S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

http://semr.math.nsc.ru

Том 20, №2, стр. 1295–1312 (2023) УДК 514.763.4
DOI 10.33048/semi.2023.20.078 MSC 53B35, 53C15, 53C55, 53D15

COMPLEX AND SYMPLECTIC GEOMETRY OF VECTOR BUNDLE MANIFOLDS

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Abstract. The aim of this paper is to explore the complex and symplectic geometries of vector bundle manifolds. We will construct an almost complex structure on total spaces of vector bundles, endowed with a complex structure, over an almost complex base. Then we give necessary and sufficient conditions for its integrability. Meanwhile, we accomplish a symplectic version of this construction. We construct almost symplectic structures on vector bundle manifolds and we characterize those which are symplectic on the total space. Finally, we apply the constructions to the case of tangent bundles and Whitney sums. In particular, we obtain an infinite family of non-compact flat Kähler manifolds.

Keywords: (almost) complex structure, symplectic structure, Kähler manifold, vector bundle, spherically symmetric metric.

Introduction

The Riemannian geometry of vector bundles has been deeply studied in various contexts. For example, many geometers have worked on the geometry of tangent bundles, starting with a systematic study of the Sasaki metric, and then introducing other metrics. An essential step in the topic was the classification of 'natural' Riemannian metrics on tangent bundles which has been accomplished using jets and natural differential operators, see [24] for details. This classification has lead to the huge class of g-natural metrics on tangent bundles. For details, we refer the reader to [1, 2, 3] and the references therein. Further, many Riemannian metrics on tangent bundles were generalized to vector bundles e.g. the Sasaki metric (cf. [22]), the Cheeger-Gromoll metric and generalized Cheeger-Gromoll metrics (cf.

Abbassi M.T.K., El Masdouri R., Lakrini I., Complex and symplectic geometry of vector bundle manifolds.

Received December, 1, 2020, published November, 25, 2023.

[12]). Recently, R. Albuquerque introduced a class of Riemannian metrics on vector bundle manifolds, namely the class of Spherically symmetric metrics (cf. [8]). For a profound study of different aspects of this class of metrics, we refer the reader to [4, 5, 8].

Nevertheless, little attention was given to complex and symplectic geometry of vector bundle manifolds. In the context of tangent bundles, using the decomposition of the tangent space to the tangent bundle, P. Dombrowski introduced an almost complex structure on tangent bundles to affine manifolds (cf. [18]). This almost complex structure is integrable if and only if the base is flat. In order to overcome the flatness, many generalizations of Dombrowski's structure have been introduced. For instance, R. Aguilar introduced isotropic almost complex structures on open sets of TM (cf. [7]), that he called isotropic almost complex structure. Their integrability have been studied in [7, 11]. In another context, M. I. Munteanu introduced another class of almost complex structures on tangent bundles and studied their integrability [30]. Other constructions were accomplished by M. Tahara, L. Vanhecke and Y. Watanabe in [36] and V. Oproiu in [31]. It is noteworthy that all of these constructions are limited only to the case of tangent bundles. On the other hand, it is well known that the cotangent bundle admits a natural symplectic structure which stems from the Liouville 1-form (cf. [9, 32]).

The class of vector bundle manifolds constitutes a large and interesting class of manifolds. Dombrowski's construction cannot be directly generalized. Nevertheless, when the base manifold is an almost complex manifold and the vector bundle possesses complex structures on fibers, one can construct almost complex structures on the total spaces. A symplectic version of those construction is also possible. More precisely, assume a vector bundle is endowed with a complex structure on fibers and the base manifold is endowed with an almost complex structure. Then, using the decomposition of the tangent spaces of the total space provided by the connection, we construct from the previous data an almost complex structure J^s (resp. an almost symplectic form Ω^s when the base is almost symplectic) which is compatible with spherically symmetric metrics (cf. [8]). Surprisingly, it turns out that the constructed almost complex structure and almost symplectic structure are associated to each other via the Sasaki metric ([1, 22, 34]). We will derive the necessary and sufficient conditions for both the integrability of J^s and closedness of Ω^s. In particular, we give necessary and sufficient conditions for the total space to be a Kähler manifold. In fact, based on a classification of compact flat Kähler manifolds of dimensions four and six, our constructions will allow us to give a whole new family of non-compact flat Kähler manifolds.

As direct applications to the constructions performed so far, we will explore the complex geometry of the Whitney sums of vector bundles, and in particular the complex geometry of tangent bundles of second order. Further, we study the case of the tangent bundle of a Kähler manifold and finally, we focus on the symplectic geometry of the tangent bundle of the cotangent bundle of a Riemannian manifold.

All geometric objects are smooth and smooth will always mean differentiable of class C^∞. All manifolds are assumed to be connected.

1. Preliminaries

Let (E, π, M) be a $\mathbb{K}$-vector bundle over an even dimensional manifold (unless otherwise stated, all dimensions are real), with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. If E is a real vector bundle, then a *complex structure* on E is a section $J \in \Gamma(\mathrm{End}(E))$ with $J^2 = -\mathrm{Id}_E$, where $\mathrm{End}(E)$ is the vector bundle of endomorphisms of E and Id_E is the identity of the vector bundle E (cf. [32]). On the other hand, a symplectic structure on a vector bundle E is a section of $E^* \otimes E^*$ (i.e. $\Omega \in \Gamma(E^* \otimes E^*)$) such that Ω_x is a symplectic form on E_x, for all $x \in M$, that is $\Omega_x : E_x \times E_x \longrightarrow \mathbb{R}$ is a non-degenerate skew-symmetric bilinear form. In this case (E, Ω) is said to be a *symplectic vector bundle* (cf. [27, 32]).

A real vector bundle admits a complex vector bundle structure if and only if it admits a complex structure (cf. [32]). Indeed, given a complex structure J on a real vector bundle E, the multiplication defined by $(a+ib).e = ae + bJe$, for $e \in E$ and $a+ib \in \mathbb{C}$, confers to E the structure of a complex vector bundle. Conversely, given a complex vector bundle, the multiplication by the complex unit i induces on the real underlying vector bundle a complex structure (cf. [32]).

Complex structures and symplectic structures on vector bundles are equivalent. Indeed, if Ω is a symplectic structure on a vector bundle, then there exists a fiber metric h on E *compatible* with Ω in the sense that the bundle isomorphism $E \longrightarrow E^*$ defined by $e \mapsto \Omega(e, .)$ is an isometry with respect to the dual metric to h, hence a complex structure is defined through $h(J_E\sigma, \delta) = \Omega(\sigma, \delta)$, for $\sigma, \delta \in \Gamma(E)$. Further, this complex structure satisfies the following properties

i) $\Omega(J_E\sigma, J_E\delta) = \Omega(\sigma, \delta)$. In this case, we say that J_E and Ω are *compatible*, and

ii) $h(J_E\sigma, J_E\delta) = h(\sigma, \delta)$. We say also that J_E and h are *compatible*.

We shall refer to this complex structure as the complex structure *associated* with Ω. Conversely, given a complex structure J_E on E, there exist a metric fiber h which is compatible with J_E, this gives a symplectic structure Ω, defined by $\Omega(\sigma, \delta) = h(J_E\sigma, \delta)$, which is compatible with h and J_E and said to be *associated with* J_E. For more details, see [32, 27].

Let (E, J_E) be a vector bundle endowed with a complex structure, a compatible fiber metric and the associated symplectic structure Ω. Define $H_x : E_x \times E_x \longrightarrow \mathbb{C}$ by $H(X, Y) := \Omega(X, J_EY) + i\Omega(X, Y)$, for $X, Y \in T_eE$. For every $e \in E$, the complex bilinear form H_e is a Hermitian inner product, hence H is said to be a *Hermitian metric* on E and (E, H) is said to be a *Hermitian vector bundle*.

If $E = TM$ is the tangent bundle to M, then a complex structure J_M on TM is said to be an *almost complex structure* on M. In this case (M, J_M) is said to be an *almost complex manifold*. The dimension of an almost complex manifold is necessarily even. Further, an almost complex structure induces a natural orientation of the underlying manifold. Accordingly, a symplectic structure on TM is a non-degenerate 2-form ω on M, and in this case ω is said to be an *almost symplectic structure* on M; furthermore, if ω is closed, then ω is said to be a *sympletic structure* and (M, ω) is said to be a *symplectic manifold*.

If (M, g) is a Riemannian manifold and J_M is an almost complex structure on it, then J_M is said to be *compatible* with g if it satisfies

$$g(J_MX, J_MY) = g(X, Y), \tag{1}$$

for all $X, Y \in T_xM$, $x \in M$. In this case the triple (M, g, J_M) is said to be an *almost Hermitian manifold*. This compatibility is the same as the compatibility in the sense of fiber metrics and complex structures on vector bundles.

Every complex manifold has a natural almost complex structure. Indeed, let M be a complex manifold and m its complex dimension. Identify $\mathbb{C}^m$ with $\mathbb{R}^{2m}$, and let (U, z^j), with $j = 1, ..., m$, a coordinate system, which yields real coordinates (x^j, y^j) such that $z^j = x^j + iy^j$. In this chart, consider the field of endomorphisms defined by $J_M : \frac{\partial}{\partial x^j} \longmapsto \frac{\partial}{\partial y^j}$ and $J_M : \frac{\partial}{\partial y^j} \longmapsto -\frac{\partial}{\partial x^j}$. Thus, J is an almost complex structure. An almost complex structure J_M on a $2m$-dimensional real manifold M is said to be *integrable* if J_M is induced by a, necessarily unique, complex structure.

The classical *Newlander-Nirenberg* theorem gives a tensorial characterization of integrable almost complex structures. Indeed, for vector fields $X, Y \in \mathfrak{X}(M)$, let

$$N(X,Y) = [X,Y] + J_M[J_M X, Y] + J_M[X, J_M Y] - [J_M X, J_M Y], \tag{2}$$

be the *Nijenhuis tensor* of an almost complex structure J_M. Then, J_M is integrable if and only if N vanishes identically (cf. [20]). An almost Hermitian manifold with an integrable complex structure is called a *Hermitian manifold*.

Let (M, g, J_M) be an almost Hermitian manifold, then we have the 2-form defined by

$$\omega(X,Y) = g(J_M X, Y) \tag{3}$$

for all vectors $X, Y \in TMX, Y \in T_xM$, $x \in M$. The 2-form ω is called the *fundamental form*. If the form ω is closed, that is a *symplectic form* on M, then (M, g, J_M) is said to be an almost Kähler manifold, and if J_M is integrable, then (M, g, J_M) is a *Kähler manifold*.

2. Constructions

Consider a real vector bundle (E, π, M) of rank k, over an n-dimensional manifold, endowed with:

(i) A fiber metric h,
(ii) A complex structure J_E compatible with h in the sense that

$$h_x(J_E(e_1), J_E(e_2)) = h_x(e_1, e_2),$$

for all $e_1, e_2 \in E_x$ and $x \in M$,
(iii) A connection ∇ which is compatible with h i.e. $\nabla h = 0$.

Denote by R^E the curvature of ∇ and by K the *connection map* (the *connector*) associated with ∇ (cf. [18, 25, 32]). Denote by Ω the symplectic structure associated to J_E given by $\Omega(\sigma, \delta) = h(J_E\sigma, \delta)$, for all $\sigma, \delta \in \Gamma(E)$. Further, assume that M is endowed with an almost complex structure J_M (we shall use the abbreviation **a.c.s**) and a Riemannian metric g such that the triple (M, g, J_M) is an almost Hermitian manifold and denote by ∇^M the Levi-Civita connection of (M, g) and by R its curvature tensor.

Denote by $\mathcal{H}$ (resp. $\mathcal{V}$) the *horizontal* (resp. *vertical*) subbundle. At each point $e \in E$, the tangent space T_eE splits as

$$T_eE = \mathcal{H}_e \oplus \mathcal{V}_e, \tag{4}$$

where $\mathcal{H}_e = \ker(K_e)$ (resp. $\mathcal{V}_e = \ker((\pi_*)_e)$ is the *horizontal* (resp. *vertical*) subspace. Further, the tangent bundle of E splits as

$$TE = \mathcal{H} \oplus \mathcal{V}.$$

The elements of $\mathcal{H}$ (resp. $\mathcal{V}$) are said to be *horizontal* (resp. *vertical*) vectors. Analogously, a vector field is said to be *horizontal* (resp. *vertical*) if it lies completely in $\mathcal{H}$ (resp. $\mathcal{V}$).

If $X \in T_xM$, with $x \in M$, then for every $e \in E$, there exist a unique horizontal vector $X^h_e \in \mathcal{H}_e$, with $\pi_* X^h_e = X$, called the *horizontal lift* of X at e. Accordingly, for every vector field $X \in \mathfrak{X}(M)$, there exists a unique horizontal vector field X^h, which is π-related to X, called the horizontal lift of X.

For $x \in M$ and $e, u \in \pi^{-1}(x)$, set $\gamma_{e,u}(t) = e + tu$ with $t \in (-\epsilon, \epsilon)$, then $\gamma_{e,u}(t) \in \pi^{-1}(x)$, for any $|t| < \epsilon$. Then, the *vertical lift of u at e* is the vertical vector given by $u^v_e = \dot{\gamma}_{e,u}(0)$. Analogously, sections of E can be lifted. Indeed, if σ is a section of E, its *vertical lift* is the vertical vector field σ^v defined by

$$\sigma^v(e) = \sigma(\pi(e))^v_e,$$

where $\sigma(\pi(e))^v_e$ is the vertical lift of $\sigma(\pi(e))$ at e.

Finally the splitting in (4) induces a decomposition of vectors. In fact, for all $X \in T_eE$, $X = X^H + X^V$, where $X^H \in \mathcal{H}_e$ (resp. $X^V \in \mathcal{V}_e$) is the horizontal (resp. vertical) component of X. This induces a decomposition of vector fields; indeed, if $X \in \mathfrak{X}(E)$, then there exist a horizontal vector field X^H and a vertical vector filed X^V such that $X = X^H + X^V$.

We shall define almost complex structures on E which are motivated by the form of some Riemannian metrics on vector bundle manifolds. The *Sasaki metric* (cf. [22]) on vector bundles is the Riemannian metric on E defined by

$$G^s_e(X,Y) = g_{\pi(e)}(\pi_* X, \pi_* Y) + h_{\pi(e)}(K_e X, K_e Y), \tag{5}$$

for every $e \in E$ and $X, Y \in T_eE$. This metric is a generalization of the classical Sasaki metric on tangent bundles of Riemannian manifolds. We refer the reader to [1, 2, 18, 23] for more details. Many generalizations were considered e.g. Cheeger-Gromoll metric and the generalized Cheeger-Gromoll metric (cf. [1, 2, 29, 12]). Recently, R. Albuquerque introduced the class of metrics defined as follows

$$G^{ss}_e(X,Y) = e^{2\varphi_1(r)} g_{\pi(e)}((d\pi)_e(X), (d\pi)_e(Y)) + e^{2\varphi_2(r)} h_{\pi(e)}(K_e(X), K_e(Y)), \tag{6}$$

for all $e \in E$ and $X, Y \in T_eE$, with φ_1, φ_2 are smooth scalar functions on E depending only on the norm $r = h(e,e)$ and smooth at $r = 0$ on the right as well as all their successive derivatives. He called such metrics *spherically symmetric metrics*. We refer the reader to [4, 5, 6, 8] for more details on this class of metrics.

Motivated by the previous considerations, one can define the following tensors. For each $e \in E$, consider the tensor defined as follows

$$J^s_e(X) = \big(J_M(\pi_*(X))\big)^h_e + (J_E(K_e X))^v_e, \tag{7}$$

for all $X \in T_eE$.

Obviously, the mapping J^s is a $(1,1)$-tensor field on E. The tensor field J^s is completely characterized by

$$\begin{cases} J(X^h) = (J_M X)^h, \\ J(\sigma^v) = (J_E \sigma)^v; \end{cases} \tag{8}$$

for all $X \in \mathfrak{X}(M)$ and $\sigma \in \Gamma(E)$.

Remark 1. *The previous construction may be generalized as follows: consider two functions φ_1 and φ_2 smooth on $(0, +\infty)$ and at zero on the right as well as all their* *successive derivatives, and set*

$$J^{ss}_e(X) = e^{\varphi_1(r)}\big(J_M(\pi_*(X))\big)^h_e + e^{\varphi_2(r)}(J_E(K_eX))^v_e. \tag{9}$$

for $e \in E$, $X \in T_eE$ and $r = h(e,e)$. Of course, for $\varphi_1 = \varphi_2 = 0$, $J^{ss} = J^s$. Analogously, J^{ss} is a $(1,1)$-tensor field on E. Moreover, $\big(J^{ss}\big)^2 = -\mathrm{Id}_{TE}$ if and only if $\varphi_1 = \varphi_2 = 0$.

Lemma 1. *Let E be a vector bundle as before, then J^s is an a.c.s on E and (E, G^{ss}, J^s) is an almost Hermitian manifold.*

Proof. For $e \in E$ and $X \in T_eE$, we have

$$\begin{aligned}(J^s_e)^2(X) &= J^s\big(\big(J_M(\pi_*(X))\big)^h_e + (J_E(K_eX))^v_e.\big)\\ &= \Big(J_M\big(J_M(\pi_*(X))\big)\Big)^h + \Big(J_E\big(J_E(K_e(X))\big)\Big)^v\\ &= -\pi_*(X)^h_e - K(X)^v_e\\ &= -X,\end{aligned}$$

hence J^s is an a.c.s. It remains to check the compatibility of J^s with spherically symmetric metrics. Let $e \in E$ and $X, Y \in T_eE$, then

$$\begin{aligned}&G^{ss}_e(J^s_eX, J^s_eY)\\ &= G^{ss}_e\Big(\big(J_M(\pi_*(X))\big)^h_e + (J_E(K_eX))^v_e, \big(J_M(\pi_*(Y))\big)^h_e + (J_E(K_eY))^v_e\Big)\\ &= e^{2\varphi_1(r)}g_{\pi(e)}\big(J_M(\pi_*(X)), J_M(\pi_*(Y))\big) + e^{2\varphi_2(r)}h\big(J_E(K_e(X)), J_E(K_e(Y))\big)\\ &= e^{2\varphi_1(r)}g_{\pi(e)}\big(\pi_*(X), \pi_*(Y)\big) + e^{2\varphi_2(r)}h\big(K_e(X), K_e(Y)\big)\\ &= G^{ss}_e(X, Y).\end{aligned}$$

□

Remark 2. *One may try to enlarge the class of metrics on the total space. One of the possible choices is to consider a class of metrics which generalizes at the same time spherically symmetric metrics and the generalized Cheeger-Gromoll metrics (cf.* [8, 12]*). Precisely, we can consider the class of metrics given by*

$$G = G^{ss} + f\xi^\flat \otimes \xi^\flat$$

where G^{ss} is a spherically symmetric metric with weights φ_1, φ_2, $f : E \longrightarrow \mathbb{R}$ is a smooth function such that $f + e^{2\varphi_2} > 0$, with $\xi : E \longrightarrow TE$ is the tautological vertical vector field on E defined by $\xi_e = e^v_e$. The '$\flat$' is taken w.r.t h. For more details on this class of metrics, see [8]*. Unfortunately, this class of metrics fails to be compatible with J^s.*

A symplectic version of the previous construction is possible for a symplectic vector bundle with an almost symplectic base. Indeed, assume now that the base manifold is endowed with an almost symplectic form ω and that Ω is a symplectic structure on E. For $e \in E$ and $X \in T_eE$, define

$$\Omega^s_e(X, Y) = \omega_{\pi(e)}(\pi_*X, \pi_*Y) + \Omega(K_eX, K_eY). \tag{10}$$

It is clear that Ω^s is a non-degenerate 2-form on E. Hence (E, Ω^s) is an almost symplectic manifold. This almost symplectic form is totally characterized by

$$\left\{\begin{array}{l}\Omega^s(X^h, Y^h) = \omega(X, Y), \\ \Omega^s(X^h, \sigma^v) = 0, \\ \Omega^s(\sigma^v, \delta^v) = \Omega(\sigma, \delta);\end{array}\right. \tag{11}$$

for all $X, Y \in \mathfrak{X}(M)$ and $\sigma, \delta \in \Gamma(E)$.

3. Integrability of J^s

Since we are dealing with tensors, we will make a suitable choice of vector fields which will simplify our computations. In fact, we will compute the components of the Nijenhuis tensor using horizontal lifts of vector fields on the base and vertical lifts of sections of E.

Let $\pi^\star R^E$ denote the curvature of the pullback connection $\pi^\star\nabla$ which is a connection on the pullback vector bundle $\pi^\star E$. We shall use the notation $\mathcal{R}^\xi(.,.) = \pi^\star R^E(.,.)\xi$. The vector bundle $\pi^\star E$ is naturally isomorphic to the vertical subbundle of E (cf. [19, 32]), then $\mathcal{R}^\xi(.,.)$ gives vertical vector fields.

The Lie brackets of the different types of vector fields (cf. [8, 19, 22]) are given by:

Lemma 2. *Let $X, Y \in \mathfrak{X}(M)$ and $\sigma, \delta \in \Gamma(E)$, then*

(1) $[X^h, Y^h] = [X, Y]^h - \mathcal{R}^\xi(X^h, Y^h)$,
(2) $[X^h, \sigma^v] = (D_X\sigma)^v$,
(3) $[\sigma^v, \delta^v] = 0$.

We shall denote by N^s the Nijenhuis tensor of J^s, then:

Proposition 1. *Let $X, Y \in \mathfrak{X}(M)$ and $\sigma, \delta \in \Gamma(E)$, then we have*

$$\begin{aligned}
(12)\quad N^s(X^h, Y^h) &= N(X, Y)^h - \Big(R^E(X, Y). + J_E R^E(J_M X, Y). \\
&\qquad + J_E R^E(X, J_M Y). - J_E R^E(J_M X, J_M Y).\Big)^v, \\
(13)\quad N^s(X^h, \sigma^v) &= \Big(\nabla_X \sigma + J_E\big(\nabla_{J_M X}\sigma\big) + J_E\big(\nabla_X J_E \sigma\big) - \nabla_{J_M X}(J_E\sigma)\Big)^v, \\
(14)\quad N^s(\sigma^v, \delta^v) &= 0,
\end{aligned}$$

where N is the Nijenhuis tensor of J_M.

Proof. By virtue of Proposition 2 and equations (8), we have:

$$
\begin{aligned}
N^s(X^h,Y^h) &= [X^h,Y^h] + J^s[J^sX^h,Y^h] + J^s[X^h,J^sY^h] - [J^sX^h,J^sY^h]\\
&= [X^h,Y^h] + J^s[(J_MX)^h,Y^h] + J^s[X^h,(J_MY)^h] - [(J_MX)^h,(J_MY)^h]\\
&= \big([X,Y]^h - \mathcal{R}^\xi(X^h,Y^h)\big)\\
&+ J^s\big([J_MX,Y]^h - \mathcal{R}^\xi((J_MX)^h,Y^h)\big)\\
&+ J^s\big([X,J_MY]^h - \mathcal{R}^\xi(X^h,(J_MY)^h)\big)\\
&- J^s\big([J_MX,J_MY]^h - \mathcal{R}^\xi((J_MX)^h,(J_MY)^h)\big)\\
&= \big([X,Y]^h - \mathcal{R}^\xi(X^h,Y^h)\big)\\
&+ (J_M[J_MX,Y])^h - \big(J_ER^E(J_MX,Y).\big)^v\big)\\
&+ (J_M[X,J_MY])^h - \big(J_ER^E(X,J_MY).\big)^v\\
&- (J_M[J_MX,J_MY])^h - \big(J_ER^E(J_MX,J_MY).\big)^v\\
&= N(X,Y)^h - \Big(R^E(X,Y). + J_ER^E(J_MX,Y). + J_ER^E(X,J_MY).\\
&- J_ER^E(J_MX,J_MY).\Big)^v,
\end{aligned}
$$

$$
\begin{aligned}
N^s(X^h,\sigma^v) &= [X^h,\sigma^v] + J^s[J^sX^h,\sigma^v] + J^s[X^h,J^s\sigma^v] - [J^sX^h,J^s\sigma^v]\\
&= \big(\nabla_X\sigma\big)^v + J^s\Big(\big(\nabla_{J_MX}\sigma\big)^v\Big) + J^s\Big(\big(\nabla_XJ_E\sigma\big)^v\Big) - \big(\nabla_{J_MX}(J_E\sigma)\big)^v\\
&= \big(\nabla_X\sigma\big)^v + \Big(J_E\big(\nabla_{J_MX}\sigma\big)\Big)^v + \Big(J_E\big(\nabla_XJ_E\sigma\big)\Big)^v - \big(\nabla_{J_MX}(J_E\sigma)\big)^v\\
&= \Big(\nabla_X\sigma + J_E\big(\nabla_{J_MX}\sigma\big) + J_E\big(\nabla_XJ_E\sigma\big) - \nabla_{J_MX}(J_E\sigma)\Big)^v,
\end{aligned}
$$

$$
\begin{aligned}
N^s(\sigma^v,\delta^v) &= [\sigma^v,\delta^v] + J^s[J^s\sigma^v,\delta^v] + J^s[\sigma^v,J^s\delta^v] - [J^s\sigma^v,J^s\delta^v]\\
&= [\sigma^v,\delta^v] + J^s[\big(J_E\sigma\big)^v,\delta^v] + J^s[\sigma^v,\big(J_E\delta\big)^v] - [\big(J_E\sigma\big)^v,\big(J_E\delta\big)^v]\\
&= 0.
\end{aligned}
$$

□

Now, we consider the integrability problem of the almost complex structure J^s. We shall analyse the expressions of the Nijenhuis tensor in order to find necessary and sufficient conditions for the integrability of J^s. The complex structure J_E is said to be *parallel* if

$$\nabla_X\big(J_E\sigma\big) = J_E\big(\nabla_X\sigma\big),$$

for all $X\in\mathfrak{X}(M)$ and $\sigma\in\Gamma(E)$. The above condition is nothing but parallelism of $J_E\in\Gamma(\mathrm{End}(E))$ with respect to the connection on $\mathrm{End}(E)$ (or $E^*\otimes E$) induced from the connection ∇.

Example 1. *If (M,g,J) is a Kähler manifold, then we have $\nabla^MJ=0$ where ∇^M is the Levi-Civita connection of (M,g).*

Proposition 2. *Assume that J_E is parallel. Then J^s is integrable if and only if the following hold*

(1) *J_M is integrable,*
(2) *$\rho(X,Y)=0$,*

where

$\rho(X,Y) = R^E(X,Y). + J_E R^E(J_M X,Y). + J_E R^E(X,J_M Y). - J_E R^E(J_M X, J_M Y).,$ *for all* $X, Y \in \mathfrak{X}(M)$.

Proof. The parallelism of J_E implies that $N^s(X^h, \sigma^v) = 0$. Indeed, for $\sigma \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$, we have

$$\begin{aligned}&\nabla_X \sigma + J_E\big(\nabla_{J_M X}\sigma\big) + J_E\big(\nabla_X J_E \sigma\big) - \nabla_{J_M X}(J_E\sigma)\\ &= \nabla_X\sigma + \nabla_{J_M X} J_E \sigma - \big(\nabla_X \sigma\big) - \nabla_{J_M X}(J_E \sigma) = 0.\end{aligned}$$

Hence, J^s is integrable if and only if $N^s(X^h, Y^h) = 0$, for all $X, Y \in \mathfrak{X}(M)$. Equivalently, J^s is integrable if and only if the following hold

- $N(X,Y) = 0$,
- $R^E(X,Y). + J_E R^E(J_M X,Y). + J_E R^E(X,J_M Y). - J_E R^E(J_M X, J_M Y). = 0$,

for all $X, Y \in \mathfrak{X}(M)$. □

Proposition 3. *If J^s is integrable, then the following hold:*

(1) *J_M is integrable,*
(2) *$R^E(X, J_M Y) = R^E(J_M X, Y)$, for all $X, Y \in \mathfrak{X}(M)$.*

In particular, if the connection ∇ is flat and J_E is parallel, then J^s is integrable if and only if J_M is integrable.

Proof. Let $X, Y \in \mathfrak{X}(M)$ and $\sigma, \delta \in \Gamma(E)$, then equation (12) implies that $N(X,Y) = 0$, and

$$\Big(R^E(X,Y). + J_E R^E(J_M X,Y). + J_E R^E(X,J_M Y). - J_E R^E(J_M X, J_M Y).\Big) = 0.$$

Taking $Y = J_M X$, one gets

$$\Big(R^E(X, J_M X). + J_E R^E(J_M X, X).\Big) = 0, \tag{15}$$

for all $X \in \mathfrak{X}(M)$, which implies that $R^E(X, J_M X) = 0$ for all $X \in \mathfrak{X}(M)$. We deduce, taking $X + Y$ instead of X into (15), that

$$R^E(X, J_M Y) = R^E(J_M X, Y),$$

for all $X, Y \in \mathfrak{X}(M)$. As matter of fact, if ∇ is flat and J_E is parallel, then J^s is integrable if and only if J_M is integrable. □

4. When Ω^s is closed?

Proposition 4. *Let $X, Y \in \mathfrak{X}(M)$ and $\sigma, \delta \in \Gamma(E)$, then the exterior differential of Ω^s is completely determined by:*

$$\begin{aligned}
d\Omega^s(X^h, Y^h, Z^h) &= d\omega(X,Y,Z), &(16)\\
d\Omega^s(\sigma^v, \delta^v, \gamma^v) &= 0, &(17)\\
d\Omega^s(X^h, Y^h, \sigma^v) &= \Omega^s(\mathcal{R}^\xi(X^h, Y^h), \sigma^v), &(18)\\
d\Omega^s(X^h, \sigma^v, Y^h) &= -\Omega^s(\mathcal{R}^\xi(X^h, Y^h), \sigma^v), &(19)\\
d\Omega^s(\sigma^v, X^h, Y^h) &= \Omega^s(\mathcal{R}^\xi(X^h, Y^h), \sigma^v), &(20)\\
d\Omega^s(X^h, \sigma^v, \delta^v) &= X^h.\Omega(\sigma,\delta) - \Omega(D_X\sigma, \delta) + \Omega(D_X\delta, \sigma), &(21)\\
d\Omega^s(\sigma^v, X^h, \delta^v) &= -X^h.\Omega(\sigma,\delta) - \Omega(D_X\sigma, \delta) + \Omega(D_X\delta, \sigma), &(22)\\
d\Omega^s(\sigma^v, \delta^v, X^h) &= -X^h.\Omega(\sigma,\delta) - \Omega(D_X\sigma, \delta) + \Omega(D_X\delta, \sigma); &(23)
\end{aligned}$$

for all $X, Y, Z \in \mathfrak{X}(M)$ *and* $\sigma, \delta, \gamma \in \Gamma(E)$.

Proof. For all $X', Y', Z' \in \mathfrak{X}(E)$, we have

(24)

$$d\Omega^s(X',Y',Z') = X'.\Omega^s(Y',Z') - Y'.\Omega^s(X',Z') + Z'.\Omega^s(X',Y') - \Omega^s([X',Y'],Z') + \Omega^s([X',Z'],Y') - \Omega^s([Y',Z'],X').$$

Putting $X' = X^h$, $Y' = Y^h$ and $Z' = Z^h$, we obtain

$$\begin{aligned} d\Omega^s(X^h,Y^h,Z^h) &= X^h.\omega(Y,Z) - Y^h.\omega(X,Z) + Z^h.\omega(X,Y) - \Omega^s([X,Y]^h \\ &\quad - \mathcal{R}^\xi(X^h,Y^h),Z^h) + \Omega^s([X,Z]^h - \mathcal{R}^\xi(X^h,Z^h),Y^h) - \Omega^s([Y,Z]^h \\ &\quad - \mathcal{R}^\xi(Y^h,Z^h),X^h) \\ &= X.\omega(Y,Z) - Y.\omega(X,Z) + Z.\omega(X,Y) - \omega([X,Y],Z) \\ &\quad + \omega([X,Z],Y) - \omega([Y,Z],X) \\ &= d\omega(X,Y,Z), \end{aligned}$$

since $\mathcal{R}^\xi(X^h,Z^h)$ is a vertical vector field, for every $X, Y \in \mathfrak{X}(M)$. On the other hand, taking $X' = \sigma^v$, $Y' = \delta^v$ and $Z' = \gamma^v$ into (24), we get

$$\begin{aligned} d\Omega^s(\sigma^v,\delta^v,\gamma^v) &= \sigma^v.\Omega^s(\delta,\gamma) - \delta^v.\Omega^s(\sigma^v,\gamma^v) + \gamma^v.\Omega^s(\sigma^v,\delta^v) \\ &\quad - \Omega^s([\sigma^v,\delta^v],\gamma^v) + \Omega^s([\sigma^v,\gamma^v],\delta^v) - \Omega^s([\delta^v,\gamma^v],\sigma^v) \\ &= \sigma^v.\Omega(\delta,\gamma) - \delta^v.\Omega(\sigma,\gamma) - \gamma^v.\Omega(\sigma,\delta) \\ &= 0, \end{aligned}$$

since $\Omega(\delta,\gamma)$ is a function on M, so all its derivatives in the direction of vertical vectors vanish. Similarly, putting $X' = X^h$, $Y' = Y^h$ and $Z' = \sigma^v$ into (24), we obtain

$$\begin{aligned} d\Omega^s(X^h,Y^h,\sigma^v) &= X^h.\Omega^s(Y^h,\sigma^v) - Y^h.\Omega^s(X^h,\sigma^v) + \sigma^v.\omega(X,Y) \\ &\quad - \Omega^s([X,Y]^h - \mathcal{R}^\xi(X^h,Y^h),\sigma^v) + \Omega^s((D_X\sigma)^v,Y^h) - \Omega^s((D_Y\sigma)^v,X^h) \\ &= \Omega^s(\mathcal{R}^\xi(X^h,Y^h),\sigma^v). \end{aligned}$$

In the same way, we have

$$\begin{aligned} d\Omega^s(X^h,\sigma^v,Y^h) &= -\Omega^s(\mathcal{R}^\xi(X^h,Y^h),\sigma^v) \\ d\Omega^s(\sigma^v,X^h,Y^h) &= \Omega^s(\mathcal{R}^\xi(X^h,Y^h),\sigma^v). \end{aligned}$$

Finally, we get

$$\begin{aligned} d\Omega^s(X^h,\sigma^v,\delta^v) &= X^h.\Omega^s(\sigma^v,\delta^v) - \sigma^v.\Omega(X^h,\delta^v) - \delta^v.\Omega^s(X^h,\sigma^v) \\ &\quad - \Omega^s([X^h,\sigma^v],\delta^v) + \Omega^s([X^h,\delta^v],\sigma^v) - \Omega^s([\sigma^v,\delta^v],X^h) \\ &= X^h.\Omega(\sigma,\delta) - \Omega(D_X\sigma,\delta) + \Omega(D_X\delta,\sigma). \end{aligned}$$

In the same way we obtain

$$\begin{aligned} d\Omega^s(\sigma^v,X^h,\delta^v) &= -X^h.\Omega(\sigma,\delta) - \Omega(D_X\sigma,\delta) + \Omega(D_X\delta,\sigma) \\ d\Omega^s(\sigma^v,\delta^v,X^h) &= -X^h.\Omega(\sigma,\delta) - \Omega(D_X\sigma,\delta) + \Omega(D_X\delta,\sigma). \end{aligned}$$

$\square$

Next, we prove that Ω^s is in fact the almost symplectic form associated to (E, G^s, J^s) provided the compatibility between J_E (resp. J_M) and Ω (resp. ω). Indeed, we have

Proposition 5. *Assume that J_E (resp. J_M) is compatible with Ω (resp. ω), then*

$$G^s(J^s Z, W) = \Omega^s(Z, W),$$

for all $Z, W \in \mathfrak{X}(E)$.

Proof. Let $X, Y \in \mathfrak{X}(M)$ and $\sigma, \delta \in \Gamma(E)$, then

$$G^s(J^s X^h, Y^h) = G^s\big((J_M X)^h, Y^h\big) = g(J_M X, Y) = \omega(X, Y) = \Omega^s(X^h, Y^h),$$

$$G^s(J^s \sigma^v, \delta^v) = G^s\big((J_E \sigma)^v, \delta^v\big) = h(J_E \sigma, \delta) = \Omega(\sigma, \delta) = \Omega^s(\sigma^v, \delta^v).$$

□

Remark 3. *We have proved in Lemma 1 that the triple (E, G^{ss}, J^s) is an almost Hermitian manifold. Denote by Ω^{ss} the almost symplectic form associated with this triple, Ω^{ss} generalizes Ω^s.*

When a vector bundle with a connection is endowed with a fiber metric, a complex structure and a symplectic structure such that the triple is pairwise compatible, if any two of them are parallel, then so is the third (cf. [27, 32]). So, assume that E is endowed with the symplectic structure Ω associated to J_E and h, then the parallelism of J_E implies that of Ω.

Theorem 1. *Assume that the complex structure J_E is parallel and let ω be an almost symplectic on M. Then (E, Ω^s) is a symplectic manifold if and only if the following conditions hold:*

(i) *ω is a symplectic form,*
(ii) *∇ is flat.*

Proof. If Ω^s is a symplectic form, then by virtue of (16) of Proposition 4, we conclude that ω is symplectic. Further, equation (18) implies that $\mathcal{R}^\xi = 0$, and hence ∇ is flat.

Conversely, if ω is symplectic and ∇ is flat, then all the components of $d\Omega^s$ vanish except for (21), (22) and (23). On the other hand, the parallelism of J_E implies that of Ω, and hence

$$X.\Omega(\sigma, \delta) = \Omega(\nabla_X \sigma, \delta) + \Omega(\sigma, \nabla_X \delta),$$

which implies at once that Ω^s is closed. □

Corollary 1. *Assume ∇ is flat, then (E, G^s, J^s) is a Kähler manifold if and only if (M, g, J_M) is a Kähler manifold.*

Remark 4. *The construction of Ω^s may be accomplished for any symplectic vector bundle with an almost symplectic base manifold and endowed with a connection ∇ without any reference to complex structures. In this case, if the symplectic structure Ω is parallel, then the manifold (E, Ω^s) is a symplectic manifold if and only if the base manifold is symplectic and the connection is flat.*

It has been shown that, in the case of flat connections, (E, G^s) is flat if and only if (M, g) is flat (cf. [4]). In conclusion, considering a vector bundle E, over a Kähler manifold, endowed with a fiber metric, a compatible flat connection and a compatible parallel complex structure, then the manifold (E, G^s, J^s) is a non-flat (unless the base is flat) Kähler manifold. This will allow the construction of a very wide class of new Kähler manifolds out of the old ones. Furthermore, if the base is not a Kähler manifold, then one gets new examples of complex manifolds, Hermitian manifolds and symplectic manifolds depending on the initial data.

5. Applications

5.1. **Whitney sums of vector bundles.** Given a vector bundle E over an almost complex manifold (M, J_M) such that E is endowed with a fiber metric h and a compatible connection ∇. The Whitney sum $(E \oplus E, \pi^{\oplus}, M)$ possesses a natural complex structure defined by

$$J_{E\oplus E}(\sigma, \delta) = (-\delta, \sigma), \tag{25}$$

for $\sigma, \delta \in \Gamma(E)$ (cf. [32]).

Thus the manifold $E \oplus E$ may be endowed with the almost complex structure given by

$$J(Z) = \big(J_M \pi_* Z\big)^{h\oplus} + \big(J_{E\oplus E} K^{\oplus} Z\big)^{v\oplus} \tag{26}$$

for all $Z \in T_{(e_1,e_2)}(E \oplus E)$ and $(e_1, e_2) \in E \oplus E$, where $K^{\oplus}$ is the connection map of the connection naturally induced on $E \oplus E$ from ∇, which we denote by $\nabla^{\oplus}$. Further, $v\oplus$ (resp. $h\oplus$) denotes the vertical lift to $E \oplus E$ (resp. horizontal lift with respect to $\nabla^{\oplus}$). This almost complex structure can be expressed in a more precise manner, but we need first to prove the following preparatory lemmas. First of all, the connection $\nabla^{\oplus}$ gives the splitting

$$T_{(e_1,e_2)}(E \oplus E) = (\mathcal{H}_{e_1} \times \mathcal{H}_{e_2}) \oplus (\mathcal{V}_{e_1} \times \mathcal{V}_{e_2}),$$

for all $(e_1, e_2) \in E \oplus E$.

Lemma 3. *Given $\sigma, \delta \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$, then*

$$(\sigma, \delta)^{v\oplus} = (\sigma^v, \delta^v)$$

$$X^{h\oplus} = (X^h, X^h),$$

where X^h (resp σ^v and δ^v) is the horizontal lift of X with respect to ∇ (resp. are the vertical lifts of sections of E).

Proof. Let $X \in \mathfrak{X}(M)$, then (X^h, X^h) is horizontal with respect to $\nabla^{\oplus}$ and $\pi^{\oplus}$-related to X, hence $X^{h\oplus} = (X^h, X^h)$. On the other hand, if $\sigma, \delta \in \Gamma(E)$, then $(\sigma, \delta) : M \longrightarrow E \oplus E$ is a section of $E \oplus E$. Thus

$$\begin{aligned}
(\sigma, \delta)^{v\oplus}(e_1, e_2) &= \frac{d}{dt}\Big|_{t=0}\Big((e_1, e_2) + t(\sigma(\pi(e_1)), \delta(\pi(e_1)))\Big) \\
&= \frac{d}{dt}\Big|_{t=0}\big(e_1 + t\sigma(\pi(e_1)), e_2 + t\delta(\pi(e_1))\big) \\
&= \big(\sigma^v(e_1), \delta^v(e_2)\big) \\
&= (\sigma^v, \delta^v)(e_1, e_2).
\end{aligned}$$

$\square$

Thus, the almost complex structure J is totally determined by the following

$$\begin{cases} J(X^{h\oplus}) = \big((J_M X)^h, (J_M X)^h\big) \\ J((\sigma,\delta)^{\oplus v}) = (-\delta^v, \sigma^v). \end{cases}$$

The complex structure $J_{E\oplus E}$ is compatible with $h \oplus h$. Furthermore, parallelism also holds. More precisely we have the following:

Lemma 4. *The complex structure $J_{E\oplus E}$ is parallel with respect to $\nabla^{\oplus}$.*

Proof. Let $\sigma, \delta \in \Gamma(E)$, then

$$\begin{aligned}(\nabla^{\oplus} J_{E\oplus E})(\sigma,\delta) &= \nabla^{\oplus}(J_{E\oplus E}(\sigma,\delta)) - J_{E\oplus E}(\nabla^{\oplus}(\sigma,\delta)) \\ &= \nabla^{\oplus}(-\delta,\sigma) - J_{E\oplus E}(\nabla\sigma,\nabla\delta) \\ &= (-\nabla\delta,\nabla\sigma) - (-\nabla\delta,\nabla\sigma) \\ &= 0.\end{aligned}$$

□

Thus by virtue of Proposition 3, the almost complex structure is integrable if and only if the following hold:

(1) J_M is integrable;
(2) $\rho^{\oplus}(X,Y) = 0$, for all $X, Y \in \mathfrak{X}(M)$, where $\rho^{\oplus}$ is the 2-form given in Proposition 2 with respect to $\nabla^{\oplus}$.

If $E = T^2M$ is the second tangent bundle of a Riemannian manifold M with Levi-Civita connection ∇. Then we have

$$\begin{array}{rccl} \zeta\colon & T^2M & \to & TM \oplus TM \\ & Z & \mapsto & \zeta(Z) = (\pi_*, K)(Z), \end{array}$$

is a vector bundle isomorphism, where K is the connection map of ∇ and π_M is the canonical projection of the tangent bundle TM of M (cf. [32]), hence T^2M is endowed with an almost complex structure which is integrable if and only if the corresponding almost complex structure on $TM \oplus TM$ is integrable.

5.2. **An infinite family of examples.** The tangent bundles to almost Hermitian manifolds constitute a wide class of examples of vector bundles with a complex structure, hence the constructions made above apply naturally, which yields new almost complex structures on the tangent bundle manifolds. More precisely, let (M, g, J_M) be an almost Hermitian manifold, then J_M gives rise to complex structures on fibers defined by

$$J(X) = \big(J_M \pi_* X\big)^h + \big(J_M K(X)\big)^v \tag{27}$$

for all $X \in T_{(x,u)}TM$ and $(x,u) \in TM$. By virtue of Lemma 1, the tensor field J is an almost complex structure on TM. Denote by ∇ the Levi-Civita connection of g. If J_M is parallel (w.r.t ∇), that is the connection ∇ is almost complex (cf. [26]), then by virtue of Proposition 2, the almost complex structure J is integrable if and only if

(i) J_M is integrable,
(ii) $\rho(X,Y) = 0$, for all $X, Y \in \mathfrak{X}(M)$;

where ρ is the 2-form defined in Proposition 2. Condition (ii) is equivalent to (M, g) being flat. Indeed, since J_M is ∇-parallel we have $R(J_MX, J_MY) = R(X, Y)$, hence condition (ii) together with Proposition 3 give the following

$$R(J_MX, Y) = R(X, J_MY) = R(J_MX, J_M^2Y) = -R(J_MX, Y),$$

which implies at once that $R = 0$, whence (M, g) is flat. Consequently, J is integrable if and only if (M, g, J_M) is a flat Hermiatian manifold. Furthermore, if we endow TM with the symplectic form Ω^s constructed in the previous section from ω, where ω is the fundamental 2-form of (M, g, J_M), then (TM, Ω^s) is a symplectic manifold if and only if (M, ω) is a flat symplectic manifold. All at all, the triple (TM, G^s, J^s) is a flat Kähler manifold if and only if (M, g, J) is a flat Kähler manifold.

Using tangent bundles, Whitney sums and the classification of compact flat Kähler manifolds in dimensions four and six, one can construct infinite families of examples of non-compact flat Kähler manifolds. Indeed, using Corollary 1, one can construct infinitely many examples of flat Kähler manifolds using a classification of four and six dimensional compact flat Kähler manifolds performed in [17]. The classification of compact flat Riemannian manifolds is a classical subject in Riemannian geometry, see for examples [13, 14, 15, 16] and the references therein. This leads to a classification of compact flat Kähler manifolds of dimensions four and six.

Let $\mathcal{M}_n = O(n) \ltimes \mathbb{R}^n$ be the group of rigid motions of $\mathbb{R}^n$. A rigid motion $(m, v) \in \mathcal{M}_n$ acts on $\mathbb{R}^n$ by

$$(m, v).x = mx + v, \text{ for } x \in \mathbb{R}^n.$$

Denote by $r : \mathcal{M}_n \longrightarrow O(n)$ (the rotational part) (resp. $t : \mathcal{M}_n \longrightarrow \mathbb{R}^n$ (the translational part)) the natural projections. Following the notations in [16], if π is a subgroup of $\mathcal{M}_n$, we denote by $\pi \cap \mathbb{R}^n$ the set of pure translations (i.e. the set of element $(m, v) \in \pi$, with $m = I_n$). The subgroup π is said to be *torsion free* if $\alpha \in \pi$ and $\alpha^k = (I, 0)$ implies that $\alpha = (I, 0)$ ($(I, 0)$ is the identity element of $\mathcal{M}_n$). In this terminology, the subgroup $\pi \cap \mathbb{R}^n$ is a torsion-free abelian normal subgroup of π.

Denote by $\mathcal{A}_n = GL_n(\mathbb{R}) \ltimes \mathbb{R}^n$ the group of affine motions. A subgroup π of $\mathcal{M}_n$ is said to be *irreducible* if $t(\alpha\pi\alpha^{-1})$ spans $\mathbb{R}^n$, for all $\alpha \in \mathcal{A}_n$. A *crystallographic subgroup* of $\mathcal{M}_n$ is a discrete irreducible subgroup of $\mathcal{M}_n$. A crystallographic subgroup which is torsion-free is called a *Bieberbach subgroup* of $\mathcal{M}_n$. For a detailed exposition and a discussion of the relevance of these notions, we refer to [16]. Bieberbach proved three fundamental theorems on crystallographic subgroups of $\mathcal{M}_n$. The first theorem assert that if π is a crystallographic subgroup of $\mathcal{M}_n$, then

(i) $\pi \cap \mathbb{R}^n$ is a free abelian group on n generators which are linearly independent translations,
(ii) $r(\pi)$ is finite;

where $r(\pi)$ is the rotational parts of elements in π, which is isomorphic to $\pi/(\pi \cap \mathbb{R}^n)$. For the other two theorems as well as related results, we refer to [16]. Furthermore, if π is a crystallographic subgroup of $\mathcal{M}_n$, then the sequence

$$0 \longrightarrow \pi \cap \mathbb{R}^n \longrightarrow \pi \longrightarrow r(\pi) \longrightarrow 1, \tag{28}$$

is exact. The algebraic analogues of those groups have been introduced, see [16]. A *crystallographic group* is a group which contains a finitely generated maximal abelian torsion-free subgroup of finite index. A *Bieberbach group* is a crystallographic group which is itself torsion-free (cf. [16]). Hence, for a crystallographic group π, there is a sequence

$$0 \longrightarrow N \longrightarrow \pi \longrightarrow \Phi \longrightarrow 1, \tag{29}$$

where Φ is a finite group and N is a finitely generated maximal abelian torsion-free subgroup of π. For such a triple of groups, π is called an *extension* of Φ by N.

The geometric relevance comes from the fact that every compact flat Riemannian manifold M is isometric to one of the form $\mathbb{R}^n/\pi$ where π is a discrete irreducible and torsion-free subgroup of $\mathcal{M}_n$. Further, the fundamental group of M is given by $\pi_1(M) = \pi$ and the holonomy of M is isomorphic to $\pi/(\pi \cap \mathbb{R}^n)$ (cf. [13, 16, 38]). Furthermore, in [10], L. Auslander and M. Kuranishi proved that every Bieberbach group is the fundamental group of a compact flat Riemannian manifold (cf. [10]).

On the other hand it has been proved that a compact Riemannian manifold is flat if and only if its holonomy group is finite. Further, every finite group Φ is the holonomy of a compact flat Riemannian manifold. Such a manifold is said to be a Φ-manifold (cf. [13]). Hence Bieberbach groups were classified by picking up a finite group, which is the holonomy, and classify all the extensions by some finitely generated maximal abelian torsion-free subgroup. Which leads to a classification of compact flat Riemannian manifolds. For a detailed study of the classification, we refer to [16].

Moreover, it has been proved that every finite group is the holonomy of a compact flat Kähler manifold (cf. [21]), hence one can choose among its extensions those that correspond to fundamental groups of compact flat Kähler manifolds (cf. [21]). So, based on results from finite group theory and their integral representations as well as the classification of Bieberbach groups, K. Dekimpe, M. Hałend fa and A. Szczepański have given a classification of compact flat Kähler manifolds of dimensions four and six. Precisely, they have proved that there exist exactly eight Kähler manifolds of dimension four. We denote the collection of those manifolds by $\mathcal{C}_4$. Further, they proved that there exist exactly 173 six dimensional compact flat Kähler manifolds. We denote their collection by $\mathcal{C}_6$. For more details, see [17].

Now, let (M, g, J) be a four (resp. six) dimensional compact flat Kähler manifold belonging to the collection $\mathcal{C}_4$ (resp. $\mathcal{C}_6$). Then, by virtue of Corollary 1, the triple (TM, G^s, J^s) is a flat Kähler manifold with twice the dimension of M, hence a collection of eight and twelve dimensional flat Kähler manifolds.

An infinite family of flat Kähler manifolds can be constructed from tangent bundles and the direct sum operation. For the sake of simplicity, denote the vector bundle $TM \oplus ... \oplus TM$ (k times) by $k(TM)$. Endow $k(TM)$ with the connection, denoted by ∇^k, induced from the Levi-Civita connection ∇ on M. Since (M, g) is flat, then ∇^k is flat. Denote by $J_{TM\oplus TM}$ the canonical complex structure on $TM \oplus TM$ (which is also $2(TM)$), then by virtue of Corollary 1, the triple $(TM \oplus TM, G^s, J^s)$ is a non-compact flat Kähler manifold of dimension $3n$, where n is the dimension of M. By induction, this construction can be carried on. More precisely, the vector bundle $2^m(TM) \longrightarrow M$ possesses a canonical complex structure since the total space is the direct sum of two copies of $2^{(m-1)}(TM)$. Thus, by Corollary 1, the triple $(2^m(TM), G^s, J^s)$, with G^s (resp. J^s) is the Sasaki metric (resp. the complex structure) on $2^m(TM)$, is a non-compact flat Kähler manifold. Hence infinitely many examples of flat Kähler manifolds with dimension $(2^m + 1)n$, where n is the dimension of M.

Consequently, we have constructed infinitely many examples of non-compact flat Kähler manifolds of dimensions of the form $(2^m + 1)n$, with $m \geq 0$ and $n = 4, 6$.

5.3. **The tangent bundle to the cotangent bundle.** Let (M, g) be a Riemannian manifold with ∇^M is its Levi-Civita connection, and let T^*M be its cotangent bundle. Denote by θ the Liouville 1-form on T^*M and $\omega = -d\theta$ the canonical symplectic structure induced by θ. The constructions made before applies in a direct manner to the vector bundle $\pi : T(T^*M) \longrightarrow T^*M$ as follows. Assume that G is a Riemannian metric on T^*M which is compatible with ω in the sense that $u \longrightarrow \omega(u, .)$ are isometries with respect to the dual metric and denote by ∇ the Levi-Civita connection of G, set

$$\Omega^s(Z, W) = \omega(\pi_* Z, \pi_* W) + \omega(KZ, KW),$$

where $Z, W \in T(T^*M)$ and K is the connection map of ∇. The 2-form Ω^s is an almost complex structure on T^*M which is integrable if and only if the following hold:

(i) ∇ is flat,
(ii) ω is parallel.

It is noteworthy that there is a very wide class of metrics on T^*M which can be constructed 'naturally' from g (cf. [28, 33, 35, 37]). For our purposes, it suffices to chose one among them which is compatible with the canonical symplectic structure on T^*M.

References

[1] M.T.K. Abbassi, *g-natural metrics: new horizons in the geometry of tangent bundles of Riemannian manifolds*, Note. Mat., **28**:suppl. 1 (2009), 6–35. Zbl 1211.53042
[2] M.T.K. Abbassi, *Métriques Naturelles Riemanniennes sur le Fibrŭé tangent à une variété Riemannienne*, Editions Universitaires Européénnes, Saarbrücken, Germany, 2012.
[3] M.T.K. Abbassi, M. Sarih, *On some hereditary properties of Riemannian g-natural metrics on tangent bundles of Riemannian manifolds*, Differ. Geom. Appl., **22**:1 (2005), 19–47. Zbl 1068.53016
[4] M.T.K Abbassi, I. Lakrini, *On the geometry of vector bundles with flat connections*, Bull. Korean Math. Soc., **56**:5 (2019), 1219–1233. Zbl 1447.53024
[5] M.T.K Abbassi, I. Lakrini, *On the completeness of total space of horizontally conformal submersions*, Commun. Math., **29**:3 (2021), 493–504. Zbl 1487.53035
[6] M.T.K Abbassi, I. Lakrini, *Conformal vector fields on vector bundle manifolds with spherically symmetric metrics*, Publ. Math. Debr., **97**:3-4 (2020), 289–311. Zbl 1499.53124
[7] R.M. Aguilar, *Isotropic almost complex structures on tangent bundles*, Manuscr. Math., **90**:4 (1996), 429–436. Zbl 0881.53020
[8] R. Albuquerque, *On vector bundle manifolds with spherically symmetric metrics*, Ann. Global Anal. Geom., **51**:2 (2017), 129–154. Zbl 1364.53023
[9] V.I. Arnold, *Mathematical methods of classical mechanics*, Springer, New York etc., 1978. Zbl 0386.70001
[10] L. Auslander, M. Kuranishi, *On the holonomy group of locally Euclidean spaces*, Ann. Math. (2), **65**:3 (1957), 441–415. Zbl 0079.38304
[11] A. Baghban, E. Abedi, *Isotropic almost complex structures and harmonic unit vector fields*, Arch. Math. Brno, **54**:1 (2018), 15–32. Zbl 1424.53098
[12] M. Benyounes, E. Loubeau, C.M. Wood, *Harmonic sections of Riemannian vector bundles, and metrics of Cheeger-Gromoll type*, Differ. Geom. Appl., **25**:3 (2007), 322–334. Zbl 1128.53037
[13] L.S. Charlap, *Compact flat Riemannian manifolds I*, Ann. Math. (2), **81**:1 (1965), 15–30. Zbl 0132.16506

[14] L.S. Charlap, A.T. Vasquez, *Compact flat Riemannian manifolds II: The cohomology of Z_p-manifolds*, Am. J. Math., **87**:3 (1965), 551–563. Zbl 0132.16601
[15] L.S. Charlap, A.T. Alphonse, *Compact flat Riemannian manifolds. III: The group of affinities*, Am. J. Math., **95**:3 (1973), 471–494. Zbl 0301.53027
[16] L.S. Charlap, *Bieberbach groups and flat manifolds*, Springer-Verlag, New York etc., 1986. Zbl 0608.53001
[17] K. Dekimpe, M. Hałenda, A. Szczepański, *Kähler flat manifolds*, J. Math. Soc. Japan, **61**:2 (2009), 363–377. Zbl 1187.53051
[18] P. Dombrowski, *On the geometry of the tangent bundles*, J. Reine Angew. Math., **210** (1962), 73–88. Zbl 0105.16002
[19] S. Dragomir, D. Perrone, *Harmonic vector fields. Variational principles and differential geometry*, Elsevier, Amsterdam, 2012. Zbl 1245.53002
[20] S.I. Goldberg, *Curvature and homology*, Academic Press, New York, London, 1962. Zbl 0105.15601
[21] F.E.A. Johnson, E.G. Rees, *Kähler groups and rigidity phenomena*, Math. Proc. Camb. Philos. Soc., **109**:1 (1991), 31–44. Zbl 0736.53058
[22] J.J. Konderak, *On sections of fibre bundles which are harmonic maps*, Bull. Math. Soc. Sci. Math. Roum., Nouv. Sér., **42(90)**:4 (1999), 341–352.
[23] O. Kowalski, *Curvature of the induced Riemannian metric on the tangent bundle of a Riemannian manifold*, J. Reine Angew. Math., **250** (1971), 124–129. Zbl 0222.53044
[24] O. Kowalski, M. Sekizawa, *Natural transformations of Riemannian metrics on manifolds to metrics on tangent bundles – a classification*, Bull. Tokyo Gakugei Univ., Sect. IV, Ser. Math. Nat. Sci., **40** (1988), 1–29. Zbl 0656.53021
[25] I. Kolář , P.W. Michor, J. Slovák, *Natural operations in differential geometry*, Springer-Verlag, Berlin, 1993. Zbl 0782.53013
[26] S. Kobayashi, K. Nomizu, *Foundations of differential geometry Vol. II*, Interscince Publishers, New York-London-Sydney, 1963. Zbl 0175.48504
[27] A. Lichnerowicz, *Généralizations de la géométrie Kählerienne globale,* Colloque de Géométrie Différentielle, Louvin, 1951.
[28] K-P. Mok, *Metrics and connections on the cotangent bundle*, Kōdai Math. Semin. Rep., **28** (1977), 226–238. Zbl 0349.53026
[29] E. Musso, F. Tricerri, *Riemannian metrics on tangent bundles*, Ann. Mat. Pura Appl., IV. Ser., **150**:4 (1988), 1–19. Zbl 0658.53045
[30] M.I. Munteanu, *Old and new structures on the tangent bundle*, in Mladenov, Ivaïlo (ed.) et al., *Proceedings of the 8th international conference on geometry, integrability and quantization*, Sts. Constantine and Elena, Bulgaria, June 9–14, 2006. Bulgarian Academy of Sciences, Sofia, 2007, 264–278. Zbl 1126.53020
[31] V. Oproiu *A generalization of natural almost Hermitian structures on the tangent bundle*, Math. J. Toyama Univ., **22** (1999), 1–14. Zbl 0955.53039
[32] W.A. Poor, *Differential geometric structures*, Dover Pubs. Inc. Mineola, New York, 2007. (1981, Zbl 0493.53027)
[33] A.A. Salimov, F. Agca, *Some properties of Sasakian metrics in cotangent bundles*, Mediterr. J. Math., **8**:2 (2011), 243–255. Zbl 1227.53058
[34] S. Sasaki, *On the differential geometry of tangent bundles of Riemannian manifolds*, Tôhoku Math. J., II. Ser., **10** (1958), 338–354. Zbl 0086.15003
[35] M. Sekizawa, *Natural transformations of affine connections on manifolds to metrics on cotangent bundles*, In: *Abstract analysis, Proc. 14th Winter Sch., Srno/Czech., 1986*, Suppl. Rend. Circ. Mat. Palermo, II. Ser., **14**, (1987), 129–142. Zbl 0635.53012
[36] M. Tahara, L. Vanhecke, Y. Watanabe, *New structures on tangent bundles*, Note Mat., **18**:1 (1998), 131–141. Zbl 0964.53021
[37] K. Yano, S. Ishihara, *Tangent and cotangent bundles. Differential geometry*, Marcel Dekker Inc., New York, 1973. Zbl 0262.53024
[38] S.-T. Yau, *Compact flat Riemannian manifolds*, J. Differ. Geom., **6** (1972), 395–402. Zbl 0238.53025

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