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APPLICATION OF A TAYLOR SERIES TO APPROXIMATE A FUNCTION WITH LARGE GRADIENTS

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Abstract: The method of approximating functions by polynomials based on Taylor series expansion is widely known. However, the residual term of such an approximation can be significant if the function has large gradients. The work assumes that the function has a decomposition in the form of a sum of regular and boundary layer components. The boundary layer component is a function of general form, known up to a factor, and is responsible for large gradients of the given function. This decomposition is valid, in particular, for the solution of a singularly perturbed problem. To approximate the function, a formula is proposed that uses the Taylor series expansion of the function and is exact for the boundary layer component. Under certain restrictions on the boundary layer component, estimates of the error in the approximation of the function are obtained. These estimates do not depend on the boundary layer component. Cases of functions of one and two variables are considered.

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1 Introduction

The method of approximating a function by polynomials based on Taylor series expansion is widely known. However, the remainder of such an expansion can be significant if the function has large gradients. Therefore, the question of interest is how to approximate such a function based on the use of Taylor series expansion so that the error is not significant. Let us consider the cases of functions with large gradients of one variable $u(x)$ on the interval $[0, 1]$ and two variables $u(x, y)$ in the rectangular domain $[0, 1]^2$.

We will assume that the function has a decomposition in the form of a sum of regular and boundary layer components. In particular, such a decomposition is valid for the solution of the singularly perturbed problem [1]. Boundary layer components are considered as functions of a general form, are known up to a factor and are responsible for large gradients of the function. In the case of a function of one variable such a decomposition was constructed to solve the singularly perturbed problem in [2]. In the case of a function of two variables, the function $u(x, y)$ corresponds to the solution of a singularly perturbed elliptic problem, and the decomposition for it was constructed in [3].

Based on the use of the Taylor series expansion of the function, we will construct an approximation to the function based on fitting to the boundary layer component and estimate the error of such approach.

Notations. By C and C_j we mean positive constants that are independent of the grid step h, of the boundary layer components $\Phi(x), \Theta(y)$ and their derivatives. We will assume that in the case of an exponential boundary layer, these constants do not depend on the small parameter ε. We will limit the various quantities to one constant C_j, if this is clear from the text.

2 Construction and proving of an approximation for a function of one variable

Let $u(x)$ be a sufficiently smooth function on the interval $[0, 1]$. The well-known Taylor series expansion with residual term $R_k(u, x)$ in integral form looks as

$$u(x) = \sum_{j=0}^{k} \frac{u^{(j)}(x_0)}{j!}(x - x_0)^j + R_k(u, x), \tag{1}$$

where

$$R_k(u, x) = \frac{1}{k!} \int_{x_0}^{x} u^{(k+1)}(t)(x - t)^k \, dt. \tag{2}$$

Let us $h = |x - x_0|$. We will assume that $x > x_0$. In the case of $x < x_0$ the interval $[x_0, x]$ is replaced by the interval $[x, x_0]$.

From (2) the estimates follow:

$$|R_k(u,x)| \le \frac{h^k}{k!} \int\limits_{x_0}^{x} |u^{(k+1)}(t)|\, dt,$$

$$|R_k(u,x)| \le \frac{h^{k+1}}{(k+1)!} \max_{s \in [x_0,x]} \left| u^{(k+1)}(s) \right|. \tag{3}$$

According to (3), for some constant C $|R_k(u,x)| \le Ch^{k+1}$, if the derivative $u^{(k+1)}(x)$ is uniformly bounded.

We will be interested in the case when this derivative is not uniformly bounded. Let the following decomposition be valid for a sufficiently smooth function $u(x)$:

$$u(x) = p(x) + \gamma\Phi(x),\ \ x \in [0,1], \tag{4}$$

where $p(x)$ is a regular component with limited derivatives up to a certain order, $\Phi(x)$ is a boundary layer component, which is a function of general form and is responsible for large gradients of the function $u(x)$. Function $\Phi(x)$ is assumed to be known, $p(x)$ and γ are not specified, and the coefficient γ is limited.

Let us note the cases of specifying $\Phi(x)$, when the function $u(x)$ has the form (4).

In the presence of a power-law boundary layer

$$\Phi(x) = (x+\varepsilon)^\alpha, 0 < \alpha < 1,\ \ x \ge 0, \varepsilon \in (0,1].$$

In the presence of a logarithmic singularity $\Phi(x) = \ln x, x \ge \varepsilon > 0$. Decomposition of a function with such a singularity was used in [4], where an elliptic equation was considered in a domain with a small hole of radius ε.

Let us dwell on the case of an exponential boundary layer, when the function $u(x)$ is the solution of a singularly perturbed boundary value problem [1]:

$$\varepsilon u''(x) + a_1(x)u'(x) - a_2(x)u(x) = f(x),\ \ \ u(0) = A,\ \ u(1) = B, \tag{5}$$

where $a_1(x) \ge \beta > 0, a_2(x) \ge 0, \varepsilon \in (0,1]$, functions a_1, a_2, f are quite smooth. The peculiarity of the problem (5) is that the equation degenerates at $\varepsilon \to 0$, due to which at small values ε the solution has large gradients at the boundary $x = 0$.

According to [2], for the solution of the problem (5) for an arbitrarily given finite n_0, the decomposition (4) is valid with the following relations:

$$\Phi(x) = e^{-\alpha x/\varepsilon},\ \ \alpha = a_1(0), \tag{6}$$

$$|p^{(n)}(x)| \le C_0\Big[1 + \frac{1}{\varepsilon^{n-1}} e^{-\beta x/\varepsilon}\Big], n \le n_0,\ \ \gamma = -\varepsilon u'(0)/a_1(0).$$

Here the constant C_0 does not depend on ε and n.

In [2], decomposition (4) was used to construct a difference scheme based on fitting to the boundary layer component $\Phi(x)$. In [5], [2] it was proven that with such approach, the convergence of the difference scheme becomes uniform with respect to the small parameter ε.

In [6] decomposition (4) is used to construct an interpolation formula that is exact on the component $\Phi(x)$. The formula contains an arbitrarily specified number of interpolation nodes, and in accordance with [6] its error is uniform in the component $\Phi(x)$. In this case, the monotonicity of the derivatives of the function $\Phi(x)$ was used. This condition is satisfied in the case of the function (6). According to (6), the following estimate is true $|\Phi^{(n)}(x)| \leq C/\varepsilon^n$. Therefore, in accordance with (3) the error $|R_k(u,x)|$ can be significant for small values of ε.

Let us show that the error is significant in the case when in (1) $k = 1$:

$$u(x) \approx u(x_0) + (x - x_0)u'(x_0). \tag{7}$$

Let in decomposition (4) $\Phi(x) = e^{-x/\varepsilon}$, where $\varepsilon \in (0, 1]$. Then for $x_0 = 0$ and $\varepsilon = h = x - x_0$ we have

$$R_1(\Phi, x) = \Phi(h) - \Phi(0) - h\Phi'(0) = e^{-1}.$$

Thus, in the presence of an exponential boundary layer, the error of the formula (7) does not decrease with decreasing h, if $\varepsilon = h$. It is of interest how to apply the Taylor series to approximate a function with decomposition (4).

To approximate the function $u(x)$ of the form (4), we correct Taylor's formula (1) so that the formula becomes exact on the component $\Phi(x)$.

Assuming that $\Phi^{(k+1)}(x_0) \neq 0$, we define the approximation:

$$u(x) \approx G_k(u, x) = \sum_{j=0}^{k} \frac{u^{(j)}(x_0)}{j!}(x - x_0)^j +$$

$$+\Big[\Phi(x) - \sum_{j=0}^{k} \frac{\Phi^{(j)}(x_0)}{j!}(x - x_0)^j\Big] \frac{u^{(k+1)}(x_0)}{\Phi^{(k+1)}(x_0)}. \tag{8}$$

From (8) it follows that $G_k(\gamma\Phi, x) = \gamma\Phi(x)$.

Taking into account (4), we get

$$u(x) - G_k(u, x) = p(x) - G_k(p, x) + \gamma(\Phi(x) - G_k(\Phi, x)). \tag{9}$$

Taking into account the relation $\gamma\Phi(x) = G_k(\gamma\Phi, x)$ and (8), from (9) we obtain

$$u(x) - G_k(u, x) = p(x) - \sum_{j=0}^{k} \frac{p^{(j)}(x_0)}{j!}(x - x_0)^j -$$

$$-\Big[\Phi(x) - \sum_{j=0}^{k} \frac{\Phi^{(j)}(x_0)}{j!}(x - x_0)^j\Big] \frac{p^{(k+1)}(x_0)}{\Phi^{(k+1)}(x_0)}. \tag{10}$$

According to (1), the relation (10) can be written as:

$$u(x) - G_k(u,x) = R_k(p,x) - \frac{R_k(\Phi,x)}{\Phi^{(k+1)}(x_0)} p^{(k+1)}(x_0). \tag{11}$$

From (3), (11) it follows

$$\Big|u(x) - G_k(u,x)\Big| \le \frac{h^{k+1}}{(k+1)!}\Big[1 + \max_{s\in[x_0,x]}\Big|\Phi^{(k+1)}(s)\Big| \Big/ \Big|\Phi^{(k+1)}(x_0)\Big|\Big] \times \max_{s\in[x_0,x]}\Big|p^{(k+1)}(s)\Big|. \tag{12}$$

From the estimate (12) it follows the next lemma.

Lemma 1. *Let the function $u(x)$ has decomposition (4), $\Phi^{(k+1)}(x_0) \ne 0$ and for some constant C_1 the following estimate is valid:*

$$\max_{s\in[x_0,x]}\Big|\Phi^{(k+1)}(s)\Big| \le C_1\Big|\Phi^{(k+1)}(x_0)\Big|. \tag{13}$$

Then for some constant C_2 there is an error estimate:

$$\Big|u(x) - G_k(u,x)\Big| \le C_2 h^{k+1} \max_{s\in[x_0,x]}\Big|p^{(k+1)}(s)\Big|, \tag{14}$$

where $G_k(u,x)$ corresponds to (8).

In contrast to the estimate (3) for formula (1), the error estimate (14) for formula (8) is uniform in the boundary layer component $\Phi(x)$ and its derivatives.

Let us look at examples of application of Lemma 1.

In the case of an exponential boundary layer at the boundary $x = 0$ function $\Phi(x)$ corresponds to (6), so for $x > x_0$ the condition (13) is satisfied when setting $C_1 = 1$.

In the case of $x < x_0$ and an exponential boundary layer at the boundary $x = 1$, the function $\Phi(x)$ in accordance with [1] has the form: $\Phi(x) = e^{\alpha(x-1)/\varepsilon}$. And in this case, the condition (13) is satisfied when setting $C_1 = 1$.

Let's consider the formula (8) for $k = 0$:

$$u(x) \approx u(x_0) + \Big[\Phi(x) - \Phi(x_0)\Big]\frac{u'(x_0)}{\Phi'(x_0)}. \tag{15}$$

Let $\Phi(x) = e^{-x/\varepsilon}$, $x > x_0$. Then for the error Δ of this formula we have:

$$\Delta = u(x) - u(x_0) - \Big[\Phi(x) - \Phi(x_0)\Big]\frac{u'(x_0)}{\Phi'(x_0)} = p(x) - p(x_0) - \varepsilon(1 - e^{-h/\varepsilon})p'(x_0),$$

where $h = x - x_0$. Using the condition $|p'(x)| \le C$, for some constant C_1 we get $|\Delta| \le C_1 h$. This corresponds to the estimate (14) at $k = 0$.

From the example it follows that the error of the formula (15) of order $O(h)$ uniformly in a small parameter ε, while the application of the corresponding classical formula (7) leads to errors of the order of $O(1)$.

Let us show that the developed formula (8) is applicable in particular, when numerically solving delay singularly perturbed differential equations.

Numerical methods for solving delay singularly perturbed problems were considered in a number of works, for example, in [7], [8].

Let us dwell on the boundary value problem

$$\varepsilon u''(x) + a(x)u'(x) - b(x)u(x-\delta) = f(x),$$

$$u(x) = g(x), -\delta \leq x \leq 0, u(1) = B, \tag{16}$$

where the functions in (16) are sufficiently smooth, $a(x) \geq a_0 > 0, \varepsilon \in (0,1], b(x) \geq 0$. To get rid of the term with a delay argument, an approach has been developed based on the expansion of this term in a Taylor series. For example, in [7],[8] approximation looks like

$$u(x-\delta) \approx u(x) - \delta u'(x). \tag{17}$$

Then the problem (16) is reduced to a singularly perturbed problem without term with a delay argument:

$$\varepsilon \tilde{u}''(x) + (a(x)+b(x)\delta)\tilde{u}'(x) - b(x)\tilde{u}(x) = f(x),\ \tilde{u}(0) = g(0),\ \tilde{u}(1) = B. \tag{18}$$

It is necessary to apply a difference scheme to find the solution $\tilde{u}(x)$ of the problem (18). Next, to the singularly perturbed problem (18) we can apply a difference scheme whose convergence is uniform in a small parameter ε. The issue of constructing difference schemes for singularly perturbed problems has been widely studied since 1969. Two main approaches have emerged: grid refinement in the boundary layer [1], [9], [10] and fitting the difference scheme to the boundary layer component on a uniform grid [5], [2], [11].

Note that using the formula (17) to move to a problem without a delay argument can lead to significant errors, since the solution to the problem (16) has large gradients in the boundary layer region. According to the example (7) the error of the formula (17) can be of the order of $O(1)$ at $\varepsilon \leq C\delta$. Because of this, when moving to the problem (18) in the region of large gradients of the function $u(x)$ the error $|u(x) - \tilde{u}(x)|$ can be of the order of $O(1)$. The case of negative delay is possible when $\delta < 0$. In the case of a singularly perturbed problem, to move to a problem without a delay argument, one can use the developed formula (8), researched in Lemma 1.

3 The case of a function of two variables

Let us dwell on the use of the Taylor series to approximate a function of two variables with large gradients. Let the function $u(x,y)$ be represented as:

$$u(x,y) = p(x,y) + \gamma_1\Phi(x) + \gamma_2\Theta(y), (x,y) \in [0,1]^2. \tag{19}$$

We assume that the regular component $p(x,y)$ is not specified and has derivatives limited to a certain order, the coefficients γ_1, γ_2 are not specified and limited, $\Phi(x)$ and $\Theta(y)$ are known boundary layer components with large gradients.

In particular, the decomposition (19) is valid for the solution of the elliptic problem with exponential boundary layers [3]:

$$\varepsilon u_{xx}+\varepsilon u_{yy}+a(x)u_x+b(y)u_y-c(x,y)u=f(x,y),\ (x,y)\in\Omega=(0,1)^2,$$
$$u(x,y)=g(x,y),\ (x,y)\in\Gamma,$$

where $\Gamma=\overline{\Omega}\diagdown\Omega$, functions a,b,c,f,g are smooth enough,

$$a(x)\geq\beta_1>0,\quad b(y)\geq\beta_2>0,\quad c(x,y)\geq 0,\ \varepsilon\in(0,1].$$

In accordance with [3] in (19)

$$\Phi(x)=e^{-\alpha x/\varepsilon},\ \Theta(y)=e^{-\beta y/\varepsilon}, \tag{20}$$

where $\alpha=a(0),\beta=b(0)$.

3.1. Formula of first order accuracy. Let

$$(x_0,y_0),(x,y)\in[0,1]^2,\ h=\sqrt{(x-x_0)^2+(y-y_0)^2}. \tag{21}$$

Let us consider the issue of approximating $u(x,y)$ based on the use of Taylor series expansion around the point (x_0,y_0).

First, let's estimate the approximation error $u(x,y)\approx u(x_0,y_0)$. Let $\Delta=u(x,y)-u(x_0,y_0)$. Due to the presence of boundary layer components in accordance with (19), the error $|\Delta|$ can be significant, despite the smallness of h. For example, if these components correspond to (20), then in the boundary layer region the case $|\Delta|\geq Ch/\varepsilon$ is possible for some constant C. The error increases with decreasing parameter ε.

Let us proceed to constructing a first-order accuracy formula whose error estimate is uniform in the components $\Phi(x),\Theta(y)$. Assuming that $\Phi'(x_0)\neq 0$, $\Theta'(y_0)\neq 0$, we define a formula that is exact on the components $\Phi(x),\Theta(y)$:

$$u(x,y)\approx G_1(u,x,y)=$$
$$=u(x_0,y_0)+\frac{\Phi(x)-\Phi(x_0)}{\Phi'(x_0)}u'_x(x_0,y_0)+\frac{\Theta(y)-\Theta(y_0)}{\Theta'(y_0)}u'_y(x_0,y_0). \tag{22}$$

From (22) we have

$$|u(x,y)-G_1(u,x,y)|=|p(x,y)-G_1(p,x,y)|\leq|p(x,y)-p(x_0,y_0)|+$$
$$+\frac{|\Phi'(s_1)|}{|\Phi'(x_0)|}|x-x_0|\,|p'_x(x_0,y_0)|+\frac{|\Theta'(s_2)|}{|\Theta'(y_0)|}|y-y_0|\,|p'_y(x_0,y_0)|. \tag{23}$$

The following lemma follows from the estimate (23).

Lemma 2. *Let the function $u(x,y)$ has the decomposition (19), for some constant C the following relations are valid:*

$$|p'_x(s_1,s_2)|\leq C,\ |p'_y(s_1,s_2)|\leq C,\ \Phi'(x_0)\neq 0,\Theta'(y_0)\neq 0.$$

Let for some constant C_1

$$|\Phi'(s_1)|\leq C_1|\Phi'(x_0)|,|\Theta'(s_2)|\leq C_1|\Theta'(y_0)|,s_1\in[x_0,x],s_2\in[y_0,y]. \tag{24}$$

Then there is the constant C_2 such that

$$|u(x,y)-G_1(u,x,y)|\leq C_2h.$$

If $\Phi(x)$ and $\Theta(y)$ correspond to (20), then condition (24) is satisfied when setting $C_1 = 1$.

In the cases $x < x_0$ or $y < y_0$, Lemma 2 remains valid with by appropriately replacing the interval, for example, $[x_0, x]$ with $[x, x_0]$.

3.2. Formula of second order accuracy. The approximation by the Taylor formula of the second order of accuracy has the form

$$u(x,y) \approx u(x_0,y_0) + \frac{\partial u}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial u}{\partial y}(x_0,y_0)(y-y_0). \tag{25}$$

As in the one-dimensional case, it can be shown that applying the formula (25) to a function with large gradients of the form (19) due to the presence of boundary layer components can lead to errors of the order of $O(1)$.

Let's modify the formula (25) so that the formula becomes exact for the boundary layer components. Assuming that $\Phi''(x_0) \neq 0$ and $\Theta''(y_0) \neq 0$, let's move from (25) to the formula

$$u(x,y) \approx G_2(u,x,y) = u(x_0,y_0) + \frac{\partial u}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial u}{\partial y}(x_0,y_0)(y-y_0) +$$
$$+ \frac{\Phi(x) - \Phi(x_0) - \Phi'(x_0)(x-x_0)}{\Phi''(x_0)} \frac{\partial^2 u}{\partial x^2}(x_0,y_0) +$$
$$+ \frac{\Theta(y) - \Theta(y_0) - \Theta'(y_0)(y-y_0)}{\Theta''(y_0)} \frac{\partial^2 u}{\partial y^2}(x_0,y_0).$$

It is easy to verify that

$$G_2(\gamma_1\Phi, x, y) = \gamma_1\Phi(x), \;\; G_2(\gamma_2\Theta, x, y) = \gamma_2\Theta(y).$$

Hence,

$$u(x,y) - G_2(u,x,y) = p(x,y) - G_2(p,x,y) =$$
$$= \Big[p(x,y) - p(x_0,y_0) - \frac{\partial p}{\partial x}(x_0,y_0)(x-x_0) - \frac{\partial p}{\partial y}(x_0,y_0)(y-y_0)\Big] -$$
$$- \frac{1}{2}\Big[\frac{\Phi''(s_1)(x-x_0)^2}{\Phi''(x_0)} \frac{\partial^2 p}{\partial x^2}(x_0,y_0) + \frac{\Theta''(s_2)(y-y_0)^2}{\Theta''(y_0)} \frac{\partial^2 p}{\partial y^2}(x_0,y_0)\Big],$$

where $s_1 \in [x_0, x], \;\; s_2 \in [y_0, y]$.

This relation implies the validity of the following lemma.

Lemma 3. *Let the function $u(x,y)$ has the decomposition (19) and for some constant C_0 for all $s_1 \in [x_0, x], \;\; s_2 \in [y_0, y]$ the wollowing estimates are performed:*

$$|p''_{xx}(s_1,s_2)| \leq C_0, \;\; |p''_{yy}(s_1,s_2)| \leq C_0, |p''_{xy}(s_1,s_2)| \leq C_0.$$

Let $\Phi''(x_0) \neq 0, \Theta''(y_0) \neq 0$ and for some constant C_1 for all $s_1 \in [x_0, x], \;\; s_2 \in [y_0, y]$ the following estimates are valid:

$$|\Phi''(s_1)| \leq C_1 |\Phi''(x_0)|, \quad |\Theta''(s_2)| \leq C_1 |\Theta''(y_0)|. \tag{26}$$

Then there is a constant C_2 such that

$$|u(x,y) - G_2(u,x,y)| \leq C_2 h^2,$$

where h is given in (21).

In the case when $\Phi(x)$ and $\Theta(y)$ correspond to (20), the conditions (26) are satisfied when given $C_1 = 1$.

In the cases $x < x_0$ or $y < y_0$, Lemma 3 remains valid with by appropriately replacing the interval, for example, $[x_0, x]$ with $[x, x_0]$.

4 Conclusion

The issue of approximating a function with large gradients based on Taylor series expansion is investigated. The problem is that the residual term can be significant if the function has large gradients. The case is considered when the function contains the boundary layer components, known up to a factor and responsible for large gradients of the function. This decomposition of the function is valid, for example, for the solution of a singularly perturbed problem. In the cases of a function of one and two variables, a modification of Taylor series expansion is proposed to achieve the fact that the remainder term does not depend on the boundary layer components. It is proven that then the estimate of the error of the remainder term depends only on the derivatives of the regular component.

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