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SPECTRUM OF A PROBLEM ABOUT THE FLOW OF A POLYMERIC VISCOELASTIC FLUID IN A CYLINDRICAL CHANNEL (VINOGRADOV-POKROVSKI MODEL)

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Abstract: We study the linear stability of a resting state for flows of incompressible viscoelastic polymeric fluid in an infinite cylindrical channel in axisymmetric perturbation class. We use structurally-phenomenological Vinogradov-Pokrovski model as our mathematical model.

We formulate two equations that define the spectrum of the problem. Our numerical experiments show that with the growth of perturbations frequency along the channel axis there appear eigenvalues with positive real part for the radial velocity component of the first spectral equation. That guarantees linear Lyapunov instability of the resting state.

Keywords: incompressible viscoelastic polymeric medium, rheological correlation, resting state, linearized mixed problem, Lyapunov stability.

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1 Introduction

To study the flows of an incompressible viscoelastic polymeric fluid in an infinite cylindrical channel we use structural-phenomenological Vinogradov-Pokrovski model as a base [1, 2]. This model interprets polymeric medium as a suspension of polymer macromolecules moving in an anysotropic fluid consisting of, e.g., solvent and other macromolecules. The environment effects on a chosen macromolecule is approximated by the impact on a chain of brownian particles, each of which is a sufficiently large part of the macromolecule. It turns out that the formulated physical model is an effective way of describing slow relaxation processes in mediums with linear polymers.

Using a mechanical analogy we call the brownian particles "beads" and the analogue of the elastic powers between the particles "springs". In the simplest case when the macromolecule is modelled as a "dumbbell"("dumbbell" is two beads connected by a spring), we formulate the system of differential correlations (Vinogradov-Pokrovski model):

$$\rho(\frac{\partial}{\partial t}v_i + v_l\frac{\partial}{\partial x_l}v_i) = \frac{\partial}{\partial x_k}\sigma_{il}, \quad \frac{\partial v_i}{\partial x_i} = 0, \tag{1}$$

$$\sigma_{il} = -p\delta_{il} + 3\frac{\eta_0}{\tau_0}a_{il}, \tag{2}$$

$$\frac{d}{dt}a_{il} - v_{ij}a_{jl} - v_{lj}a_{ji} + \frac{1+(k-\beta)I}{\tau_0}a_{il} = \frac{2}{3}\gamma_{il} - \frac{3\beta}{\tau_0}a_{ij}a_{jl}, \tag{3}$$

$$I = a_{11} + a_{22} + a_{33}, \quad \gamma_{il} = \frac{v_{il}+v_{li}}{2}, \quad i,l = 1,2,3. \tag{4}$$

Here ρ is polymer density, v_i is i-th velocity component, σ_{il} is stress tensor, p is pressure; η_0, τ_0 are initial values of shear viscosity and relaxation time for viscoelastic component, v_{ij} is velocity gradient tensor, a_{il} is symmetric anisotropy stress tensor; γ_{il} is symmetrized velocity gradient tensor, where components of the velocity gradient tensor $\nabla \otimes v$ are calculated as follows: $v_{il} = \frac{\partial v_i}{\partial x_l}$, $i, l = 1, 2, 3$; k and β are phenomenological parameters that take into account the size and the form of a macromolecule ball. Equations (1) are motion equation and incompressibility condition, and equations (2)-(3) are rheological correlation, that connects kinematic characteristics of the flow with its thermodynamic parameters; for each component a_{il} the sum of the first three terms in the left part of equality (3) is the so called upper convective derivative or Oldroyd derivative [3], $\frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{v}, \nabla)$ is material derivative.

Note that the accepted physical representation of a polymeric medium allows us to describe its main rheological properties: the decrease of viscosity and the first difference of normal stresses with the growth of shear velocity, the growth of stretching viscosity to a certain limit with the growth of deformation velocity.

Moreover, unlike the known models FENE-R [4], FENE-CR [5] that take into account additional physical mechanisms reflecting the behaviour features of a studied material: boundedness and nonlinearity of a spring elongation, connected to the finite length of a macromolecule and the existence of weaves and engagements in it, which obstruct its uniform and infinite elongation (instead of a Hooke law the nonlinear law of a spring elasticity is used); or RHL-model [6] that takes into account potential barriers, that slow down the transition from one equilibrium configuration to the other (additional force of an inner resistance is introduced), the Pokrovski-Vinogradov model allows us to acquire nonzero values of the second difference for normal stresses. Specifically, it tries to take into account the anisotropy effect of the chosen molecule environment that is caused by its elongation and orientation in space during the flow process of its macromolecule chains.

Rheological properties, predicted by the Pokrovski-Vinogradov model with parameters $k = 1, 2\beta$, that guarantee monotone of a flow curve, are qualitatively and quantitatively agree with the experimental data for melts and solutions of polymers [7, 8, 9].

A number of works [10, 11] studied the linear Lyapunov stability of Poiseuille-type flows in an infinite plane channel (the pressure drop on a segment doesn't depend on time) for the model (1)-(4), as well as for its generalization on the case of nonisometric flow of an incompressible weakly conducting polymeric fluid with the existence of a negative space charge [12, 13, 14, 15] and on the case of nonisothermic model with the additional external interaction of a uniform magnetic field [16, 17, 18, 19].

In particular, in work [10] it was proven that the Poiseuille-type flow for the model (1)-(4) is linearly unstable in a perturbation class of generalized functions from the [20, 21] Schwartz space S' of the functions of slow growth with respect to the x variable, that changes along the channel side. I.e. the solution of a linearized problem is growing as an exponential function with the power st, $s = i\xi\hat{u} + \sqrt[3]{Q(y)}\xi^{\frac{2}{3}} + o(\xi^{\frac{2}{3}})$, for $|\xi| \to \infty$ (ξ is a dual variable for variable x with respect to the Fourier transform, $\hat{u}$ is a component of the base Poiseuille-type flow, $Q(y) \neq 0$, is a function, that depends on the problem parameters and its base solution, "o" is small o).

The question of stability of the resting state for nonisothermic model of the polymeric fluid flow in an infinite plane channel under the influence of an external magnetic field was studied in works [22, 23, 24]. The main result being that the resting state in the case of an absolute conductivity, i.e. vanishing of the parameter inversly proportional to the magnetic Reynolds number, and additionally vanishing of one of the dissipative coefficients is linearly unstable by Lyapunov.

For the analysis of the applicability of mathematical models for the description of real flows of polymeric fluid of special interest is the question about the stability of the resting state of the model. From the physical point of view this property is a necessary one.

The result of the work [25] was refined in the works [26, 27]. It states that the spectrum of a linearized with respect to the resting state mixed problem for the system (1)–(4) does not lie in an open right half-plane. One of the main results of these works is that the mixed problem has solutions with more than exponential growth $e^{Re\lambda t}$, $Re\lambda > 0$, $t \to +\infty$.

In the current work we study the location of the spectrum of a linear problem about the flow of a polymeric fluid in an infinite cylindrical channel. As a base solution we again chose the resting state and as a perturbation class we choose perturbations with axial symmetry and periodic with respect to the variable varying along the channel axis.

Note that the study of the flows of fluids of different nature in domains with cylindrical boundaries is fundamentally important not only from the point of view of a boundary geometry influencing the formation of characteristic features of the flow (alike the flows of viscous fluid (Navier-Stokes model) between the two coaxial rotating cylinders in the classic Taylor work [28]), but also to have the ability to experimentally check the picture of a flow predicted by the model.

The work is structured in the following way.

In the second paragraph we formulate the initial quasilinear and linearized with respect to the resting state models written in a cylindrical coordinate system. In the axisymmetric case the spectral problem essentially splits into two problems. The first is for determining the velocity components u (along the radius), w (along the cylinder axis) and the generalized pressure $\hat{\Omega}$. The second is for determining the velocity component v (along the angle coordinate).

It turns out that the first problem can be reduced to the spectral problem for the component u. In the end of the paragraph we formulate the main theorem of this work. The theorem statement is about two spectral problems.

The third paragraph is the proof of the main theorem.

Finally the last fourth paragraph is the graphical interpretation of a possible spectrum placement and graphs of the eigenfunctions for different values of the Reynolds (Re) and Weisenberg (Wi) numbers.

The main result of numerical experiments is that for a wide range of values of numbers Re and Wi the resting state is unstable in the class of axisymmetric flows of polymeric fluid.

2 Quasilinear and linearized models. Formulation of the main results

Following the monographs [1, 2, 29, 30, 31] and works [32, 33], we formulate the mathematical model for describing flows of an incompressible polymeric fluid in an infinite cylinder channel with round section (see Fig. 1).

We can write the model in a dimensionless form and in a cylindrical coordinate system as follows:

$$div\mathbf{u} = \frac{1}{r}\frac{\partial(ru)}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \varphi} + \frac{\partial w}{\partial z} = 0, \tag{5}$$

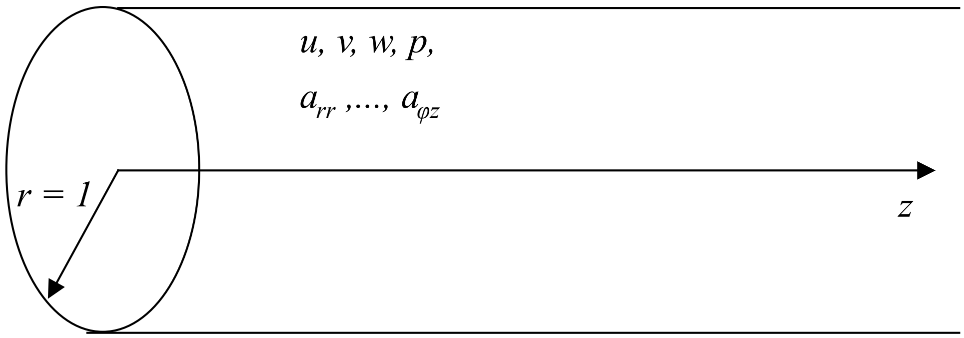

FIG. 1. Cylindrical channel and main parameters of the polymeric fluid flow

$$\frac{du}{dt} - \frac{v^2}{r} + \frac{\partial p}{\partial r} = \frac{1}{Re}\left(\frac{\partial a_{rr}}{\partial r} + \frac{1}{r}\frac{\partial a_{r\varphi}}{\partial \varphi} + \frac{\partial a_{rz}}{\partial z} + \frac{a_{rr} - a_{\varphi\varphi}}{r}\right), \tag{6}$$

$$\frac{dv}{dt} + \frac{uv}{r} + \frac{1}{r}\frac{\partial p}{\partial \varphi} = \frac{1}{Re}\left(\frac{\partial a_{r\varphi}}{\partial r} + \frac{1}{r}\frac{\partial a_{\varphi\varphi}}{\partial \varphi} + \frac{\partial a_{\varphi z}}{\partial z} + \frac{2a_{r\varphi}}{r}\right), \tag{7}$$

$$\frac{dw}{dt} + \frac{\partial p}{\partial z} = \frac{1}{Re}\left(\frac{\partial a_{rz}}{\partial r} + \frac{1}{r}\frac{\partial a_{\varphi z}}{\partial \varphi} + \frac{\partial a_{zz}}{\partial z} + \frac{a_{rz}}{r}\right), \tag{8}$$

$$\frac{da_{rr}}{dt} - 2\left(A_r\frac{\partial u}{\partial r} + \frac{a_{r\varphi}}{r}\frac{\partial u}{\partial \varphi} + a_{rz}\frac{\partial u}{\partial z}\right) + L_{rr} = 0, \tag{9}$$

$$\frac{da_{\varphi\varphi}}{dt} + 2\left(\frac{v}{r} - \frac{\partial v}{\partial r}\right)a_{r\varphi} - 2\left(\frac{1}{r}(u + \frac{\partial v}{\partial \varphi})A_\varphi + a_{\varphi z}\frac{\partial v}{\partial z}\right) + L_{\varphi\varphi} = 0, \tag{10}$$

$$\frac{da_{zz}}{dt} - 2\left(a_{rz}\frac{\partial w}{\partial r} + \frac{a_{\varphi z}}{r}\frac{\partial w}{\partial \varphi} + A_z\frac{\partial u}{\partial z}\right) + L_{zz} = 0, \tag{11}$$

$$\frac{da_{r\varphi}}{dt} + \left(\frac{v}{r} - \frac{\partial v}{\partial r}\right)A_r + \left(a_{r\varphi}\frac{\partial w}{\partial z} - a_{rz}\frac{\partial v}{\partial z} - \frac{A_\varphi}{r}\frac{\partial u}{\partial \varphi} - a_{\varphi z}\frac{\partial u}{\partial z}\right) + L_{r\varphi} = 0, \tag{12}$$

$$\frac{da_{rz}}{dt} - a_{rz}\left(\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z}\right) - \left(A_r\frac{\partial w}{\partial r} + \frac{a_{r\varphi}}{r}\frac{\partial w}{\partial \varphi} + \frac{a_{\varphi z}}{r}\frac{\partial u}{\partial \varphi} + A_z\frac{\partial u}{\partial z}\right) + L_{rz} = 0, \tag{13}$$

$$\frac{da_{\varphi z}}{dt} + \left(\frac{v}{r} - \frac{\partial v}{\partial r}\right)a_{rz} - \left(a_{\varphi z}\frac{\partial u}{\partial r} + A_z\frac{\partial v}{\partial z} + a_{r\varphi}\frac{\partial w}{\partial r} + \frac{A_\varphi}{r}\frac{\partial w}{\partial \varphi}\right) + L_{\varphi z} = 0. \tag{14}$$

In equations (5)–(14) t is time, u, v, w are components of a velocity vector $\mathbf{u}$ in a cylindrical coordinate system; p is hydrodynamic pressure; $a_{rr}, \dots, a_{\varphi z}$ are components of a symmetrical anisotropy tensor Π of a second rank [1, 2];

$$L_{rr} = K_I a_{rr} + \beta\|a_r\|^2, \quad L_{\varphi\varphi} = K_I a_{\varphi\varphi} + \beta\|a_\varphi\|^2, \quad L_{zz} = K_I a_{zz} + \beta\|a_z\|^2,$$

$$L_{r\varphi} = K_I a_{r\varphi} + \beta(a_r, a_\varphi),\ L_{rz} = K_I a_{rz} + \beta(a_r, a_z),\ L_{\varphi z} = K_I a_{\varphi z} + \beta(a_\varphi, a_z),$$

$$a_r = (a_{rr}, a_{r\varphi}, a_{rz}), \quad a_\varphi = (a_{r\varphi}, a_{\varphi\varphi}, a_{\varphi z}), \quad a_z = (a_{rz}, a_{\varphi z}, a_{zz}),$$

$$A_r = a_{rr} + Wi^{-1}, \quad A_\varphi = a_{\varphi\varphi} + Wi^{-1}, \quad A_z = a_{zz} + Wi^{-1},$$

$$K_I = Wi^{-1} + \bar{k}I/\beta, \quad I = a_{rr} + a_{\varphi\varphi} + a_{zz}, \quad \bar{k} = k - \beta,$$

the square of the vector norm $\|\cdot\|^2$ is the sum of its components, k, β, $0 < \beta < 1$ are phenomenological parameters of a rheological model [1, 2], $Re = (\rho u_H l)/\eta_0$ is the Reynolds number, $Wi = (\tau_0 u_H)/l$ is the Weisenberg number, $\rho(= const)$ is the density of the medium, η_0, τ_0 are initial values of shear viscosity and relaxation time [1, 2], l is the characteristic length, u_H is the characteristic velocity, $\triangle_{r,\varphi,z} = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$ is the Laplace operator, $\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \varphi} + w\frac{\partial}{\partial z}$.

The system (5)–(14) is written in a dimensionless form: variables t, r, z, u, v, w, p, $a_{rr}, \dots, a_{\varphi z}$ are related to $l/u_H, l, u_H, \rho u_H^2, \frac{Wi}{3}$ correspondingly.

We state that the no-slip condition hold on the boundary $r = 1$

$$\mathbf{u} = 0, \tag{15}$$

and on the cylinder axis, i.e. for $r = 0$, there are boundedness conditions on all the unknown variables $u, v, w, p, \alpha_{rr}, \dots, \alpha_{\varphi z}$.

As a base solution we choose the resting state

$$\mathbf{u} = 0, \quad p = p_0 - const, \quad \alpha_{rr} = 0, \dots, \alpha_{r\varphi} = 0.$$

Linearizing the boundary problem (5)–(15) with respect to the chosen solution results in a following problem (small perturbations of the components of the solution are written the same as initial variables)

$$Ru + \frac{1}{r}v_\varphi + w_z = 0, \tag{16}$$

$$u_t + \Omega_r = \frac{1}{r}(\alpha_{r\varphi})_\varphi + (\alpha_{rz})_z + \frac{\alpha_{rr} - \alpha_{\varphi\varphi}}{r}, \tag{17}$$

$$v_t + \frac{1}{r}\Omega_\varphi = (\alpha_{r\varphi})_r + \frac{1}{r}(\alpha_{\varphi\varphi} - \alpha_{rr})_\varphi + (\alpha_{\varphi z})_z + \frac{2\alpha_{r\varphi}}{r}, \tag{18}$$

$$w_t + \Omega_z = (\alpha_{rz})_r + \frac{1}{r}(\alpha_{\varphi z})_\varphi + (\alpha_{zz} - \alpha_{rr})_z + \frac{\alpha_{rz}}{r}, \tag{19}$$

$$\Lambda\alpha_{rr} = 2æ^2 u_r, \tag{20}$$

$$\Lambda\alpha_{\varphi\varphi} = \frac{2}{r}(u + v_\varphi)æ^2, \tag{21}$$

$$\Lambda\alpha_{zz} = 2æ^2 u_z, \tag{22}$$

$$\Lambda\alpha_{r\varphi} = \frac{æ^2}{r}u_\varphi - æ^2\left(\frac{v}{r} - v_r\right) \tag{23}$$

$$\Lambda\alpha_{rz} = æ^2(w_r + u_z), \tag{24}$$

$$\Lambda\alpha_{\varphi z} = æ^2(v_z + \frac{w_\varphi}{r}). \tag{25}$$

$$\mathbf{u} = 0, \quad \text{for } r = 1.$$

Here $\alpha_{rr} = \frac{a_{rr}}{Re}, \dots, \alpha_{\varphi z} = \frac{a_{\varphi z}}{Re}$, $æ^2 = \frac{1}{W Re}$, $R = \frac{\partial}{\partial r} + \frac{1}{r}$, $\Lambda = \frac{\partial}{\partial t} + \frac{1}{W}$, $\Omega = p - \alpha_{rr}$.

Remark 1. *For the function Ω, i.e. generalized "pressure", the following holds*

$$D_0\Omega = \left(\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} - \frac{1}{r}\frac{\partial}{\partial r}\right)(\alpha_{\varphi\varphi} - \alpha_{rr}) + \\ + 2R\left(\frac{\partial}{\partial z}\alpha_{rz} + \frac{1}{r}\frac{\partial}{\partial\varphi}\alpha_{r\varphi} + \frac{2}{r}\frac{\partial}{\partial\varphi}\left(\frac{\partial}{\partial z}\alpha_{\varphi z} + \frac{1}{r}\alpha_{r\varphi}\right)\right), \tag{26}$$

where $D_0 = \triangle = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2} = R^2 + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}$.

We will be looking for a solution of the problem (16)–(25) in the special form:

$$\begin{aligned} u(t, r, \varphi, z) &= u(r)\exp\{\lambda t + inz + im\varphi\}, \dots, \\ \alpha_{\varphi z}(t, r, \varphi, z) &= \alpha_{\varphi z}(r)\exp\{\lambda t + inz + im\varphi\}, \end{aligned} \tag{27}$$

where $\lambda = \eta + i\xi$, $\xi, \eta \in R^1$, $n, m \in Z$ are some parameters.

Then for the components of the anisotropy tensor under the additional condition

$$\lambda \neq -\frac{1}{Wi} \tag{28}$$

we get the following:

$$\begin{cases} \alpha_{rr} = \dfrac{u'}{\bar\lambda}, \quad \alpha_{\varphi\varphi} = \dfrac{u + imv}{r\bar\lambda}, \quad \alpha_{zz} = \dfrac{inu}{\bar\lambda}, \\ \alpha_{r\varphi} = \dfrac{1}{2\bar\lambda}\left\{\dfrac{imu}{r} + v' - \dfrac{v}{r}\right\}, \quad \alpha_{rz} = \dfrac{w' + inu}{2\bar\lambda}, \\ \alpha_{\varphi z} = \dfrac{i(nv + \frac{1}{r}mw)}{2\bar\lambda}, \quad \bar\lambda = \dfrac{\lambda + Wi^{-1}}{2æ^2}. \end{cases} \tag{29}$$

The other four equations of the system (16)–(24) are rewritten as follows:

$$Ru + i(\frac{m}{r}v + nw) = 0, \tag{30}$$

$$\Omega' = -\left(\frac{m^2+2}{2r^2\bar{\lambda}} + \frac{n^2}{2\bar{\lambda}} + \lambda\right)u + \frac{1}{r\bar{\lambda}}u' + \frac{im}{2\bar{\lambda}r}(v' - \frac{3v}{r}) + \frac{in}{2\bar{\lambda}}w', \tag{31}$$

$$R^2 v - (n^2 + \frac{1+3m^2}{r^2} + 2\lambda\bar{\lambda})v = \frac{2nm}{r}w - \frac{4im}{r^2}u + \frac{2im\bar{\lambda}}{r}\Omega, \tag{32}$$

$$R^2 w - (n^2 + \frac{m^2}{r^2} + 2\lambda\bar{\lambda})w = 2in\bar{\lambda}\Omega + 2mnv + 2n^2 u - \frac{2inu}{r}. \tag{33}$$

From (17), (26) and (29) it follows that

$$\begin{aligned} d_0\Omega = &\left(-\frac{m^2}{r^2} - \frac{1}{r}\frac{d}{dr}\right)\left(\frac{u+imv}{r\bar{\lambda}} - \frac{u'}{\bar{\lambda}}\right) + \\ &+ R\left(in\frac{w'+inu}{\bar{\lambda}} + \frac{im}{r}\frac{\frac{imu}{r} + v' - \frac{v}{r}}{\bar{\lambda}}\right) + \\ &+ \frac{im}{r}\left(-n\frac{nv + \frac{1}{r}mw}{\bar{\lambda}} + \frac{1}{r}\frac{\frac{imu}{r} + v' - \frac{v}{r}}{\bar{\lambda}}\right) - n^2\left(\frac{inu}{\bar{\lambda}} - \frac{u'}{\bar{\lambda}}\right), \end{aligned} \tag{34}$$

$$\widetilde{\Omega}' = \frac{imv'}{2\bar{\lambda}} - \frac{inw'}{2\bar{\lambda}} \quad \text{for } r = 1, \tag{35}$$

where

$$\widetilde{\Omega} = \Omega - \frac{u}{r\bar{\lambda}} - \frac{in}{\bar{\lambda}}w. \tag{36}$$

Here

$$d_0 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} - n^2 = R^2 - \frac{m^2}{r^2} - n^2.$$

In an axisymmetric case, when $m = 0$, which is the main interest to us, the system (30)–(33) is simplified by splitting into two independent subsystems. That and the boundary conditions (25) leads us to the following two spectral boundary problems:

$$\begin{cases} (d_1 - 2\lambda\bar{\lambda})v = 0, \\ v = 0 \quad \text{for } r = 1, \\ |v(0)| < \infty. \end{cases} \tag{37}$$

$$\begin{cases} Ru + inw = 0, \\ (d_0 - 2\lambda\bar{\lambda})w - 2in\bar{\lambda}\hat{\Omega} - 2n^2 u + n^2 w = 0, \\ \hat{\Omega}' + \dfrac{2\lambda\bar{\lambda} + n^2}{2\bar{\lambda}}u = 0, \\ u = w = 0 \quad \text{for } r = 1, \\ |u(0)| < \infty, \quad |w(0)| < \infty. \end{cases} \tag{38}$$

Here

$$d_1 = R^2 - n^2 - \frac{1}{r^2}, \quad d_0 = R^2 - n^2, \quad \hat{\Omega} = \widetilde{\Omega} + \frac{in}{2\bar{\lambda}} w,$$

function $\widetilde{\Omega}$ can be represented through Ω due to (35). The correlation (34) takes the following form:

$$d_0 \widetilde{\Omega} = \frac{in^3}{\bar{\lambda}}(w - u).$$

Assume the parameter n is non-zero. Than by differentiating the second equation from the system (38) and replacing $\hat{\Omega}'$ by using the third equation from the system (38) we can also replace w using the first equation and get to the spectral problem for one component u:

$$\begin{cases} u^{IV} + \frac{2}{r}u''' + u''\left(-\frac{3}{r^2} - 2\lambda\bar{\lambda}\right) + u'\left(\frac{3}{r^3} - \frac{2\lambda\bar{\lambda}}{r} + 2in^3\right) + \\ + u\left(-\frac{3}{r^4} + \frac{2\lambda\bar{\lambda}}{r^2} + n^2(2\lambda\bar{\lambda} + n^2)\right) = 0, \\ |u(0)| < \infty, \quad \left|(u' + \frac{1}{r}u)\big|_{r=0}\right| < \infty, \\ u(1) = 0, \quad u'(1) + u(1) = 0. \end{cases} \tag{39}$$

When $n = 0$ the problem (38) is significantly simplified.
Firstly (up to a constant)

$$u = 0, \quad \Omega = p = 1, \tag{40}$$

and secondly, the component w is the solution to the following boundary problem

$$\begin{cases} \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - 2\lambda\bar{\lambda}\right) w = 0, \\ w(1) = 0, \\ |w(0)| < \infty. \end{cases} \tag{41}$$

By introducing a new variable

$$\xi = i\sqrt{2\lambda\bar{\lambda}}r \tag{42}$$

into the equation of the boundary problem (41) and by using the boundary condition for $r = 0$ we get [34]

$$w = J_0(\xi) = J_0(i\sqrt{2\lambda\bar{\lambda}}r), \tag{43}$$

where J_0 is the Bessel function of the zeroth order. The second boundary condition leads to the equality

$$2\lambda\bar{\lambda} = -\mu_k^2, \; k = 1, 2, \dots \tag{44}$$

where μ_k are roots of the equation

$$J_0(\mu) = 0.$$

Using their symmetry about the origin we assume $\mu_k > 0$.

From the equation (44) we get

$$\lambda_{1,2}^{k,0} = \frac{-\frac{1}{Wi} \pm \sqrt{\frac{1}{Wi^2} - 4æ^2\mu_k^2}}{2}, \; k = 1, 2, \dots \tag{45}$$

It is obvious that $Re\lambda_{1,2}^{k,0} \leq -\sigma < 0$ for some constant σ.

In its turn the boundary problem (37) takes the following form:

$$\begin{cases} \left(\dfrac{d^2}{dr^2} + \dfrac{1}{r}\dfrac{d}{dr} - \dfrac{1}{r^2} - 2\lambda\bar{\lambda}\right) v = 0, \\ v(1) = 0, \\ |v(0)| < \infty. \end{cases} \tag{46}$$

By using the new variable (42) again we get the solution of the equation of the boundary problem (46)

$$v = J_1(/xi) = J_1(i\sqrt{2\lambda\bar{\lambda}}r), \tag{47}$$

where J_1 is the Bessel function of the first order.
Consequently

$$\lambda_{1,2}^{k,1} = \frac{-\frac{1}{Wi} \pm \sqrt{\frac{1}{Wi^2} - 4æ^2\nu_k^2}}{2}, \; k = 2, 3, \dots \tag{48}$$

where $J_1(\nu_k) = 0$.

In the equation (48) we exclude the case $\nu_1 = 0$ since in that case the variable change (42) is degenerate due to (28), namely $\lambda = 0$.

Remembering (29) we can find other components of the spectral vector functions. Note that these vector functions tend to zero with the growth of time due to (45), (48).

This leaves us with one special case when

$$\lambda = -\frac{1}{Wi} \quad (\text{see } (28)) \tag{49}$$

Lets return to the system (16)–(25). Assume that $n \neq 0$. Then we get the following solution [35]

$$\mathbf{u} = 0, \tag{50}$$

$$\alpha_{r\varphi} = -\frac{in}{r^2}\int_0^r \xi^2 \alpha_{\varphi z} d\xi, \tag{51}$$

$$\alpha_{rz}(\sqrt{n}r) = CJ_1(\sqrt{n}r) + \frac{i}{\sqrt{n}}J_1(\sqrt{n}r)\int_0^r \frac{Y_1(\sqrt{n}\eta)}{W(\sqrt{n}\eta)}\Big[(\alpha_{rr}-\alpha_{zz})_\eta + \\ + \frac{\alpha_{rr}\alpha_{zz}}{\eta}\Big]d\eta - \frac{i}{\sqrt{n}}Y_1(\sqrt{n}r)\int_0^r \frac{J_1(\sqrt{n}\eta)}{W(\sqrt{n}\eta)}\Big[(\alpha_{rr}-\alpha_{zz})_\eta + \frac{\alpha_{rr}\alpha_{zz}}{\eta}\Big]d\eta, \tag{52}$$

$$W(\eta) = \big[Y_1(\eta)J_1'(\eta) - Y_1'(\eta)J_1(\eta)\big]\Big|_{\eta=1}\frac{1}{\eta},$$

$$p = \frac{1}{in}\Big[(\alpha_{rz})' + \frac{\alpha_{rz}}{r}\Big] + \alpha_{zz}, \tag{53}$$

where $\alpha_{\varphi z}, \alpha_{rr}, \alpha_{zz}, \alpha_{\varphi\varphi}$ are arbitrary functions, $Y_1(\xi)$ is Bessel function of the second kind, C is some constant.

In case of $n = 0$, the solution has a simpler form:

$$\mathbf{u} = 0, \tag{54}$$

$$\alpha_{rz} = \alpha_{r\varphi} = 0, \tag{55}$$

$$p = \alpha_{rr} + \int_0^r \frac{\alpha_{rr} - \alpha_{\varphi\varphi}}{\xi}d\xi, \tag{56}$$

$\alpha_{\varphi z}, \alpha_{rr}, \alpha_{zz}, \alpha_{\varphi\varphi}$ are arbitrary functions.

Of course, components of the anisotropy tensor in the righthand parts of the formulae (51), (52), (53) and (56) should be such that the values in the lefthand side are bounded.

Remark 2. *There is no point in detailing the conditions on the components of the anisotropy tensor required to bound the solution for $r = 0$ since it is possible that $\lambda = -\frac{1}{Wi}$ is separated from other points of the spectrum which allows us to use the reverse Laplace transform due to (29). It is indeed so for $n = 0$: it is sufficient to remember the representation (45), (48) for the points of the spectrum. The general case when $n \neq 0$ is a subject for the special analysis.*

Returning to the general problems (39), (37) lets formulate the main theorem of this work.

Theorem 1. *Let $\lambda \neq -\frac{1}{Wi}$. Then the spectral equation for the eigenvalues of the boundary problem (39) has the following form*

$$u_2'(1)u_1(1) - u_1'(1)u_2(1) = 0,$$

the functions $u_1(r)$ and $u_2(r)$ are defined in (58), (59); (60), (61) correspondingly. The spectrum of the boundary problem (37) is defined by the correlation formula (64), i.e.

$$\lambda_{1,2}^{k,3} = \frac{-\frac{1}{Wi} \pm \sqrt{\frac{1}{Wi} - 4æ^2(\sigma_k^2 + n^2)}}{2}, \; k = 1, 2, 3, \dots$$

where σ_k are roots of the Bessel function J_1.

3 Proof of the main theorem

We begin by analysing the boundary problem (39), namely by studying the fundamental system of the differential equation. The main idea here is the information about the basis of a defining Euler equation [36, 37]

$$\tilde{u}^{IV} + \frac{2}{r}\tilde{u}''' - \frac{3}{r^2}\tilde{u}'' + \frac{3}{r^3}\tilde{u}' - \frac{3}{r^4}\tilde{u} = 0. \quad (57)$$

The basis of the equation (57) consists of the following functions:

$$\tilde{u}_1 = r^3, \quad \tilde{u}_2 = r, \quad \tilde{u}_3 = r\ln r, \quad \tilde{u}_4 = \frac{1}{r}.$$

The function $\tilde{u}_1$ defines the basis element of the equation from problem (39) [37]:

$$u_1 = \sum_{k=0}^{\infty} a_k r^{k+3} = a_0 r^3 + a_1 r^4 + a_2 r^5 + \dots, \quad (58)$$

where u_1 is an entire function, $-\infty < r < +\infty$.

Coefficients a_i can be found:

$$\begin{aligned} &a_0 = 1, \quad a_1 = 0, \quad a_2 = \frac{1}{24}\lambda\bar{\lambda}, \quad a_3 = -\frac{2}{175}in^3, \\ &a_k = \frac{2\lambda\bar{\lambda}}{(k+2)(k+4)}a_{k-2} - \frac{2in^3}{(k+2)^2(k+4)}a_{k-3} - \\ &-\frac{n^2(2\lambda\bar{\lambda}+n^2)}{k(k+2)^2(k+4)}a_{k-4},\ k = 4, 5, 6, \dots \end{aligned} \quad (59)$$

The next element of basis u_2 corresponds to the function $\tilde{u}_2$, the other elements don't satisfy the boundary conditions for $r = 0$.
Here is a characteristic equation for (57):

$$\alpha(\alpha-1)(\alpha-2)(\alpha-3) + 2\alpha(\alpha-1)(\alpha-2) - 3\alpha(\alpha-1) + 3\alpha - 3 = 0.$$

It's roots are $\alpha_1 = 3$, $\alpha_{2,3} = 1$. They define three solutions $\tilde{u}_1$, $\tilde{u}_2$ and $\tilde{u}_3$, but there is a problem for calculating u_2 due to the fact that the difference between roots is integer and non-negative $\alpha_1 - \alpha_2 = 2$. In this case, generally speaking, the method of indeterminate coefficients doesn't give us the recurrent correlation similar to (59). So we should use the Frobenius method to construct u_2 [37]. But in our case the coefficient c_1 before r^2 equals zero and the method of indeterminate coefficients can be applied.

So,

$$u_2 = \sum_{k=0}^{\infty} c_k r^{k+1} = c_0 r + c_1 r^2 + c_2 r^3 + \dots, \quad (60)$$

$$c_0 = 1, \quad c_1 = 0, \quad c_2 = 0, \quad c_3 = -\frac{2}{27} i n^3,$$
$$c_k = \frac{2\lambda\bar{\lambda}}{k(k-2)} c_{k-2} - \frac{2in^3}{k^2(k-2)} c_{k-3} - \frac{n^2(2\lambda\bar{\lambda} + n^2)}{k^2(k-2)(k+2)} c_{k-4}, \; k = 4, 5, 6 \dots \tag{61}$$

That means that the set of solutions of the equation (39) which satisfy the boundary conditions for $r = 0$ has the following representation:

$$u(r) = C_1 u_1(r) + C_2 u_2(r), \tag{62}$$

where C_1, C_2 are arbitrary constants, $u_1(r)$, $u_2(r)$ are entire functions of the form (58) with coefficients (59) or (60) with coefficients (61).

The boundary conditions for $r = 1$ lead us to the spectral equation for the eigenvalues λ:

$$u_2'(1)u_1(1) - u_1'(1)u_2(1) = 0, \tag{63}$$

proving the first statement of the theorem 1.

Analysis of the spectral boundary problem (37) can be done by analogy with the analysis of the boundary problem (46). That gives us

$$\lambda_{1,2}^{k,3} = \frac{-\frac{1}{Wi} \pm \sqrt{\frac{1}{Wi} - 4æ^2(\sigma_k^2 + n^2)}}{2}, \; k = 1, 2, 3, \dots \tag{64}$$

where σ_k are the roots of the equation

$$J_1(\sigma_k) = 0.$$

Similar to the problem (37) (see inequality (45)) it is clear that

$$Re\lambda_{1,2}^{k,3} \leq \sigma_1 < 0, \tag{65}$$

where σ_1 is some constant. Which completes the proof of theorem 1.

Remark 3. *Due to the inequalities (65), the component v tends to zero with the growth of variable t.*

4 Numerical study of the spectrum of the problem (39)

To illustrate the results formulated in paragraph 2 we study the problem (39), using numerical methods.

It is known, that spectra of discretized operators are depend significantly on the way they are discretized. If the discretization is unfortunate, then the calculated spectrum can be significantly different from the spectrum of the initial operator.

Monograph [38] uses matrix collocation derivatives to approximate the derivatives in the segment $[-1, 1]$ and uses Gauss-Lobatto points as nodes

$$\xi_k = \cos\left(\pi\left(1 - \frac{k-1}{N}\right)\right), \qquad k = 1, 2, \dots, N+1.$$

If the discrete analogue of the function $u(\xi)$ is defined in this nodes $u_k = u(\xi_k)$ and can be represented as $u = (u_1, \dots, u_{N+1})^T$, then its derivarive $u'(\xi_k)$ is approximated as a multiplication $D_1 u$, where D_1 is a matrix of size $(N+1)\times(N+1)$, and its elements are found through the formula

$$d_{ij}^{[1]} = \begin{cases} (-1)^{l+j} c_l/(c_j(\xi_l - \xi_j)), & l \neq j, \\ -\xi_j/(2(1-\xi_j^2)), & 2 \leq l = j \leq N, \\ (2N^2+1)/6, & l = j = 1, \\ -(2N^2+1)/6, & l = j = N+1, \end{cases}$$

$$c_j = \begin{cases} 2, & \text{for } j = 1, N+1, \\ 1 & \text{for } j = 2, \dots, N. \end{cases}$$

Higher order derivatives are obtained by multiplying $D_k = D_1^k$.

To adapt the collocation method for the spectral problem (39), we need to take into account that the problem is stated on the segment $0 < r < 1$. That means it needs to be linearly transformed $\xi = 2r - 1$ into the segment $-1 < \xi < 1$. Using the complex derivative formula $d/dr = 2d/d\xi$ we get that the approximation of the problem (39) derivative has the form $\hat{D}_1 = 2D_1$ for functions defined in nodes $r_k = (\xi_k + 1)/2$, where ξ_k are Gauss-Lobatto points.

Turning to discretezation of homogeneous boundary conditions we should note that the combination of the homogeneous Dirichlet and Neumann conditions can often be found in boundary problems and different applications. The mechanism for accounting for these conditions in collocation matrices is detailed in [38]. As an example the monograph uses the spectrum of the Orr-Sommerfeld operator for the plane-parallel Poiseuille flow.

In case of the problem (39) the function u satisfies homogeneous Dirichlet conditions on both boundaries $u(0) = u(1) = 0$. To take that into account we need to delete the first and the last row and column of the matrix D. As a result we get the matrix $\hat{D}$ of size $(N-1)\times(N-1)$, which is applied to the vector $(u_2, \dots, u_N)^T$.

Similar to the method described in [38] for the Neumann condition on both boundaries we will represent the function in the form $u(r) = (r-1)\upsilon(r)$, where $\upsilon(1) = 0$. From the obvious formula

$$\frac{d^k}{dr^k}u = k\frac{d^{k-1}}{dr^{k-1}}\upsilon + (r-1)\frac{d^k}{dr^k}\upsilon = k\frac{d^{k-1}}{dr^{k-1}}\left(\frac{1}{r-1}u\right) + (r-1)\frac{d^k}{dr^k}\left(\frac{1}{r-1}u\right)$$

we obtain correlations on the collocation derivatives for the function that satisfies the Dirichlet and Neumann conditions on the right boundary:

$$\tilde{D}_k = k\hat{D}_{k-1}\mathrm{diag}\left(\frac{1}{r_j - 1}\right) + \mathrm{diag}(r_j - 1)\hat{D}_k\mathrm{diag}\left(\frac{1}{r_j - 1}\right). \qquad (66)$$

To take into account the boundedness in zero condition for the collocation derivatives (see (39)), we use the preliminary information about the form of the solutions, that satisfy them. In the third paragraph we acquired the solutions to the equation of the problem (39) in the form of power series (see (58) and (60)). From them it follows that the following equivalence relations hold:

$$1 = \lim_{r\to 0} \frac{u_1(r)}{r} = \lim_{r\to 0} \frac{u_2(r)}{r^3}.$$

or

$$u_1(r) = r(1 + r\varphi_1(r)), \qquad u_2(r) = r^3(1 + r\varphi_2(r)),$$

where $\varphi_i(0) \neq 0$, $i = 1, 2$.

Note that functions r and r^3 satisfy the Euler equation

$$Lq = 0, \quad \text{where} \quad L = r^2 \frac{d^2}{dr^2} - 3r\frac{d}{dr} + 3 = 0,$$

and after we substitute the functions u_1 and u_2 in it the relative error has the form

$$\frac{Lu_i}{u_i} = r\Phi_i(r) \to 0 \text{ for } r \to 0.$$

Based on these facts we get that the satisfaction of boundary conditions on the left boundary for the solution u of the problem (39) is equivalent to the fact that u satisfies the equality

$$r^2 \frac{d^2u}{dr^2} - 3r\frac{du}{dr} + 3u = 0 \text{ for } r \to 0. \tag{67}$$

We will use this equality as the boundary condition.

Let's move on to the construction of the necessary matrix bundle. We have the equality $Lu = 0$, that holds asymptotically in zero (67), and the equation (see (39))

$$\left[\frac{d^4}{dr^4} + \frac{2}{r}\frac{d^3}{dr^3} - \frac{3}{r^2}\frac{d^2}{dr^2} + \left(\frac{3}{r^3} + 2in^3\right)\frac{d}{dr} + \left(\frac{-3}{r^4} + n^4\right) - \right.$$
$$\left. - 2\mu\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \left(\frac{1}{r^2} + n^2\right)\right)\right] u = 0, \tag{68}$$

where $\mu = \lambda\bar{\lambda}$, which is true for $0 < r < 1$.

We discretize both differential operators using matrices (66), which already take into account the homogeneous Dirichlet and Neumann conditions. This gives us

$$\left[\tilde{D}_2 - \text{diag}\frac{3}{r_k}\tilde{D}_1 + \text{diag}\frac{3}{r_k^2}\right] u = 0, \tag{69}$$

$$\left[\tilde{D}_4 + \operatorname{diag}\frac{2}{r_k}\tilde{D}_3 - \operatorname{diag}\frac{3}{r_k^2}\tilde{D}_2 + \left(\operatorname{diag}\frac{3}{r_k^3} + 2in^3 I\right)\tilde{D}_1 + \right.$$
$$\left. + \left(\operatorname{diag}\frac{-3}{r_k^4} + n^4 I\right) - 2\mu\left(\tilde{D}_2 + \operatorname{diag}\frac{1}{r_k}\tilde{D}_1 - \left(\operatorname{diag}\frac{1}{r_k^2} + n^2 I\right)\right)\right] u = 0, \tag{70}$$

where in this case u is the vector such that $u_k = u(r_k)$, I is the identity matrix of size $(N-1)\times(N-1)$.

We introduce the following matrices

$$A_0 = \tilde{D}_2 - \operatorname{diag}\frac{3}{r_k}\tilde{D}_1 + \operatorname{diag}\frac{3}{r_k^2}, \qquad B_0 = 0,$$

$$A_1 = D_4 + \operatorname{diag}\frac{2}{r_k}\tilde{D}_3 - \operatorname{diag}\frac{3}{r_k^2}\tilde{D}_2 + \left(\operatorname{diag}\frac{3}{r_k^3} + 2in^3 I\right)\tilde{D}_1 + \left(\operatorname{diag}\frac{-3}{r_k^4} + n^4 I\right),$$

$$B_1 = 2\left(\tilde{D}_2 + \operatorname{diag}\frac{1}{r_k}\tilde{D}_1 - \left(\operatorname{diag}\frac{1}{r_k^2} + n^2 I\right)\right).$$

Here B_0 is a zero matrix of size $(N-1)\times(N-1)$. Then the discretized equations (69), (70) can be written as

$$(A_0 - \mu B_0)u = 0, \qquad (A_1 - \mu B_1)u = 0.$$

correspondingly.
Since the equality (67) has an asymptotic character, we can unite both matrix equalities by changing first rows of matrices A_1, B_1 into first rows of matrices A_0, B_0. Thereby we point out that the closest to zero point r_1 satisfies condition (67), and every other one satisfies (68). We will denote this matrix bundle as $A - \mu B$.

Additionally we balance the bundle, which helps with the quality of the spectral calculations. The balancing procedure consists of the calculation of the nondegenerate matrix X such that

$$A = X\hat{A}, \;\; B = X\hat{B}, \;\; \hat{A}\hat{A}^* + \hat{B}\hat{B}^* = I.$$

To do this it is sufficient to make a QR decomposition

$$\operatorname{qr}\begin{pmatrix} A^* \\ B^* \end{pmatrix} = \begin{pmatrix} \hat{A}^* & * \\ \hat{B}^* & * \end{pmatrix}\begin{pmatrix} X^* \\ 0 \end{pmatrix}.$$

This operation doesn't change the bundle spectrum and the set of right eigenvectors.

Here is a further analysis plan.

We will use the standard software to calculate the spectrum of the matrix bundle $\hat{A} - \mu\hat{B}$ we got. Then we calculate approximate eigenvalues λ_j of the problem (39) as roots of the quadratic equation $\mu = \lambda(\lambda + 1/Wi)/2æ^2$. Figures 2, 3 illustrate the location of the spectrum points for large and small parameters Wi and Re. On this figures we can see that despite the several orders of magnitude difference between the parameters, the structure of the spectrum is very similar in both cases, the significant difference being the scale of the structure. It is important to note that if for $n = 1$ the whole spectrum lies strictly in the left half-plane, then already for $n = 50$ several eigenvalues lie in the right half-plane.

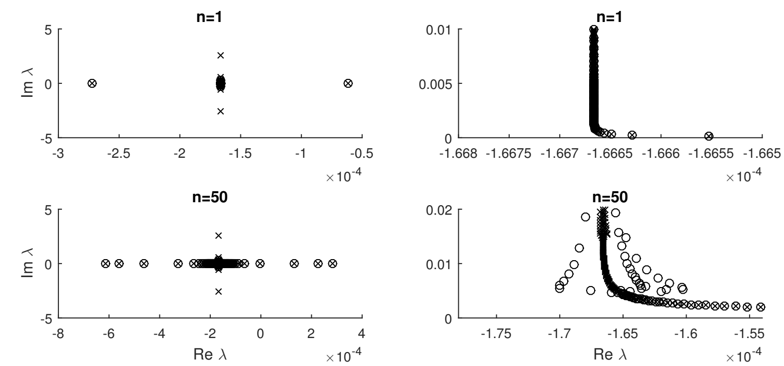

FIG. 2. Location of the spectrum points for $Wi = 3 \cdot 10^3$, $Re = 5 \cdot 10^3$, general picture to the right, local picture in in the zone of concentration of points of the spectrum to the left; eigenvalues are circles for $N = 100$, crosses for $N = 300$.

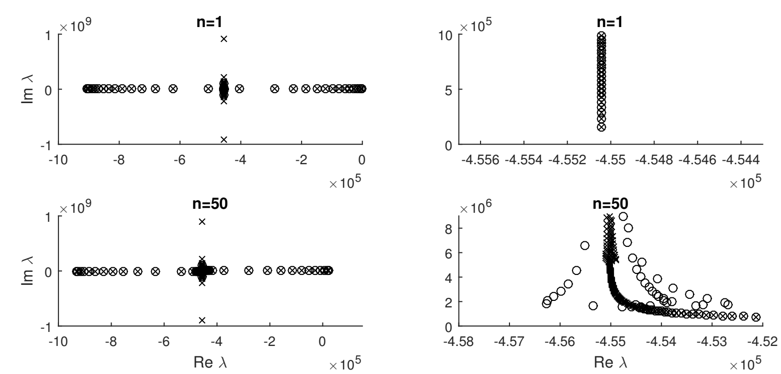

FIG. 3. Location of the spectrum points for $Wi = 1.1 \cdot 10^{-6}$, $Re = 10^{-2}$, general picture to the right, local picture in in the zone of concentration of points of the spectrum to the left; eigenvalues are circles for $N = 100$, crosses for $N = 300$.

Fig. 4 shows graphs of the maximum of the real parts of eigenvalues for different N and values of Re and Wi. Graphs are in the double logarithmic scale. All graphs have a pronounced minimum which corresponds to the moment of the transition of the eigenvalue with the maximal real part through the imaginary axis.

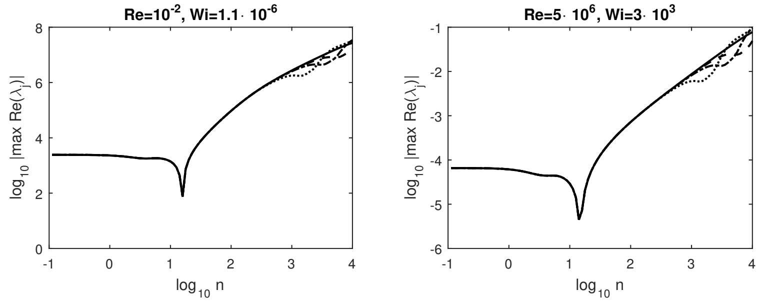

FIG. 4. Graph of the module of the maximum real part of the eigenvalue depending on n. Dotted line is $N = 100$, dash-dotted line – $N = 150$, dashed line is $N = 200$, solid line is $N = 300$.

Figures 5 and 6 show eigenvectors corresponding to eigenvalue with the maximum real part. These vectors are calculated for $Wi = 1.1 \cdot 10^{-6}$, $Re = 10^{-2}$, $N = 300$. For other values of Wi and Re there are almost no difference in eigenvectors compared to the illustrated ones.

We should note that the oscillation frequency of the eigenfunction significantly increases with the growth of n. Moreover, starting from a certain moment the fixed number of the collocation points N is not enough to represent the function that oscillates so frequently. If that happens then the numerical results for chosen N and n can't be considered correct. It seems that precisely that effect influences the difference between graphs on Fig. 4 for large n and different N.

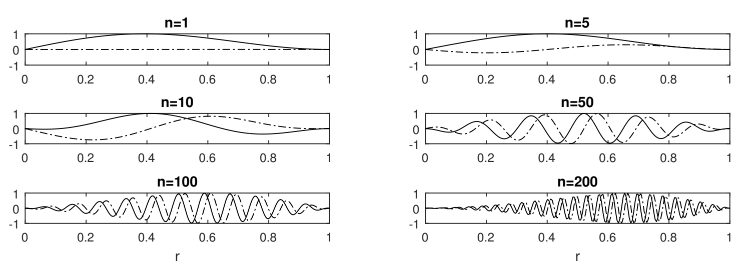

FIG. 5. Right eigenvectors, corresponding to the eigenvalue with the largest real part for $N = 300$. Solid line is the real part, dash-dotted line is the imaginary part.

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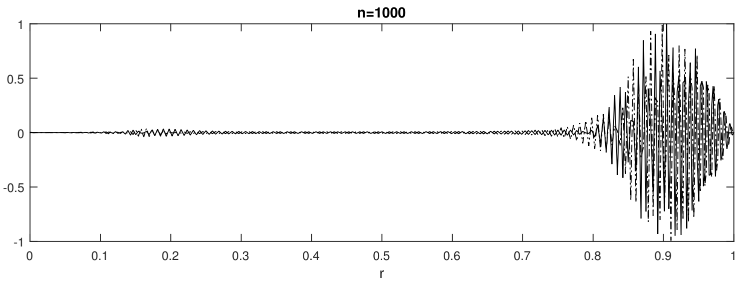

FIG. 6. Right eigenvectors, corresponding to the eigenvalue with the largest real part for $N = 300$, $n = 1000$. Solid line is the real part, dash-dotted line is the imaginary part.

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