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THE EXACT SOLUTIONS FOR THE FLOW OF LIQUID POLYMER WITH VARIABLE DISCHARGE IN THE FLAT CHANNEL WITH PERMEABLE WALLS

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Abstract: We have studied the problem of steady-state flow of viscoelastic liquid in the flat channel for the modified Vinogradov–Pokrovskii rheological model. It was shown that the problem has a set of solutions which could be calculated exactly. These type of solutions correspond to the flow with permeable walls and variable discharge along the flat channel. The solutions include the cases of constant and linear pressure gradient in the channel.

Keywords: Vinogradov–Pokrovskii rheological model, Poiseuille flow, steady-state solutions.

1 Introduction

Liquid polymers are the viscoelastic fluid medias which consist of the long entangled macromolecules. Under the flows with non-zero velocity gradients this macromolecules interact with each other in a complex way, stretch and rotate over time. This molecular structure gives the polymeric medias certain distinctive features such as memory of deformations, anisotropy and

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shear-thinning. The mathematical description of such a complex matter is a difficult problem which implies many assumptions and simplifications. Some of them are often not very well justified from the physical point of view. Any mathematical model of viscoelastic media have to take into account some features of the liquid and ignore others, since it is practically impossible to include all of the significant features in single model simple enough to be of any use. The result of it is the number of different models of polymer dynamics which utlize various approaches and assumptions. It has to be noted that these models are usually quite complex mathematically and many of their mathematical properties are not studied yet.

The core of any rheological model of viscoelastic fluid is the constitutive equation which connects the tensor of stresses with the tensor of velocity gradients. The exact form of this expression depends on the assumptions made and thus is different in different models. Generally speaking it makes sense to highlight two approaches for the derivation of constitutive equation. The first one is the phenomenological approach which is focused on the experimental measurements of real liquid polymeric solutions and melts received by rheometric devices [1, 2]. Using the data, the one can make certain assumptions based on general understanding of the behavior of the liquid in study and receive the constitutive equation which fits the experiments. The second approach is the mesoscopic one which relies on the stochastic modeling of the dynamics and interaction of the macromolecules themselves [3, 4, 5]. The macroscopic constitutive equation is thus received by ensemble average of Brownian motion of molecules. In this paper we are using the mesoscopic rheological Vinogradov–Pokrovskii model (mVP) [6, 7].

As it was mentioned, the rheological models are quite complex mathematically, which makes the study of their properties challenging. In particular, the number of known exact solutions for the models is limited by several simple cases. Even more so, the mVP model, as being relatively recent, is not well studied yet in that regard. So the goal of this paper is to look for one group of solutions for this model which can be found exactly. At the same time the exact solutions provide a lot of information about properties of the model which makes the search of such solutions the important goal, especially if the model is not thoroughly studied yet. The search of the exact partial solutions for rheological models is a popular modern subject of research and number of recent papers introduced several cases when these solutions could be obtained for different models [8, 9, 10, 11]. In particular, there are some exact solutions for mVP models representing rectilinear and rotating steady-state flows [13, 14].

One of the most well-known type of stationary flow of viscous liquid is the Pouseuille flow in flat or cylindrical channel, meaning the stationary flow under constant pressure gradient and under non-slip boundary conditions. The similar type of flow for viscoelastic polymeric liquids is the popular subject of the research due to both relative simplicity of its geometry and important applications for both technological processes and study if the stability of the viscoelastic flows. The exact solutions of plain Poiseuille-type were found for several widely known models such as Oldroyd-b model [9, 10, 11] and Doi-Edwards model [8] under certain simplifications. There are also couple of the papers dedicated to Pouseuille-type flow for mVP model, including the papers written by the authors of the current one [13, 15]. It was shown that exact solutions for plain Pouseuille flow for this model can be obtained even if it was required to use numerics to construct this solutions for practical purposes. It was also revealed that this solutions are not always unique. The analysis of this potential non-uniqueness of the stationary solutions was later performed by assuming that the walls of the flat channel are permeable. It was shown that of all possible stationary solutions for Pouseuille flow only one is close to the solutions of the problem with small flow through the walls, which makes only one solution feasible from that point of view [15].

The current paper develops the idea of the steady-state flow in the flat channel with permeable (perforated) walls. This type of flow has its own interest since the perforated walls with forced pumping of the gas or liquid through them is the proven way of stabilization of the flow in the tube or channel. It appears that the mVP model has a class of solutions for that type of flow which could be found exactly under certain assumptions. It is shown in the paper that these solutions correspond to the flow with variable discharge and variable pressure gradient, which expand the set of known exact solutions for stationary two-dimensional flows of mVP model.

2 Governing equations

We are looking for the two-dimensional flow of incompressible viscoelasic liquid in the flat channel of the width l. Let us introduce the equations of the Vinogradov–Pokrovskii rheological model (mVP) as it was described in [7]. The notations for the dimensionless variables are the time t, Carthesian coordinates (x, y), velocity vector $\boldsymbol{u} = (u, v)$, pressure p. The dimension scale for these variables are l/u_H, l, u_H and ρu_H^2 respectively, where u_H is the characteristic velocity and ρ is the constant pressure of the liquid. The continuity equation has the standard form

$$\mathrm{div}\boldsymbol{u} = u_x + v_y = 0, \tag{1}$$

The general form of the momentum equations could be expressed in the following form

$$\frac{d\boldsymbol{u}}{dt} + \bigtriangledown p = div\Pi, \tag{2}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\boldsymbol{u}, \bigtriangledown) = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}.$$

The right-hand side of the equations contain the dimensionless second-order tensor $\Pi = (\alpha_{ij}), i, j = 1, 2$, which represents the additional viscoelastic stresses of the system. In mVP this tensor is called the tensor of anisotropy.

The equations which define the components of this tensor through the hydrodynamical variables are the constitutive equations of the model. Generally speaking, such tensor is defined through the tensor of velocity gradients (tensor of the speed of deformation) [5]. The difficulty of the formulation of rheological constitutive equation is that the dependence of tensor Π from velocity gradient tensor is nonlinear and non-local in time. Liquid polymers demonstrate the memory effects which means that the tensor Π depends on the history of the flow, not just of the state of the flow at the current moment. It follows that the constitutive equation could not be the straightforward algebraic relation but should be the system of differential or integral equations instead. The mVP model utilizes the mesoscopic approach for the derivation of constitutive equation, meaning that the dynamics of the marcomolecules is modeled by reducing it to movement of one molecule in the anisotropic viscous media. The molecule itself is modeled as two beads connected with the elastic spring. This viscous media represents the influence of the neibouir molecules on the moving one. The molecule is moving according to the stochastic generalized Langevin equation [3]. The assumption is made that the viscoelastic stresses in the liquid are caused by deviation from the equilibrium state and that deviation are determined by tensor of anisotropy and two phenomenological constants k and β, representing the effect of the size and orientation of the macromolecules in the flow. Thus the fixed velocity field $\boldsymbol{u}$ could correspond to different stresses. In two-dimensional case the constitutive equations are three differential equations:

$$\frac{d\alpha_{11}}{dt} - 2\alpha_1 u_x - 2\alpha_{12} u_y + \Lambda_{11} = 0, \tag{3}$$

$$\frac{d\alpha_{12}}{dt} - \alpha_1 v_x - \alpha_2 u_y + \tilde{K_I}\alpha_{12} = 0, \tag{4}$$

$$\frac{d\alpha_{22}}{dt} - 2\alpha_2 v_y - 2\alpha_{12} v_x + \Lambda_{22} = 0. \tag{5}$$

Here
$\alpha_i = \alpha_{ij} + \varkappa^2, i, j = 1, 2; \varkappa^2 = 1/(ReW)$;
$\Lambda_{ii} = K_I \alpha_{ii} + \beta Re(\alpha_{ii}^2 + \alpha_{12}^2), i = 1, 2$;
$K_I = Re(\varkappa^2 + \bar{k}/3I), I = \alpha_{11} + \alpha_{22}$,
$\tilde{K_I} = K_I + \beta ReI = Re(\varkappa^2 + \hat{k}/3I), \hat{k} = \overline{k} + 3\beta, \overline{k} = k - \beta$,
$Re = \rho u_H l/\eta_0$ is the Reynolds number , $W = \tau_0 u_H/l$ is the Weissenberg number (see[7]), η_0, τ_0 are the initial values of sheer viscosity and relaxation time respectively (see [7, 16]). The system (1)-(5) is the mVP model in two-dimensional case.

Now let us introduce the boundary conditions of the problem. Assume that the velocity at the walls of the channel ($y = 0$ and $y = 1$) is under control, that is the velocity vector is set. In this paper we will use the most simplified approach by ignoring the size and position of the holes in the walls of the channel and making the walls permeable at any point (see [18]). Assuming that the continuity eqiation is true up to the walls, we are getting the boundary conditions of the following form:

$$\begin{cases} y = 0 : u = q_0(t,x), v_y = -(q_0)_x(t,x), t > 0, x \in R^1; \\ y = 1 : u = q_1(t,x), v_y = -(q_1)_x(t,x), t > 0, x \in R^1. \end{cases} \quad (6)$$

Here $q_0(t,x)$ и $q_0(t,x)$ are known functions.

3 Steady-state flows in the flat channel

In this section we are showing the solutions of the problem in several partial cases where the exact solutions are available. First we will assume that the solution is steady, that is the unknown functions are independent on time. Also let the vertical velocity component v be independent on x. It immediately follows from (1) that the horizontal velocity component u is

$$u = f(y) - xv'(y),$$

where $f(y)$ is some smooth function. Similar to [18], we will look for the steady-state solutions of the following form for the equations (1)-(5):

$$\begin{cases} \alpha_{ij} = \alpha_{ij}(y),\ i,j = 1,2, \\ p = G(y) + \hat{P}_0 - \hat{A}x + \frac{\hat{B}x^2}{2}, \\ u = f(y) - xv'(y), \\ v = v(y). \end{cases} \quad (7)$$

Here $G(y)$ is an unknown function, $G(0) = 0$, $\hat{P}_0$ is the pressure at $y = x = 0$, $\hat{A}$ is the dimensionless pressure gradient at $x = 0$ along the axis of the channel, $\hat{B}$ is some constant, $F_{0,1}(y)$ are the unknowns.

The expression for the tensor of anisotropy in (7) implies the natural assumption that the internal stresses for the stabilized flow should be independent on the x coordinate. The expression for p for this type of flows commonly assumes that $\hat{B} = 0$ because it thus match the pressure profile for the well-known plain Poiseuille flow of viscous liquid [17]. Here we use more general form for pressure, more typical for the rotating flows [14]. We will show that precise solutions of the problem are obtainable even for this slightly more general case.

Next, assume that

$$\hat{\alpha}_{22}(y) = -\varkappa^2 = const,\ \hat{\alpha}_{12} \equiv 0, \quad (8)$$

which appears to be a special case greatly simplifying the equations of the model. We will also assume that $\bar{k} = 0, \beta = 1$.

For the functions v, f, $\hat{\alpha}_{11}$ we have the following equations from (1)-(5):

$$vv'' - (v')^2 - \hat{B} = 0; \quad (9)$$

$$vf' - fv' - \hat{A} = 0; \quad (10)$$

$$v\hat{\alpha}_{11}' + 2v'(\hat{\alpha}_{11} + \varkappa^2) + \varkappa^2 Re\hat{\alpha}_{11} + Re\hat{\alpha}_{11}^2, \quad (11)$$

$$\hat{\alpha}_{11}(0) = \alpha_0.$$

Let us define the boundary conditions for the velocity components. Assume that the v is known at the walls of the channel, then from (7) we have

$$u = \nu_0 - v'(y)x,\ v(y) = \mu_0,\ y = 0,$$
$$u = \nu_1 - v'(y)x,\ v(y) = \mu_1,\ y = 1,$$

or

$$\begin{aligned} f = \nu_0,\ v = \mu_0, y = 0,\\ f = \nu_1,\ v = \mu_1, y = 1. \end{aligned} \tag{12}$$

Here $\nu_{0,1}$ and $\mu_{0,1}$ are some constants.

Let us first rewrite the (11) such as:

$$v\gamma' + 2v'\gamma - \varkappa^2 Re\gamma + Re\gamma^2 = 0, \quad \gamma|_{y=0} = \gamma(0),$$

or

$$(v^2\gamma)' - \frac{\varkappa^2 Re}{v}(v^2\gamma) = -\frac{Re}{v^3}(v^2\gamma)^2. \tag{13}$$

Here $\gamma = \hat{\alpha}_{11}(y) + \varkappa^2$.

The equation (13) is the Bernoulli differential equation. We will get the solution by the well-known procedure [19]. Let

$$w = \frac{1}{v^2\gamma} \quad w' = \frac{(v^2\gamma)'}{(v^2\gamma)^2},$$

Then we have:

$$w' + \frac{\varkappa^2 Re}{v} w = \frac{Re}{v^3},$$

It follows that

$$v^2\gamma(y) = \frac{v^2(0)\gamma(0)\exp\left(Re\int_0^y \varkappa^2/v(\xi)d\xi\right)}{1 + v^2(0)\gamma(0)\int_0^y Re/v^3(\xi)\exp\left(Re\int_0^\xi \varkappa^2/v(\eta)d\eta\right)d\xi},$$

where $\gamma(0) = \hat{\alpha}_0 + \varkappa^2$. Then

$$\alpha_{11}(y) = \frac{\mu_0^2(\hat{\alpha}_0 + \varkappa^2)\exp\left(Re\int_0^y \varkappa^2/v(\xi)d\xi\right)}{v^2(y)(1 + \mu_0^2(\hat{\alpha}_0 + \varkappa^2)\int_0^y Re/v^3(\xi)\exp\left(Re\int_0^\xi \varkappa^2/v(\eta)d\eta\right)d\xi)} - \varkappa^2. \tag{14}$$

The formula (14) provides the value of $\alpha_{11}(y)$ for known $v(y)$.

Function G satisfies the equation (see (7)):

$$(G + \frac{v^2}{2})' = 0,$$

or

$$G(y) = \frac{\mu_0}{2} - \frac{v^2(y)}{2}. \tag{15}$$

The equation (9) could be rewritten as:

$$\left(\frac{v'}{v}\right)' = \frac{\hat{B}}{v^2}$$

or

$$Y'' = \hat{B} exp(-2Y), Y = \ln v. \tag{16}$$

By reducing the order of the equation (16), we get the following:

$$\frac{\partial Z^2}{\partial exp(-2Y)} = \hat{B}, \tag{17}$$

where $Z(Y) = Y'$.

It follows from (17) that:

$$(v')^2 = Cv^2 - \hat{B}. \tag{18}$$

Let us gradually calculate the solutions of (18) for several possible cases. First we will assume that $\hat{B} < 0$ and $C < 0$. Then

$$v'(y) = c_1 \sqrt{k^2 - v^2(y)}. \tag{19}$$

Here:
$k^2 = |\hat{B}|/c_1^2$, c_1 is some constant.

By taking the boundary conditions (9) into account we have:

$$\begin{cases} v'^2(0) = c_1^2(k^2 - \mu_0^2), \\ v'^2(1) = c_1^2(k^2 - \mu_1^2). \end{cases}$$

From (19) we finally have:

$$\arcsin \frac{v(y)}{k} = c_1 y + c_2$$

or

$$v(y) = k \sin(c_1 y + c_2), k = \frac{|\hat{B}|^{\frac{1}{2}}}{c_1}, \tag{20}$$

where c_2 is constant.

From the boundary condition at (20) $y = 0$ we have:

$$c_2 = \arcsin \frac{\mu_0}{k}, \tag{21}$$

and at $y = 1$:

$$c_1 = -\arcsin \frac{\mu_0}{k} + \arcsin \frac{\mu_1}{k}. \tag{22}$$

From (20)-(22) it follows that

$$v(y) = v(y) = k \sin(\arcsin \frac{\mu_0}{k} + y[\arcsin \frac{\mu_1}{k} - arcsin \frac{\mu_0}{k}]).$$

We are assuming that (see (21), (22)):

$$0 < c_2 \leq \frac{\pi}{2}, \quad 0 < c_1 + c_2 \leq \frac{\pi}{2}.$$

Now let us look at (10). This equation can be rewritten in the form

$$\left(\frac{f}{v}\right)' = \frac{\hat{A}}{v^2}$$

therefore

$$f(y) = v(y)\int_0^y \frac{\hat{A}}{v^2(\xi)}d\xi + c_3,$$

where c_3 is constant.

It follows from the boundary conditions (10) that

$$f(0) = c_3 = \nu_0,$$

$$f(1) = v(1)\int_0^1 \frac{\hat{A}}{v^2(1)}d\xi + \nu_0 = \nu_1.$$

Finally,

$$\hat{A}v(1)\int_0^1 \frac{d\xi}{v^2(\xi)} = \nu_1 - \nu_0. \tag{23}$$

Note that the equation (24) connects the value $\hat{A}$ with parameters ν_0 and ν_1. We have to assume that (24) is true for the problem to have the solutions of the form (7). But that means we cannot set the values of $\hat{A}$, $\hat{B}$, μ_0, μ_1, ν_0, ν_1 independently, because the boundary conditions of (10) would not be guaranteed in that case. The representation (7) and the assumption (8) forces us to treat the equation (24) as the definition of $\hat{A}$:

$$\hat{A} = \frac{\nu_1 - \nu_0}{\mu_0\int_0^1 1/(v^2(\xi))d\xi}.$$

The integral in the last equation can be calculated:

$$\int_0^y \frac{d\xi}{v^2(\xi)} = \int_0^y \frac{d\xi}{(k\sin(\hat{l}(\xi)))^2},$$

$$(\hat{l}(y) = c_1y + c_2, d\hat{l} = c_1dy,$$

$$\frac{1}{k^2c_1}\int_{c_2}^{c_1y+c_2} \frac{d\hat{l}}{sin^2\hat{l}} = -\frac{1}{k^2c_1}(\mathrm{ctg}(c_1y + c_2) - \mathrm{ctg}c_2).$$

Then the equation (24) will take the form:

$$\nu_1 - \nu_0 - \hat{A}\frac{k\sin(c_1 + c_2)}{k^2c_1}\frac{\sin(c_1)}{sin(c_1 + c_2)\sin c_2} = 0.$$

Then

$$\hat{A} = \frac{(\nu_1 - \nu_0)c_1k\sin c_2}{\sin c_1}. \tag{24}$$

The solutions if the problem are:

$$\begin{cases} v(y) = k\sin(c_1y + c_2), \\ u(y) = \dfrac{\hat{A}v(y)(\mathrm{ctg}(c_2) - \mathrm{ctg}(c_1y + c_2))}{c_1k^2} - xkc\cos(c_1y + c_2) + \nu_0, \\ p(x, y) = G(y) + \hat{P}_0 - \hat{A}x + \dfrac{\hat{B}x^2}{2}. \end{cases} \tag{25}$$

Here
$$c_1 = -\arcsin\frac{\mu_0}{k} + \arcsin\frac{\mu_1}{k},\ c_2 = \arcsin\frac{\mu_0}{k},$$
$k = |\hat{B}|/c_1$, $\hat{A}$ is determined by (24) and $G(y)$ is calculated by (15). The values $\hat{B}$, μ_0, μ_1, ν_0, ν_1, $\hat{P}_0$ are the parameters of the problem.

Now let us assume that $\hat{B} < 0$ and $C > 0$ in (18). Then

$$v'(y) = c_1\sqrt{k^2 + v^2(y)}. \tag{26}$$

where $k^2 = |\hat{B}|/c_1^2$. We have
$$\int \frac{dv}{\sqrt{v^2+k^2}} = \ln|v + \sqrt{v^2+k^2}| = c_1 y + c_2,$$
$$v + \sqrt{v^2+k^2} = c_2 e^{c_1 y}$$
From the boundary conditions we have

$$v(y) = \frac{c_2^2 e^{2c_1 y} - k^2}{2c_2 e^{c_1 y}},$$

where
$$c_1 = \ln\frac{\mu_1 + \sqrt{\mu_1^2+k^2}}{\mu_0 + \sqrt{\mu_0^2+k^2}} \qquad c_2 = \mu_0 + \sqrt{\mu_0^2+k^2}.$$
Now from (10) we can determine f:

$$f(y) = v(y)\int_0^y \frac{\hat{A}}{v^2(\xi)}d\xi + c_3.$$

Then, using the boundary conditions (12)

$$f(y) = \frac{-\hat{A}}{4c_1c_2^3 e^{c_1 y}}\left(1 - \frac{c_2^2 e^{2c_1 y} - k^2}{c_2^2 - k^2}\right) + \nu_0. \tag{27}$$

Using the last equation at $y = 1$ as definition for $\hat{A}$, we have
$$\hat{A} = \frac{4(\nu_1 - \nu_0)c_1 c_2^3 e^{c_1}(c_2^2 - k^2)}{c_2^2 e^{2c_1} - c_2^2}. \tag{28}$$

Finally we have
$$\begin{cases} v(y) = \dfrac{c_2^2 e^{2c_1 y} - k^2}{2c_2 e^{c_1 y}}, \\ u(y) = \dfrac{-\hat{A}}{4c_1c_2^3 e^{c_1 y}}\left(1 - \dfrac{c_2^2 e^{2c_1 y} - k^2}{c_2^2 - k^2}\right) - \dfrac{x}{2}\left(\dfrac{c_1 c_2 e^{c_1 y}}{2} + \dfrac{k^2 c_1}{2c_2 e^{c_1 y}}\right), \\ p(x,y) = G(y) + \hat{P}_0 - \hat{A}x + \dfrac{\hat{B}x^2}{2}. \end{cases} \tag{29}$$
Here
$$c_1 = \ln\frac{\mu_1 + \sqrt{\mu_1^2+k^2}}{\mu_0 + \sqrt{\mu_0^2+k^2}}, \qquad c_2 = \mu_0 + \sqrt{\mu_0^2+k^2}, \quad k = |\hat{B}|/c_1,$$

$\hat{A}$ is determined by (24) and $G(y)$ is calculated by (15). As before, the components of the tensor of anisotropy are determined by (8), (14).

It may have look like the case $\hat{B} < 0$ corresponds to the boundary-value problem with multiple solutions. Indeed, the problem (10)-(12) provides two sets of solutions (25), (29) with identical boundary conditions (12). But in fact the solutions of type (25) and type (29) have different pressure profiles. To show that, let us denote $\hat{B}_1$ as a coefficient for the solution (25) and $\hat{B}_2$ for (29). We have

$$\hat{B}_1(k) = c_1^2 k^2 = -\left(\arcsin\frac{\mu_0}{k} + \arcsin\frac{\mu_1}{k}\right)^2 k^2,$$

$$\hat{B}_2(k) = c_1^2 k^2 = \left(\ln\frac{\mu_1 + \sqrt{\mu_1^2 + k^2}}{\mu_0 + \sqrt{\mu_0^2 + k^2}}\right)^2 k^2.$$

It can be shown that

$$\lim_{k\to+\infty} \hat{B}_1(k) = \lim_{k\to+\infty} \hat{B}_2(k) = -(\mu_0 - \mu_1)^2,$$

but

$$\frac{\partial}{\partial k}\hat{B}_1(k) > 0,\ \frac{\partial}{\partial k}\hat{B}_2(k) < 0,\ k > \max(|\mu_0|, |\mu_1|).$$

It follows that for all possible k we have $\hat{B}_1(k) < -(\mu_0 - \mu_1)^2$, $\hat{B}_2(k) > -(\mu_0 - \mu_1)^2$. That means the solutions (25) and (29) correspond to different values of $\hat{B}$ and thus any given pressure profile of the form (7) provides either solution (25) or solution (29) but not both of them at the same time.

Now we will look for the solutions with $\hat{B} = 0$, which corresponds to the case with constant pressure gradient along the axis of the channel.

From $\hat{B} = 0$ and (9) it follows that:

$$\left(\frac{v'}{v}\right)' = 0$$

and

$$v = c_1 e^{c_2 y},$$

here c_1, c_2 are constants.

From boundary conditions at $y = 0$ we have

$$c_1 = \mu_0, \tag{30}$$

and at $y = 1$:

$$c_2 = \ln\frac{\mu_1}{\mu_0}. \tag{31}$$

From (30) and (31) we have

$$v(y) = \mu_0 \left(\frac{\mu_1}{\mu_0}\right)^y.$$

Now from (10) we have

$$\left(\frac{f}{v}\right)' = \frac{\hat{A}}{v^2},$$

and

$$\frac{f}{v} = \frac{\hat{A}}{\mu_0^2}\int_0^y \left(\frac{\mu_1}{\mu_0}\right)^{-2\xi} d\xi + c_3,$$

where c_3 is constant. From the boundary conditions at $y = 0$ we have

$$c_3 = \frac{\nu_0}{\mu_0}.$$

Thus, the $f(y)$ is equal to

$$f(y) = \left(\frac{1}{2\ln(\mu_1/\mu_0)}\frac{\hat{A}}{\mu_0^2}\left(1 - \left(\frac{\mu_1}{\mu_0}\right)^{-2y}\right) + \frac{\nu_0}{\mu_0}\right)\mu_0\left(\frac{\mu_1}{\mu_0}\right)^y =$$

$$= \frac{1}{2\ln(\mu_1/\mu_0)}\frac{\hat{A}}{\mu_0}\left(\left(\frac{\mu_1}{\mu_0}\right)^y - \left(\frac{\mu_1}{\mu_0}\right)^{-y}\right) + \nu_0\left(\frac{\mu_1}{\mu_0}\right)^y.$$

and from the condition at $y = 0$ we have

$$f(1) = \nu_1 = \frac{\hat{A}}{2\mu_0\ln(\mu_1/\mu_0)}\left(\frac{\mu_1^2 - \mu_0^2}{\mu_0\mu_1}\right) + \frac{\nu_0\mu_1}{\mu_0}.$$

As before, we will use the last equation to define $\hat{A}$:

$$\hat{A} = \frac{2\mu_0\mu_1\ln(\mu_1/\mu_0)}{\mu_1^2 - \mu_0^2}(\nu_1\mu_0 - \nu_0\mu_1).$$

The expressions for the rest of the unknowns are similar to the case of $\hat{B} < 0$. So the final expression of the solution is

$$\begin{cases} v(y) = \mu_0\left(\frac{\mu_1}{\mu_0}\right)^y, \\ u(y) = \frac{1}{2\ln(\mu_1/\mu_0)}\frac{\hat{A}}{\mu_0}\left(\left(\frac{\mu_1}{\mu_0}\right)^y - \left(\frac{\mu_1}{\mu_0}\right)^{-y}\right) + \\ \quad + \nu_0\left(\frac{\mu_1}{\mu_0}\right)^y - x\mu_0\ln\frac{\mu_1}{\mu_0}\left(\frac{\mu_1}{\mu_0}\right)^y, \\ p(x,y) = \frac{\mu_0^2}{2} - \frac{\mu_0^2}{2}\left(\frac{\mu_1}{\mu_0}\right)^{2y} + \hat{P}_0 - \hat{A}x. \end{cases} \tag{32}$$

The last case to study is $\hat{B} > 0$. As before, we have the following equation from (18):

$$v'(y) = c_1\sqrt{v^2(y) - k^2}. \tag{33}$$

where $k^2 = |\hat{B}|/c_1^2$. Then

$$\int \frac{dv}{\sqrt{v^2 - k^2}} = \ln|v + \sqrt{v^2 - k^2}| = c_1 y + c_2$$

$$v + \sqrt{v^2 - k^2} = c_2 e^{c_1 y}$$

From boundary conditions we have

$$v(y) = v = \frac{c_2^2 e^{2c_1 y} + k^2}{2c_2 e^{c_1 y}}$$

where

$$c_1 = \ln\frac{\mu_1 + \sqrt{\mu_1^2 - k^2}}{\mu_0 + \sqrt{\mu_0^2 - k^2}} \qquad c_2 = \mu_0 + \sqrt{\mu_0^2 - k^2}.$$

Finally we have:

$$f(y) = \frac{-\hat{A}}{4c_1 c_2^3 e^{c_1 y}}(1 - \frac{c_2^2 e^{2c_1 y} + k^2}{c_2^2 + k^2}) + \nu_0.$$

Using the last equation at $y = 1$ as definition for $\hat{A}$, we have

$$\hat{A} = \frac{4(\nu_1 - \nu_0)c_1 c_2^3 e^{c_1}(c_2^2 + k^2)}{c_2^2 e^{2c_1} - c_2^2}.$$

$$u(y) = \frac{-\hat{A}}{4c_1 c_2^3 e^{c_1 y}}(1 - \frac{c_2^2 e^{2c_1 y} + k^2}{c_2^2 + k^2}) - \frac{x}{2}(\frac{c_1 c_2 e^{c_1 y}}{2} - \frac{k^2 c_1}{2c_2 e^{c_1 y}}).$$

$$\begin{cases} v(y) = v = \dfrac{c_2^2 e^{2c_1 y} + k^2}{2c_2 e^{c_1 y}}, \\ u(y) = \dfrac{-\hat{A}}{4c_1 c_2^3 e^{c_1 y}}(1 - \dfrac{c_2^2 e^{2c_1 y} + k^2}{c_2^2 + k^2}) - \dfrac{x}{2}(\dfrac{c_1 c_2 e^{c_1 y}}{2} - \dfrac{k^2 c_1}{2c_2 e^{c_1 y}}), \\ p(x, y) = G(y) + \hat{P}_0 - \hat{A}, \end{cases} \tag{34}$$

where

$$c_1 = \ln\frac{\mu_1 + \sqrt{\mu_1^2 - k^2}}{\mu_0 + \sqrt{\mu_0^2 - k^2}} \quad c_2 = \mu_0 + \sqrt{\mu_0^2 - k^2}, \quad k = |\hat{B}|/c_1,$$

$\hat{A}$ is determined by (24) and $G(y)$ is calculated by (15). As before, the components of the tensor of anisotropy are determined by (8), (14).

Fig. 1 illustrates the solution of type (25) at $x = 0$. This solution is calculated for the case that the flow into the channel exceeds the flow out of it ($\mu_0 = 2,\ \mu_1 = 1$). The vertical velocity decreases along the line perpendicular to the axis of the channel. Also the pressure at the boundary $y = 0$ is lower than at the boundary $y = 1$. It is clear from (25) that the pressure along the Ox axis reaches maximum at the point $x = \hat{A}/\hat{B}$. It is also easy to see that the horizontal component of the flow changes direction. The liquid is moving left for the $x < -L$ and moving right for $x > L$ where L is some constant. Solutions of type (29) are shown on Fig. 2. As it can be seen, the overall shape of these profiles is quite similar to the case of type (25). The profiles for the cases (32) and (34) could be plotted in the same manner but we are not showing them here since their appearence is not much different from the one shown on the Fig. 1 and Fig. 2.

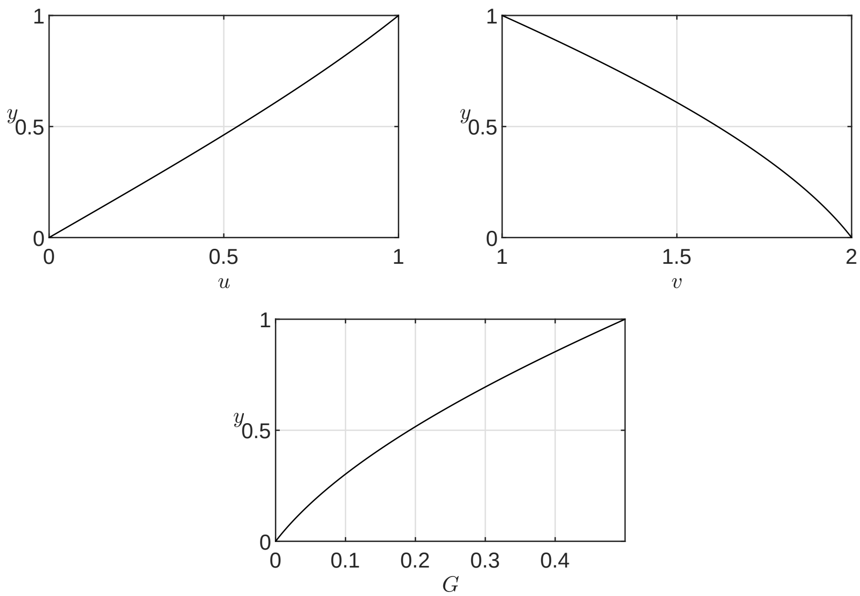

Рис. 1. Profiles for the solution at $x = 0$. Here $\mu_0 = 2$, $\mu_1 = 1$, $\nu_0 = 0$, $\nu_1 = 1$, $\hat{B} = -2.58$.

The variable flow discharge of the liquid polymer in the channel is equal to

$$Q = \int_0^1 u(x,y)dy = \int_0^1 f(y)dy - x(\mu_1 - \mu_0).$$

It can be seen from the obtained formulas that the discovered class of the solutions has some distinctive features. This class does not include the plain Poiseuille-type of the flows, that is the flows with no-slip conditions at the walls and constant pressure gradient. Indeed, if $\hat{B} = 0$ and $\mu_0 = \mu_1 = 0$, then the equation (9) has the solution equal to zero, which in turn means that $\hat{A} = 0$, meaning the solution of the problem is the trivial state of the rest. Even more so, this class of the solutions does not include the solution with $\mu_0 = \mu_1$ in general. That means, the quantity of the liquid pumped through the one wall should be different from the quantity if liquid drained through the other, which in turn does not allow us to have the solutions with constant flow discharge along the channel. All of it implies that our assumptions about perforated walls and various discharge are essential to have the solutions of introduced type. It has to be noted though, that the mVP model has the Pouseuille-type solutions and the solutions with constant

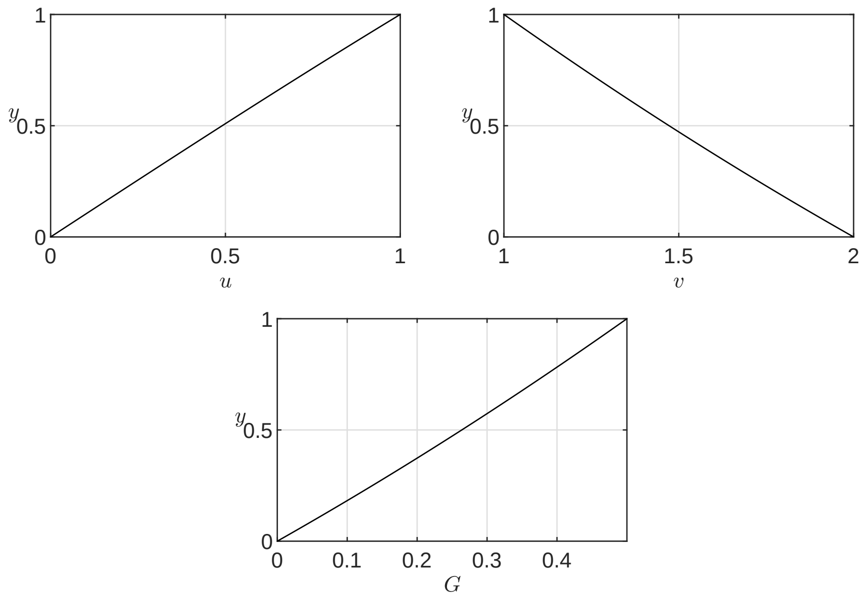

Рис. 2. Profiles for the solution at $x = 0$. Here $\mu_0 = 2$, $\mu_1 = 1$, $\nu_0 = 0$, $\nu_1 = 1$, $\hat{B} = -0.66$.

discharge in general [12, 13], but their form is differnt from the expressions introduced in this paper.

Conclusions

In this paper we have studied the problem of steady-state flow of liquid polymer in the flat channel with permeable walls within the rheological mVP model. Under some assumptions we have introduced the set of exact solutions of that problem. These solutions represent the flows with variable discharge along the axis of the channel and with variable pressure gradient, which differ these solutions from the classical Pouseuille flows.

References

[1] A.I. Leonov, A.N. Prokunin, *Nonlinear phenomena in flows of viscoelastic polymer fluids*, Springer, Berlin, 1994.
[2] J.G. Oldroyd, *On the formulation of rheological equations of state*, Proc. R. Soc. Lond., Ser. A, **200** (1950), 523–541. Zbl 1157.76305
[3] M. Doi, S.F. Edwards, *The theory of polymer dynamics*, Clarendon, Oxford, 1986.
[4] R.B. Bird, C.F. Curtiss, R.C. Armstrong, O. Hassager, *Dynamics of polymeric liquids. Vol. 2 Kinetic theory*, Wiley, New York, 1987.
[5] G. Astarita, G. Marrucci, *Principles of non-Newtonian fluid mechanics*, McGraw-Hill, New York, 1974.
[6] V.N. Pokrovskii, *The mesoscopic theory of polymer dynamics*, 2nd Ed. London: Springer. 2010. 256 p.

[7] Yu.A. Altukhov, A.S. Gusev, G.V. Pyshnograi, *Introduction into the mesoscopic theory of fluid polymer systems*, AltGPA, Barnaul, 2012.

[8] S.N. Aristov, O.I. Skul'skii, *Exact solution of the problem of flow of a polymer solution in a plane channel*, J. Engin. Phys. Thermophys., **76** (2003), 577–585.

[9] T. Hayat, S. Zaib, S. Asghar, A.A. Hendi, *Exact solutions in generalized Oldroyd-B fluid*, Appl. Math. Mech., Engl. Ed., **33**:4 (2012), 411–426. Zbl 1266.76067

[10] K.R. Rajagopal, R.K. Bhatnagar, *Exact solutions for some simple flows of an Oldroyd-B fluid*, Acta Mech., **113**:1-4 (1995), 233–239. Zbl 0858.76010

[11] N. Shahid, M. Rana, I. Siddique, *Exact solution for motion of an Oldroyd-B fluid over an infinite flat plate that applies an oscillating shear stress to the fluid*, Bound. Value Probl., **2012**, Paper No. 48. Zbl 1273.76027

[12] Yu.L. Kuznetsova, O.I. Skulskiy, G.V. Pyshnograi, *Presure driven flow of a nonlinear viscoelastic fluid in a plane channel*, Computational Continuum Mechanics, **3**:2 (2010), 55–69.

[13] A.M. Blokhin, R.E. Semenko, *Stationary magnetohydrodynamic flows of non-isothermal non-compressible polymer liquid in a flat channel*, Vestnik YuUrGU. Ser. Mat. Model. Progr., **11**:4 (2018), 41–54.

[14] A.M. Blokhin, R.E. Semenko, *Vortex motion of an incompressible polymer liquid in the cylindrical near-axial zone*, Fluid Dyn., **53**:2 (2018), 177–188. Zbl 1395.76006

[15] A.M. Blokhin, R.E. Semenko, *Search for stationary Poiseuille flows for an incompressible polymer fluid in channels with perforated walls*, J. Appl. Mech. Tech. Phys., **63**:1 (2022), 26–33. Zbl 1515.76010

[16] N.V. Bambaeva, A.M. Blokhin, *Stationary solutions of equations of incompressible viscoelastic polymer liquid*, Comput. Math. Math. Phys., **54**:5 (2014), 874–899. Zbl 1313.76021

[17] L.D. Landau, E.M. Lifshits *Fluid mechanics. 2nd ed. Volume 6 of Course of Theoretical Physics*, Pergamon Press, Oxford etc., 1987. Zbl 0655.76001

[18] A.B. Vatazhin, G.A. Lyubimov, S.A. Regirer, *Magnetohydrodynamical flows in the channels*, Nauka, Moscow, 1970. Zbl 0245.76086

[19] N.M. Matveev, *Methods of integration of ordinary differential equations*, Vysshaya Shkola, Moscow, 1967. Zbl 0153.40201

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