S©MR

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ
Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru
ISSN 1813-3304

Vol. 20, No. 2, pp. 1464–1473 (2023) УДК 512.542
https://doi.org/10.33048/semi.2023.20.090 MSC 20D10; 20D40

ON WEAKLY tcc-SUBGROUPS OF FINITE GROUPS

A. TROFIMUK

Communicated by A.V. Vasil'ev

Abstract. The subgroups A and B are said to be cc-*permutable*, if A is permutable with B^x for some $x \in \langle A, B\rangle$. A subgroup A of a finite group G is called *weakly* tcc-*subgroup* (wtcc-*subgroup, for brevity)* in G, if there exists a subgroup Y of G such that $G = AY$ and A has a chief series $1 = A_0 \leq A_1 \leq \ldots \leq A_{s-1} \leq A_s = A$ such that every A_i is cc-permutable with all subgroups of Y for all $i = 1, \ldots, s$. In this paper, we studied the influence of given systems of wtcc-subgroups on the structure of a group G.

Keywords: Finite group, cc-permutable subgroups, Sylow subgroups, maximal subgroups, supersoluble group.

1 Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. We use the standard notations and terminology of [1, 2]. The notation $H \leq G$ means that H is a subgroup of a group G. If $H \leq G$ and $H \neq G$, we write $H < G$. The notation $H \trianglelefteq G$ means that H is a normal subgroup of a group G.

We say that the subgroups A and B of a group G are *permutable* if $AB = BA$. Note that the equality $AB = BA$ is equivalent to $AB \leq G$. In accordance with [2], subgroups A and B of a group G are called *totally permutable* if every subgroup of A is permutable with every subgroup of B.

Trofimuk, A., On weakly tcc-subgroups of finite groups.

This work was supported by the BRFFR (grant No. F23RNF237).
Received February, 16, 2023, Published December, 12, 2023.

In [3] the following concept was introduced: the subgroups A and B are said to be cc-*permutable*, if A is permutable with B^x for some $x \in \langle A, B\rangle$. It is clear that if A and B are permutable, then A and B are cc-permutable. Naturally total permutability was generalized to tcc-permutability, i.e. the subgroups A and B of G are said to be tcc-*permutable* [4] if every subgroup of A is cc-permutable with every subgroup of B. Results related to the study of groups with given systems of cc-permutable or tcc-permutable subgroups were provided in [5], see also the literature in [4].

We follow [6, 7] and collect the following concepts:

(1) A subgroup A of a group G is called tcc-*subgroup* in G, if there exists a subgroup T of G such that:

(1.1) $G = AT$;

(1.2) A and T are tcc-permutable.

(2) A subgroup A of a group G is said to be *NS-supplemented* in G, if there exists a subgroup B of G such that:

(2.1) $G = AB$;

(2.2) whenever X is a normal subgroup of A and $p \in \pi(B)$, there exists a Sylow p-subgroup B_p of B such that $XB_p = B_pX$.

By [6, Lemma 3.1 (3)], every tcc-subgroup of G is NS-supplemented in G, but the converse does not hold. For example, the alternating group $G = A_4$ has NS-supplemented Sylow 3-subgroup P and P is not tcc-subgroup in G. In [6, 7] proved the supersolubility of a group G under the condition that factors, Sylow subgroups, maximal subgroups, minimal subgroups, maximal subgroups of every Sylow subgroups of G are NS-supplemented or tcc-subgroups in G.

In the present paper, we study another generalization of tcc-subgroup in a new way.

Definition.

A subgroup A of a group G is called weakly tcc-*subgroup* (wtcc-*subgroup, for brevity) in G, if there exists a subgroup Y of G such that:*

(1) *$G = AY$;*

(2) *A has a chief series*

$$1 = A_0 \leq A_1 \leq \ldots \leq A_{s-1} \leq A_s = A \tag{1}$$

such that every A_i is cc-*permutable with all subgroups of Y for all $i = 1, \ldots, s$.*

In this definition, we say that Y is a wtcc-*supplement* to A in G and a chief series (1) is a wtcc-*series* of A.

It is clear that every tcc-subgroup of G is wtcc-subgroup in G, but the converse does not hold. For example, the group $G = Z_3 \times S_3$ has Sylow 3-subgroup P such that P is a wtcc-subgroup in G, but P is not tcc-subgroup in G. Besides, the groups A_4 and $Z_3 \times S_3$ show that there is no inclusion-relationship between the concepts to be wtcc-subgroup and to be NS-supplemented.

In Section 3, we give some properties of wtcc-subgroups. In Section 4, we study the structure of a group G in which the maximal subgroups, Sylow subgroups, minimal subgroups, 2-maximal subgroups, maximal subgroups of every Sylow subgroup, factors are wtcc-subgroups. Besides, we obtain the generalization of some result of [6] without using the properties of tcc-permutability.

Theorem. *Let G be a group.*

1. *Then G is supersoluble if one of the following statements holds:*

(1.1) *every maximal subgroup of G is a* wtcc-*subgroup in G.*

(1.2) *every Sylow subgroup of G is a* wtcc-*subgroup in G.*

(1.3) *let H be a normal subgroup of G. Suppose that G/H is supersoluble and every cyclic subgroup of prime order or order 4 of H is a* wtcc-*subgroup in G.*

(1.4) *every 2-maximal subgroup of G is a* wtcc-*subgroup in G.*

(1.5) *all maximal subgroups of every non-cyclic Sylow subgroup of G are* wtcc-*subgroups in a soluble group G.*

2. *Let A and B be* wtcc-*subgroups in a group G and $G = AB$. If A and B are supersoluble, then G is supersoluble.*

The following example shows that we cannot omit the condition «every cyclic subgroup of order 4 of H is a wtcc-subgroup in G» in (1.3) of main theorem.

Example. The non-supersoluble group $G = Q_8 \rtimes Z_9$ (IdGroup=[72,3], see [8]) has a cyclic 2-subgroup H of order 4 such that H is not a wtcc-subgroup in G and every subgroup of prime order of G is a wtcc-subgroup in G.

2 Preliminaries

In this section, we give some definitions and basic results which are essential in the sequel.

Denote by $Z(G)$, $F(G)$ and $\Phi(G)$ the centre, Fitting and Frattini subgroups of G respectively, and by $\mathrm{O}_p(G)$ and $\mathrm{O}_{p'}(G)$ the greatest normal p- and p'-subgroups of G respectively. Denote by $\pi(G)$ the set of all prime divisors of order of G. We use E_{p^t} to denote an elementary abelian group of order p^t, Z_m to denote a cyclic group of order m, Q_m to denote a quaternion group of order m. The semidirect product of a normal subgroup A and a subgroup B is written as follows: $A \rtimes B$. If H is a subgroup of G, then $H_G = \bigcap_{x\in G} H^x$ is called *the core* of H in G. Recall that $H^G = \langle H^g \mid g \in G\rangle$ is the smallest normal subgroup of G containing H.

A group whose chief factors have prime orders is called *supersoluble*. Recall that a *p-closed* group is a group with a normal Sylow p-subgroup and a *p-nilpotent* group is a group with a normal Hall p'-subgroup. If a group G contains a maximal subgroup M with trivial core, then G is said to be *primitive* and M is its *stabilizer*.

The class of all supersoluble groups is denoted by $\mathfrak{U}$. A simple check proves the following lemma.

Lemma 1. *Let G be a soluble group. Assume that $G \notin \mathfrak{U}$, but $G/K \in \mathfrak{U}$ for every non-trivial normal subgroup K of G. Then:*

(1) *G contains a unique minimal normal subgroup N, $N = F(G) = \mathrm{O}_p(G) = C_G(N)$ for some $p \in \pi(G)$;*

(2) *$Z(G) = \mathrm{O}_{p'}(G) = \Phi(G) = 1$;*

(3) *G is primitive; $G = N \rtimes M$, where M is a stabilizer;*

(4) *N is an elementary abelian subgroup of order p^n, $n > 1$.*

Lemma 2. *([11, Theorem 2]) Let G be a group with $p \in \pi(G)$ and $p \neq 3$. If G has a Hall $\{p, r\}$-subgroup for every $r \in \pi(G)$, then G is p-soluble.*

Lemma 3. *Let G be soluble. If G has a subgroup H of prime index, then G/H_G is supersoluble.*

Proof Suppose that H is not normal in G. Then $H_G \neq H$ and G/H_G is primitive with stabilizer H/H_G. By [12, Theorem 15.6],

$$G/H_G = (P/H_G) \rtimes (H/H_G), \quad P/H_G = C_{G/H_G}(P/H_G).$$

Let $|G : H| = p$, where p is prime. Then

$$|G/H_G : H/H_G| = |G : H| = p, \quad |P/H_G| = p.$$

The subgroup H/H_G is cyclic, as the automorphism group of P/H_G of prime order. Hence G/H_G is supersoluble. If H is normal in G, then $H = H_G$ and G/H_G is supersoluble. Lemma is proved.

A subgroup A of a group G is called *seminormal* in G, if there exists a subgroup B such that $G = AB$ and AX is a subgroup of G for every subgroup X of B, see [5].

Lemma 4. *([13, Lemma 10]) If A is a seminormal 2-nilpotent subgroup of G, then A^G is soluble.*

Lemma 5. *([14, Theorem]) Let G be a finite group, and H a subgroup of G. Suppose that for every prime p dividing the order of G there exists a Sylow p-subgroup G_p of G such that H is subnormal in $\langle H, G_p \rangle$. Then H is subnormal in G.*

3 Properties of wtcc-subgroups

Lemma 6. *Let A be a* wtcc*-subgroup in G and Y be a* wtcc*-supplement to A in G and $1 = A_0 \leq A_1 \leq \ldots \leq A_{s-1} \leq A_s = A$ be a* wtcc*-series of A. Then the following statements hold:*

(1) *A is a* wtcc*-subgroup in H for any subgroup H of G such that $A \leq H$;*

(2) *if A is supersoluble, then AN/N is a* wtcc*-subgroup in G/N for any $N \trianglelefteq G$;*

(3) *for every* $i = 1, \ldots, s$ *and arbitrary* $X \leq Y$ *there exists an element* $y \in Y$ *such that* $A_i X^y \leq G$. *In particular,* $A_i M \leq G$ *for some maximal subgroup* M *of* Y *and* $A_i H \leq G$ *for some Hall* π*-subgroup* H *of soluble* Y *and any* $\pi \subseteq \pi(G)$*;*

(4) $A_i K \leq G$ *for every* $i = 1, \ldots, s$ *and every subnormal subgroup* K *of* Y*;*

(5) *for every* $i = 1, \ldots, s$ *and every subnormal subgroup* K *of* Y *we have* $A_i K^g \leq G$ *for any* $g \in G$*;*

(6) *if* Y *is soluble and* A *is* r*-closed, then Sylow* r*-subgroup* A_r *of* A *is subnormal in* G*, where* r *is the greatest prime in* $\pi(G)$*;*

(7) *if* A *is* 2*-nilpotent, then the derived subgroup* A' *is subnormal in* G*.*

Proof 1. Since Y is a wtcc-supplement to A in G, it follows that $G = AY$. By Dedekind's identity, $H = H \cap AY = A(H \cap Y)$. Since $H \cap Y \leq Y$, then for any $i = 1, \ldots, s$ and any $Z \leq H \cap Y$ there exists an element $u \in \langle A_i, Z\rangle$ such that $A_i Z^u \leq G$. Hence A is a wtcc-subgroup in H.

2. Since $G = AY$, it follows that $G/N = (AN/N)(YN/N)$. Let X/N be an arbitrary subgroup of YN/N. Since $N \leq X \leq YN$, we have by Dedekind's identity, $X = X \cap YN = (X \cap Y)N$.

Consider the series

$$1 = A_0N/N \leq A_1N/N \leq \ldots \leq A_{s-1}N/N \leq A_sN/N = AN/N \qquad (2)$$

of AN/N. Since A_i is normal in A, it follows that $A \leq N_G(A_iN)$ and A_iN/N is normal in AN/N. Obviously,

$$(A_iN/N)/(A_{i-1}N/N) \simeq A_iN/A_{i-1}N \simeq A_i/A_i \cap A_{i-1}N \simeq A_i/A_{i-1}(A_i \cap N).$$

Since A is supersoluble, $\mid A_i/A_{i-1} \mid$ is a prime and therefore $\mid (A_iN/N)/(A_{i-1}N/N) \mid$ is a prime. Hence the series (2) is a chief series of AN/N.

Because $X \cap Y$ is a group of Y, we have for any $i = 1, \ldots, s$ there exists an element $u \in \langle A_i, X \cap Y\rangle$ such that $A_i(X \cap Y)^u \leq G$. Hence

$$(A_iN/N)(X/N)^{uN} = A_i(X \cap Y)^u N/N \leq G/N$$

for $uN \in \langle A_i, X \cap Y\rangle N/N \subseteq \langle A_i, X\rangle N/N = \langle A_iN/N, X/N\rangle$. Consequently, AN/N is a wtcc-subgroup in G/N.

3. Since A is a wtcc-subgroup in G, for every $i = 1, \ldots, s$ and $X \leq Y$ there exists an element $u \in \langle A_i, X\rangle$ such that $A_iX^u \leq G$. Because $u \in G = AY = YA$, it follows that $u = ya$ for some $y \in Y$ and $a \in A$. Then

$$A_iX^u = A_iX^{ya} = A_i(X^y)^a = A_i^a(X^y)^a = (A_iX^y)^a \leq G.$$

Hence there is a subgroup A_iX^y in G for some $y \in Y$. Clearly, that if X is a Hall π-subgroup of Y, then $H = X^y$ is a Hall π-subgroup of Y. Thus $A_iH \leq G$. Similarly, for maximal subgroup X of Y. Then $M = X^y$ is a maximal subgroup of Y and $A_iM \leq G$.

4. Since K is subnormal in Y, there is a chain of subgroups

$$Y = K_0 \geq K_1 \geq \ldots \geq K_{n-1} \geq K_n = K$$

such that K_{i+1} is normal in K_i for all i. We use induction by n. By (3), there exists an element $y \in Y$ such that $A_j K_1^y = A_j K_1 \leq G$ for every $j = 1, \dots, s$. Hence the statement holds for $n = 0$ and $n = 1$. Therefore $n \geq 2$. By (1), A is a wtcc-subgroup in AK_1 and K_1 is a wtcc-supplement to A in AK_1. Since the length of subnormal chain between K and K_1 less than n, it follows that by induction, there is a subgroup A_iK of AK_1. Consequently $A_iK \leq G$.

5. Since $g \in G = AY = YA$, it follows that $g = ya$ for some $y \in Y$ and $a \in A$. Then

$$A_iK^g = A_iK^{ya} = A_i(K^y)^a = (A_iK^y)^a.$$

Since K is subnormal in Y, we have K^y is subnormal in Y. By (4), $A_iK^y \leq G$. Theredore, $AK^g \leq G$.

6. We proceed by induction on $\mid G \mid$. By (3), $AY_1 \leq G$ for some Hall r'-subgroup Y_1 of Y. If $AY_1 < G$, then by (1), A is a wtcc-subgroup in AY_1 and by induction, A_r is subnormal in AY_1. Besides, A_r is subnormal in some Sylow r-subgroup G_r of G. Let $Y_r \leq R$, where R is a Sylow r-subgroup of G and $R^g = G_r$ for some $g \in G$. By [9, Theorem 1], A_r is subnormal in $G = AY = AY_1Y_r = (AY_1)Y_r^g = (AY_1)G_r$.

Hence we consider that $G = AY_1$. By (3), $AQ \leq G$ for some Sylow q-subgroup Q of Y_1. If $AQ < G$, then A is a wtcc-subgroup in AQ and by induction, A_r is subnormal in AQ. Therefore A_r is normal in AQ and $Q \leq N_G(A_r)$. Since it is true for any $q \in \pi(Y_1)$, it follows that A_r is normal in $G = AY_1$.

Hence $G = AQ$. By (4), Q is a minimal wtcc-supplement to A in G. By (3), $AM < G$ for some maximal subgroup M of Q. Because A is a wtcc-subgroup in AM, we have by induction, A_r is subnormal in AM and hence A_r is normal in AM. Since $\mid G : AM \mid = q$, it follows that $G/(AM)_G$ is isomorphic to a subgroup of symmetric group S_q. Hence $G_r \leq (AM)_G \leq AM$ and $A_r = G_r$ is subnormal in G.

7. We proceed by induction on $|G|$. By (3), for every $p \in \pi(Y)$ there exists a Sylow p-subgroup Y_p of Y such that $AY_p \leq G$. Suppose that $AY_p < G$ for every $p \in \pi(Y)$. Then by (1), A is a wtcc-subgroup in AY_p and by induction, A' is subnormal in AY_p. It is clear that for every $p \in \pi(G)$ there exists a Sylow p-subgroup R of G such that $R \leq AY_p$. Since $A' \leq \langle A', R \rangle \leq AY_p$, we have A' is subnormal in $\langle A', R \rangle$. By Lemma 5, A' is subnormal in G.

Hence we consider that $G = AY_q$ for some $q \in \pi(Y)$. By (4), A is seminormal in G. Since A is 2-nilpotent, A^G is soluble by Lemma 4. Hence $G = AY_q = A^GY_q$ is soluble. By (4), Y_q is a minimal wtcc-supplement to A in G and $AT < G$ for some maximal subgroup T of Y_q. Because A is a wtcc-subgroup in AT, we have by induction, A' is subnormal in AT. Since $\mid G : AT \mid = q$, it follows that by Lemma 3, $G/(AT)_G$ is supersoluble and hence the derived subgroup

$$(G/(AT)_G)' = G'(AT)_G/(AT)_G$$

is nilpotent. Since $A' \le G'$, we have

$$A'(AT)_G/(AT)_G \le G'(AT)_G/(AT)_G$$

and hence $A'(AT)_G$ is subnormal in G. It is clear that $A' \le A'(AT)_G \le AT$. Since A' is subnormal in AT, A' is subnormal in $A'(AT)_G$ and A' is subnormal in G. Lemma is proved.

4 Main results

(1.1) Let M be an arbitrary maximal subgroup of G. By Lemma 6 (3), $MY_p \le G$ for some Sylow p-subgroup Y_p of Y. Since M is maximal in G, it follows that either $MY_p = M$ or $MY_p = G$. If $MY_p = M$ for all $p \in \pi(Y)$, then $Y \le M$ and $G = MY = M$, a contradiction. Therefore there exists $q \in \pi(Y)$ such that $MY_q = G$ and Y_q is a wtcc-supplement to M in G. By Lemma 6 (4), we can consider that Y_q is a minimal wtcc-supplement to M in G. By Lemma 6 (3), $MS < G$ and $\mid G : MS \mid= q$ for some maximal subgroup S of Y_q. Since M is a maximal subgroup of G, we have
$\mid G : M \mid=\mid G : MS \mid= q$. By [1, VI.9.5], G is supersoluble.

(1.2) We show that G is soluble. Let R be a Sylow r-subgroup of G. Then R is a wtcc-subgroup in G. Let T be a wtcc-supplement to R in G. By Lemma 6 (3), $RQ \le G$ for some Sylow q-subgroup Q of T and for any $q \in \pi(T) \setminus \{p\}$. The subgroup RQ is a Hall $\{r, q\}$-subgroup of G. By Lemma 2, G is r-soluble for $r \neq 3$. Let t be the smallest prime in $\pi(G)$. If $t > 2$, then G is soluble. If $t = 2$, then by the above, G is t-soluble and consequently, G is soluble.

Next we show that G is supersoluble. Assume that the claim is false and let G be a minimal counterexample. Let N be a non-trivial normal subgroup of G and RN/N be a Sylow r-subgroup of G/N. By Lemma 6 (2), RN/N is a wtcc-subgroup in G/N. Then G/N is supersoluble by the choice of G.

Let P be a Sylow p-subgroup of G, where p is the greatest prime in $\pi(G)$. By Lemma 6 (6), P is subnormal in G and consequently, P is normal in G. By Lemma 1, G has a unique minimal normal subgroup N such that $N = C_G(N) = O_p(G) = F(G) = P$ and N is an elementary abelian subgroup of order p^n, $n > 1$.

Let T be a wtcc-supplement to P in G and

$$1 = P_0 \le P_1 \le \ldots \le P_{s-1} \le P_s = P$$

be a wtcc-series of P. It is clear $\mid P_1 \mid= p$. By Lemma 6 (3), for every $r \in \pi(T)$ there exists a Sylow r-subgroup R of T such that $P_1R \le G$. If $p \neq r$, then

$$P \cap P_1R = P_1(P \cap R) = P_1$$

is normal in P_1R and $R \le N_G(P_1)$. Since this inclusion holds for any $r \in \pi(T) \setminus \{p\}$, we have $T_1 \le N_G(P_1)$ for some Hall p'-subgroup T_1 of T. Hence P_1 is normal in $G = PT = PT_pT_1 = PT_1$, a contradiction.

(1.3) Assume that the theorem is false and let G be a minimal counterexample. Let K be a proper subgroup of G. It clear that $K \cap H \trianglelefteq K$ and $K/K \cap H \simeq KH/H$ is supersoluble. By Lemma 6 (1), every cyclic subgroup of prime order or order 4 of $K \cap H$ is a wtcc-subgroup in K. Then by induction, K is supersoluble and hence G is a minimal non-supersoluble group. Suppose that H is a proper subgroup of G. Hence H is supersoluble. Let q be the greatest prime in $\pi(H)$. Then by [1, VI.9.1], a Sylow q-subgroup Q of H is normal in H and consequently, Q is normal in G. Let $\overline{Q_1} = Q_1/Q$ be a cyclic subgroup of prime order or order 4 of H/Q. Then

$$\overline{Q_1} = \langle xQ \rangle = \langle x \rangle Q/Q \simeq \langle x \rangle / \langle x \rangle \cap Q \simeq \langle x \rangle,$$

because $x \notin Q$. Since $x \in H$, by Lemma 6 (2), $\overline{Q_1}$ is a wtcc-subgroup in G/Q. Since $(G/Q)/(H/Q)$ is supersoluble, by induction, G/Q is supersoluble.

By [10], G is soluble, G has a unique normal Sylow p-subgroup P and $P = G^{\mathfrak{U}}$, $\overline{P} = P/\Phi(P)$ is a minimal normal subgroup of $\overline{G} = G/\Phi(P)$ and $\mid P/\Phi(P) \mid > p$. Besides, P has exponent p if $p \neq 2$ and exponent at most 4 if $p = 2$.

If $p \neq q$, then $G \simeq G/Q \cap P$ is supersoluble, because G/Q and G/P are supersoluble. So $p = q$ and $Q \leq P$. Since $Q\Phi(P)/\Phi(P) \leq \overline{P}$ and $\overline{P}$ is a minimal normal subgroup of $\overline{G}$, we have $Q \leq \Phi(P)$ or $Q\Phi(P) = P$. If $Q \leq \Phi(P)$, then G is supersoluble, because $Q \leq \Phi(G)$ and G/Q is supersoluble, a contradiction. If $Q\Phi(P) = P$, then $Q = P$. Therefore, we can consider that $P \leq H$.

Suppose that $p = 2$. Let $x \in P$ and $P_1 = \langle x \rangle$. Then $\mid P_1 \mid = 2$ or $\mid P_1 \mid = 4$. By the hypothesis, P_1 is a wtcc-subgroup in G. By Lemma 6 (3), G has a Hall $2'$-subgroup S such that $P_1 S \leq G$. By [1, IV.2.8], $P_1 \leq N_G(S)$ and consequently, $P \leq N_G(S)$ and S is normal in G, a contradiction.

Assume that $p > 2$. Let $\overline{K} = K/\Phi(P)$ be a subgroup of order p in $\overline{P}$. Then

$$\overline{K} = \langle x\Phi(P) \rangle = \langle x \rangle \Phi(P)/\Phi(P).$$

Since $x \in P$, it follows that $\mid \langle x \rangle \mid = p$ and hence by Lemma 6 (2), $\overline{K}$ is a wtcc-subgroup in $\overline{G}$ and $\overline{T} = T/\Phi(P)$ is a wtcc-supplement to $\overline{K}$ in $\overline{G}$. Hence by Lemma 6 (3), for every $r \in \pi(\overline{T})$, $r \neq p$ there exists a Sylow r-subgroup $\overline{R}$ of $\overline{T}$ such that $\overline{KR} \leq \overline{G}$. It is clear that $\overline{R}$ is a Sylow r-subgroup in $\overline{G}$. We have that

$$\overline{P} \cap \overline{KR} = \overline{K}(\overline{P} \cap \overline{R}) = \overline{K}$$

is normal in $\overline{KR}$ and $\overline{R} \leq N_{\overline{G}}(\overline{K})$. Since $\overline{P}$ is abelian, $\overline{K}$ is normal in $\overline{G}$. Therefore $\overline{K} = \overline{P}$, a contradiction.

(1.4) Assume that the claim is false and let G be a minimal counterexample. By Lemma 6 (1) and by (1.1), every maximal subgroup M of G is supersoluble. Hence G is a minimal non-supersoluble group. Then by [10], G is soluble, $\mid \pi(G) \mid \leq 3$ and G has a unique normal subgroup $P = G^{\mathfrak{U}}$. It is clear that $\Phi(G) = 1$. Hence P is a minimal normal subgroup of order p^n, $n > 1$ and $G = P \rtimes M$ for some maximal subgroup M of G.

If $|\ \pi(G)\ |= 3$, then G has an ordered Sylow tower of supersoluble type and $M = T \rtimes R$, where $|\ T\ |= t$, $|\ R\ |= r$ and $t, r \in \pi(G)$. The subgroups T and R are 2-maximal subgroups of G. Then by hypothesis, $TY_1 = G = RY_2$, where Y_1 and Y_2 are wtcc-supplements in G. Besides, $P \leq Y_1$ and $P \leq Y_2$. Let P_1 be a minimal normal subgroup of P. Then by Lemma 6 (4), $T \leq N_G(P_1)$ and $R \leq N_G(P_1)$. Then P_1 is normal in $G = PM = PTR$, a contradiction.

So, $|\ \pi(G)\ |= 2$. Then M is a q-subgroup. If $|\ M\ |> q$, then M has a maximal subgroup M_1 such that $M_1 \neq 1$. It is clear that $H = P \rtimes M_1$ is a maximal subgroup of G. Since H is supersoluble, it follows that H has a maximal subgroup H_1 such that $M_1 \leq H_1$ and $|\ H : H_1\ |= p$. By hypothesis, H_1 is a wtcc-subgroup in G. Then $H_1V = G$, where V is a wtcc-supplement to H_1 in G. Let

$$1 = K_0 \leq K_1 \leq \ldots \leq K_{s-1} \leq K_s = H_1$$

be a wtcc-series of H_1. Since G is p-closed, H_1 is p-closed and $|\ K_1\ |= p$. By Lemma 6 (3), V has a Sylow q-subgroup V_q such that $K_1V_q \leq G$. Hence $V_q \leq N_G(K_1)$ and K_1 is normal in $G = H_1V = H_1PV_q$, a contradiction.

Therefore, $|\ M\ |= q$ and P is a maximal subgroup of G. Let P_1 be a maximal subgroup of P. Then by hypothesis, $P_1K = G$, where K is a wtcc-supplement to P_1 in G. By Lemma 6 (3), K has a Sylow q-subgroup K_1 such that $P_1K_1 \leq G$ and $K_1 \leq N_G(P_1)$. Hence P_1 is normal in $G = P_1K = PK_1$. Hence $|\ P\ |= p$, a contradiction.

(1.5) Let P be a Sylow p-subgroup of G. If P is cyclic, then G is p-supersoluble. Let P be non-cyclic. Then by Lemma 6 (3), for every maximal subgroup P_i of P and every $q \in \pi(G) \setminus \{p\}$ there exists a Sylow q-subgroup Q of G such that $P_iQ \leq G$. By [7, Theorem 3.4], G is p-supersoluble. Since it is true for any $p \in \pi(G)$, we have G is supersoluble.

2. Assume that the claim is false and let G be a minimal counterexample. Let N be a non-trivial normal subgroup of G. The subgroups $AN/N \simeq A/A{\cap}N$ and $BN/N \simeq B/B{\cap}N$ are wtcc-subgroups in G/N by Lemma 6 (2), $AN/N \simeq A/A \cap N$ and $BN/N \simeq B/B \cap N$ are supersoluble. Hence $G/N = (AN/N)(BN/N)$ is supersoluble by induction.

We show that G is soluble. By Lemma 6 (7), A' and B' are subnormal in G. If A and B are abelian, then by Theorem Itô, G is soluble. Hence we consider that either $A' \neq 1$ or $B' \neq 1$. Suppose that $A' \neq 1$. Since A is supersoluble, $(A')^G$ is nilpotent. If $(A')^G = G$, then G is soluble. If $(A')^G < G$, then $G/(A')^G$ is supersoluble. Hence G is soluble.

Since by hypothesis, A and B are supersoluble wtcc-subgroups of soluble group G, by Lemma 6 (6), A_p and B_p are subnormal in G for the greatest prime $p \in \pi(G)$. Because $P = A_pB_p$ is a Sylow p-subgroup of G, we have G is p-closed. By Lemma 1, G has a unique minimal normal subgroup N such that $N = C_G(N) = \mathrm{O}_p(G) = F(G) = P$ and N is an elementary abelian subgroup of order p^n, $n > 1$.

Without loss of generality, we assume that $A_p \neq 1$. Let

$$1 = A_0 \leq A_1 \leq \ldots \leq A_{s-1} \leq A_s = A$$

be a wtcc-series of A. Since A is p-closed, $\mid A_1 \mid= p$. By Lemma 6 (3), $A_1Y_1 \leq G$ for some Hall p'-subgroup Y_1 of Y. Then

$$A_1 = P \cap A_1Y_1 = A_1(P \cap Y_1)$$

is normal in A_1Y_1. Hence $Y_1 \leq N_G(A_1)$. Since P is abelian, a Sylow p-subgroup Y_p of Y centralizes A_1 and A_1 is normal in $G = AY_pY_1$, a contradiction.

The theorem is proved.

References

[1] B. Huppert, *Finite groups. I*, Springer-Verlag, Berlin etc., 1967. Zbl 0217.07201

[2] A. Ballester-Bolinches, R. Esteban-Romero, M. Asaad, *Products of finite groups*, de Gruyter Expositions in Mathematics, **53**, Walter de Gruyter, Berlin, 2010. Zbl 1206.20019

[3] W. Guo, K.P. Shum, A.N. Skiba, *Criterions of supersolubility of products of supersoluble groups*, Publ. Math. Debr., **68**:3-4 (2006), 433–449. Zbl 1102.20022

[4] M. Arroyo-Jordá, P. Arroyo-Jordá, *Conditional permutability of subgroups and certain classes of groups*, J. Algebra, **476** (2017) 395–414. Zbl 1362.20018

[5] W. Guo, *Structure theory for canonical classes of finite groups*, Springer, Berlin, 2015. Zbl 1343.20021

[6] A.A. Trofimuk, *On the supersolubility of a group with some tcc-subgroups*, J. Algebra Appl., **20**:2 (2021), Article ID 2150020. Zbl 1475.20030

[7] V.S. Monakhov, A.A. Trofimuk, *On the supersolubility of a finite group with NS-supplemented subgroups*, Acta Math. Hung., **160**:1 (2020) 161–167. Zbl 1474.20031

[8] *GAP – Groups, Algorithms, Programming – a System for Computational Discrete Algebr*, Ver. GAP 4.12.2 released on 18 December 2022.

[9] H. Wielandt, *Subnormalität in faktorisierten endlichen Gruppen*, J. Algebra, **69** (1981) 305–311. Zbl 0454.20027

[10] K. Doerk, *Minimal nicht überauflösbare, endliche Gruppen*, Math. Z., **91** (1966) 198–205. Zbl 0135.05401

[11] V.N. Tyutyanov, V.N. Kniahina, *Finite groups with biprimary Hall subgroups*, J. Algebra, **443** (2015), 430–440. Zbl 1325.20016

[12] K. Doerk, T. Hawkes, *Finite soluble groups*, W. de Gruyter, Berlin etc., 1992. Zbl 0753.20001

[13] V.N. Knyagina, V.S. Monakhov, *Finite groups with seminormal Schmidt subgroups*, Algebra Logic, **46**:4 (2007), 244–249. Zbl 1155.20016

[14] C. Casolo, *A criterion for subnormality and Wielandt complexes in finite groups*, J. Algebra, **169**:2 (1994) 605–624. Zbl 0812.20007

Alexander Trofimuk
Brest State A.S. Pushkin University,
Kosmonavtov Boulevard, 21,
224016, Brest, Belarus
Email address: alexander.trofimuk@gmail.com