the arbitrary choice of points $y_0, y_1 \in F(X)$ with $\sigma(y_0, y_1) \leq \delta^*$ the set $F(X)$ has bounded turning property (C^*, δ^*)-BT.

Theorem 3.1. is proved.

4 Main result

In this section the following result will be posed.

4.1. Theorem. *Let D be a domain in the extended complex plane $\bar{\mathbf{C}}$ and $F : D \to 2^{\bar{\mathbf{C}}}$ be a hyperinjective mapping with ω-BAD property. If all thick points of F are continua with bonded turning property (C, δ)-BT, then F is a single-valued η-quasimöbius mapping with the distortion function η depending only on the control function ω.*

The proof of the theorem is divided into several consecutive steps.

Recall that a point p in a connected space X is called *separating point* if the set $X \setminus \{p\}$ is not connected (see [11], §46.VIII).

4.2. Proposition. *Let X, Y be metric spaces and $F : X \to 2^Y$ be a hyperinjective continuous mapping with all thick points being continua. If X has no separating points then $F(X)$ has no separating points as well.*

The proof will be given in paragraph 5.4.

4.3. Proposition. *Under the conditions in Theorem 4.1. let the mapping F be continuous. If a circle $\gamma \subset D$ bounds an open ball $B_0 \subset D$, then $F(\gamma)$ is a Jordan curve of bounded turning BT.*

Proof. By Proposition 2.3. $F(\gamma)$ is a continuum. By Theorem 3.1(i) $F(\gamma)$ is locally connected, and by Proposition 4.2. it has no separating points.

The set $B_1 := D \setminus \bar{B}_0$ is a domain (see [13], Theorem 10-2). By Proposition 2.3. the sets $F(B_0)$, $F(B_1)$ are connected and therefore each of them is contained in a component of $\bar{\mathbf{C}} \setminus F(\gamma)$.

Let us assume that a point $y_0 \in F(\gamma)$ is not a limit point for $F(B_j)$ where $j \in \{0, 1\}$. Then $y_0 \in F(x_0)$ for $x_0 \in \gamma$ and there exists an open ball U centered at y_0 such that $U \cap F(B_j) = \emptyset$. Since F is continuous, it follows that the set $V := \{x \in D : F(x) \cap U \neq \emptyset\}$ is an open neighbourhood of x_0. Hence there exists a point $x_1 \in V \cap B_j$ such that $F(x_1) \cap U \neq \emptyset$ and $F(x_1) \cap F(B_j) \neq \emptyset$. It contradicts to $U \cap F(B_j) = \emptyset$.

Thus every point $y_0 \in F(\gamma)$ is a limit point both for $F(B_0)$ and for $F(B_1)$.

Recall (see [11], §52.IV) that *θ-curve* is a join $\tau_1 \cup \tau_2 \cup \tau_3$ of three Jordan arcs with common ends a, b such that $\tau_i \cap \tau_j = \{a, b\}$ for all $i, j \in \{1, 2, 3\}$, $i \neq j$. Let us show that $F(\gamma)$ contains no θ-curves.

If a θ-curve $\tau = \tau_1 \cup \tau_2 \cup \tau_3$ contains in $F(\gamma)$, then it divides $\bar{\mathbf{C}}$ into three non-intersecting domains G_{12}, G_{13}, G_{23} with boundaries $\partial G_{ij} = \tau_i \cup \tau_j$ (see [11], §61.II, Theorem 2). Then for some indices $j, k \in \{i, j, k\} = \{1, 2, 3\}$ the set $F(B_0) \cup F(B_1)$ intersects with G_{jk}. Since every point of the arc τ_i is a limit point for each of connected sets $F(B_0)$ and $F(B_1)$, we obtain the inclusion $F(B_0) \cup F(B_1) \subset G_{ij} \cup G_{ik}$. But $G_{ij} \cup G_{ik}$ does not intersect G_{jk}, and it contradicts to $F(B_0) \cup F(B_1) \cap G_{jk} \neq \emptyset$. Thus there is no θ-curves in $F(\gamma)$.

It is known (see [11], §52.IV, Theorem 1), that if a non-degenerate locally connected continuum in $\bar{\mathbf{C}}$ has no separating points and does not contain θ

curves, then it is a Jordan curve. It means that $F(\gamma)$ is a Jordan curve, and it has bounded turning BT by Theorem 3.1(ii).

Proposition 4.3. is proved.

4.4. Proposition. *If under the assumptions in Theorem 4.1. the mapping $F(\gamma)$ is continuous, then $F(x)$ is a singleton for every $x \in D$.*

Proof. For an arbitrary given point $x_0 \in D$ there exists an open disk $B \subset D$ with the boundary $\partial B = \gamma \subset D$. According to Proposition 4.3. the set $F(\gamma) \subset \bar{\mathbf{C}}$ is a Jordan curve. Then there exists a homeomorphism $g : \bar{\mathbf{C}} \to \bar{\mathbf{C}}$ such that $g(F(\gamma)) = \gamma$ (see, e.g., [11], §61.V, Corollary 2). The multivalued mapping $G = g \circ F : \gamma \to 2^\gamma$ is continuous and all its thick points are compact. According to Lemma 5.5. there exists an integer $N > 0$ such that $\#G(z) = N$ for each of thick points $G(z)$ with $z \in \gamma$. Since every thick point $G(z)$ is a continuum, $N = 1$. It means that all thick points $G(z)$ and hence all thick points $F(y)$, $y \in \gamma$, are singletons. In particular, $\#F(x_0) = 1$. Thus the mapping $F : D \to 2^{\bar{\mathbf{C}}}$ is actually a single-valued mapping $F : D \to \bar{\mathbf{C}}$ since y_0 was an arbitrary point in D.

Proposition 4.4. is proved.

4.5. Proof of Theorem 4.1. There is no assumption on F to be continuous in Theorem 4.1. However, the mapping $F^0 : D \to 2^{\bar{\mathbf{C}}}$ in Lemma 5.6. is continuous and possesses ω-BAD property with the same control function ω. Since the topological limit of continua with bounded turning (C, δ)-BT is either a single point or a continuum with bounded turning (C, δ)-BT, the mapping F^0 satisfies the conditions in Proposition 4.4. Thus F^0 is a single-valued mapping $F^0 : D \to \bar{\mathbf{C}}$ with ω-BAD property. Then the mapping F^0 is η-quasimöbius mapping where the control function η depends only on ω (see [5], Theorem 6.2). In particular, F^0 is K-quasiconformal mapping whith the coefficient of quasiconformality K depending only on ω.

For every point $z_0 \in D$ the continuum $F(z_0)$ contains the point $w_0 = F^0(z_0)$. If there exists a point $w \in F(z_0) \setminus \{w_0\}$, then $w = F^0(z)$ where $z \neq z_0$. This contradicts the hyperinjectivity of F. It means that $F(z_0) = \{F^0(z_0)\}$ and that $F(z) \equiv F^0(z)$ at every point $z \in D$.

Hence F is a single-valued mapping and it is η-quasimöbius with η depending only on ω.

The proof of Theorem 4.1. is complete.

5 Appendix

5.1. *Proof for Lemma 2.2.* For connected spaces it was noticed in [10], §2, p.562. Let the metric space (X, ρ) has the bounded turning property (c, δ)-BT and $U \subset X$ be an open neighbourhood of an arbitrary given point $p \in X$. Consider a closed ball $\bar{B}(p, r)$ with radius $r < \delta$. Let U_0 be the component of U such that $p \in U_0$. Because of (c, δ)-BT property every point $x \in B(p, r/c)$ may be connected to p by a continuum $\tau \subset \bar{B}(p, r)$. Since U_0 is the maximal connected set in U that contains the point p, we have $x \in \tau \subset U_0$. Thus

$B(p, r/c) \subset U_0$, and p is the inner point in U_0. It means that $U_0 \subset U$ is the desired connected neibourhood of p.

5.2. (see [11], §17.III). In topological spaces X and Y, the miltivalued mapping $F : X \to 2^Y$ with closed thick points is called to be *lower semicontinuous* if the set $\{x \in X : F(x) \cap U \neq \emptyset\}$ is open for an arbitrary open set $U \subset Y$. This mapping is said to be *upper semicontinuous* if the set $\{x \in X : F(x) \subset U\}$ is open for an arbitrary open set $U \subset Y$. The mapping $F : X \to 2^Y$ is said to be *continuous* if it is both upper and lower semicontinuous.

If Y is a compact metric space, then the exponential topology (i.e. Vietoris topology) in 2^Y (see [11], §17.I) coinsides with the metric topology defined by the Hausdorff distance (see [11], §21.7 and §42.II) as well as with topology generated in 2^Y by the operation of topological limit Lim (see [11], §42.II, note 1).

5.3. *Proof for Proposition 2.3(i).* Let an open covering $\mathcal{U}$ of the set $F(K)$ be given. Then for each point $x_0 \in K$ the thick point $F(x_0)$ being compact has a finite subcovering $\mathcal{U}(x_0)$. The set $U(x_0) = \cup\{U \in \mathcal{U}(x_0)\}$ is an open neibourhood of thick point $F(x_0)$. Then the upper semicontinuous property of the mapping F implies the existence of an open neibourhood $V(x_0)$ of x_0 such that $F(V(x_0)) \subset U(x_0)$. Thus we obtain the open covering $\{V(x_0) : x_0 \in K\}$ of the compact K, and then there exists a finite subcovering $\{V(x_1), ..., V(x_k)\}$ of K. Then $\{\mathcal{U}(x_1), ..., \mathcal{U}(x_k)\} \subset \mathcal{U}$ is a desired finite subcovering of $F(K)$. Thus the compactness of $F(K)$ is proved.

Let the space Y be Hausdorff, and let us consider the left inverse mapping $f : F(K) \to K$. An arbitrary given close subset $E \subset K$ is compact (see [11], §41.II, Theorem 2]). Then $F(E)$ is compact in Y by above proof. Since every compact set in a Hausdorff space is closed (see [11], §41.II, Theorem 1]), then the set $f^{-1}(E) = F(E)$ is closed. It means (see [11], §13.IV) that the mapping f is continuous.

Proof for Proposition 2.3(ii). The restriction $F|\gamma$ remains to be hyperinjective and continuous. Suppose on the contrary that $F(\gamma)$ is not connected. Then $F(\gamma) = U_1 \sqcup U_2$ for nonempty open subsets $U_1, U_2 \subset F(\gamma)$ with $U_1 \cap U_2 = \emptyset$. Since $F|\gamma$ is continuous, we get three pairwise non-intersecting open sets $V_1 = \{x \in \gamma : F(x) \subset U_1\}$, $V_2 = \{x \in \gamma : F(x) \subset U_2\}$, and $V_{12} = \{x \in \gamma : F(x) \cap U_1 \neq \emptyset \neq F(x) \cap U_2\}$. Since the set γ is connected and $\gamma = V_1 \sqcup V_2 \sqcup V_{12}$, just only one of the sets V_1, V_2, V_{12} is not empty. There exists at least one thick point $F(x_0)$ which is connected and therefore it cannot intersect both U_1 and U_2. It means that $F(x_0) \subset U_j$ for some $j \in \{1, 2\}$. Clearly, we may assume that $F(x_0) \subset U_1$. Then $x_0 \in V_1 \neq \emptyset$ and $V_2 = V_{12} = \emptyset$. Thus $\gamma = V_1$ and $F(\gamma) \cap U_2 = \emptyset$. It means that $U_2 = \emptyset$, and it contradicts our assumption. It follows that $F(\gamma)$ is a connected set.

5.4. *Proof for Proposition 4.2.* The set $F(X)$ is compact (by Proposition 2.3.), so we can assume that $Y = F(X)$. Let us assume (on the contrary) that a point $y_0 \in F(x_0)$ is separating point in Y.

The point x_0 is not separating in X, so the set $X \setminus \{x_0\}$ is connected. Then (by Proposition 2.3.) the set $F(X \setminus \{x_0\}) = Y \setminus F(x_0)$ is also connected and is contained in a component U_0 of the set $Y \setminus \{y_0\}$. Being a component the set U_0 is closed with respect to $Y \setminus \{y_0\}$ (see [11], §46.III, Theorem 1). Then the set $V_0 := (Y \setminus \{y_0\}) \setminus U_0$ is open with respect to $Y \setminus \{y_0\}$ and non-empty according to our assumption. Since $Y \setminus \{y_0\}$ is open in Y, then V_0 is open in Y as well. Besides, $V_0 = (Y \setminus \{y_0\}) \setminus U_0 \subset (Y \setminus \{y_0\}) \setminus (Y \setminus F(x_0)) = F(x_0) \setminus \{y_0\}$. Since the mapping F is lower semicontinuous, then the set $\{x \in X : F(x) \cap V_0 \neq \emptyset\} = \{x_0\}$ must be open in X. But the connected set X does not have isolated points. This contradiction completes the proof of Proposition 4.2.

5.5. Lemma (see [9], Lemma 1.3). *Let $X = Y = \{z \in \bar{\mathbf{C}} : |z| = 1\}$ and $F : X \to 2^Y$ be a hyperinjective mapping from a circle $X \subset \bar{\mathbf{C}}$ into a circle $Y \subset \bar{\mathbf{C}}$. If F is continuous and all thick points $F(x)$ are compact, then there exists an integer $N > 0$ and a homeomorphism $\varphi : Y \to Y$ such that the left inverse mapping $f : Y \to X$ is $f(y) = (\varphi(y))^N$.*

5.6. Lemma (see [7], Theorems 3.2 and 3.5). *Let X, Y be Ptolemaic Möbius spaces. Assume that a hyperinjective mapping $F : X \to 2^Y$ possesses the ω-BAD property and all it's thick points $F(x)$ are closed in Y. Then at every limit point $x_0 \in X$ there exists a topological limit*

$$L(x_0) := \mathrm{Lim}_{\ x \to x_0;\ x \neq x_0} F(x) \subset F(x_0).$$

Moreover, the mapping

$$F^0(x) := \begin{cases} F(x), & \text{if } x \text{ is an isolated point in } X; \\ L(x), & \text{if } x \text{ is a limit point} \end{cases}$$

is continuous and possesses the ω-BAD property witn the same control function ω.

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