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MINIMALITY CONDITIONS, TOPOLOGIES, AND RANKS FOR SPHERICALLY ORDERED THEORIES

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Abstract. The class of ordered structures is productively studied both in order to classify them and in various applications connected with comparing of objects and information structuring. Important particular kinds of ordered structures are represented by o-minimal, weakly o-minimal and circularly minimal ones as well as their variations including definable minimality. We show that the well developed powerful theory for o-minimality, circular minimality, and definable minimality is naturally spread for the spherical case. Reductions of spherical orders to linear ones, called the linearizations, and back reconstructions, called the spherifications, are examined. Neighbourhoods for spherically ordered structures and their topologies are studied. It is proved that related topological spaces can be T_0-spaces, T_1-spaces and Hausdorff ones. These cases are characterized by the cardinality estimates of the universe. Definably minimal linear orders, their definably minimal extensions and restrictions as well as spherical ones are described. The notion of convexity rank is generalized for spherically ordered theories, and values for the convexity rank are realized in weakly spherically minimal theories which are countably categorical.

Keywords: spherical order, weak spherical minimality, definable minimality, topology, convexity rank.

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1. Introduction

Ordered structures and their theories are broadly used in Mathematics and applications both for studying various structural aspects of reality and for ranking objects of different nature in order to compare and structure them. There are many classification results for partially [1, 2, 3, 4], linearly [5], circularly [6, 7] ordered structures, their elementary theories [7, 8] and automorphism groups [9, 10]. There is a deep theory classifying o-minimal structures [16, 11], weakly o-minimal structures [17] and their variations [18, 19, 20].

Recall [16] that a linearly ordered structure $\mathcal{M} = \langle M, =, <, \ldots \rangle$ is said to be *o-minimal* if any definable (with parameters) subset of $\mathcal{M}$ is the union of finitely many intervals and points in $\mathcal{M}$. We also recall [17] that such a structure $\mathcal{M}$ is *weakly o-minimal* if any definable (with parameters) subset of M is a finite union of convex sets in $\mathcal{M}$.

In the present paper we propose an approach classifying spherical generalizations of o-minimal and weakly o-minimal theories and structures via minimal conditions studied in [7, 8, 21] and of related topologies [23] connected with tame ones [11].

The paper is organized as follows. Preliminary notions on circular and spherical orders, and topological terminology are represented in Section 2. In Section 3, notions on spherical minimality are introduced, links between linear and spherical orders are described. Topological properties related to spherical orders are studied in Section 4. In Section 5, we describe links between linear and spherical definable minimalities. Convexity rank and its realizations for spherically ordered theories are described in Section 6.

2. Preliminaries

We denote by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$ structures and by $A, B, C, \ldots$ correspondent universes of these structures. By $\mathrm{Th}(\mathcal{A})$ we denote the complete first-order theory of the structure $\mathcal{A}$. Throughout we use standard model-theoretic [24, 25, 26] and topological [23] notions and notations.

Recall [7, 30, 31] that a *circular*, or *cyclic* order relation is described by a ternary relation K_3 satisfying the following conditions:
(co1) $\forall x \forall y \forall z (K_3(x, y, z) \to K_3(y, z, x))$;
(co2) $\forall x \forall y \forall z (K_3(x, y, z) \land K_3(y, x, z) \leftrightarrow x \approx y \lor y \approx z \lor z \approx x)$;
(co3) $\forall x \forall y \forall z (K_3(x, y, z) \to \forall t [K_3(x, y, t) \lor K_3(x, t, z) \lor K_3(t, y, z)])$;
(co4) $\forall x \forall y \forall z (K_3(x, y, z) \lor K_3(y, x, z))$.

In fact circular orders are obtained from linear ones by their representations on circles.

Following [33] for a natural number $n \geq 1$, a formula $\varphi(\overline{x})$ of a theory T is called *n-ary*, or an *n-formula*, if $\varphi(\overline{x})$ is T-equivalent to a Boolean combination of T-formulae, each of which is of n free variables.

For a natural number $n \geq 2$, an elementary theory T is called *n-ary*, or an *n-theory*, if any T-formula $\varphi(\overline{x})$ is n-ary.

A theory T is called *binary* if T is 2-ary, it is called *ternary* if T is 3-ary, etc.

We will admit the case $n = 0$ for n-formulae $\varphi(\overline{x})$. In such a case $\varphi(\overline{x})$ is just T-equivalent to a sentence $\forall \overline{x} \varphi(\overline{x})$.

If T is a theory such that T is n-ary and not $(n-1)$-ary then the value n is called the arity of T and it is denoted by $\mathrm{ar}(T)$. If T does not have any arity we put $\mathrm{ar}(T)=\infty$.

Similarly, for a formula φ of a theory T we denote by $\mathrm{ar}_T(\varphi)$ the natural value n if φ is n-ary and not $(n-1)$-ary. If a theory T is fixed we write $\mathrm{ar}(\varphi)$ instead of $\mathrm{ar}_T(\varphi)$.

Clearly, $\mathrm{ar}(K_3(x,y,z))=3$ if the relation has at least three element domain, i.e., $K_3(x,y,z)$ is not reduced to Boolean combinations of formulae with at most two free variables. Hence, theories with infinite circular order relations are at least 3-ary.

The following generalization of a circular order produces a *n-ball*, or *n-spherical*, or *n-circular* order relation [33, 34, 35], for $n\geq 3$, which is described by a n-ary relation K_n satisfying the following conditions:

(nso1) $\forall x_1,\ldots,x_n(K_n(x_1,x_2,\ldots,x_n)\to K_n(x_2,\ldots,x_n,x_1))$;

(nso2) $\forall x_1,\ldots,x_n\Bigg((K_n(x_1,\ldots,x_i,\ldots,x_j,\ldots,x_n)\wedge$

$$\wedge K_n(x_1,\ldots,x_j,\ldots,x_i,\ldots,x_n))\leftrightarrow\bigvee_{1\leq k<l\leq n}x_k\approx x_l\Bigg)$$

for any $1\leq i<j\leq n$;

(nso3) $\forall x_1,\ldots,x_n\Bigg(K_n(x_1,\ldots,x_n)\to$

$$\to\forall t\left(\bigvee_{i=1}^{n}K_n(x_1,\ldots,x_{i-1},t,x_{i+1},\ldots,x_n)\right)\Bigg);$$

(nso4) $\forall x_1,\ldots,x_n(K_n(x_1,\ldots,x_i,\ldots,x_j,\ldots,x_n)\vee$

$$\vee K_n(x_1,\ldots,x_j,\ldots,x_i,\ldots,x_n)),\ 1\leq i<j\leq n.$$

Clearly, the axioms for n-spherical orders can be naturally adapted for 2-spherical, i.e., linear ones producing a linear order K_2. Here (nso2) gives the reflexivity: $\forall xK_2(x,x)$, and the antisymmetry: $\forall x_1,x_2(K_2(x_1,x_2)\wedge K_2(x_2,x_1)\to x_1\approx x_2)$, (nso1) is replaced by the transitivity:

$$\forall x_1,x_2,x_3(K_2(x_1,x_2)\wedge K_2(x_2,x_3)\to K_2(x_1,x_3)),$$

and the axioms (nso3) and (nso4) give the linearity:

$$\forall x_1,x_2(K_2(x_1,x_2)\vee K_2(x_2,x_1)).$$

The only case $n=2$, i.e. a linear order, can admit endpoints, since for the cases $n\geq 3$ each element lays between other ones.

Structures $\mathcal{A}=\langle A,K_n\rangle$ with n-spherical orders K_n, where the domain A is the set of all coordinates for tuples in K_n, will be called *n-spherical orderings*, or *n-spherical orders*, too.

We have $\mathrm{ar}(K_n(x_1,\ldots,x_n))=n$ for spherical orders with at least n-elements domains [33] producing at least n-ary theories with infinite n-spherical order relations.

Remark 2.1. Along with a n-spherical order K_n in a structure $\mathcal{A} = \langle A, K_n\rangle$ a *dual* n-spherical order $\overline{K}_n$ is definable which is defined by the formula

$$\neg K_n(x_1, \dots, x_n) \vee \bigvee_{1\le k<l\le n} x_k \approx x_l.$$

In particular, any linear order $\le = K_2$ has the dual linear order $\ge = \overline{K}_2$, any circular order $\le = K_3$ has the dual circular order $\overline{K}_3$, etc.

Definition [23]. A *topological space* is a pair $(X, \mathcal{O})$ consisting of a set X and a family $\mathcal{O}$ of *open* subsets of X satisfying the following conditions:
(O1) $\varnothing \in \mathcal{O}$ and $X \in \mathcal{O}$;
(O2) If $U_1 \in \mathcal{O}$ and $U_2 \in \mathcal{O}$ then $U_1 \cap U_2 \in \mathcal{O}$;
(O3) If $\mathcal{O}' \subseteq \mathcal{O}$ then $\cup\mathcal{O}' \in \mathcal{O}$.

Definition [23]. A topological space $(X, \mathcal{O})$ is a T_0*-space* if for any pair of distinct elements $x_1, x_2 \in X$ there is an open set $U \in \mathcal{O}$ containing exactly one of these elements.

Definition [23]. A topological space $(X, \mathcal{O})$ is a T_1*-space* if for any pair of distinct elements $x_1, x_2 \in X$ there is an open set $U \in \mathcal{O}$ such that $x_1 \in U$ and $x_2 \notin U$.

Definition [23]. A topological space $(X, \mathcal{O})$ is a T_2*-space*, or *Hausdorff* if for any pair of distinct points $x_1, x_2 \in X$ there are open sets $U_1, U_2 \in \mathcal{O}$ such that $x_1 \in U_1$, $x_2 \in U_2$, and $U_1 \cap U_2 = \varnothing$.

3. Linear and spherical orders and their minimalities

The following assertion connects linear orders and n-spherical orders.

Fact 1. (cf. [7, Fact 2.2], [32, Theorem 11.9])
(i) *If $\langle M, \le\rangle$ is a linear ordering and*

$$K_n = \{(a_{i_1}, a_{i_2}, \dots a_{i_n}) \mid a_i = a_j \textit{ for some } i \ne j,$$

$$\textit{or } a_1 < a_2 < \dots < a_n \textit{ and } (i_1, i_2, \dots, i_n) \textit{ is obtained}$$

$$\textit{by an even permutation of } (1, 2, \dots, n)\},$$

then K_n is a definable n-spherical order on M.
(ii) *If $\langle M, K_n\rangle$ is a n-spherical ordering and $a \in M$, then the relation $K_{n-1,a}$ defined on $M_a \rightleftharpoons M \setminus \{a\}$ by the rule*

$$K_{n-1,a}(x_1, \dots, x_{n-1}) \rightleftharpoons K_n(a, x_1, \dots, x_{n-1})$$

is a $(n-1)$-spherical order. Moreover, the relations $K_{n-m,a_1,\dots,a_m}$ defined on

$$M_{a_1,\dots,a_m} \rightleftharpoons M \setminus \{a_1, \dots, a_m\},$$

for pairwise distinct $a_1, \dots, a_m \in M$, by the rule

$$K_{n-m,a_1,\dots,a_m}(x_1, \dots, x_{n-m}) \rightleftharpoons K_n(a_1, \dots, a_m, x_1, \dots, x_{n-m})$$

are $(n-m)$-spherical orders including the linear order $K_{2,a_1,\dots,a_{n-2}}(x_1, x_2)$, where $n - m \ge 2$. Furthermore, extending this linear order to one, denoted by $\le'$ and by $\le'_{a_1,\dots,a_{n-2}}$, on M by specifying that $a_1 \le' \dots \le' a_{n-2} \le' b$ for all $b \in M_{a_1,\dots,a_m}$ then the derived n-spherical order relation is the original n-spherical order K_n.

Definition. The orders $K_{n-m,a_1,\ldots,a_m}(x_1,\ldots,x_{n-m})$ are called the $(n-m)$-*reductions* of the n-spherical order K_n, with respect to $a_1,\ldots,a_m$, and the 2-reductions $K_{2,a_1,\ldots,a_{n-2}}(x_1,x_2)$ are called the *linearizations* of K_n (with respect to $a_1,\ldots,a_{n-2}$), denoted by $\widehat{L}(K_n,a_1,\ldots,a_{n-2})$ or simply $\widehat{L}(K_n)$.

Thus we obtain the operations reducing n-spherical orders to linear ones. Converse actions transforming linear orders L to n-spherical orders are called the n-*spherifications* $\widehat{K}_n(L)$.

The following identities hold, with respect to fixed elements $a_1,\ldots,a_{n-2}$:

$$\widehat{L}(\widehat{K}_n(L))=L, \widehat{K}_n(\widehat{L}(K_n))=K_n.$$

Clearly, the operators $\widehat{L}$ and $\widehat{K}_n$ preserve the structural complexity, in particular, the ω-categoricity and rank values for definable sets.

Definition. Let $A\subseteq M$, where $\mathcal{M}=\langle M,K_n,\ldots\rangle$ is a n-spherically ordered structure. The set A is called *convex* if for any $a_1,a_2,\ldots,a_{n-1}\in A$ the following holds: for any $b\in M$ with $\models K_n(a_1,\ldots,a_{n-1},b)$ we have $b\in A$ or for any $b\in M$ with $\models\neg K_n(a_1,\ldots,a_{n-1},b)$ we have $b\in A$.

Notation. 1. For a linearly ordered structure $\mathcal{M}=\langle M,\leq,\ldots\rangle$ we denote by $s_n(\mathcal{M})$ the structure $\langle M,K_n,\ldots\rangle$, where we replace the linear order $\leq$ by a n-ary relation K_n, which is derived from $\leq$ following Fact 1(i). Following [7] the operator $s_3(\cdot)$ is denoted by $c(\cdot)$.

By Fact 1 we also introduce structures $s_n^m(\mathcal{M})$, for $m\leq n$, where the predicate K_m for $\mathcal{M}$ is replaced by the predicate K_n. In particular, the operator $s_3^2(\cdot)$ produces a circular order for a given linear order.

2. Let $\mathcal{M}_0,\mathcal{M}_1,\ldots,\mathcal{M}_{k-1}$ be linear orderings. Replacing $\leq$ by K_n as above we denote by $s_n(\mathcal{M}_0+\mathcal{M}_1+\ldots+\mathcal{M}_{k-1})$ the n-spherically ordered sum $\mathcal{M}$ of $\mathcal{M}_0,\mathcal{M}_1,\ldots,\mathcal{M}_{k-1}$ such that for each $0\leq i\leq k-1$, considered $(\mathrm{mod}k)$, $\mathcal{M}_i$ is the immediate predecessor of $\mathcal{M}_{i+1}$: that is, for $a_1,\ldots,a_j\in M_i$, $a_{j+2},\ldots,a_n\in M_{i+1}$, and $b\in M$, if $\models K_n(a_1,\ldots,a_j,b,a_{j+2},\ldots,a_n)$, then $b\in M_i\cup M_{i+1}$.

3. We put $K_n^0(x_1,\ldots,x_n)\rightleftharpoons K_n(x_1,\ldots,x_n)\wedge\bigwedge\limits_{i\neq j}\neg x_i\approx x_j$.

Remark 1. Notice that the operator s_n^2 differs from n-spherification $\widehat{K}_n$ since in the second case $n-2$ new elements are added. At the same time the results of applications of operators s_n^2 and $\widehat{K}_n$ can be isomorphic or non-isomorphic depending on positions of new elements $a_1,\ldots,a_{n-2}$.

Indeed, $s_3^2(\omega+\omega^*)$ is isomorphic to $\widehat{K}_3(\omega+\omega^*)$ if new element a_1 is situated between elements of ω or of ω^*, and these structures are not isomorphic if new element a_1 is greater than ω and less than ω^*, obtaining $\widehat{K}_3(\omega+\omega^*)=c(\omega+1+\omega^*)$.

Definition. Let $S\subseteq M$, where $\mathcal{M}=\langle M,K_n,\ldots\rangle$ is a n-spherically ordered structure. The set S is said to be an *open n-segment surface* if $S=\{b\in M\mid \mathcal{M}\models K_n^0(a_1,b,a_3,\ldots,a_n)\}$ for some pairwise distinct $a_1,a_3,\ldots,a_n\in M$. It has endpoints $a_1,a_3,\ldots,a_n$. For an open n-segment surface S of a n-spherically ordered set, sometimes we will write $S=(a_1,a_3,\ldots,a_n)$ if we wish to indicate the endpoint frame of S. Similarly, we may define *closed*, *partially open*, etc., n-segment surfaces in $\mathcal{M}$ including all/some coordinates a_i. By a *n-segment surface* in $\mathcal{M}$ we shall mean, ambiguously, any of the above types of n-segment surfaces in $\mathcal{M}$.

It is obvious that both a n-segment surface and a point are convex sets.

Definition. Let $\mathcal{M} = \langle M, K_n, \ldots\rangle$ be a n-spherically ordered structure. The structure $\mathcal{M}$ is said to be *n-spherically minimal* if any definable (with parameters) subset of M is a positive Boolean combination of segment surfaces and points in $\mathcal{M}$. The structure $\mathcal{M}$ is said to be *weakly n-spherically minimal* if any definable (with parameters) subset of M is a finite union of convex sets.

A complete theory T is said to be (*weakly*) *n-spherically minimal* if all its models are (weakly) n-spherically minimal.

Following [7] for $n = 3$ we say on circular minimality instead of 3-spherical one.

The following proposition immediately follows by Fact 1.

Proposition 1. *Let $\mathcal{M} = \langle M, K_n, \ldots\rangle$ be a (weakly) n-spherically minimal structure, and $a_1, \ldots, a_{n-2}$ be arbitrary pairwise distinct elements of M. Take the relation $\leq = \leq_{a_1,\ldots,a_{n-2}} = K_{n-2,a_1,\ldots,a_{n-2}}$ on*

$$M_{a_1,\ldots,a_{n-2}} = M \setminus \{a_1, \ldots, a_{n-2}\}$$

and $\leq' = \leq'_{a_1,\ldots,a_{n-2}}$ on M as in Fact 1.

(i) *Let $\mathcal{M}_{a_1,\ldots,a_{n-2}}$ be the structure with domain $M_{a_1,\ldots,a_{n-2}}$, the order $\leq$, and a relation symbol for each $\{a_1, \ldots, a_{n-2}\}$-definable relation of $\mathcal{M}$ on powers of $M_{a_1,\ldots,a_{n-2}}$. Then $\mathcal{M}_{a_1,\ldots,a_{n-2}}$ is a (weakly) o-minimal structure.*

(ii) *Let $\mathcal{M}'_{a_1,\ldots,a_{n-2}}$ be the structure with domain M, the order $\leq'$, and a relation symbol for each $\{a_1, \ldots, a_{n-2}\}$-definable relation of $\mathcal{M}$ on powers of M. Then $\mathcal{M}'_{a_1,\ldots,a_{n-2}}$ is (weakly) o-minimal.*

Since o-minimal structures have o-minimal theories, Proposition 1 implies:

Corollary 1. *Any n-spherically minimal structure has n-spherically minimal theory.*

By the definition of $s_n(\mathcal{M})$ we have:

Proposition 2. *Let $\mathcal{M} = \langle M, \leq, \ldots\rangle$ be a weakly o-minimal structure. Then $s_n(\mathcal{M})$ is a weakly n-spherically minimal structure.*

The following examples illustrate possibilities of spherical minimality.

Example 1. (cf. [7, Example 2.9]) Taking an n-dimensional sphere $S \subseteq \mathbb{R}^n$ of radius 1 and a n-spherical order K_n on S preserving orientations of directed n-tetrahedrons we obtain the rotation group $\mathrm{SO}(n)$ which preserves K_n. The expanded group $\mathcal{G} = \langle \mathrm{SO}(n), K_n\rangle$ is *n-spherically ordered.* Since K_n is dense it is ω-categorical, with quantifier elimination in view of [35, Section 4]. Similarly [18, Theorem 5.1] the structure $\mathcal{G}$ is n-spherically minimal.

Example 2. (cf. [7, Example 2.10]) Let $\mathcal{R}$ be an o-minimal expansion of a real closed field $\langle R, \leq, +, \cdot\rangle$. For an ultrapower ${}^*\mathcal{R}$ of $\mathcal{R}$ and a set $V := \{x \in {}^*\mathcal{R} \mid |x| < n$ for some $n \in \omega\}$ the structure $\langle {}^*\mathcal{R}, V\rangle$ is weakly o-minimal by [36]. Replacing the relation $<$ by a correspondent n-spherical order K_n on ${}^*\mathcal{R}$ and using Proposition 2, we obtain a weakly n-spherically minimal structure $\langle {}^*\mathcal{R}, V, K_n\rangle$.

Example 3. It is known [16, 37] that o-minimal structures $\langle M, \leq\rangle$ are elementary equivalent to ordered sets of form $C_1 + \ldots + C_m$, where C_i are elementary equivalent to one of the following ordered sets: a finite ordered set, ω, ω^*, $\omega + \omega^*$, $\omega^* + \omega$, $\mathbb{Q}$, and if C_i does not have a last element, then C_{i+1} has a first element.

Generalizing constructions in [7] and using Proposition 1 we spread possibilities for o-minimal structures $\mathcal{M}$ both for circularly minimal structures $c(\mathcal{M}) = s_3(\mathcal{M})$ and for n-spherically minimal structures $s_n(\mathcal{M})$, $n \geq 4$.

Example 2.12 in [7] illustrates that for the circularly minimal structure $c(\omega + \omega^* + \mathbb{Q} + \omega + \omega^* + \mathbb{Q})$, $\mathrm{acl}(\emptyset)$ is a proper superset of $\mathrm{dcl}(\emptyset)$ contrasting with the o-minimal case, where $\mathrm{acl}(A) = \mathrm{dcl}(A)$ for any subset A of a universe.

Similarly, the n-spherical structures, for $n \geq 4$, produce algebraic closures $\mathrm{acl}(\emptyset)$ with $\mathrm{acl}(\emptyset) \supsetneqq \mathrm{dcl}(\emptyset)$. These structures are based on possibilities of distributions of discrete suborders, of dense n-spherical orders described in [35] with respect to finitely many accumulation points for "increasing/decreasing" sequences of elements generalizing the order $\omega + \omega^*$.

4. Topologies for spherically ordered theories

Definition. For a n-spherically ordered structure $\mathcal{M} = \langle M, K_n, \ldots \rangle$, $n \geq 4$, and for elements $a_1, \ldots, a_n \in M$ we say that a_n lays *between* $a_1, \ldots, a_{n-1}$, or *inside* with respect to $a_1, \ldots, a_{n-1}$ if

$$\mathcal{M} \models K_n^0(a_1, \ldots, a_{n-1}, a_n). \tag{1}$$

If $n = 2$ then the relation (1) denotes that a_2 is *greater* than a_1, and for $n = 3$ that relation denotes that a_2 lays *between* a_1 and a_3, or *inside* with respect to a_1, a_3.

Similarly we say that a_i lays *between* or *inside* with respect to $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$ if a_i lays *between* or *inside* with respect to the circular permutation $a_{i+1}, \ldots, a_n, a_1, \ldots, a_{i-1}$ of $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$.

If a_i does not lay between $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$ we say that a_i lays *outside* with respect to $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$.

By the axiom (nso1) of circular permutations the general case of (non-)betweenness is reduced to that relation with respect to $a_1, \ldots, a_{n-1}$.

For a sequence $a_1, \ldots, a_{n-1}$ of pairwise distinct elements the set $\mathrm{Int}(a_1, \ldots, a_{n-1})$ (respectively, $\mathrm{Ext}(a_1, \ldots, a_{n-1})$) of all elements inside (outside) with respect to $a_1, \ldots, a_{n-1}$ is called the *interior* (*exterior*) with respect to $a_1, \ldots, a_{n-1}$.

The sets $\mathrm{Int}(a_1, \ldots, a_{n-1})$ and $\mathrm{Ext}(a_1, \ldots, a_{n-1})$ are considered as neighbourhoods U_{int} and U_{ext} of elements belonging to them.

By the definition we have $U_{\mathrm{int}} \cap U_{\mathrm{ext}} = \emptyset$ for the neighbourhoods U_{int} and U_{ext} with respect to a fixed sequence $a_1, \ldots, a_{n-1}$.

Assuming $|M| \geq n$ each element $a \in M$ belongs to some neighbourhoods U_{int} and U_{ext} with respect to elements in $M \setminus \{a\}$. Thus the family $\mathcal{O}$ generated by the neighbourhoods with respect to unions and finite intersections produces a topological space $\mathcal{X}_{\mathrm{Ext}}^{\mathrm{Int}}(\mathcal{M}) = (M, \mathcal{O})$. It is called *$n$-spherical*.

The n-spherical topological space $\mathcal{X}_{\mathrm{Ext}}^{\mathrm{Int}}(\mathcal{M})$ has two natural restrictions $\mathcal{X}^{\mathrm{Int}}(\mathcal{M})$ and $\mathcal{X}_{\mathrm{Ext}}(\mathcal{M})$ generated by neighbourhoods U_{int} and U_{ext}, respectively. These restrictions are called *n-spherical*, too.

Remark 2. Since neighbourhoods U_{int} and U_{ext} with respect to a fixed sequence $a_1, \ldots, a_{n-1}$ are disjoint and $U_{\mathrm{int}} \cup U_{\mathrm{ext}} = M \setminus \{a_1, \ldots, a_{n-1}\}$ these neighbourhoods are complement each other. Thus $\mathcal{X}^{\mathrm{Int}}(\mathcal{M})$ and $\mathcal{X}_{\mathrm{Ext}}(\mathcal{M})$ are dual: one of them is uniquely defined by another one, via complements. Moreover, both $\mathcal{X}^{\mathrm{Int}}(\mathcal{M})$ and $\mathcal{X}_{\mathrm{Ext}}(\mathcal{M})$, separately and together, define $\mathcal{X}_{\mathrm{Ext}}^{\mathrm{Int}}(\mathcal{M})$.

Proposition 3. *For any n-spherically ordered structure $\mathcal{M}$ with $|M| \geq n$ the space $\mathcal{X}^{\rm Int}_{\rm Ext}(\mathcal{M})$ is Hausdorff.*

Proof. Let a and b be two distinct elements in M.

If $n = 2$ then we have $a < b$ or $b < a$. Without loss of generality we assume that $a < b$. Having $c \in M$ with $a < c < b$ we obtain disjoint neighbourhoods ${\rm Int}(c)$ and ${\rm Ext}(c)$ separating $a \in {\rm Ext}(c)$ and $b \in {\rm Int}(c)$. If c does not exist, i.e., b is the successor of a we take separating neighbourhoods ${\rm Int}(a)$ and ${\rm Ext}(b)$ with $a \in {\rm Ext}(b)$ and $b \in {\rm Int}(a)$. Thus $\mathcal{X}^{\rm Int}_{\rm Ext}(\mathcal{M})$ is Hausdorff.

Now let $n \geq 3$. We fix pairwise distinct elements $d_1, \ldots, d_{n-2} \in M \setminus \{a, b\}$. We have two possibilities for a and b with respect to $d_1, \ldots, d_{n-2}$:

$$\models K^0_n(d_1, \ldots, d_{n-2}, a, b),$$

or $\models \neg K^0_n(d_1, \ldots, d_{n-2}, a, b)$, i.e., $\models K^0_n(d_1, \ldots, d_{n-2}, b, a)$. Following Fact 1 the relation $K^0_n(d_1, \ldots, d_{n-2}, x, y)$ defines a linear order $<$ on $M \setminus \{d_1, \ldots, d_{n-2}\}$ implying $a < b$ or $b < a$. Now repeating the arguments for $n = 2$ we obtain a pair of disjoint neighbourhoods ${\rm Int}(d_1, \ldots, d_{n-2}, c)$ and ${\rm Ext}(d_1, \ldots, d_{n-2}, c)$, or ${\rm Int}(d_1, \ldots, d_{n-2}, a)$ and ${\rm Ext}(d_1, \ldots, d_{n-2}, b)$, or ${\rm Ext}(d_1, \ldots, d_{n-2}, a)$ and ${\rm Int}(d_1, \ldots, d_{n-2}, b)$, separating a and b and witnessing the Hausdorffness of $\mathcal{X}^{\rm Int}_{\rm Ext}(\mathcal{M})$.

Remark 3. Any 2-spherically ordered structure $\mathcal{M}$ with $|M| \geq 2$, i.e., a linearly ordered structure with an order $K_2 = \leq$ of cardinality at least 3, produces the T_0-spaces $\mathcal{X}^{\rm Int}(\mathcal{M})$ and $\mathcal{X}_{\rm Ext}(\mathcal{M})$ which are not T_1-spaces. Indeed, there are $a_1, a_2 \in M$ with $a_1 < a_2$, and for any $a \in M$ we have ${\rm Int}(a) = \{b \in M \mid a < b\}$, ${\rm Ext}(a) = \{b \in M \mid b < a\}$. Thus, the neighbourhoods $U_{\rm int}$, respectively $U_{\rm ext}$, form chains isomorphic to $<$ and witnessing that $\mathcal{X}^{\rm Int}(\mathcal{M})$ and $\mathcal{X}_{\rm Ext}(\mathcal{M})$ are T_0-spaces, not T_1-ones. In particular, a_2 is separated from a_1 by ${\rm Int}(a_1)$ but a_1 is not separated from a_2 by interiors. Similarly, a_1 is separated from a_2 by ${\rm Ext}(a_2)$ but a_2 is not separated from a_1 by exteriors.

Remark 4. Any 3-spherically ordered structure $\mathcal{M}$ with $|M| \geq 3$, i.e., a circularly ordered structure with an order K_3, produces the T_2-spaces $\mathcal{X}^{\rm Int}(\mathcal{M})$ and $\mathcal{X}_{\rm Ext}(\mathcal{M})$. Indeed, taking distinct elements $a_1, a_2 \in M$ we have either $\models K^0_3(a_1, a_2, b)$ or $\models K^0_3(a_2, a_1, b)$ for some b.

In the first case we have $a_2 \in {\rm Int}(a_1, b)$. If there is an element c between a_1 and a_2 we choose an element d between b and a_1, or taking $d = b$ if there are no elements between b and a_1, producing distinct neighbourhoods ${\rm Int}(d, c)$ and ${\rm Int}(c, b)$ containing a_1 and a_2, respectively, and witnessing that $\mathcal{X}^{\rm Int}(\mathcal{M})$ is Hausdorff. If there are no elements between a_1 and a_2 we repeat the choice of d and obtain distinct neighbourhoods ${\rm Int}(d, a_2)$ and ${\rm Int}(a_1, b)$ containing a_1 and a_2, respectively, witnessing again the Hausdorffness of $\mathcal{X}^{\rm Int}(\mathcal{M})$.

In the second case we have similarly $a_1 \in {\rm Int}(a_2, b)$ and there are $c, d \in M$ with distinct neighbourhoods ${\rm Int}(d, c)$ and ${\rm Int}(c, b)$ containing a_2 and a_1, respectively, or distinct neighbourhoods ${\rm Int}(d, a_1)$ and ${\rm Int}(a_2, b)$ containing a_2 and a_1, respectively, and witnessing again the Hausdorffness of $\mathcal{X}^{\rm Int}(\mathcal{M})$.

The Hausdorffness of $\mathcal{X}_{\rm Ext}(\mathcal{M})$ is implied by its duality with respect to the Hausdorff space $\mathcal{X}^{\rm Int}(\mathcal{M})$.

Remark 5. Any n-spherically ordered structure $\mathcal{M}$ with $|M| \geq n \geq 4$ produces the T_2-spaces $\mathcal{X}^{\rm Int}(\mathcal{M})$ and $\mathcal{X}_{\rm Ext}(\mathcal{M})$. Indeed, taking distinct elements $a_1, a_2 \in M$ we choose pairwise distinct elements $b_1, \ldots, b_{n-3} \in M \setminus \{a_1, a_2\}$ and obtain a circular order $K_n^0(b_1,\dots,b_{n-3},x,y,z)$ on $M\setminus\{a_1,a_2\}$ implying the Hausdorffness for $\mathcal{X}^{\rm Int}(\mathcal{M})$ and $\mathcal{X}_{\rm Ext}(\mathcal{M})$ in view of Remark 4.

Remark 6. Any n-spherically ordered structure $\mathcal{M}$ with $|M|<n$ produces the spaces $\mathcal{X}^{\rm Int}(\mathcal{M})$, $\mathcal{X}_{\rm Ext}(\mathcal{M})$, $\mathcal{X}^{\rm Int}_{\rm Ext}(\mathcal{M})$ which are not T_0-spaces. Indeed, if $|M|<n$ then $K_n=M^n$ implying that there are no separated elements at all.

Summarizing the assertions and remarks above we obtain the following theorems.

Theorem 1. *For any n-spherically ordered structure $\mathcal{M}$ the following conditions are equivalent:*
(1) $\mathcal{X}^{\rm Int}(\mathcal{M})$ *is a T_0-space;*
(2) $\mathcal{X}_{\rm Ext}(\mathcal{M})$ *is a T_0-space;*
(3) $\mathcal{X}^{\rm Int}_{\rm Ext}(\mathcal{M})$ *is Hausdorff;*
(4) $|M|\geq n$.

Proof. (1) $\Rightarrow$ (4), (2) $\Rightarrow$ (4), (3) $\Rightarrow$ (4) follow by Remark 6. (4) $\Rightarrow$ (1) and (4) $\Rightarrow$ (2) are implied by Remark 3, (4) $\Rightarrow$ (3) holds by Proposition 3.

Theorem 2. *For any n-spherically ordered structure $\mathcal{M}$ the following conditions are equivalent:*
(1) $\mathcal{X}^{\rm Int}(\mathcal{M})$ *is a T_1-space;*
(2) $\mathcal{X}_{\rm Ext}(\mathcal{M})$ *is a T_1-space;*
(3) $\mathcal{X}^{\rm Int}(\mathcal{M})$ *is Hausdorff;*
(4) $\mathcal{X}_{\rm Ext}(\mathcal{M})$ *is Hausdorff;*
(5) $|M|\geq n\geq 3$.

Proof. (1) $\Rightarrow$ (5), (2) $\Rightarrow$ (5), (3) $\Rightarrow$ (5), (4) $\Rightarrow$ (5) follow by Remark 6. (5) $\Rightarrow$ (3) and (5) $\Rightarrow$ (4) are implied by Remarks 4 and 5, and (3) $\Rightarrow$ (1), (4) $\Rightarrow$ (2), (1) $\Leftrightarrow$ (2) hold by the definition.

5. Definable minimality for linearly ordered and spherically ordered theories

Definition [21, 8]. A structure $\mathcal{M}$ is called *definably minimal* if each definable, with parameters, subset of M is either finite or cofinite.

Definition [22]. A structure $\mathcal{M}$ is called *strongly minimal* if each structure $\mathcal{N}\equiv\mathcal{M}$ is definably minimal.

We denote by $\mathrm{Mod}(T)$ the class of all T-models and by $\mathrm{Min}(T)$ the class of all definably minimal T-models.
By the definition we have the following:

Proposition 4. *A theory T is strongly minimal if and only if* $\mathrm{Mod}(T)=\mathrm{Min}(T)$.

The linear order ω represents a definably minimal structure which is not strongly minimal, since the theory $\mathrm{Th}(\omega)$ is unstable, with the strict order property.
Following [21, 8] definably minimal linear orders are exhausted by expansions of m, ω, ω^*, $\omega+\omega^*$, $\omega+m$, $m+\omega^*$ for natural m.
Here the only orders m are minimal models of their theories, under inclusion, since $\omega\simeq 1+\omega$ and $\omega^*\simeq\omega^*+1$. Thus unstable definably minimal linear orders do not have minimal models under inclusion. These minimal models and their hypergraphs are studied both in general context [27] and for theories of abelian groups [28, 29].

Using the description of definably minimal linear orders we observe that there are no uncountable definably minimal linear orders whereas by Löwenheim-Skolem theorem strongly minimal theories with infinite models have definably minimal models of arbitrarily large infinite cardinality.

The following theorem gives a description for definably minimal extensions and restrictions of definably minimal linear orders.

Theorem 3. *For any linear definably minimal order $\mathcal{M}$ a linear order $\mathcal{N} \supset \mathcal{M}$ with finite $N \setminus M$ is definably minimal if and only if $\mathcal{N}$ does not have the form $\omega + m + \omega^*$ for some natural $m > 0$.*

Proof. Let $\mathcal{N} \supset \mathcal{M}$ be a linear order with finite $Z = N \setminus M$. We have the following possibilities:

i) $\mathcal{M} = m \in \omega$ and elements of Z are situated before, between and/or after elements of m producing the finite definably minimal linear order with $m + |Z|$ elements;

ii) $\mathcal{M} = \omega$ and elements of Z are situated before, between and/or after elements of ω obtaining the definably minimal order $\omega + m$, where $m > 0$ iff there are elements of Z after all elements of ω;

iii) $\mathcal{M} = \omega^*$ and elements of Z are situated before, between and/or after elements of ω^* obtaining the definably minimal order $m + \omega^*$, where $m > 0$ iff there are elements of Z before all elements of ω^*;

iv) $\mathcal{M} = \omega + \omega^*$ and elements of Z are situated before, between and/or after elements of $\omega + \omega^*$ producing either isomorphic definably minimal order $\omega + \omega^*$, if Z does not contain elements greater than all elements of ω and less than all elements of ω^*, or an order $\omega + m + \omega^*$, if Z contains an element a greater than all elements of ω and less than all elements of ω^*; this element a divides M into two infinite parts by the formulae $x < a$ and $a \leq x$ violating the definable minimality;

v) $\mathcal{M} = \omega + m$ and elements of Z are situated before, between and/or after elements of $\omega + m$ obtaining the definably minimal order $\omega + m'$, where $m' \geq m$ and $m' > m$ iff there are elements of Z after all elements of $\omega + m$;

vi) $\mathcal{M} = m + \omega^*$ and elements of Z are situated before, between and/or after elements of $m + \omega^*$ obtaining the definably minimal order $m' + \omega^*$, where $m' \geq m$ and $m' > m$ iff there are elements of Z before all elements of $m + \omega^*$.

The only case iv) with $\omega + m + \omega^*$ fails the definable minimality for $\mathcal{N}$.

Theorem 4. *For any linear definably minimal order $\mathcal{M}$ a linear order $\mathcal{N} \supset \mathcal{M}$ with countable $N \setminus M$ is definably minimal if and only if $\mathcal{N}$ does not contain forms $\omega^* + \omega$, $\omega^* + \omega^*$, $\omega + \omega$, $\omega + m + \omega^*$ for some natural $m > 0$.*

Proof. Clearly each type $\omega^* + \omega$, $\omega^* + \omega^*$, $\omega + \omega$, $\omega + m + \omega^*$, for some natural $m > 0$, fails the definable minimality. Indeed, we have the formulae $x < a$ and $a \leq x$ with infinitely many solutions, where $a \in \omega$ for the type $\omega^* + \omega$, a belongs to the first ω^* in $\omega^* + \omega^*$, a belongs to the second ω in $\omega + \omega$, $a \in m$ for $\omega + m + \omega^*$. Conversely, if $\mathcal{N}$ is definably minimal then taking an arbitrary element $a \in N$ we have either finite $\mathcal{N} < a$ and countable $a < \mathcal{N}$, or countable $\mathcal{N} < a$ and finite $a < \mathcal{N}$.

In the first case for any $b > a$ either there are finitely many elements between a and b and countably many elements greater than b, or there are countably many elements between a and b and finitely many elements greater than b. Therefore $\mathcal{N}$

has the type $m+\omega^*$, if $a\in m$, or ω, for $a<b\in\omega$, or $\omega+m$, if $a\in\omega$, and $b\in\omega$ or $b\in m$, or $\omega+\omega^*$, if $a\in\omega$, and $b\in\omega$ or $b\in\omega^*$.

In the second case we replace $<$ by $>$ and obtain the following possibilities: $m+\omega^*$, ω^*, $\omega+m$, $\omega+\omega^*$.

In any case $\mathcal{N}$ does not contain forms $\omega^*+\omega$, $\omega^*+\omega^*$, $\omega+\omega$, $\omega+m+\omega^*$.

Theorem 5. *For any linear definably minimal order $\mathcal{M}$ each restriction $\mathcal{N}\subset\mathcal{M}$ is definably minimal, too.*

Proof. We have the following possibilities for $\mathcal{N}$:

i) $\mathcal{M}=m\in\omega$ and $\mathcal{N}\simeq m'$ for $m'\leq m$;

ii) $\mathcal{M}=\omega$, and $\mathcal{N}\simeq\omega$, if N is infinite, or $\mathcal{N}\simeq m\in\omega$, if N is finite;

iii) $\mathcal{M}=\omega^*$, and $\mathcal{N}\simeq\omega^*$, if N is infinite, or $\mathcal{N}\simeq m\in\omega$, if N is finite;

iv) $\mathcal{M}=\omega+\omega^*$, and $\mathcal{N}\simeq\omega+\omega^*$, if N has infinitely many elements of ω and infinitely many elements of ω^*; $\mathcal{N}\simeq\omega+m$, if N has infinitely many elements of ω and finitely many elements of ω^*; $\mathcal{N}\simeq m+\omega^*$, if N has finitely many elements of ω and infinitely many elements of ω^*; $\mathcal{N}\simeq m$, if N has finitely many elements of ω and finitely many elements of ω^*;

v) $\mathcal{M}=\omega+m$, and $\mathcal{N}\simeq\omega+m'$ for $m'\leq m$, if N is infinite, or $\mathcal{N}\simeq m'\in\omega$, if N is finite;

vi) $\mathcal{M}=m+\omega^*$, and $\mathcal{N}\simeq m'+\omega^*$ for $m'\leq m$, if N is infinite, or $\mathcal{N}\simeq m'\in\omega$, if N is finite.

Thus all possible restrictions $\mathcal{N}$ of $\mathcal{M}$ are again definably minimal.

Theorems 3, 4, 5 are naturally spread for theories of n-spherical orders, via the n-spherifications.

The following definition modifies the notion of definable minimality for n-spherical theories.

Definition. A n-spherically ordered structure $\mathcal{M}$ is called K_n*-minimal* if for any pairwise distinct elements $a_1,\ldots,a_{n-1}\in M$ either $\mathrm{Int}(a_1,\ldots,a_{n-1})$ or $\mathrm{Ext}(a_1,\ldots,a_{n-1})$ is finite, i.e., M is divided by K_n into a finite and a cofinite parts with respect to any distinct $a_1,\ldots,a_{n-1}$.

We say that $\mathcal{M}$ produces a *limit point*, or an *accumulation point* a, in a model of $\mathrm{Th}(\mathcal{M})$, with respect to K_n if there is a sequence $(\overline{a}_m)_{m\in\omega}$, where each $\overline{a}_m$ consists of $n-1$ pairwise distinct coordinates, and a sequence $(U_m)_{m\in\omega}$ of neighbourhoods of the form $\mathrm{Int}(\overline{a}_m)$ or $\mathrm{Ext}(\overline{a}_m)$ such that the sets $V_m=U_0\cap\ldots\cap U_m$ form a strictly decreasing chain with $\mathrm{tp}(a/M)\ni\varphi_m(x,\overline{a}_m)$, where $\varphi_m(x,\overline{a}_m)$ is a formula $K^0_n(x,\overline{a}_m)$ or $\neg K^0_n(x,\overline{a}_m)\wedge\bigwedge\limits_{a'\in\overline{a}_m}\neg x\approx a'$ with the set U_m of solutions, $m\in\omega$.

Remark 7. Following [21, 8] K_2-minimal structures are exhausted by expansions of m, ω, ω^*, $\omega+\omega^*$, $\omega+m$, $m+\omega^*$ for natural m. In every one of these cases accumulation points are equivalent: they have the same type over the model, i.e., the quotient with respect to the equivalence relation $x\equiv y$ is a singleton. Here the accumulation points are produced by the neighbourhoods $U_{\rm int}$ for ω, $U_{\rm ext}$ for ω^*, $U_{\rm ext}\cap U_{\rm int}$ for $\omega+\omega^*$, $\omega+m$, $m+\omega^*$. Thus each accumulation point a is a limit of an increasing or a decreasing sequence.

The same possibilities replacing the linear orders m, ω, ω^*, $\omega+\omega^*$, $\omega+m$, $m+\omega^*$ by the circular ones $c(\cdot)$ are valid for K_3-minimal structures. In fact we have three non-isomorphic possibilities $c(\omega)$, $c(\omega^*)$, and $c(\omega+\omega^*)$ with accumulation points, since $c(\omega+m) \simeq c(\omega)$ and $c(m+\omega^*) \simeq c(\omega^*)$. Moreover, $c(\omega)$ and $c(\omega^*)$ are mutually dual and connected by an anti-isomorphism between ω and ω^*, and $c(\omega+\omega^*)$ is self-dual: it is anti-isomorphic to itself.

K_n-minimal structures have similar possibilities combining interiors and exteriors. Here for $K_n^0(a_1,\ldots,a_{i-1},x,a_{i+1},\ldots,a_n)$ both some elements a_j may be fixed and some a_k may be moving producing cases above, if $n-2$ elements are fixed, and additional cases with at least two moving points.

For instance, taking $K_4^0(a_1^0,a_2^0,a_3^0,x)$ we can consider a sequence

$$(\langle a_1^m,a_2^m,a_3^m\rangle)_{m\in\omega}$$

with $a_1^{m+1},a_2^{m+1},a_3^{m+1}\in \mathrm{Int}(a_1^m,a_2^m,a_3^m)$ (respectively, with $a_1^{m+1},a_2^{m+1},a_3^{m+1}\in \mathrm{Ext}(a_1^m,a_2^m,a_3^m)$) and an accumulation point $a\in \mathrm{Int}(a_1^m,a_2^m,a_3^m)$ ($a'\in \mathrm{Int}(a_1^m,a_2^m, a_3^m)$), $m\in\omega$, illustrating the K_n-minimalities.

Applying the correspondence between linear and spherical orders described in Fact 1 and Hausdorffness of topology with respect to $\mathcal{X}^{\mathrm{Int}}_{\mathrm{Ext}}(\mathcal{M})$ in Theorem 1 we obtain the following possibilities for K_n-minimal structures with $n\geq 2$.

Theorem 6. *For any infinite n-spherically ordered structure $\mathcal{M}=\langle M,K_n,\ldots\rangle$ the following conditions are equivalent:*

(1) *$\mathcal{M}$ is K_n-minimal;*

(2) *$\mathcal{M}$ produces unique accumulation point with respect to K_n;*

(3) *any linearization $\leq$ of $\mathcal{M}$ is $\leq$-minimal;*

(4) *any linearization $\leq$ of $\mathcal{M}$ produces unique limit point with respect to $\leq$.*

Proof. (1) $\Rightarrow$ (2). Since $\mathcal{M}$ is infinite there are infinitely many neighbourhoods producing at least one accumulation point. If $\mathcal{M}$ produces at least two accumulation points a and b with respect to K_n. Since the topology $\mathcal{X}^{\mathrm{Int}}_{\mathrm{Ext}}(\mathcal{M})$ is Hausdorff by Theorem 1, there are disjoint infinite neighbourhoods U_1 and U_2 separating a and b. Both the formula $K_n(\ldots)$ for U_1 and its negation $\neg K_n(\ldots)$ have infinitely many solutions contradicting the K_n-minimality of $\mathcal{M}$.

(2) $\Rightarrow$ (1). If $\mathcal{M}$ is not K_n-minimal then there are infinite neighbourhoods U_1 and U_2 defined by a formula $K_n(\ldots)$ and its negation $\neg K_n(\ldots)$, respectively. Since the topology $\mathcal{X}^{\mathrm{Int}}_{\mathrm{Ext}}(\mathcal{M})$ is Hausdorff both the neighbourhood U_1 and the neighbourhood U_2 produce accumulation points a and b. These points are distinct since U_1 and U_2 are disjoint.

(3) $\Leftrightarrow$ (4) follows by (1) $\Leftrightarrow$ (2) for $n=2$.

(1) $\Rightarrow$ (3). Let $\mathcal{M}$ be K_n-minimal, $\leq$ be a linearization of $\mathcal{M}$ with respect to

$$K_n(a_1,\ldots,a_{n-2},x,y).$$

Taking an arbitrary element $b\in M\setminus\{a_1,\ldots,a_{n-2}\}$, by the K_n-minimality of $\mathcal{M}$, we have finitely or cofinitely many solutions for $K_n(a_1,\ldots,a_{n-2},b,y)$ and for $K_n(a_1,\ldots,a_{n-2},x,b)$, i.e., for $b\leq y$ and for $x\leq b$ confirming that the linearization is $\leq$-minimal.

(3) $\Rightarrow$ (1). Let any linearization $\leq$ of $\mathcal{M}$ be $\leq$-minimal. Taking a formula

$$K_n(a_1,\ldots,a_{n-2},a_{n-1},y)$$

and a $\leq$-minimal linearization $\leq$ defined by the formula $K_n(a_1,\ldots,a_{n-2},x,y)$ we have finitely or cofinitely many solutions for $a_{n-1}\leq y$ and for $x\leq a_{n-1}$ implying that $K_n(a_1,\ldots,a_{n-2},a_{n-1},y)$ has finitely or cofinitely many solutions and confirming that $\mathcal{M}$ is K_n-minimal.

Remark 8. It is essential in Theorem 6 that *all* linearizations $\leq$ of $\mathcal{M}$ are $\leq$-minimal. Indeed, the linear order $\omega+1+\omega^*$ is not definably minimal producing n-spherical orders K_n which are not K_n-minimal, whereas the linearization $\leq'$ removing the element 1 becomes $\leq'$-minimal.

We observe a similar effect taking a n-spherical order K_n with $m<n$ accumulation points and transforming it into a $\leq$-minimal linearization by removing $n-2$ elements, say with $\widehat{K}_n(\omega+(n-2)+\omega^*)$.

6. Convexity rank for spherically ordered theories

The following notion of convexity rank generalizes that known one from the class of linearly ordered theories [38] to the classes of n-spherically ordered theories. It produces a measure of minimality with respect to convex sets.

Definition. Let T be a weakly n-spherically minimal theory, $\mathcal{M}$ be a sufficiently saturated model of T, $\varphi(x)$ be a M-definable formula with one free variable. The *convexity rank of the formula* $\varphi(x)$ $(\mathrm{RC}(\varphi(x)))$ is defined as follows:

0) $\mathrm{RC}(\varphi(x))=-1$ if $\varphi(M)$ is empty, and $\mathrm{RC}(\varphi(x))\geq 0$ if $|\varphi(M)|\in\omega\setminus\{0\}$;

1) $\mathrm{RC}(\varphi(x))\geq 1$ if $\varphi(M)$ is infinite;

2) $\mathrm{RC}(\varphi(x))\geq\alpha+1$ if there is a parametrically definable equivalence relation $E(x,y)$ and infinitely many elements $b_i, i\in\omega$, such that:

- for any distinct $i,j\in\omega$, $\mathcal{M}\models\neg E(b_i,b_j)$;
- for any $i\in\omega$, $\mathrm{RC}(E(x,b_i))\geq\alpha$ and $E(\mathcal{M},b_i)$ is a convex subset of $\varphi(\mathcal{M})$;

3) $\mathrm{RC}(\varphi(x))\geq\delta$ if $\mathrm{RC}(\varphi(x))\geq\alpha$ for all $\alpha<\delta$ (δ is a limit ordinal).

If $\mathrm{RC}(\varphi(x))=\alpha$ for some α then we say that $\mathrm{RC}(\varphi(x))$ is *defined*. Otherwise, i.e., if $\mathrm{RC}(\varphi(x))\geq\alpha$ for all α, we put $\mathrm{RC}(\varphi(x))=\infty$.

The *convexity rank of* 1-*type* p $(\mathrm{RC}(p))$ is the infimum of the set $\{\mathrm{RC}(\varphi(x))\mid\varphi(x)\in p\}$, i.e., $\mathrm{RC}(p):=\inf\{\mathrm{RC}(\varphi(x))\mid\varphi(x)\in p\}$.

For the theory T the *convexity rank* $\mathrm{RC}(T)$ is the supremum of values $\mathrm{RC}(p)$ of convexity ranks of 1-types p.

Remark 9. The operator $\widehat{L}$ of linearization with respect to elements $a_1,\ldots,a_{n-2}$ preserves the convexity rank $\mathrm{RC}(T)$ of a theory T, if $\mathrm{RC}(T)$ is defined without these elements. Conversely, under that condition the operator $\widehat{K}_n$ of n-spherification preserves the convexity rank, too.

At the same time, extensions of ordered structures by new elements can produce additional definable relations increasing the convexity rank.

Remark 10. Similarly the connection between o-minimality and weak o-minimality, n-spherically minimality of a theory T means that T is weakly n-spherically minimal with $\mathrm{RC}(T)=1$.

Theorem 7. *For any natural $m\geq 1$ and $n\geq 2$ there is a countably categorical weakly n-spherically minimal theory $T_{m,n}$ such that $\mathrm{RC}(T_{m,n})=m$.*

Proof. There are two ways to construct a required theory $T_{m,n}$. The first one is based on examples of countably categorical weakly o-minimal theories $T_{m,2}$, [39, Section 2], [40, Example 1.2], being expansions of dense linear orders by refinements of definable equivalence classes using new binary symbols. The theory $T_{m,2}$ is transformed into $T_{m,n}$ by n-spherification $\widehat{K}_n$ preserving the convexity rank in view of Remark 9.

Another way is based directly on dense n-spherical order K_n [35]. We introduce m new symbols $E_1, \ldots, E_m$ for equivalence relations, with $E_1 \supset E_2 \supset \ldots \supset E_m$ having convex equivalence classes only. We divide the universe into countably many E_1-classes, each E_1-class is divided into countably many E_2-classes, etc., continuing the process m times and preserving the ω-categoricity for the resulting theory $T_{m,n}$.

7. Conclusion

We studied natural modifications of o-minimality, weak o-minimality, definable minimality adapted for spherical theories and producing n-spherical minimality, weak n-spherical minimality, definable minimality for the class of n-spherical structures and their theories. Topological and definable properties for various spherical minimalities are described. The values of convexity rank for countably categorical weakly n-spherically minimal theories are found. Thus the possibilities of minimality conditions, of topologies and of ranks are described for the spherical case.

We show that there are links between linear and spherical orders via the operators of linearization and of spherification. At the same time spherical orders produce new properties with respect to linear ones. In particular, spherical orders admit non-trivial algebraic closures which differ from definable one whereas these closures coincide for linearly ordered structures.

It would be natural to transform the well developed, rich and productive classification theory of o-minimality and its variation for linearly and partially ordered structures to the class of spherically ordered theories using possibilities of minimality conditions and the convexity rank. Thus the problem arises on a transformation of the descriptions of linearly and circularly ordered structures with minimality, topology and rank conditions till spherically ordered ones.

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