S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

http://semr.math.nsc.ru

Том 20, № 2, стр. 1341–1360 (2023) УДК 510.5

DOI 10.33048/semi.2023.020.081 MSC 03D45, 03D80, 68Q15

ON THE COMPUTABILITY OF ORDERED FIELDS

M.V.KOROVINA AND O.V.KUDINOV

Abstract. In this paper we develop general techniques for structures of computable real numbers generated by classes of total computable (recursive) functions with special requirements on basic operations in order to investigate the following problems: whether a generated structure is a real closed field and whether there exists a computable copy of a generated structure. We prove a series of theorems that lead to the result that there are no computable copies for $\mathcal{E}^n$-computable real numbers, where $\mathcal{E}^n$ is a level in Grzegorczyk hierarchy, $n \geq 3$. We also propose a criterion of computable presentability of an archimedean ordered field.

Keywords: computable analysis, computability, index set, computable model theory, complexity.

1. Introduction

In the framework of computable model theory originated in [14, 17, 5] there have been investigated conditions on the existence of computable copies for countable homogeneous boolean algebras [1, 16], for superatomic boolean algebras [8], for ordered abelian groups [10] among others and established several negative results for archimedean ordered fields [15, 12]. Nevertheless, till now there where no natural criteria on the existence of computable presentations of ordered fields even in an archimedean case. In this paper we try to fill this gap.

We are also going dipper to revile relations between a class of computable (recursive) functions K and structures $\widetilde{K}$ and K^* of computable real numbers generated by

---

Korovina, M.V., Kudinov, O.V., On the Computability of Ordered Fields.

M.V. Korovina's research was carried out within the framework of the state contract of the A.P. Ershov Institute of Informatics Systems (project no. FWNU-2021-0003). O.V. Kudinov's research was carried out within the framework of the state contract of the Sobolev Institute of Mathematics (project no. FWNF-2022-0011).

Received August, 5, 2020, published November, 30, 2023.

K corresponding Cauchy approximations and sign-digit expansions. We propose natural restrictions on a class K under which the structures $\widetilde{K}$ and K^* coincide. We also revile requirements on K when K^* is a real closed field.

Further we investigate a natural problem whether there exist computable copies of generated structures for popular classes of computable functions such as the Grzegorczyk classes $\mathcal{E}^n$, $n \geq 3$. We establish that the corresponding real closed fields do not have computable copies. In order to do that we develop techniques of index sets and multiple m-completeness. On this way we have to establish a criterion of m-completeness for tuples of c.e. sets and Σ_2^0-sets. From our point of view this criterion is an interesting result itself and can be used for different purposes.

The paper is organised as follows: Section 2 contains preliminaries and basic background. In Section 3 we propose the notions of $\widetilde{K}$ and K^* generated structures. We show under which requirements on K the corresponding generated structures $\widetilde{K}$ and K^* coincide and under which requirements on K the corresponding generated structure K^* is a real closed field. Further we prove a criterion of the computable presentability of an archimedean ordered field. In Section 4 we define 3-tuple of index sets (A_0, A_1, A_2) depending on K such that $A_i \in \Sigma_2^0$ with the following embedding property $A_0 \subseteq A_1 \subseteq A_2$. In the Theorem 1 we show that if the corresponding $\widetilde{K}$ as a structure, in particular as an abelian group, has a computable copy then $A_0 \cup (A_2 \setminus A_1) \in \Sigma_2^0$. In Sections 4.1, 4.2 and 4.3 we develop techniques to establish that under natural assumptions on a class K the 3-tuple (A_0, A_1, A_2) is m-complete in the class of 3-tuples of Σ_2^0-sets with the embedding property. It is well-known that in this case $A_0 \cup (A_2 \setminus A_1) \notin \Sigma_2^0$ and therefore for the corresponding $\widetilde{K}$ there is no a computable copy. It is worth noting than these classes contain computable real numbers generated by Grzegorczyk classes, in particular $\mathcal{E}^3$ and beyond.

2. Preliminaries

We refer the reader to [19, 22] for basic definitions and fundamental concepts of recursion theory, [11, 23] for computable analysis, [7] for computable model theory, [9, 18] for Grzegorczyk classes $\mathcal{E}^n$, $n \geq 2$, of computable (recursive) functions. We recall that, in particular, φ_e denotes the partial computable (recursive) function with an index e in the Kleene numbering. For simplicity of descriptions we identify a function with its graph. We also use notations $W_e = \mathrm{dom}(\varphi_e)$, $\overline{W_e} = \omega \setminus W_e$, $\pi_e = \mathrm{im}(\varphi_e)$, $c : \omega^2 \to \omega$ for Cantor numbering of pairs and $l : \omega \to \omega$, $r : \omega \to \omega$ for the corresponding functions such that $n = c(l(n), r(n))$. We fix the set BF of standard basic functions $\lambda x.0$, $s(x)$ and I_m^n, where $I_m^n(x_1, \dots, x_n) = x_m$ for $1 \leq m \leq n$ and denote the total computable numerical functions as $\mathbb{T}$ and $\mathrm{Tot} = \{n \mid \varphi_n \in \mathbb{T}\}$. We fix the following computable numbering $q : \omega \to \mathbb{Q}$ of the rational numbers:

$$
\begin{aligned}
&\text{for } m \geq 0, k > 0 : \\
&q(2c(m,k)) = +\frac{m}{k}, \\
&q(2c(m,k)+1) = -\frac{m}{k}, \\
&q(c(m,0)) = q(c(m,1)) = m \text{ and} \\
&q(l) = 1 \text{ for the rest arguments } l.
\end{aligned}
$$

When it is clear from a context we use the notation q_n for $q(n)$. We denote $I = \{2c(m,k) \mid m \in \omega, k \in \omega$ and $\frac{m}{k}$ is irriducible$\} \cup \{2c(m,k)+1 \mid m \in \omega, k \in \omega$ and $\frac{m}{k}$ is irriducible$\} \cup \{c(m,0) \mid m \in \omega\}$. For tuples $z_1 \dots, z_k$ of numbers or functions we use the notation $\bar{z}$ when it is clear from a context. For the positive rational numbers we use the notation $\mathbb{Q}^+ = \{q \in \mathbb{Q} \mid q > 0\}$, for the dyadic numbers we use the notation $Dyad = \{\frac{m}{2^{i+1}} \mid m \in \mathbb{Z}, i \geq 0\}$ and $B(\alpha, r) = \{x \in \mathbb{R} \mid |x - \alpha| < r\}$, for an open ball with the center $\alpha \in \mathbb{R}$ and the radius $r \in \mathbb{R}^+$.

2.1. **Computable Presentations.** We say that a structure $\mathcal{A} = \langle A, \sigma\rangle$ with a finite language σ admits a computable presentation (copy) if there is a numbering $\nu : \omega \to A$ such that the relations and operations from σ including equality are computable with respect to the numbering ν. The pair $(\mathcal{A}, \nu)$ is called a computable structure and the numbering ν is called its computable presentation.

If only operations are computable with respect to the numbering ν, a structure $(\mathcal{A}, \nu)$ is called a numbered (effective) algebra.

2.2. **Grzegorczyk classes.** In this paper we use the following properties of $\mathcal{E}^n$ for $n \geq 2$:

(1) Every class contains BF and the functions $+$, $\cdot$, $c(x, y)$, $l(x)$, $r(x)$.
(2) Every class is closed under composition and the standard bounded recursion scheme.
(3) Ritchie's characterisation of $\mathcal{E}^2$ [18]: $f \in \mathcal{E}^2$ if and only if an computation on some TM of $f(x_1, \dots, x_m)$ requires the number of cells bounded from above by $c_f \cdot \sum_{i=1}^m \Big(L(x_i) + 1\Big)$, where $L(x)$ is the word length in binary notation of the number x.

3. Generated Computable Reals

3.1. **Definitions.** Let K be a class of total numerical functions. We associate with K the classes

$$K^{\{0,1,2\}} = \{f \in K \mid \mathrm{im}(f) \subseteq \{0,1,2\} \text{ and } f \text{ is a unary function}\}$$

and

$$K_1 = \{f \mid f \in K \text{ and } f \text{ is a unary function}\}.$$

Let us define a subset K^* of the computable real numbers as follows:

$$x \in K^* \leftrightarrow (\exists \phi \in K_1)(\forall n \in \omega)|q_{\phi(n)} - x| < \frac{1}{2^n}.$$

We proceed with the definition of the corresponding subset $\widetilde{K}$ of the computable real numbers. For $f \in K^{\{0,1,2\}}$ let us denote

$$\bar{f} = \sum_{i=0}^{\infty} \frac{f(i) - 1}{2^{i+1}}.$$

Then we define

$$\widetilde{K} = \{m + \bar{f} \mid m \in \mathbb{Z}, f \in K^{\{0,1,2\}}\}.$$

One of the trivial examples, where K is the set of the almost constant functions illustrates that in general $K^* \neq \widetilde{K}$. Indeed, in this case the generated class K^* coincides with $\mathbb{Q}$ while $\widetilde{K}$ coincides with Dyad. Below we will show under which requirements on K we have $K^* = \widetilde{K}$.

In this paper we use requirements **Req** on K:

Req 1 it contains the basic functions BF, and $\lambda x.2x$,

Req 2 it is closed under composition and $=^*$, i.e., if $f \in K_1$ and $g =^* f$ then $g \in K_1$, where

$$g =^* f \leftrightarrow (\exists n \in \omega)(\forall m \geq n)\, g(m) = f(m),$$

Req 3 it contains computable functions $\mathrm{Add} : \omega \times \omega \to \omega$, $\mathrm{Mult} : \omega \times \omega \to \omega$ and $\mathrm{Inv} : \omega \to \omega$ such that

$$\forall n\, \forall m\, q_n + q_m = q_{\mathrm{Add}(n,m)},$$
$$\forall n\, \forall m\, q_n \cdot q_m = q_{\mathrm{Mult}(n,m)},$$
$$\forall n\, (q_n)^{-1} = q_{\mathrm{Inv}(n)} \text{ if } q_n \neq 0.$$

Assume K satisfies **Req**, for all n, $m \in \omega$, $\mathrm{Add}(n,m)$, $\mathrm{Mult}(n,m), \mathrm{Inv}(n) \in I$, where the set I is defined in Preliminaries and

Req 4 it is closed under the following bounded primitive recursion scheme: if $\alpha\ g, \psi \in K$ and f is defined by

$$f(\bar{x}, y) = \begin{cases} \alpha(\bar{x}) & \text{if} \quad y = 0 \\ \psi(\bar{x}, y, f(\bar{x}, y-1)) & \text{if} \quad y \geq 1 \end{cases}$$

and

$$f(\bar{x}, y) \leq g(\bar{x}, y)$$

then $f \in K$.

Then we tell that K satisfies the requirements **Req***.

Lemma 1. *Let K satisfy the requirements* **Req*** *and contain $+$, $\cdot$, $\lambda x.2^x$ then*

- *all constant functions are in K.*
- *$c(x, y)$, $l(x)$, $r(x) \in K$.*
- *Let*

$$g(n, m) = \begin{cases} 2 & \mathit{if} \quad q_n < q_m \\ 1 & \mathit{if} \quad q_n = q_m \\ 0 & \mathit{if} \quad q_n > q_m. \end{cases}$$

Then $g \in K$.

Lemma 2. *Let K satisfy the requirements* **Req** *and $\phi' \in K_1$ such that, for a real number x, $(\exists N \in \omega)(\forall n \geq N)|q_{\phi'(n)} - x| < \frac{1}{2^n}$. Then there exists $\phi \in K_1$ such that $(\forall n \in \omega)|q_{\phi(n)} - x| < \frac{1}{2^n}$, in other words, $x \in K^*$.*

3.2. When $\widetilde{K} = K^*$.

Proposition 1. *Let K satisfy the requirements* **Req*** *and contain $+$, $\cdot$, $\lambda x.2^x$ then $\widetilde{K} = K^*$.*

Proof. First we show that $\widetilde{K} \subseteq K^*$. W.l.o.g. we assume $f \in K^{\{0,1,2\}}$ and $x = \bar{f}$. It is clear that a required function $\phi \in K$ can be defined by

$$q_{\phi(n)} = \sum_{i=0}^{n} \frac{f(i) - 1}{2^{i+1}}.$$

The numerator of this fraction is bounded by above by $2^{n+1} - 1$ and the denominator is 2^{n+1}. It is worth noting that the number of a rational q, where $|q| = \frac{i}{j}$ is bounded from above by $2c(i,j)+1$. By definition $\lambda x.2^x \in K$ and K is closed under composition so by Lemma 1 $\phi(n)$ is bounded from above by some function from K. We have that $\phi \in K$ and $x \in K^*$.

To show that $\widetilde{K} \supseteq K^*$ w.l.o.g we assume that $x \in [-1,1] \cap K^*$ and ϕ is given. We are going to construct simultaneously $f \in K^{\{0,1,2\}}$ such that $\bar{f} = x$ and a supporting function $\psi \in K_1$ satisfying the equation $q_{\psi(n)} = \sum_{i=0}^{n} \frac{f(i)-1}{2^{i+1}}$. For that using bounded primitive recursion scheme we construct a function $h \in K$ that satisfies $h(i) = c(f(i), \psi(i))$ and finally permits us to define $f \in K^{\{0,1,2\}}$ such as $f(i) = l(h(i))$. Denote $J_n = [q_{\phi(n)} - \frac{1}{2^n}, q_{\phi(n)} + \frac{1}{2^n}]$. In our construction we want to meet the following properties:

- $q_{\psi(n)} = \sum_{i=0}^{n} \frac{f(i)-1}{2^{i+1}}$,
- $q_{\psi(n+1)} = q_{\psi(n)} + \frac{f(n+1)-1}{2^{n+2}}$ and
- in the case study:
  - if $q_{\psi(n)} > J_{n+3}$ then $f(n+1) = 0$,
  - if $q_{\psi(n)} \in J_{n+3}$ then $f(n+1) = 1$,
  - if $q_{\psi(n)} < J_{n+3}$ then $f(n+1) = 2$.

We define $f(0)$ and $\psi(0)$ as follows:

- if $0 > J_2$ then $f(0) = 0$, $q_{\psi(0)} = -\frac{1}{2}$,
- if $0 \in J_2$ then $f(0) = 1$, $q_{\psi(0)} = 0$,
- if $0 < J_2$ then $f(0) = 2$, $q_{\psi(0)} = \frac{1}{2}$.

Before formality let us note that the properties above guarantee $\bar{f} = x$. Indeed, by induction we show that $(\forall n \in \omega)|x - q_{\psi(n)}| \leq \frac{1}{2^{n+1}}$.

For $n = 0$ it follows from the definition of $f(0)$, $\psi(0)$ and the assumption that $x \in [0,1]$. For the inductive transition $n \to n+1$ we consider three cases:

- If $q_{\psi(n)} > J_{n+3}$ then $q_{\psi(n+1)} = q_{\psi(n)} - \frac{1}{2^{n+2}}$. By induction assumption, $|x - q_{\psi(n)}| \leq \frac{1}{2^{n+1}}$, therefore $|x - q_{\psi(n+1)}| \leq \frac{1}{2^{n+2}}$.
- If $q_{\psi(n)} < J_{n+3}$ the $q_{\psi(n+1)} = q_{\psi(n)} + \frac{1}{2^{n+2}}$. By induction assumption, $|x - q_{\psi(n)}| \leq \frac{1}{2^{n+1}}$, therefore $|x - q_{\psi(n+1)}| \leq \frac{1}{2^{n+2}}$.
- If $q_{\psi(n)} \in J_{n+3}$ then $q_{\psi(n+1)} = q_{\psi(n)}$. In this case the length of J_{n+3} is equal to $\frac{1}{2^{n+2}}$. We have x and $q_{\psi(n+1)} \in J_{n+3}$, so $|x - q_{\psi(n+1)}| \leq \frac{1}{2^{n+2}}$.

Therefore $\bar{f} = x$.

We define h as follows: First $h(0) = c(f(0), \psi(0))$. Assume $h(n)$ is already constructed. First we find $e \in \{0,1,2\}$ by the following rules:

- if $q_{r(h(n))} > J_{n+3}$ then $e = 0$,
- if $q_{r(h(n))} \in J_{n+3}$ then $e = 1$,
- if $q_{r(h(n))} < J_{n+3}$ then $e = 2$.

Then we find d_0, d_1, $d_2 \in K$ according the following rules: $\frac{e-1}{2^{n+1}} = q_{d_e(n)}$. After that we have the scheme $l(h(n+1)) = e$, $r(h(n+1)) = \mathrm{Add}(r(h(n)), d_e(n))$. Finally, $h(n+1) = c(e, \mathrm{Add}(r(h(n)), d_e(n)))$ and $f(n) = l(h(n))$. It is clear that the defined function f meets the required properties. The same as in the proof of the inclusion $\widetilde{K} \subseteq K^*$ it is easy to see that ψ is bounded from above by a function from K. The function f is bounded from above by $\lambda x.2$. So by Lemma 1 h is bounded from above by a function from K. Finally, $f \in K$. □

We propose to use the phrase 'x is a K-number' when $\widetilde{K} = K^*$ and $x \in K^*$ since in this case different intuitive approaches such as Cauchy approximations and sign-digit expansions give exactly the same set of computable reals. In particular, below we mention $\mathcal{E}^n$-numbers in some results.

3.3. When K^* is a real closed field.

Remark 1. *If K satisfies the requirements* **Req** *then $\mathbb{Q} \subseteq K^*$. As an example: let $-1 = q_\alpha$ for some α then the constant function $\lambda x.\alpha \in K$ so $-1 \in K^*$. The same holds for any $q \in \mathbb{Q}$.*

Let $f : \omega \to \omega$ be a computable function. The sequence $\{q_{f(n)}\}_{n\in\omega}$ is called fast Cauchy if $|q_{f(n+1)} - q_{f(n)}| \leq \frac{1}{2^n}$ for $n > 0$.

Lemma 3. *If K satisfies the requirements* **Req** *and $x \in K^*$ then there is a function $f \in K_1$ such that $\{q_{f(n)}\}_{n\in\omega}$ is a fast Cauchy sequence converging to x such that $|x - q_{f(n)}| < \frac{1}{2^{2n}}$ for all $n \in \omega$.*

Proof. Indeed, since $x \in K^*$ there exists $f^* \in K_1$ such that $|x - q_{f^*(n)}| < \frac{1}{2^n}$. Then put $f(n) = f^*(2n)$. The sequence $\{q_{f(n)}\}_{n\in\omega}$ is fast Cauchy one since

$$|q_{f(n+1)} - q_{f(n)}| < \frac{1}{2^{2n}} + \frac{1}{2^{2(n+1)}} < \frac{1}{2^n}$$

for $n > 0$. □

Proposition 2. *Let K satisfy the requirements* **Req**. *Then $(K^*, +, \cdot, \leq)$ is a field.*

Proof. Let $|x| < A$ and $|y| < A$ for x, $y \in K^*$ and $|x - q_{f(n)}| < \frac{1}{2^{2n}}$, $|y - q_{g(n)}| < \frac{1}{2^{2n}}$ for f, $g \in K_1$ that exist by Lemma 3. We show that K^* contains their product. Indeed, $|x \cdot y - q_{f(n)} \cdot q_{g(n)}| < A \cdot \frac{1}{2^{2n}} + (A + \frac{1}{2^{2n}}) \cdot \frac{1}{2^{2n}} < \frac{1}{2^n}$ where $n > N$ for some large N. By definition, $q_{f(n)} \cdot q_{g(n)} = q_{\mathrm{Mult}(f(n),g(n))}$. Since $x \cdot y$ is a real number there exists a sequence $\{q_{i_s}\}_{s\leq N}$ such that $|x \cdot y - q_{i_s}| < \frac{1}{2^s}$ for $s \leq N$. Then we define a new function as follows:

$$\chi(n) = \begin{cases} \mathrm{Mult}(f(n), g(n)) & \text{if} \quad n > N \\ i_s & \text{if} \quad n \leq N. \end{cases}$$

By definition for all $n \in \omega$ we have $|x \cdot y - q_{\chi(n)}| < \frac{1}{2^n}$ and $\chi =^* \mathrm{Mult}(f, g)$. So $\chi \in K_1$ and defines $x \cdot y$.

For addition it is even easier: since $|x + y - (q_{f(n)} + q_{g(n)})| < \frac{1}{2^{2n}} + \frac{1}{2^{2n}} \leq \frac{1}{2^n}$ for $n > 0$ the function $\eta = \mathrm{Add}(f, g)$ defines $x + y$ and belongs to K_1. For inverses elements: $-x = -1 \cdot x$. Then let $x \neq 0$, $x \in K^*$ and $|x - q_{f(n)}| < \frac{1}{2^{2n}}$ for $f \in K_1$ that exists by Lemma 3. Since there is $B \in \mathbb{Q}^+$ such that $|x| > B > 0$ without loss of generality we can assume that $q_{f(n)} \neq 0$ for all $n \in \omega$ or use the construction as for product. We have

$$|x^{-1} - (q_{f(n)})^{-1}| = \frac{|x - q_{f(n)}|}{|x| \cdot |q_{f(n)}|} < \frac{1}{2^{2n}} \cdot B^{-1} \cdot (B + \frac{1}{2^{2n}})^{-1} < \frac{1}{2^n}$$

for $n > M$ for some large M. And now we use a construction as for product. □

Proposition 3. *Let K satisfy the requirements* **Req*** *and contains $\lambda x.2^x$. Then $(K^*, +, \cdot, \leq)$ is a real closed field.*

Proof. The claim that K^* is a field follows from Proposition 2. To complete the proof, we show that the real roots of unitary polynomials with coefficients in K^* are also in K^*. Assume contrary that there exists a unitary polynomial $p(x) = \sum_{i=0}^{n} a_i x^i \in K^*[x]$ of a minimal degree which has a root x_0 in $\mathbb{R}$ but not in K^*. The polynomial p does not have multiple roots since in opposite case it is possible to compute $g = \text{G.C.D.}(p(x), p'(x))$ and $h = \frac{p}{q}$ which are in $K^*[x]$. The polynomial h has exactly the same roots as p and $\deg(h) \le \deg(p)$ however h does not have multiple roots. Therefore since $a_n = 1$ the coefficients $\bar{a} = (a_0, \dots, a_{n-1})$ of p satisfy the following formula:

$$\Psi(\bar{a}) = \Psi_1(\bar{a}) \vee \Psi_2(\bar{a}),$$

where

$$\Psi_1(\bar{a}) \leftrightharpoons \exists A\, \exists B\, \exists c > 0 \exists \epsilon > 0 \Big(A < B \wedge p(A) < -\epsilon \wedge p(B) > \epsilon \wedge \forall x \in [A, B]\, c > p'(x) > \epsilon \Big)$$

and

$$\Psi_2(\bar{a}) \leftrightharpoons \exists A\, \exists B\, \exists c < 0 \exists \epsilon < 0 \Big(A < B \wedge p(A) > -\epsilon \wedge p(B) < \epsilon \wedge \forall x \in [A, B]\, c < p'(x) < \epsilon \Big).$$

It is clear that the sets defined by $\Psi_1(\bar{a})$ and $\Psi_2(\bar{a})$ are not overlapping. W.l.o.g. we assume $\mathbb{R} \models \Psi_1(\bar{a})$. By continuity arguments, we can chose some A, B, $\epsilon > 0$, $c > 0$ and balls $B(a_i, r_i)$, $i = 1, \dots, n-1$, such that $|B - A| < 1$ and for all $\bar{b} \in \prod_{i=0}^{n-1} B(a_i, r_i)$, where $\bar{b} = (b_0, \dots, b_{n-1})$ and $b_n = 1$, we have

$$\sum_{i=0}^{n} b_i A^i < -\epsilon \wedge \sum_{i=0}^{n} b_i B^i > \epsilon \wedge (\forall x \in [A, B])\, c > \sum_{i=1}^{n} i \cdot b_i \cdot x^{i-1} > \epsilon.$$

Now we are going to show that x_0 being a unique root in $[A, B]$ belongs to K^*. For that we fix m and the precision $\frac{1}{2^m}$ assuming that m is quite large. Our goal is to construct a function $\psi \in K_1$ such that $y_m = q_{\psi(m)}$ and $|x_0 - y_m| < \frac{1}{2^m}$ for $m \gg 1$. Since $a_i \in K^*$ there exists $\phi_i \in K_1$ such that for all $s \in \omega$, $|a_i - q_{\phi_i(s)}| < \frac{1}{2^s}$. Put $s = 3m$ and $b_i = q_{\phi_i(3m)}$. Then $b_i \in B(a_i, r_i) \cap \mathbb{Q}$, $i \le n-1$, for $m \gg 1$ and for all $m \in \omega$, $|b_i - a_i| \le \frac{1}{2^{3m}}$. Let $\widetilde{p}(x) = \sum_{i=0}^{n} b_i x^i$. Since $\bar{b}$ satisfies the formula Ψ_1, the polynomial $\widetilde{p}(x)$ has a unique root z in $[A, B]$. We show that $z \in K^*$, i.e., we construct a function $\phi \in K$ such that for all $k \in \omega$, $|z - q_{\phi(m,k)}| < \frac{1}{2^k}$. We describe the standard bisection method for finding the root z of $\widetilde{p}$ on $[A, B]$. Let

$$q_{\phi(m,0)} = \frac{A + B}{2}.$$

Assume $\phi(m, k)$ is already constructed. Then we define $\phi(m, k+1)$ by the following rules:

- if $\widetilde{p}(q_{\phi(m,k)}) = 0$ then $\phi(m, k+1) = \phi(m, k)$,
- if $\widetilde{p}(q_{\phi(m,k)}) > 0$ then $\phi(m, k+1) = \phi(m, k) - \frac{B-A}{2^{k+2}}$,
- if $\widetilde{p}(q_{\phi(m,k)}) < 0$ then $\phi(m, k+1) = \phi(m, k) + \frac{B-A}{2^{k+2}}$.

By induction it is straightforward that for all $k \in \omega$, $|z - q_{\phi(m,k)}| < \frac{|B-A|}{2^{k+1}}$. The function $\phi(m, k)$ is constructed by the recursive scheme. Let us show that $\phi(m, k)$ is bounded. It is clear that the endpoints of the interval with the center $q_{\phi(m,k)}$ and the radius $\frac{B-A}{2^{k+1}}$, where $A = \frac{d_1}{l}$ and $B = \frac{d_2}{l}$, have a form

$$\frac{2^k d_1 + (d_2 - d_1) \cdot i \cdot l}{2^k \cdot l}$$

for appropriate $i \le 2^k$. Therefore $q_{\phi(m,k)}$ has a form

$$\frac{2^{k+1} d_1 + (d_2 - d_1) \cdot (2i + 1) \cdot l}{2^{k+1} \cdot l}$$

and it is clear that $\phi(m,k)$ is bounded by some function from K and therefore $\phi \in K$. Let us define $\psi(m) = \phi(m, 4m)$. Since for all $x \in [A,B]$ we have $|\widetilde{p}'(x)| > \epsilon > 0$, by the mean value theorem $\widetilde{p}(q) - \widetilde{p}(z) = \widetilde{p}'(\theta)(q-z)$ for some $\theta \in [A,B]$. Put $y_m = q_{\psi(m)}$. Then

$$| \widetilde{p}(y_m) | = | \widetilde{p}(q_{\psi(m)}) | \leq \frac{c}{2^{4m}} < \frac{1}{2^{3m}}$$

for $m \gg 1$. We show that $y_m = q_{\psi(m)}$ is required. Let $M \in \mathbb{Q}^+$ be a bound on a_i, A, B and ϵ, i.e., $|a_i| < M$ for $0 \leq i \leq n$, $|A| < M, |B| < M$ and $\frac{1}{\epsilon} < M$. One can assume that m is sufficiently big , i.e., $2^m > M^n + \cdots + 1 = \sum_{i=0}^{n} M^i$. It is worth noting that $2^m \cdot \epsilon > 1$ and we already establish above $| b_i - a_i | < \frac{1}{2^{3m}}$. So for all x, $y \in [A,B]$ we have $| p(y) - \widetilde{p}(y) | < (1 + \cdots + M_n) \cdot \frac{1}{2^{3m}}$. As a corollary for $y = y_m$ taking into account $\widetilde{p}(y_m) < \frac{1}{2^{3m}}$ we get

$$| p(y_m) | \leq \frac{1}{2^{3m}} \cdot (1 + \dots |y|^n) \leq \frac{1}{2^{3m}} \cdot 2^m = \frac{1}{2^{2m}}.$$

By the mean value theorem, for all x, $y \in [A,B]$ there exists $\theta \in [A,B]$ such that $p(x) - p(y) = (x-y) \cdot p'(\theta)$. If x_0 is the root of p in the interval $[A,B]$ and $y_m = q_{\psi(m)}$ then

$$|x_0 - y_m| \leq \frac{|p(y)|}{\epsilon} < \frac{1}{2^{2m}} \cdot 2^m \leq \frac{1}{2^m} \text{ for } m \gg 1.$$

So $y_m \in \mathbb{Q}$ is an approximation of the root x_0 with the precision $\frac{1}{2^m}$ for $m \gg 1$. By Lemma 2, $x_0 \in K^*$, a contradiction. □

Corollary 1. *Let K satisfy the requirements* **Req*** *and contain $\lambda x.2^x$. Then the set of K-numbers forms a real closed field.*

Corollary 2. *For $n \geq 3$ the set of $\mathcal{E}^n$–numbers forms a real closed field.*

Remark 2. *It is worth noting that Proposition 3 is closely related to well-known K. Ko's Theorem on real closedness of the polynomial time computable real numbers however this result is not particular case of our proposition. At the same time the particular case when K is the set of all primitive recursive functions has been considered by P. Hertling (handwritten notes) and it has been proven by him that the primitive recursive real numbers is a real closed field.*

3.4. **Criterion of Computable Presentability of Archimedean Ordered Fields.** Let $L = (L, \leq)$ be linearly ordered and $\mathbb{Q} \subseteq L$. Assume $\mu : \omega \to L$ is a numbering. With L we associate 2 families of non-strict Dedekind cuts:

$$A_k = \{n \mid q_n \leq \mu(k)\}$$
$$B_k = \{n \mid q_n \geq \mu(k)\}$$

and naturally define $S_k = A_k \oplus B_k = \{2n \mid n \in A_k\} \cup \{2n+1 \mid n \in B_k\}$ and $S_L = \{S_k \mid k \in \omega\}$. The family S_L is endowed with the standard numbering $\beta(k) = S_k$. The following proposition provides a criterion of computable presentability of an archimedean ordered field.

Proposition 4. *Let $F = (F, +, \cdot, \leq)$ be an archimedean ordered field, $\mu : \omega \to F$ be its numbering such that (F, μ) is an effective algebra. Then (F, μ) is a computable copy if and only if the family (S_F, β) is computable.*

Proof. The claim $\to$ follows from the observation that if the numbering μ is a computable presentation of an ordered field F then $\mu \geq q$, i.e., $q_n = \mu(h(n))$ for a computable function $h : \omega \to \omega$ and the family (S_F, β) is computable.
For the claim $\leftarrow$ we assume that (S_F, β) is computable. Let $0 = q_i$ and $-1 = \mu(a)$ for some $a \in \omega$. So the substraction is defined as $\mu(n) - \mu(m) = \mu(n) + \mu(a) \cdot \mu(m)$.

It is clear that $\mu(n) = \mu(m)$ iff $\mu(n) - \mu(m) = \mu(k) \wedge 2i \in S_k \wedge 2i+1 \in S_k$ and $x \neq 0$ iff $(\exists y)\, y \cdot x = 1$. Therefore equality is computable. It is easy to see that order is also computable. Indeed,

$$\mu(n) < \mu(m) \text{ iff } (\exists k)(\exists l)\, \mu(n) \leq q_k < q_l \leq \mu(m),$$
$$\mu(n) \leq \mu(m) \text{ iff } \mu(n) < \mu(m) \vee \mu(n) = \mu(m),$$
$$\mu(n) \not\leq \mu(m) \text{ iff } \mu(m) < \mu(n).$$

□

Corollary 3. *Let K satisfy the requirements* **Req** *and $\mu : \omega \to K^*$ be its numbering such that (K^*, μ) is an effective algebra. Then (K^*, μ) is a computable copy if and only if the family (S_{K^*}, β) is computable.*

4. Index sets vs. Computable Presentability

In this section we assume AC denotes the almost constant functions, i.e., $\mathrm{AC} = \{f : \omega \to \omega \mid (\exists c \in \omega)(\exists x \in \omega)(\forall y > x) f(y) = c\}$ and $\mathrm{AC} \subseteq K$.

Definition 1. *Suppose K is a class of total computable numerical functions, $\widetilde{K}$ is the set of reals generated by K. Then we associate with K a structure $(\widetilde{K}, \sigma_{\mathrm{str}}) = (\widetilde{K}, 0, Q^3_+, Q^3_-, \leq)$, where*

$$\widetilde{K} \models Q^3_+(x, y, z) \leftrightarrow x + y \leq z$$
$$\widetilde{K} \models Q^3_-(x, y, z) \leftrightarrow x + y \geq z.$$

It is easy to see that if a structure $\widetilde{K}$ has a computable copy $(\widetilde{K}, \mu)$, then the graph of addition is computable and the set $\mu^{-1}(\mathrm{Dyad})$ is computably enumerable:

$$\mu(n) \in \mathrm{Dyad} \leftrightarrow (\exists k \in \omega)(\exists l \in \mathbb{Z})\, 2^k \cdot \mu(n) + l = 0.$$

To proceed further we define index sets:

$$A_0 = \{n \mid \pi_n \subseteq \{0,1,2\} \wedge n \notin \mathrm{Tot}\},$$
$$A_1 = \{n \mid \pi_n \subseteq \{0,1,2\} \wedge (n \notin \mathrm{Tot} \vee \varphi_n \in \mathrm{AC})\},$$
$$A_2 = \{n \mid \pi_n \subseteq \{0,1,2\} \wedge (n \notin \mathrm{Tot} \vee \varphi_n \in K')\} =$$
$$\{n \mid \pi_n \subseteq \{0,1,2\} \wedge (n \notin \mathrm{Tot} \vee \overline{\varphi_n} \in \widetilde{K})\}$$

where $K' = \{\varphi_n \mid \overline{\varphi_n} \in \widetilde{K} \cap [-1, 1]\}$.

Theorem 1. *Suppose K is a class of total computable numerical functions. If the structure $(\widetilde{K}, \sigma_{\mathrm{str}})$ generated by K has a computable presentation then $A_0 \cup (A_2 \setminus A_1) \in \Sigma^0_2$.*

Proof. Let $\mu : \omega \to \widetilde{K}$ be a computable presentation. Since the set $E = \{n \mid -1 \leq \mu(n) \leq 1\}$ is computable, there exists a computable function h such that $\mathrm{im}(h) = E$ and $\tilde{\mu} = \mu \circ h$ is a computable numbering of $\widetilde{K} \cap [-1, 1]$. Assume $x = \tilde{\mu}(n)$. Now we construct a map $\nu : \omega \to \mathbb{T}$ by induction:

$$\nu(n)(0) = 1$$
$$\nu(n)(s+1) = \begin{cases} 0 & \text{if} \quad x < x_s \\ 1 & \text{if} \quad x = x_s \\ 2 & \text{if} \quad x > x_s, \end{cases}$$

where $x_s = \sum_{i \le s} \frac{\nu(n)(i)-1}{2^i}$. Since $|x_s - x| \le \frac{1}{2^s}$, $|x_{s+1} - x| \le \frac{1}{2^{s+1}}$. From $\mathrm{AC} \subseteq K$ it follows that $x_s \in \widetilde{K}$. We have the following properties: $\nu(n)$ is total, $\nu(n) \in \mathbb{T}$ and $\nu(n)$ provides a sign-digit representation of x. It is worth noting that Dyad is Σ^0_1–relation on $\widetilde{K}$ and for any $f : \omega \to \{0, 1, 2\}$ it holds that $f \in \mathrm{AC} \leftrightarrow \overline{f} \in \mathrm{Dyad}$ (see c.f. [11]). As a corollary, $Y = \{n \mid \nu(n) \in AC\}$ is computably enumerable. Now we show that $A_0 \cup A_2 \setminus A_1 \in \Sigma^0_2$. Let us note that

$$n \in A_0 \cup (A_2 \setminus A_1) \leftrightarrow$$
$$n \in A_0 \vee \Big(n \in A_2 \wedge \Big((\exists m \in \omega \setminus Y)\, \overline{\nu(m)} = \overline{\varphi_n} \vee n \notin \mathrm{Tot}\Big)\Big).$$

We have the following:

- The relation $n \notin Y$ is Π^0_1.
- The relation $\overline{\nu(m)} = \overline{\varphi_n}$ is Π^0_1. It follows from the following observations. Let

$$\Phi(f, g) \leftrightharpoons (\exists s > 0) \Big| \sum_{k=0}^{s} \frac{f(k)-1}{2^{k+1}} - \sum_{k=0}^{s} \frac{g(k)-1}{2^{k+1}} \Big| > \frac{1}{2^{s-1}}.$$

  Then for $f,\, g \in \mathbb{T}^{\{0,1,2\}}$, $\overline{f} \ne \overline{g} \leftrightarrow \Phi(f, g)$. So, $\overline{\nu(m)} = \overline{\varphi_n} \leftrightarrow \neg\Phi(\nu(m), \varphi_n)$.
- The relation $n \in A_2$ is Σ^0_2 since

$$n \in A_2 \leftrightarrow n \notin \mathrm{Tot} \vee (\exists m \in \omega) \neg\Phi(\nu(m), \varphi_n).$$

- The relation $n \notin Tot$ is Σ^0_2.

Therefore $A_0 \cup (A_2 \setminus A_1) \in \Sigma^0_2$. □

It is worth noting that the same proof is valid when one consider just a computable presentation μ of a linear ordered $(\widetilde{K}, \le)$ with the requirement that $\mu \ge q$.

4.1. **Criterion of m-completeness for tuples of Σ^0_1 and Σ^0_2 sets.** In this section for $s \ge 1$ we consider s-tuples $(A_0, \dots, A_{s-1})$, where either all A_i are Σ^0_1-sets or all A_i are Σ^0_2–sets.

For uniformity of a presentation we introduce a symbol l where $l \in \{1, 2\}$, a relation $\sim_l$ on sets and an oracle z_l that have the following interpretation. If $l = 1$ then $A \sim_l B$ means that A and B are equal and the oracle $z_l = \emptyset$. If $l = 2$ then $A \sim_l B$ means that $(A \setminus B) \cup (B \setminus A)$ is finite, i.e., A and B are almost equal, denoted $A =^* B$. The oracle $z_l = Kw$, where $Kw = \{n \mid \varphi_n(n)\downarrow\}$ or could be any creative set. We generalise ideas of the criterion of m-completeness of Σ^0_1-sets and Σ^0_2-sets in [2] to fit m-completeness of s-tuples of Σ^0_1-sets and Σ^0_2-sets that requires modifications of concepts and definitions.

Definition 2. *Let $(A_0, \dots, A_{s-1})$ and $(B_0, \dots, B_{s-1})$ be s-tuples of subsets of ω. We say that $(A_0, \dots, A_{s-1})$ is m-reducible to $(B_0, \dots, B_{s-1})$ ($(A_0, \dots, A_{s-1}) \le_m (B_0, \dots, B_{s-1})$) if there exists a computable function f such that $f^{-1}(B_i) = A_i$ for all $0 \le i < s$.*

Definition 3. *Let A belong to a class L, where $L \subseteq (\Sigma^0_n[\omega])^s$. We say that A is m-complete in the class L if any element from this class is m-reducible to it.*

Remark 3. *It is well known (see c.f. [3]) that given any (partial) Σ^0_2-function f one can effectively construct a total computable function F such that $(\forall x \in \mathrm{dom}(f))\, W_{f(x)} =^* W_{F(x)}$, moreover $\varphi_{f(x)} =^* \varphi_{F(x)}$.*

Definition 4. *Let $(F_0,\dots,F_{s-1})$ be an s-tuple of functions, where $F_i:\omega^s\to\omega$, $0\le i\le s-1$ and $(A_0,\dots,A_{s-1})$ be an s-tuple of Σ^0_l-sets. We say that $(F_0,\dots,F_{s-1})$ is m-reducible to $(A_0,\dots,A_{s-1})$, denoted as $(F_0,\dots,F_{s-1})\le_m(A_0,\dots,A_{s-1})$, if there exist computable functions $h:\omega^s\to\omega$, $a_i:\omega^s\to\omega$, $b_i:\omega^s\to\omega$, $0\le i\le s-1$, such that*

$$F_i(\bar{x})=\begin{cases} a_i(\bar{x}) & \text{if} \quad h(\bar{x})\in A_i\\ b_i(\bar{x}) & \text{if} \quad h(\bar{x})\notin A_i.\end{cases}$$

It is easy to see that this definition is a generalisation of the corresponding definition from [3].

Lemma 4. *For s-tuples X and A of Σ^0_l-sets if $(F_0,\dots,F_{s-1})\le_m(X_0,\dots,X_{s-1})$ and $(X_0,\dots,X_{s-1})\le_m(A_0,\dots,A_{s-1})$ then $(F_0,\dots,F_{s-1})\le_m(A_0,\dots,A_{s-1})$.*

Proposition 5. *(Smullyan's Fixed Point Theorem)*[21] *Let $\lambda_0,\dots,\lambda_{s-1}$ be partial computable functions of arity $n+s$. Then there exist computable functions*

$$h_0,\dots,h_{s-1}$$

of arity n such that the following equality simultaneously holds

$$\varphi_{\lambda_i(\bar{z},\bar{h}(\bar{z}))}=\varphi_{h_i(\bar{z})}$$

for all $0\le i\le s-1$ and for all $\bar{z}$ under the standard agreement: if the index in the left part of the equality is undefined then the function on the right is undefined anywere.

Proposition 6. *Let $(A_0,\dots,A_{s-1})$ be a s-tuple of Σ^0_l-sets. The following claims are equivalent.*

(1) $(A_0,\dots,A_{s-1})$ is m-complete in the class of s-tuples of Σ^0_l-set.
(2) There exists a computable function (its productive function) $H:\omega^s\to\omega$ such that for all $x_0,\dots x_{s-1}\in\omega$

$$H(x_0,\dots x_{s-1})\in\bigcap_{i=0}^{s-1}\Big((A_i\cap W^{z_l}_{x_i})\cup(\overline{A_i}\cap\overline{W}^{z_l}_{x_i})\Big).$$

(3) There exists a s-tuple of functions $(F_0,\dots,F_{s-1})$, where $F_i:\omega^s\to\omega$, $0\le i<s$, such that
 (a) $(F_0,\dots,F_{s-1})\le_m(A_0,\dots,A_{s-1})$,
 (b) $W_{F_i(\bar{x})}\not\sim_l W_{x_i}$ for all $0\le i<s$ and for all $\bar{x}$.
(4) There exists a s-tuple of functions $(F_0,\dots,F_{s-1})$, where $F_i:\omega^s\to\omega$, $0\le i<s$, such that
 (a) $(F_0,\dots,F_{s-1})\le_m(A_0,\dots,A_{s-1})$,
 (b) $\varphi_{F_i(\bar{x})}\not\sim_l\varphi_{x_i}$ for all $0\le i<s$ and for all $\bar{x}$.

Proof. 1) $\leftrightarrow$ 2). For $l=1$ the equivalents of the statements can be found in [6]. The existence of m-complete s-tuple of computably enumerable sets has been also established there. For $l=2$ we only need a relativisation to the oracle z_2 which also could be found in [6].
1) $\to$ 3). Without loss of generality we assume $s=3$.

Case 1: l=1. First we take the following m-complete 3-tuple:

$$X_0 = \{n \mid \varphi_n(0)\downarrow\},$$
$$X_1 = \{n \mid \varphi_n(1)\downarrow\},$$
$$X_2 = \{n \mid \varphi_n(2)\downarrow\}.$$

Further on by By Lemma 4 the considerations below will hold for any m-complete 3-tuple. By Graph theorem [19] we construct a computable sequence

$$B = \{B_{x_0x_1x_2}\}_{x_0x_1x_2\in\omega}$$

of partial computable functions such that

$$B_{x_0x_1x_2}(i) = \varphi_{x_i}(i) \text{ if } i = 0, 1, 2$$
$$B_{x_0x_1x_2}(k)\uparrow \text{ if } k > 2.$$

Then there exists a computable function $h : \omega^3 \to \omega$ such that

$$B_{x_0x_1x_2} = \varphi_{h(x_0,x_1,x_2)}.$$

By definition of B we have $h(x_0, x_1, x_2) \in X_i \leftrightarrow x_i \in X_i$ for $i \leq 2$. To finish the construction of F_i we take a_i and b_i for $i \leq 2$ as follows.

$a_0 = a_1 = a_2$ is an index of $\perp$,
b_0 is an index of the function $\{< 0, 0 >\}$,
b_1 is an index of the function$\{< 1, 0 >\}$,
b_2 is an index of the function$\{< 2, 0 >\}$,

We define for $i \leq 2$

$$F_i(\bar{x}) = \begin{cases} a_i & \text{if} \quad h(\bar{x}) \in X_i \\ b_i & \text{if} \quad h(\bar{x}) \notin X_i. \end{cases}$$

By construction (F_0, F_1, F_2) is m-reducible to (X_0, X_1, X_2). Let us show that

$$W_{F_i(\bar{x})} \neq W_{x_i}.$$

Fix i and consider two cases:

- Assume $h(\bar{x}) \in X_i$. Since $F_i(\bar{x}) = a_i$, $W_{F_i(\bar{x})} = \emptyset$. At the same time $W_{x_i} \neq \emptyset$ since $x_i \in X_i$ and $\varphi_{x_i}(i)\downarrow$.
- Assume $h(\bar{x}) \notin X_i$. Since $F_i(\bar{x}) = b_i$, $\varphi_{b_i}(i)\downarrow$ and $i \in W_{F_i(\bar{x})} \neq \emptyset$. At the same time $\varphi_{x_i}(i)\uparrow$, i.e., $i \notin W_{x_i}$.

Therefore (F_0, F_1, F_2) is a required 3-tuple.

Case 2: l=2. First we take the following 3-tuple:

$$Z_0 = \{n \mid W_n \cap 3\omega \text{ is finite}\},$$
$$Z_1 = \{n \mid W_n \cap (3\omega + 1) \text{ is finite}\},$$
$$Z_2 = \{n \mid W_n \cap (3\omega + 2) \text{ is finite}\}.$$

By analogy to the case 1 we chose $h : \omega^3 \to \omega$ such that for all $\bar{x} = (x_0, x_1, x_2)$ and every $k \in \omega$ we have $\varphi_{h(\bar{x})}(3k+i) = \varphi_{x_i}(3k+i)$ for $i \leq 2$. To finish the construction of F_i we take a_i and b_i for $i \leq 2$ as follows.

$a_0 = a_1 = a_2$ is an index of constant zero function,
$b_0 = b_1 = b_2$ is an index of $\perp$.

We define for $i \leq 2$

$$F_i(\bar{x}) = \begin{cases} a_i & \text{if} \quad h(\bar{x}) \in Z_i \\ b_i & \text{if} \quad h(\bar{x}) \notin Z_i. \end{cases}$$

By construction (F_0, F_1, F_2) is m-reducible to (Z_0, Z_1, Z_2).

Let us show that $W_{F_i(\bar{x})} \neq^* W_{x_i}$. Fix i. Assume contrary $W_{F_i(\bar{x})} =^* W_{x_i}$ and consider two cases:

- Let $h(\bar{x}) \in Z_i$. Since $F_i(\bar{x}) = a_i$, $W_{F_i(\bar{x})} = \omega$. At the same time $W_{x_i} \cap (3\omega + i)$ is finite from the choice of h and finiteness of $W_{h(\bar{x})} \cap (3\omega + i)$, a contradiction.
- Let $h(\bar{x}) \notin Z_i$. Since $F_i(\bar{x}) = b_i$, $W_{F_i(\bar{x})} = \emptyset$. By assumption, W_{x_i} is finite. At the same time $W_{x_i} \cap (3\omega + i)$ is infinite from the choice of h and infiniteness of $W_{\bar{x}} \cap (3\omega + i)$, a contradiction.

Therefore (F_0, F_1, F_2) is a required 3-tuple.

3) $\to$ 4). The implication is straightforward since $\varphi_{F_i(\bar{x})} \not\sim_l \varphi_{x_i}$ follows from $W_{F_i(\bar{x})} \not\sim_l W_{x_i}$.

4) $\to$ 2). Without loss of generality we assume $s = 3$. The following construction is uniform for both l. First for $i \leq 2$ we define functions G_i and T_i which are computable with the oracle z_l:

$$G_i(\bar{x}, \bar{y}) = \begin{cases} b_i(\bar{y}) & \text{if} \quad h(\bar{y}) \in W_{x_i}^{z_l} \\ \uparrow & \text{if} \quad h(\bar{y}) \notin W_{x_i}^{z_l}, \end{cases}$$

$$T_i(\bar{x}, \bar{y}) = \begin{cases} a_i(\bar{y}) & \text{if} \quad h(\bar{y}) \in A_i \\ \uparrow & \text{if} \quad h(\bar{y}) \notin A_i. \end{cases}$$

By Reduction principle for function graphs [19] we find a function $E_i(\bar{x}, \bar{y})$ with the following properties:

- $E_i(\bar{x}, \bar{y})$ is computable with the oracle z_l, therefore for some computable function $g_i : \omega^3 \to \omega$, $E_i(\bar{x}, \bar{y}) = K^{4,z_l}(g_i(\bar{x}), \bar{y})$, where K^{4,z_l} is Kleene universal function for 3-arity functions computable with the oracle z_l.
- If $h(\bar{y}) \in W_{x_i}^{z_l} \setminus A_i$ then $E_i(\bar{x}, \bar{y}) = b_i(\bar{y})$. If $h(\bar{y}) \in A_i \setminus W_{x_i}^{z_l}$ then $E_i(\bar{x}, \bar{y}) = a_i(\bar{y})$.

If $l = 1$ by Proposition 5 for $i < 3$ there exist three computable functions n_0, n_1, $n_2 : \omega^3 \to \omega$ such that

$$\varphi_{K^4(g_i(\bar{x}), n_0(\bar{g}(\bar{x})), n_1(\bar{g}(\bar{x})), n_2(\bar{g}(\bar{x})))} = \varphi_{n_i(\bar{g}(\bar{x}))}.$$

If $l = 2$ there exist three computable functions n_0, n_1, $n_2 : \omega^3 \to \omega$ such that

$$\varphi_{K^{4,z_2}(g_i(\bar{x}), n_0(\bar{g}(\bar{x})), n_1(\bar{g}(\bar{x})), n_2(\bar{g}(\bar{x})))} =^* \varphi_{n_i(\bar{g}(\bar{x}))}$$

under the condition that $K^{4,z_2}(g_i(\bar{x}), \bar{n}(\bar{g}(\bar{x}))) \downarrow$. The last statement requires more details which we show below. It is easy to see that, for $i \leq 2$, $f_i(\bar{z}, \bar{y}) \leftrightharpoons K^{4,z_2}(z_i, \bar{y})$ is Σ_2^o-function. Therefore by Remark 3 there exist computable functions λ_i such that for $(\bar{z}, \bar{y}) \in \mathrm{dom}(f_i)$

$$\varphi_{f_i(\bar{z}, \bar{y})} =^* \varphi_{\lambda_i(\bar{z}, \bar{y})}.$$

By Proposition 5,

$$\varphi_{\lambda_i(\bar{z},\bar{n}(\bar{z}))} = \varphi_{n_i(\bar{z})}.$$

Therefore

$$\varphi_{f_i(\bar{z},\bar{n}(\bar{z}))} =^* \varphi_{n_i(\bar{z})}$$

for $(\bar{z}, \bar{n}(\bar{z})) \in \mathrm{dom}(f_i)$. Now we are ready do define $H : \omega^3 \to \omega$:

$$H(\bar{x}) = h(n_0(\bar{g}(\bar{x})), n_1(\bar{g}(\bar{x})), n_2(\bar{g}(\bar{x}))).$$

Let us show that $H(\bar{x}) = h(n_0(\bar{g}(\bar{x})), n_1(\bar{g}(\bar{x})), n_2(\bar{g}(\bar{x})))$ is a required computable function. Assume contrary that for some x_0, x_1 and x_2

$$H(x_0, x_1, x_2) \notin \Big((A_i \cap W^{z_l}_{x_i}) \cup (\overline{A_i} \cap \overline{W}^{z_l}_{x_i})\Big)$$

for some $i \in \{0, 1, 2\}$. We have two cases:

(a) $H(\bar{x}) \in A_i \setminus W^{z_l}_{x_i}$

(b) $H(\bar{x}) \in W^{z_l}_{x_i} \setminus A_i$.

It is worth noting that in both cases $K^{4,z_l}(g_i(\bar{x}), \bar{n}(\bar{g}(\bar{x}))) \downarrow$. In the case (a),

$\varphi_{F_i(\bar{n}(\bar{g}(\bar{x})))} = \varphi_{a_i(\bar{n}(\bar{g}(\bar{x})))}$ by the definition of F_i,

$\varphi_{a_i(\bar{n}(\bar{g}(\bar{x})))} = \varphi_{K^{4,z_l}(g_i(\bar{x}),\bar{n}(\bar{g}(\bar{x})))}$ by the definition of g_i,

$\varphi_{K^{4,z_l}(g_i(\bar{x}),\bar{n}(\bar{g}(\bar{x})))} = \varphi_{n_i(\bar{g}(\bar{x}))}$ by the choise of $\bar{n}$.

This contradicts the condition on F_i. In the case (b),

$\varphi_{F_i(\bar{n}(\bar{g}(\bar{x})))} = \varphi_{b_i(\bar{n}(\bar{g}(\bar{x})))}$ by the definition of F_i,

$\varphi_{b_i(\bar{n}(\bar{g}(\bar{x})))} = \varphi_{K^{4,z_l}(g_i(\bar{x}),\bar{n}(\bar{g}(\bar{x})))}$ by the definition of g_i,

$\varphi_{K^{4,z_l}(g_i(\bar{x}),\bar{n}(\bar{g}(\bar{x})))} = \varphi_{n_i(\bar{g}(\bar{x}))}$ by the choise of $\bar{n}$.

This contradicts the condition on F_i. Therefore H is a required productive function. □

4.2. ***m*-complete 3-tuple of Σ^0_2-sets.** Let us fix the following index sets:

$$E_0 = \{n \mid \pi_n \subseteq \{0,1,2\} \wedge \big(\varphi_n \cap \omega \times \{0\} =^* \emptyset\big)\},$$
$$E_1 = \{n \mid \pi_n \subseteq \{0,1,2\} \wedge \big(\varphi_n \cap \omega \times \{1\} =^* \emptyset\big)\},$$
$$E_2 = \{n \mid \pi_n \subseteq \{0,1,2\} \wedge \big(\varphi_n \cap \omega \times \{2\} =^* \emptyset\big)\}.$$

Using Proposition 6 we show that the 3-tuple (E_0, E_1, E_2) is m-complete in the class of 3 -tuples of Σ^0_2-sets.

In order to define (F_0, F_1, F_2) we first take a computable function $h : \omega^3 \to \omega$ such that

$$\varphi_{h(x_0,x_1,x_2)}(3k) = \begin{cases} 0 & \text{if } \varphi_{x_0}(k) \downarrow \\ \uparrow & \text{otherwise,} \end{cases}$$

$$\varphi_{h(x_0,x_1,x_2)}(3k+1) = \begin{cases} 1 & \text{if } \varphi_{x_1}(k) \downarrow \\ \uparrow & \text{otherwise,} \end{cases}$$

$$\varphi_{h(x_0,x_1,x_2)}(3k+2) = \begin{cases} 2 & \text{if } \varphi_{x_2}(k) \downarrow \\ \uparrow & \text{otherwise.} \end{cases}$$

Now we pick up appropriate functions a_i and b_i for $i \leq 2$. The functions b_i is defined by $\varphi_{b_i(\bar{x})}(k) = \varphi_{x_i}(k) + 1$, where a_i is an index of id function. Then we define for $i \leq 2$

$$F_i(\bar{x}) = \begin{cases} a_i & \text{if} \quad h(\bar{x}) \in E_i \\ b_i(\bar{x}) & \text{if} \quad h(\bar{x}) \notin E_i. \end{cases}$$

Let us show that $\varphi_{F_i(\bar{x})} \not\sim_2 \varphi_{x_i}$ for $i \leq 2$. Without loss of generality it is sufficient to consider $i = 0$. Assume $h(\bar{x}) \notin E_0$. Then $W_{h(\bar{x})}$ is infinite and infinitely often a value of $\varphi_{h(\bar{x})}$ is zero. By construction, W_{x_0} is infinite. Since for $k \in \omega$,

$$\varphi_{F_0(\bar{x})}(k) = \varphi_{b_0(\bar{x})}(k) = \varphi_{x_0}(k) + 1,$$

there exist infinitely many $k \in \omega$ such that $\varphi_{F_0(\bar{x})}(k) \neq \varphi_{x_0}(k)$. As a corollary, $\varphi_{F_0(\bar{x})} \neq^* \varphi_{x_0}$. Assume $h(\bar{x}) \in E_0$. Then φ_{x_0} is a finite function. Since for $k \in \omega$,

$$\varphi_{F_0(\bar{x})}(k) = \varphi_{a_0(\bar{x})}(k) = k,$$

we have $\varphi_{F_0(\bar{x})} \neq^* \varphi_{x_0}$.

Lemma 5. *Let a 3-tuple (E_0, E_1, E_2) be m-complete in the class of 3-tuples of Σ^0_2-set. Then the following 3-tuple*

$$\begin{aligned} Y_0 &= E_0 \cap E_1 \cap E_2, \\ Y_1 &= E_1 \cap E_2, \\ Y_2 &= E_2 \end{aligned}$$

is m-complete in the class of 3-tuples (X_0, X_1, X_2) of Σ^0_2-set with the additional condition $X_0 \subseteq X_1 \subseteq X_2$.

By the choice of (E_0, E_1, E_2) at the beginning of this section we have $Y_0 \subseteq Y_1 \subseteq Y_2$, where

$$\begin{aligned} Y_0 &= \{n \mid \pi_n \subseteq \{0,1,2\} \wedge W_n \text{ is finite}\}, \\ Y_1 &= \{n \mid \pi_n \subseteq \{0,1,2\} \vee (\exists N)(\forall k \geq N)(\varphi_n(k)\downarrow \rightarrow \varphi_n(k) = 0\}, \\ Y_2 &= \{n \mid \pi_n \subseteq \{0,1,2\} \vee \varphi_n \cap \omega \times \{2\} =^* \emptyset)\}. \end{aligned}$$

Remark 4. *It is worth noting that if 3-tuple (Z_0, Z_1, Z_2) is m-complete in the class of 3-tuple (X_0, X_1, X_2) of Σ^0_2-sets with the additional condition $X_0 \subseteq X_1 \subseteq X_2$ then the set $Z_0 \cup (Z_2 \setminus Z_1) \notin \Sigma^0_2$ since this combination is the m-greatest 3-tuple among the similar combinations of Σ^0_2-sets.*

4.3. **Classes K without computably presentable $\widetilde{K}$.** Below we list important restrictions on a class K of total computable numerical functions:

(1) Together with the basic functions BF the class contains $+, \dot{-}, \cdot, [\frac{x}{2}]$ and $[\sqrt{x}]$.
(2) The class has a computable universal function for all unary functions i.e. the sequence $\{\mathcal{F}_n\}_{n\in\omega}$ of all unary functions from K is computable.
(3) There exists a computable function $H : \omega \to \omega$ such that for all $n \in \omega$ $\mathrm{im}(\mathcal{F}_{H(n)}) = \{0\} \cup \{x+1 \mid x \in W_n\}$. Moreover, for all $i \in W_n$, $\mathcal{F}^{-1}_{H(n)}(i+1)$ is an infinite set.

(4) It is closed under composition and either under the standard bounded recursion scheme or the following second bounded recursion scheme: If $\alpha,\ g, \psi \in K$ and f is defined by

$$f(\bar{x}, y) = \begin{cases} \alpha(\bar{x}) & \text{if} \quad y = 0 \\ \psi(\bar{x}, y, f(\bar{x}, [\frac{y}{2}])) & \text{if} \quad y \geq 1 \end{cases}$$

and

$$f(\bar{x}, y) \leq g(\bar{x}, y)$$

then $f \in K$.

Remark 5. *It is worth noting that the class K satisfying the restrictions above contains all almost constant functions, Cantor 3-tuple (c, l, r), $\mathrm{sg}|\mathrm{x} - \mathrm{y}|$.*

Remark 6. *Some of natural examples that satisfy the restrictions above are all $\mathcal{E}^n$, $n \geq 2$. For $\mathcal{E}^2$ the verification of satisfiability based on Ritchie's characterisation (see Preliminaries) and the existence of a computable universal function for $\mathcal{E}^2$ and for $\mathcal{E}^n$, $n \geq 3$ it is straightforward. Another good example is the class P of functions computable in polynomial time under binary notations of arguments and values.*

Characterization of P in terms of the second primitive recursion scheme was obtained by A. Cobham in [4]. In particular, his result provides the item (4) for P, other items are straightforward.

Theorem 2. *Let K satisfy the requirements above and $(\widetilde{K}, \sigma_{\mathrm{str}})$ be the structure generated by K. Then $\widetilde{K}$ does not have a computable presentation.*

The proof follows from Theorem 1 and the claim $A_0 \cup (A_2 \setminus A_1) \notin \Sigma^0_2$ which is based on the following proposition and Section 4.2.

Proposition 7. $(Y_0, Y_1, Y_2) \leq_m (A_0, A_1, A_2)$.

Proof. We are going to construct a computable function f such that $n \in Y_i \leftrightarrow f(n) \in A_i$. In order to do that we will construct a computable sequence $\{F_n\}_{n \in \omega}$ of computable functions by steps and then effectively find a required reduction f.

We take

- a standard computable reduction function $\alpha : \omega \to \omega$ for $\mathrm{Fin} \leq_{\mathrm{m}} \omega \setminus \mathrm{Tot}$ (c.f. [20]) with the following properties:
  - if W_n is finite then $W_{\alpha(n)}$ is finite,
  - if W_n is infinite then $\varphi_{\alpha(n)}$ is total,
  - $\pi_n = \pi_{\alpha(n)}$,
  - If $(\exists^\infty a)\, \varphi_n(a) = x$ then $(\exists^\infty b)\, \varphi_{\alpha(n)}(b) = x$ and vice versa.
- a computable function t such that
  - $W_{t(n)} = \{c(k, d) \mid \varphi_{\alpha(n)}(k) = d\}$.

Now we point out the requirements on a step s which we want to meet in our construction:

- $F_n^{s+1} \supseteq F_n^s$,
- $\mathrm{dom}(F_n^s) = [0, \ldots, m_n^s]$ is a proper initial segment of ω,
- if $\mathcal{F}_{H(t_s(n))}(s+1) = c(c, d) + 1 \wedge d > 2$ then $(\exists j)\, F_n(j) > 2$,
- if $\mathcal{F}_{H(t_s(n))}(s+1) = c(i, 2) + 1$ then in the process of the construction we provide the following: $\overline{F_n} \notin \{\overline{\mathcal{F}_0}, \ldots, \overline{\mathcal{F}_i}\}$,

- $F_n = \bigcup_{s\in\omega} F_n^s$.

W.l.o.g. we assume now that K is closed under the second bounded recursion scheme since the case when K is closed under the standard bounded recursion scheme is much more easy so it is left to a reader.

Description of the construction of m_n^s, $\{F_n^s\}_{n,s\in\omega}$, $t_s(n)$ and I_n^s:

Step 0

$m_0^n = 0$, $F_n^0(0) = 0$, $t_0(n) = t(n)$ and $I_n^0 = 0$.

Step s+1

Case 1 If $\mathcal{F}_{H(t_s(n))}(s+1) = c(i,0)+1$ or $\mathcal{F}_{H(t_s(n))}(s+1) = 0$ then we proceed as follows:

$$m_n^{s+1} = \begin{cases} 2m_n^s & \text{if} \quad m_n^s > 0 \\ 1 & \text{if} \quad m_n^s = 0 \end{cases}$$

and for all $j \le m_n^s$, $F_n^{s+1}(j) = F_n^s(j)$, for all $m_n^s \le j \le m_n^{s+1}$ we put $F_n^{s+1}(j) = F_n^s(m_n^s)$, $t_{s+1}(n) = t_s(n)$ and $I_n^{s+1} = I_n^s$.

Case 2 $\mathcal{F}_{H(t_s(n))}(s+1) = c(i,1)+1$ we consider the following subcases:

Subcase 2.1 $i > s+1$. We proceed as in Case 1.

Subcase 2.2 $i \le s+1$ and $i \le I_n^s$. We proceed as in Case 1.

Subcase 2.3 $I_n^s < i \le s+1$. We proceed as follows:

$$m_n^{s+1} = \begin{cases} 2m_n^s & \text{if} \quad m_n^s > 0 \\ 1 & \text{if} \quad m_n^s = 0 \end{cases}$$

and for all $j \le m_n^s$, $F_n^{s+1}(j) = F_n^s(j)$, for all $m_n^s \le j < m_n^{s+1}$ we put $F_n^{s+1}(j) = F_n^s(m_n^s)$ and for $F_n^{s+1}(m_n^{s+1})$ we chose the least value from $\{0,1\}$ such that $F_n^{s+1}(m_n^{s+1}) \ne F_n^s(m_n^s)$, $t_{s+1}(n) = t_s(n)$ and $I_n^{s+1} = i$.

Case 3 If $\mathcal{F}_{H(t_s(n))}(s+1) = c(i,2)+1$ then for all $j \le m_n^s$, $F_n^{s+1}(j) = F_n^s(j)$ and we proceed as follows: $m_n^{s+1} = m_n^s + 2(i+1)$ and for $k \le i$ we chose the values of F_n^{s+1} on the arguments $m_s^n + 2k + 1$ and $m_s^n + 2k + 2$ according with the following table:

$\mathcal{F}_k(m_n^s+2k+1)$	$\mathcal{F}_k(m_n^s+2k+2)$	$F_n^{s+1}(m_s^n+2k+1)$	$F_n^{s+1}(m_s^n+2k+2)$
2	2	0	0
2	0	0	0
2	1	0	0
1	2	0	0
1	1	0	0
1	0	2	2
0	2	2	2
0	1	2	2
0	0	2	2

The equality $W_{t_{s+1}(n)} = W_{t_s(n)} \setminus \{c(i,2)\}$ defines the value of $t_{s+1}(n)$ and $I_n^{s+1} = I_n^s$.

Case 4 If $\mathcal{F}_{H(t_s(n))}(s+1) = c(i,d)+1$ and $d > 2$ then for all $j \le m_n^s$, $F_n^{s+1}(j) = F_n^s(j)$ and we proceed as follows:

$$m_n^{s+1} = \begin{cases} 2m_n^s & \text{if} \quad m_n^s > 0 \\ 1 & \text{if} \quad m_n^s = 0 \end{cases}$$

and for all $j \le m_n^s$, $F_n^{s+1}(j) = F_n^s(j)$, for all $m_n^s \le j < m_n^{s+1}$ we put $F_n^{s+1}(j) = F_n^s(m_n^s)$ and $F_n^{s+1}(m_n^{s+1}) = d$, $t_{s+1}(n) = t_s(n)$ and $I_n^{s+1} = I_n^s$.

We put $F_n = \bigcup_{s\in\omega} F_n^s$ and effectively find computable function $f : \omega \to \omega$ such that $\varphi_{f(n)} = (F_n \cap (W_{\alpha(n)} \times \omega)) \cup \{c(x, d) \mid F_n(x) = d \wedge d > 2\}$.

Now we show that f is a required reduction.

If $n \in Y_0$ then $W_{\alpha(n)}$ is finite, so is $\varphi_{f(n)}$ and $f(n) \in A_0$. If $n \notin Y_0$ then there are two cases:
1) $W_{\alpha(n)}$ is ω, by construction, $\varphi_{f(n)} = F_n$ and F_n is total, so $f(n) \notin A_0$.
2) $\pi_n \not\subseteq \{0, 1, 2\}$. Then for some $j \in \omega$ $F_n(j) > 2$, so $\pi_{f(n)} \not\subseteq \{0, 1, 2\}$. Again $f(n) \notin A_0$. So we have $f^{-1}(A_0) = Y_0$.

If $n \in Y_1 \setminus Y_0$ then $\varphi_{\alpha(n)}$ is total and $\varphi_{\alpha(n)} =^* 0$. In this case $\varphi_{f(n)} = F_n$ and by construction $F_n \in \mathrm{AC}$ since $(\exists s_1 \in \omega)(\forall s \geq s_1)\, I_s^n = I_{s_1}^n$. So $f(n) \in A_1 \setminus A_0$.

If $n \notin Y_2$ then $\varphi_{\alpha(n)}$ is total and $\varphi_{f(n)} = F_n$. The case $\mathcal{F}_{H(t_s(n))}(s+1) = c(i, 2)+1$ arises infinitely often and the collection of the corresponding numbers i is infinite too. So, for infinitely many i, either $\mathrm{im}(F_n) \not\subseteq \{0, 1, 2\}$ or $\overline{F_n} \notin \{\overline{\mathcal{F}_0}, \ldots, \overline{\mathcal{F}_i}\}$. Hence either $\mathrm{im}(F_n) \not\subseteq \{0, 1, 2\}$ or $\overline{F_n} \notin \widetilde{K}$. That means $f(n) \notin A_2$.

If $n \in Y_2 \setminus Y_1$ then $\varphi_{\alpha(n)} = F_n$ is total. By the choice of n, $(\exists N)(\forall i \geq N)\, \varphi_{\alpha(n)}(i) \neq 2$. Hence $(\exists^\infty i)\varphi_{\alpha(n)}(i) = 1$.

Let us note that after some step s_0 for all i we have $c(i, 2) \notin W_{t_s(n)}$ for $s \geq s_0$. We define $t_\infty(n) = t_{s_0}(n)$. It is easy to see that, for $s \geq s_0$, $t_\infty(n) = t_s(n)$. On the step $s + 1$, when $\mathcal{F}_{H(t_\infty(n))}(s + 1) = c(i, 1) + 1$, $F_n^{s+1}(m_n^{s+1}) \neq F_n^{s+1}(m_n^{s+1} - 1)$. Hence $F_n \notin \mathrm{AC}$, so $f(n) \notin A_1$. So we have $f^{-1}(A_1) = Y_1$.

Using s_0 and N from above we explain that $F_n \in K$.

Let $m_n = m_n^{s_0+1}$. It is easy to see that the following functions belongs to K:

- the characteristic function of the set $A = \{m_n \cdot 2^i \mid i \geq 0\}$,
- the function $g(x)$, that computes $\max\{y \in A \mid y \leq x\}$ for $x \geq m_n$ and for $x < m_n$ it is equal to 0,
- the function $S(x) = \mu(s')(F_n^{s'}(x)\downarrow)$.

In order to meet our goal we construct the function $I(x) = I_n^{S(x)-1}$ by the following rules:

Assume $I_0 = I_n^{s_0}$. Then we define

- for $x < m_n$, $I(x) = 0$,
- for $x = m_n$, $I(x) = I_0$,
- for $x > m_n$ and $g(x) > m_n$

$$I(x) = \begin{cases} l(\mathcal{F}_{H(t_\infty(n))}(S(x) - 1) - 1) \text{ if} \\ \qquad I([\frac{x}{2}]) \leq i \leq S(x) - 1 \wedge r(\mathcal{F}_{H(t_\infty(n))}(S(x) - 1) - 1) = 1 \\ I([\frac{x}{2}]) \text{ otherwise,} \end{cases}$$

- for $x > m_n$ and $g(x) = m_n$, $I(x) = I_0$.

From above we can see that $\lambda x.I(x) \in K$. We can assume that $x \geq N$ and $S(x) > s_0$. Let us denote $i \leftrightharpoons l(\mathcal{F}_{H(t_\infty(n))}(S(x) - 1) - 1)$ and consider two cases.
1. Suppose $x \in A$. If $\mathcal{F}_{H(t_\infty(n))}(S(x)) = c(i, 0) + 1$ then $F_n(x) = F_n([\frac{x}{2}])$. The same is done if $\mathcal{F}_{H(t_\infty(n))}(S(x)) = c(i, 1) + 1$ but $i \leq I(x)$ or $i > S(x)$. Otherwise, i.e., if $\mathcal{F}_{H(t_\infty(n))}(S(x)) = c(i, 1) + 1$ and $I(x) \leq i \leq S(x)$ then for the value of $F_n(x)$ we chose the first one from $\{0, 1\}$ which differs from $F_n([\frac{x}{2}])$.
2. Suppose $x \notin A$. If $\mathcal{F}_{H(t_\infty(n))}(S([\frac{x}{2}]+1) = c(i, 0)+1$or 0 then $F_n(x) = F_n([\frac{x}{2}])$. The same is done if $\mathcal{F}_{H(t_\infty(n))}(S([\frac{x}{2}]+1) = c(i, 0)+1$ but $i \leq I([\frac{x}{2}]+1)$ or $i > S([\frac{x}{2}]+1)$. Otherwise, i.e., if $\mathcal{F}_{H(t_\infty(n))}(S([\frac{x}{2}])) = c(i, 1) + 1$ and $I([\frac{x}{2}]) \leq i \leq S([\frac{x}{2}])$ then for the value of $F_n(x)$ we chose the first one from $\{0, 1\}$ which differs from $F_n([\frac{x}{2}])$. Therefore the scheme above shows that $F_n \in K$. So we have $f^{-1}(A_2) = Y_2$.

□

Corollary 4. *The structures $\widetilde{P}$, $\widetilde{\mathcal{E}}^n$, $n \geq 2$ do not have computable copies and the fields of $\mathcal{E}^n$-numbers, $n \geq 3$ do not have computable copies.*

Proof. The claim follows from Corollary 2 and Theorem 2. □

References

[1] P.E. Alaev, *Computable homogeneous Boolean algebras and a metatheorem*, Algebra Logic, **43**:2 (2004), 73–87. Zbl 1062.03034

[2] M.M. Arslanov, *Local theory of degrees of unsolvability and Δ^0_2-sets*, Kazan' University Press, Kazan', 1987. Zbl 0875.03082

[3] M.M. Arslanov, *On some generalisations of a fixed point theorem*, Sov. Math., **25**:5 (1981), 1–10. Zbl 0523.03029

[4] A. Cobham, *The intrinsic computational difficulty of functions*, in Y. Bar-Hillel ed., *Logic, Methodology and Philosophy of Science: Proceedings of the 1964 International Congress*, North Holland, Amsterdam, 1964, 24–30.

[5] Yu.L. Ershov, *Decision problems and constructivizable models*, Nauka, Moscow, 1980. Zbl 0495.03009

[6] Yu.L. Ershov, *Theory of numberings*, in: E.R. Griffor, ed., *Handbook of computability theory*, Elsevier, Amsterdam, 1999, 473–503. Zbl 0948.03040

[7] Yu.L. Ershov, S.S. Goncharov, *Constructive models*, Consultants Bureau, New York, 2000. Zbl 0954.03036

[8] S.S. Goncharov, *Countable Boolean algebras and decidability*, Plenum, New York, 1997. Zbl 0912.03019

[9] A. Grzegorczyk, *Some classes of recursive functions*, Rozprawy Mathemaczne IV, Warszawa, 1953.

[10] N. Khisamiev, *Constructive abelian groups*, in Ershov, Yu. L. (ed.) et al., *Handbook of recursive mathematics, Vol. 2*, Elsevier. Stud. Logic Found. Math., **139**, Elsevier, Amsterdam, 1998, 1177–1231. Zbl 0940.03044

[11] Ker-I, Ko, *Complexity theory of real functions*, Birkhäuser, Boston etc., 1991. Zbl 0791.03019

[12] M. Korovina, O. Kudinov, *Spectrum of the field of computable real numbers*, Algebra Logic, **55**:6 (2017), 485–500. Zbl 1420.03113

[13] M. Korovina, O. Kudinov, *The uniformity principle for Σ-definability*, J. Log. Comput., **19**:1 (2009), 159–174. Zbl 1157.03013

[14] A.I. Mal'cev, *Constructive algebras*, Russ. Math. Surv., **16**:3 (1961), 77–129. Zbl 0129.25903

[15] M. Miller, V. Ocasio González, *Degree spectra of real closed fields*, Arch. Math. Logic, **58**:3-4 (2019), 387–411. Zbl 1477.03169

[16] A.S. Morozov, *Countable homogeneous Boolean algebras*, Algebra Logic, **21** (1983), 181–190. Zbl 0539.03011

[17] M.O. Rabin, *Computable algebra, general theory and theory of computable fields*, Trans. Am. Math. Soc., **95**:2 (1960), 341–360. Zbl 0156.01201

[18] R.W. Ritchie, *Classes of predictably computable functions*, Trans. Am. Math. Soc., **106** (1963), 139–173. Zbl 0107.01001

[19] H. Rogers, *Theory of recursive functions and effective computability*, McGraw-Hill, Maidenhead, Berksh., 1967. Zbl 0183.01401

[20] J.R. Shoenfield, *Degrees of unsolvability*, North-Holland Publ., Amsterdam-London, 1971. Zbl 0245.02037

[21] R.M. Smullyan, *Theory of formal systems*, Annals of Mathematics Studies, **47**, Princeton University Press, Princeton, 1961. Zbl 0097.24503

[22] R.I. Soare, *Recursively enumerable sets and degrees. A study of computable functions and computably generated sets*, Springer, Berlin etc., 1987. Zbl 0667.03030

[23] K. Weihrauch., *Computable analysis. An introduction*, Springer, Berlin, 2000. Zbl 0956.68056

Margarita Vladimirovna Korovina
A.P. Ershov Institute of Informatics Systems,
pr. Acad. Lavrentjev, 6,
630090, Novosibirsk, Russia
Email address: rita.korovina@gmail.com

Oleg Victorovich Kudinov
Sobolev Institute of Mathematics,
pr. Koptyug, 4,
630090, Novosibirsk, Russia
Email address: kud@math.nsc.ru