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MULTIVALUED GROUPS AND NEWTON POLYHEDRON

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Abstract: On the set of complex number $\mathbb{C}$ it is possible to define n-valued group for any positive integer n. The n-multiplication defines a symmetric polynomial $p_n = p_n(x, y, z)$ with integer coefficients. By the theorem on symmetric polynomials, one can present p_n as polynomial in elementary symmetric polynomials e_1, e_2, e_3. V. M. Buchstaber formulated a question on description coefficients of this polynomial. Also, he formulated the next question: How to describe the Newton polyhedron of p_n? In the present paper we find all coefficients of p_n under monomials of the form $e_1^i e_2^j$ and prove that the Newton polyhedron of p_n is a right triangle.

Keywords: multi-set, multivalued group, symmetric polynomial, Newton polyhedron.

One branch of Abstract Algebra is studying algebraic systems with multivalued operations. Solutions of the Yang-Baxter equation (2-simplex equation) and its generalization, n-simplex equations, $n \geq 3$, are examples of multivalued operations. In 1971, V. M. Buchstaber and S. P. Novikov [2]

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introduced a construction, suggested by the theory of characteristic classes of vector bundles, in which the product of each pair of elements is an n-multi-set, the set of n points with multiplicities. This construction leads to the notion of n-valued group.

A good survey on n-valued groups and its applications can be found in [1]. In Section 5 of this paper, n-valued groups were constructed on the set of complex numbers $\mathbb{C}$ for any natural n. The n-valued multiplication is described by the polynomials $p_n = p_n(z; x, y)$ which are x, y, z–symmetric polynomials with integer coefficients. If we introduce elementary symmetric polynomials

$$e_1 = x + y + z, \quad e_2 = xy + yz + zx, \quad e_3 = xyz,$$

then $p_n = P_n(e_1, e_2, e_3)$ is a polynomial with integer coefficients in variables e_1, e_2, e_3. In [1] two questions were formulated on the description the coefficients of P_n as well as a question on the Newton polyhedron of p_n.

In the present paper we find the coefficients for monomials of the form $e_1^i e_2^j$ in P_n. It gives particular answer to the first two questions. Also, we prove that if

$$f = f(x_1, x_2, \ldots, x_n) \in \mathbb{Z}[x_1, x_2, \ldots, x_n]$$

is a symmetric homogeneous polynomial of degree k, which contains a monomial ax_1^k for some non-zero a, then its Newton polyhedron is the $k\Delta^{n-1}$-simplex. From this theorem follows that the Newton polyhedron of p_n is the right triangle with side which depend on n. This is the complete answer to the third question.

At the end of the paper we formulate some open questions.

1 Multivalued groups and Buchstaber's questions

1.1. Multivalued groups. Recall definitions and some facts from the theory of multivalued groups (see, for example, [1]).

Let X be a non-empty set. An *n-valued multiplication* on X is a map

$$\mu\colon X \times X \to (X)^n = Sym^n X, \; \mu(x, y) = x * y = [z_1, z_2, \ldots, z_n], \; z_k = (x * y)_k,$$

where $(X)^n = Sym^n X$ is the n-th symmetric power of X, that is the quotient X^n/S_n of the Cartesian power X^n under the action of S_n by permutations of components. The next axioms are natural generalizations of the classical axioms of group multiplication.

Associativity. The n^2-multi-sets:

$$[x*(y*z)_1, x*(y*z)_2, \ldots, x*(y*z)_n], \quad [(x*y)_1*z, (x*y)_2*z, \ldots, (x*y)_n*z]$$

are equal for all $x, y, z \in X$.

Unit. An element $e \in X$ such that

$$e * x = x * e = [x, x, \ldots, x]$$

for all $x \in X$.

Inverse. A map $inv\colon X \to X$ such that

$$e \in inv(x) * x \text{ and } e \in x * inv(x)$$

for all $x \in X$.

The map μ defines *n-valued group structure* $\mathcal{X} = (X, \mu, e, inv)$ on X if it is associative, has a unit and an inverse.

Let μ be the multiplication

$$\mu\colon \mathbb{C} \times \mathbb{C} \to (\mathbb{C})^n$$

that is defined by the formula

$$\mu(x, y) = x * y = [(\sqrt[n]{x} + \epsilon^r \sqrt[n]{y})^n, \quad 1 \leq r \leq n],$$

where ϵ is a primitive n-th root of unity. This multiplication endows $\mathbb{C}$ with the structure of an n-valued group with the unit $e = 0$. The inverse element is given by the map $inv(x) = (-1)^n x$.

The n-valued multiplication is described by the polynomials

$$p_n = p_n(z; x, y) = \prod_{k=1}^{n} (z - (inv(x) * inv(y))_k),$$

whence the product $x * y$ is defined by z-roots of the equation $p_n = 0$. The polynomials $p_n(z; x, y)$ are x, y, z–symmetric polynomials with integral coefficients, e.g.,

$$p_1 = x + y + z, \quad p_2 = (x + y + z)^2 - 4(xy + yz + zx).$$

Set

$$e_1 = x + y + z, \quad e_2 = xy + yz + zx, \quad e_3 = xyz.$$

Then

$p_1 = e_1,$

$p_2 = e_1^2 - 2^2\, e_2,$

$p_3 = e_1^3 - 3^3\, e_3,$

$p_4 = e_1^4 - 2^3\, e_1^2 e_2 + 2^4\, e_2^2 - 2^7\, e_1 e_3,$

$p_5 = e_1^5 - 5^4\, e_1^2 e_3 + 5^5\, e_2 e_3,$

$p_6 = e_1^6 - 2^2 \cdot 3\, e_1^4 e_2 + 2^4 \cdot 3\, e_1^2 e_2^2 - 2^6\, e_2^3 - 2 \cdot 3^4 \cdot 17\, e_1^3 e_3 - 2^3 \cdot 3^4 \cdot 19\, e_1 e_2 e_3 + 3^3 \cdot 19^3\, e_3^2,$

$p_7 = e_1^7 - 5 \cdot 7^4\, e_1^4 e_3 + 2 \cdot 7^6\, e_1^2 e_2 e_3 - 7^7\, e_2^2 e_3 + 7^8\, e_1 e_3^2.$

The following questions were formulated in [1].

(1) What is the relationship between prime factors of n and prime factors of the coefficients of the polynomials p_n?

(2) How to distinguish the monomials that have zero coefficient?
(3) How to describe the Newton polyhedron of p_n?

2 Coefficients and the Newton polyhedron of p_n

Since p_n is a symmetric homogeneous polynomial of degree n, by the theorem on symmetric polynomials we can present p_n as a polynomial on the elementary symmetric polynomials e_1, e_2, and e_3,

$$p_n = \sum_{\substack{k_1 \geq k_2 \geq k_3 \geq 0 \\ k_1+k_2+k_3=n}} A_{k_1,k_2,k_3} e_1^{k_1-k_2} e_2^{k_2-k_3} e_3^{k_3} \in \mathbb{Z}[e_1, e_2, e_3].$$

The main problem is to find the coefficients A_{k_1,k_2,k_3}.

We can write p_n in the form

$$p_n = \prod_{k=1}^{n} (z - ((inv(x) * inv(y))_k) = \prod_{k=1}^{n} (z - ((-1)^n x * (-1)^n y)_k) =$$

$$= \prod_{k=1}^{n} \left(z - \left(\sqrt[n]{(-1)^n x} + \epsilon^k \sqrt[n]{(-1)^n y} \right)^n \right).$$

If $y = 0$, then

$$\bar{p}_n = p_n(z; x, 0) = \prod_{k=1}^{n} \left(z - (\sqrt[n]{(-1)^n x})^n \right) = \prod_{k=1}^{n} (z - (-1)^n x) = (z - (-1)^n x)^n.$$

Denote by

$$\bar{e}_1 = e_1(z; x, 0) = x + z, \quad \bar{e}_2 = e_2(z; x, 0) = zx.$$

We see that $e_3(z; x, 0) = 0$.

The next proposition gives particular answers to the first two questions.

Proposition 1. *1) If n is odd, then all $A_{k_1,k_2,0}$, $k_2 \neq 0$, are zero, i.e. in this case p_n does not contains monomials $e_1^i e_2^j$, $j > 0$.*

2) If $n = 2k$ is even, then the coefficient $A_{2k-i,i,0}$ at $e_1^{2(k-i)} e_2^i$, is equal to

$$A_{2k-i,i,0} = (-4)^i C_k^i = (-4)^i \frac{k!}{i!(k-i)!}, \quad i = 1, 2, \ldots, k.$$

Proof. 1) If n is odd, then

$$\bar{p}_n = (z + x)^n = \bar{e}_1^n.$$

It means that in p_n all coefficients $A_{k_1,k_2,0}$, where $k_1 \geq k_2 > 0$ and $k_1+k_2 = n$ are zero.

2) If $n = 2k$ is even, then

$$\bar{p}_n = (z - x)^n = (\bar{e}_1^2 - 4\bar{e}_2)^k = \sum_{i=0}^{k} (-4)^i C_k^i (\bar{e}_1^2)^{k-i} (\bar{e}_2)^i.$$

Hence, we have found the following coefficients in p_n.

$$A_{2k-i,i,0} = (-4)^i C_k^i, \quad i = 1, 2, \dots, k.$$

□

Example. From this proposition follows that for even n hold

$$\bar{p}_2 = \bar{e}_1^2 - 2^2\,\bar{e}_2,$$

$$\bar{p}_4 = \bar{e}_1^4 - 2^3\,\bar{e}_1^2\bar{e}_2 + 2^4\,\bar{e}_2^2,$$

$$\bar{p}_6 = \bar{e}_1^6 - 2^2\cdot 3\,\bar{e}_1^4\bar{e}_2 + 2^4\cdot 3\,\bar{e}_1^2\bar{e}_2^2 - 2^6\,\bar{e}_2^3,$$

$$\bar{p}_8 = \bar{e}_1^8 - 2^4\,\bar{e}_1^6\bar{e}_2 + 2^5\cdot 3\,\bar{e}_1^4\bar{e}_2^2 - 2^8\,\bar{e}_1^2\bar{e}_2^3 + 2^8\,\bar{e}_2^4.$$

It is easy to see that for even n all coefficients of $\bar{p}_n$ except the coefficient at $\bar{e}_1^n$ are even. This is not true for polynomials p_n, as example p_6 shows. We can formulate

Conjecture. *1) If $n = p^m$ is a power of a prime p, then all coefficients, except the coefficient at e_1^n are divided into p. 2) If n is even, then all coefficients A_{k_1,k_2,k_3} are non-zero.*

2.1. Newton polyhedron. In this subsection we give a complete answer to the third question in [1]. Recall the necessary definition. Let

$$f = f(x_1, x_2, \dots, x_n) = \sum a_{i_1\dots i_n} x_1^{i_1} \dots x_n^{i_n} \in \mathbb{Z}[x_1, x_2, \dots, x_n]$$

be a polynomial with integer coefficients. Denote by I_f the set of multi indexes $(i_1, \dots, i_n)$ such that $a_{i_1\dots i_n} \neq 0$. The convex hull

$$N_f = Conv(I_f) \subset \mathbb{R}^n$$

is said to be a *Newton polyhedron* of f.

To find Newton polyhedra for polynomials p_n, consider them for small n,

$$p_1 = x + y + z,$$

$$p_2 = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx,$$

$$p_3 = (z + x + y)^3 - 27xyz,$$

$$p_4 = ((x+y+z)^2 - 4(xy+yz+zx))^2 - 2^7(x+y+z)xyz = p_2^2 - 2^7 p_1 xyz.$$

Denote by $N_i \subset \mathbb{R}^3$ the Newton polyhedron for p_i. Then

– N_1 is the right triangle $A_1B_1C_1$ with the vertices $A_1 = (1,0,0)$, $B_1 = (0,1,0)$, $C_1 = (0,0,1)$;

– N_2 is the right triangle $A_2B_2C_2$ with the vertices $A_2 = (2,0,0)$, $B_2 = (0,2,0)$, $C_2 = (0,0,2)$;

– N_3 is the right triangle $A_3B_3C_3$ with the vertices $A_3 = (3,0,0)$, $B_3 = (0,3,0)$, $C_3 = (0,0,3)$;

– N_4 is the right triangle $A_4B_4C_4$ with the vertices $A_4 = (4, 0, 0)$, $B_4 = (0, 4, 0)$, $C_4 = (0, 0, 4)$.

To describe N_k for $k > 2$ we introduce the next definition.

Definition 1. *Let k be a positive integer. The standard n-simplex of size k is the subset of $\mathbb{R}^{n+1}$ given by*

$$k\Delta^n = \left\{ (t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = k \text{ and } t_i \geq 0 \text{ for } i = 0, 1, \ldots, n \right\}.$$

For simplicity we shall call the standard n-simplex of size k by $k\Delta^n$-simplex.

For $k = 1$ we get the definition of the standard n-simplex (or unit simplex).

The $k\Delta^n$-simplex has $n + 1$ vertices,

$$E_0 = (k, 0, 0, \ldots, 0, 0), E_1 = (0, k, 0, \ldots, 0, 0), \ldots E_n = (0, 0, 0, \ldots, 0, k).$$

Now we are ready to prove the main result of the present subsection.

Theorem 1. *Let $f = f(x_1, x_2, \ldots, x_n) \in \mathbb{Z}[x_1, x_2, \ldots, x_n]$ be a symmetric homogeneous polynomial of degree k, which contains a monomial ax_1^k for some non-zero a. Then its Newton polyhedron N_f is the $k\Delta^{n-1}$-simplex.*

Proof. Since ax_1^k is a monomial of f and f is symmetric, it contains monomials ax_i^k for all $i = 1, 2, \ldots, n$. Hence, N_f contains the vertices

$$E_0 = (k, 0, 0, \ldots, 0, 0), E_1 = (0, k, 0, \ldots, 0, 0), \ldots E_{n-1} = (0, 0, 0, \ldots, 0, k) \in \mathbb{R}^n$$

and hence it contains $k\Delta^{n-1}$-simplex. Let us show that any other vertex of N_f, which corresponds a monomial in f lies in this simplex. Indeed, any such monomial has the form

$$bx_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}, \quad b \in \mathbb{R}, \quad b \neq 0.$$

Since

$$k_1 + k_2 + \ldots + k_n = k, \quad k_i \geq 0 \text{ for } i = 1, 2, \ldots, n,$$

the corresponding vertex lies in $k\Delta^{n-1}$-simplex. □

We seen that the polynomial p_k is homogeneous and has the form $p_k = e_1^k + \ldots$. Hence, the answer to the third question of V. M. Buchstaber follows from Theorem 1.

Corollary. *The Newton polyhedron that corresponds to the polynomial $p_k(x, y, z)$, $k \geq 1$, is the $k\Delta^2$-simplex that is a right triangle with sides of length $\sqrt{2}\,k$.*

3 Some open questions

The following questions seem interesting:

(1) Let $f = f(x_1, x_2, \dots, x_n) \in \mathbb{Z}[x_1, x_2, \dots, x_n]$ be a symmetric polynomial, N_f is its Newton polyhedron. Let us present f as a polynomial in elementary symmetric polynomial, $f = F[e_1, \dots, e_n] \in \mathbb{Z}[e_1, \dots, e_n]$ and construct its Newton polyhedron N_F. What is the relation between N_f and N_F?

(2) Let $f = f(x_1, x_2, \dots, x_n) \in \mathbb{Z}[x_1, x_2, \dots, x_n]$ be a symmetric polynomial of degree k, which does not contain x_1^k. What can we say about its Newton polyhedron N_f?

(3) Since there is a homomorphism of one multivalued group to another multivalued group and the kernel of this homomorphism is defined, we can talk about extensions of multivalued groups. Construct a theory of extensions of multivalued groups.

(4) Is it possible to define (co)homology for multivalued groups?

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