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RESONANCE IN OSCILLATORS WITH NONLINEARITY MANIFESTED AT INTERMEDIATE AMPLITUDES

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Abstract. The present paper discusses a method for finding self-consistent external influences on a nonlinear oscillator that lead to the phenomenon of resonance as in the linear case. It is shown that for bounded nonlinear systems it is possible to find such a self-consistent external force. To illustrate the search for self-consistent external influences, the simplest system with a nonlinear term represented by the saturation function is chosen. The resonant solution stability with a small amplitude deviation of the obtained self-consistent external force is investigated.

Keywords: Nonlinear resonance, self-consistent source, oscillatory systems with bounded nonlinearity

1. Introduction

Resonant phenomena in linear oscillatory systems are well studied and described in all books on general physics and oscillation theory. If the external force frequency coincides with one of the partial natural frequencies of the linear system, the oscillation amplitude in the absence of attenuation increases according to the linear law and can reach significant values, thus, leading to the structure destruction. In the case of nonlinear systems, the monochromatic effect does not lead to a significant increase in the oscillation amplitude, since their frequency depends on the amplitude and, consequently, the equality of the frequencies of the external force and natural oscillations is violated. This problem was encountered during the construction of the first cyclotrons [Veksler, 1945], [McMilan, 1945]. As a result, the resonance curve becomes limited and asymmetric with respect to the linear oscillation frequency.

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This process in weakly nonlinear systems is also well described in literature; see, for example, [Andronov, Witt, Khaykin, 1959], [Landau & Lifshitz, 1976], [Kartashova, 2010], [Rajasekar & Sanjuan, 2016].

One of the ways to overcome the movement of the natural frequency of a nonlinear system from resonance is to control the external force by adjusting its frequency to the local natural frequency. Such a mechanism is called autoresonance, it has become widespread in discrete [Fajans & Friedland, 2001] and distributed systems [Aranson et al, 1992], [Friedland, 2009], [Kalyakin, 2008]. It should be particularly noted that in this way it is possible to excite solitons and breathers, which are actively studied in the framework of nonlinear evolution equations, such as the Korteweg-de Vries equation and the nonlinear Schrodinger equation. In this direction, you can read [Garifullin et al, 2007], [Maslov et al, 2007], [Kiselev, 2016] and [Kalyakin, 2019]. It is not yet possible to find solutions to the forced nonlinear equations analytically accurately in the general case, therefore, attempts are being made to modify the external force for it to correspond to certain properties of the nonlinear system. Such external force is called self-consistent, and here we will give several references to works where self-consistent versions of the integrable nonlinear evolutionary equations are studied [Melnikov, 1990], [Zeng et al, 2003], [Chvartatskyi et al, 2016], [Focas & Latifi, 2022]. This approach is very interesting, but so far the external force is presented in a rather complex way and the possibility of its physical implementation has not been studied yet.

Finally, it is important to note that there are nonlinear systems in which the natural oscillation frequencies do not depend on the amplitude. Such systems are called isochronous, and here we will refer only to the work [Calogero, 2008], [Calogero, 2011], [Parkavi et al, 2022]. However, we could not find papers devoted to the resonant phenomena in isochronous systems.

Along with isochronous systems, there are systems with limited nonlinearity, when the nonlinearity manifests itself only in a certain range of amplitudes [Dancer, 1982], [Krasnosel'skii, 2013], [Benediktsson, 1965], [Walcott & Zak, 1986].

The purpose of this study is to search for self-consistent external influences that make it possible to swing oscillations in nonlinear systems. Here we will limit ourselves to the simplest nonlinear oscillator model and show that it is possible to select limited external influences that lead to the resonant phenomena similar to those existing in the linear system.

In Section 2 a general method for finding self-consistent external influences to excite resonance is given. Further, as an example of this method, in Section 3 we demonstrate this approach by using the example of the nonlinear system with saturation. The stability of the obtained solution is investigated in Section 4. The results obtained are summarized in the conclusion.

2. Self-consistent source in a bounded nonlinear oscillator

The oscillations of the nonlinear oscillator in the conservative case are described by the equation:

$$\frac{d^2u}{dt^2} + g(u) = 0 \tag{1}$$

Let us consider the following nonlinear system in which

$$g(u) = u + F(u) \tag{2}$$

where $F(u)$ is some continuous nonlinear function. We will consider the nonlinearity bounded by $|F(u)| < F_0$, $F_0 \in \mathbb{R}$, $F(0) = 0$ and $F(u) \to \mu u^2$ for $u \to 0$, $\mu \in \mathbb{R}$, for the nonlinearity to be infinitely small of a higher order than u, in order for a linear resonance to be obtained in this neighborhood with a sinusoidal effect with a unit frequency. Thus, in the limit of small and large amplitudes, the ODE describing the system will be linear, and nonlinearity will play a role only at intermediate amplitudes.

Is it possible to find such a self-consistent external force in order to excite the oscillator strongly? The equation of the system is represented as (3), where we specifically identified the harmonic force leading to the linear resonance:

$$\frac{d^2u}{dt^2} + u + F(u) = 2\mathrm{cos}t + f(t) \tag{3}$$

Is it feasible to swing this system? Instead of solving this problem directly, that is, by the given force $f(t)$ to look for the equation solution (3) and see if our system enters the resonance state, we will solve the problem from the opposite - assume that we have the resonant response as in the linear case, that is:

$$u(t) = t\mathrm{sin}t \tag{4}$$

Now from the equation (3) we find $f(t)$, which should cause it:

$$f(t) = F[t\mathrm{sin}(t)] \tag{5}$$

The external force function is defined and continuous on $\mathbb{R}_+$, it is also bounded. Therefore, the question whether it is possible to sway such system can be answered in the affirmative. It is also worth noting that in our case, the oscillator will be accelerated by a non-monochromatic external force, as it happens in the linear case, but by a wide spectrum force. Let us consider a specific example of the dynamic system.

3. The saturation type nonlinearity system

As an example, let us consider the system which motion is described by the equation:

$$\frac{d^2u}{dt^2} + u + \frac{au^2}{1 + b^2u^2} = 0 \tag{6}$$

a, $b \in \mathbb{R}$, $b \neq 0$, a is the positive parameter determining the proximity degree of the system in question to the harmonic oscillator [Andronov, Witt, Khaykin, 1959]. The saturation nonlinearity systems are very common in technical applications [Mellodge, 2015]. Also, with the help of the nonlinear function in the left side of the equation (6), the medium with saturation can be approximated [Vakhitov, Kolokolov, 1973].

The equilibrium states of the given oscillator are found from the equation:

$$u + \frac{au^2}{1 + b^2u^2} = 0 \tag{7}$$

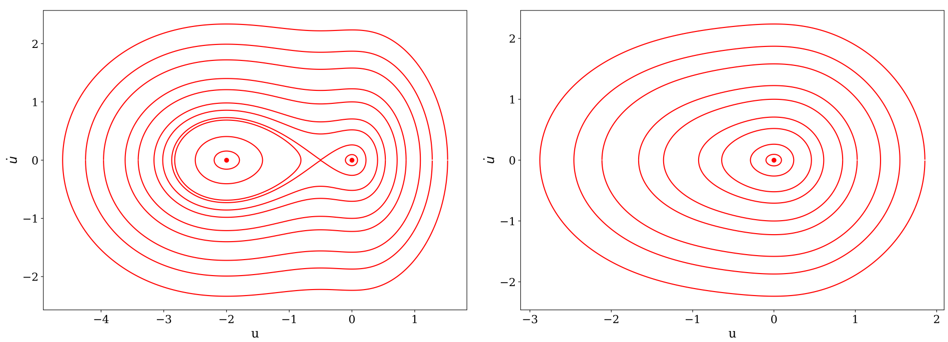
The trivial solution $u = 0$ of the equation (7) corresponds to the center. And under the condition $a > 2b$, two more equilibrium states arise

$$u = \frac{-a \pm \sqrt{a^2 - 4b^2}}{2b^2} \tag{8}$$

one of which is a saddle, and the other is also a focus. The phase planes for this oscillator are shown in the Fig. 1. Therefore, for further study, without limiting generality, we will consider only two cases (for the parameter values $a = b = 1$, as well as for $a = 2.5$, $b = 1$).

Analytically, the phase trajectories of the system are given by the following one-parameter family of functions:

$$\frac{du}{dt} = \pm\sqrt{2\left(\frac{a}{b^3}\text{arctan}u - \frac{a}{b^2}u - \frac{u^2}{2}\right) + c},\ c \in \mathbb{R} \tag{9}$$



a) $a = 2.5$, $b = 1$ b) $a = 1$, $b = 1$

FIG. 1. The phase plane of the equation (6)

The nonlinearity for this oscillator plays its role only at intermediate amplitudes - with sufficiently large amplitudes, phase trajectories are curved ellipses.

To illustrate the above-mentioned method of obtaining resonance in nonlinear systems, we will swing this oscillator, namely, we will affect the system with an external force $2\text{cos}t + f(t)$, where:

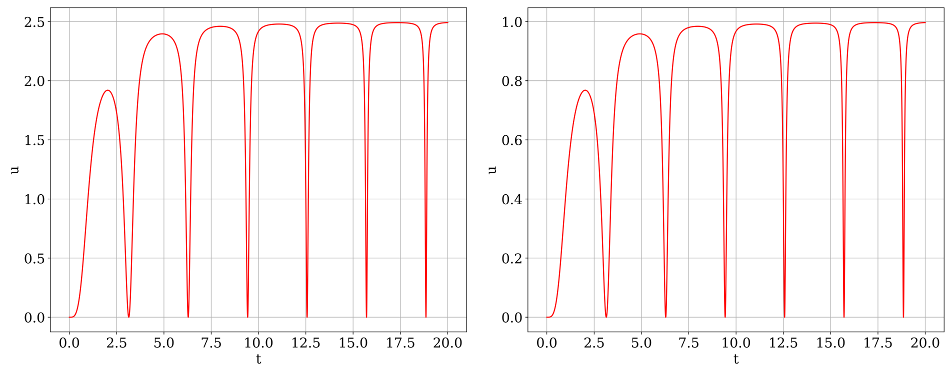
$$f(t) = F(t\text{sin}t) = \frac{at^2\text{sin}^2 t}{1 + b^2t^2\text{sin}^2 t} \tag{10}$$

With such an impact, the solution to the Cauchy problem with zero initial conditions will be the function (4).

In order to understand how to implement this impact on the system, we will study the external force behavior $f(t)$. The maximum value of $f(t)$ will tend to the ratio $\frac{a}{b^2}$. At the same time, the jumps from zero to the maximum at the points πk, $k \in \mathbb{N}$ become approximately the same over time, which can be seen in the Fig.2.

Let's determine how much energy should be added, in addition to the sinusoidal action cost, to maintain the system in the resonance state for the finite time T:

$$E(T) = \int_0^T f^2(t)dt = \int_0^T \left(\frac{at^2\text{sin}^2 t}{1 + b^2t^2\text{sin}^2 t}\right)^2 dt \tag{11}$$



a) $a = 2.5$, $b = 1$ b) $a = 1$, $b = 1$

FIG. 2. The external force $f(t)$ in the equation (10)

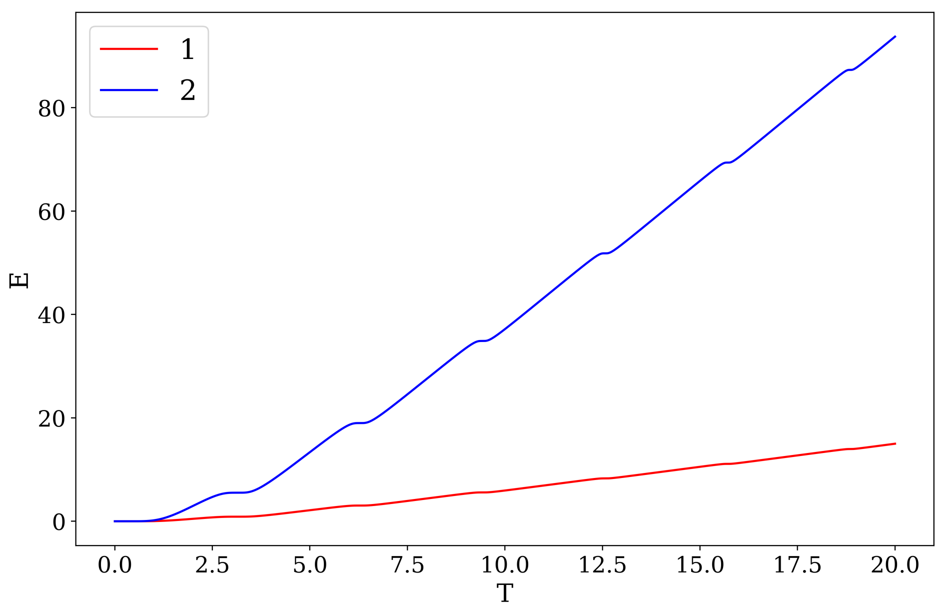


FIG. 3. Resonant excitation of vibrations. (1) - $a = 1$, $b = 1$, (2) - $a = 2.5$, $b = 1$

The external force $f(t)$ takes zero value only at points of the form πn, $n \in \mathbb{N}$, and in the vicinity of these points this function increases sharply to its maximum value. Therefore, only the maximum value of the function $f^2(t)$ can be left under the integral, which in turn gives the linear growth of the integral with a variable upper limit. Consequently, the increase in the energy amount is determined by the value of $\frac{a^2}{b^4}$ (Fig. 3). So, the amount of energy entering the (6) system turns out to be approximately directly proportional to the time it is in resonance.

4. Stability of the resonant solution with a small change in the amplitude of a self-consistent external force

It is obvious that resonant conditions can be fulfilled only for self-consistent conditions for external force. Let us see what happens if the external force differs from the self-consistent one by a small amount. To do this, let us consider the equation (12), in which the small parameter epsilon is introduced.

$$\frac{d^2u}{dt^2} + u + \frac{au^2}{1+b^2u^2} = 2\mathrm{cos}t + \frac{(a+\varepsilon)t^2\mathrm{sin}^2t}{1+b^2t^2\mathrm{sin}^2t} \tag{12}$$

To plot the graphs, we will use the Runge—Kutta numerical method (with a step of $h = 0.0005$), implemented in Python. It is worth noting that the nonlinear term in the left side of the equation (12), having passed intermediate amplitudes, practically does not change and tends to the value of $\frac{a}{b^2}$, which, of course, is the specifics of this system with saturation; other nonlinear functions satisfying the conditions on $F(u)$ may have a more complicated behavior [Pelinovsky & Melnikov, 2022]. Let us take the parameters a and b so that the oscillator nonlinear effect is observed as long as possible. With $a = 10$ and $b = 1$, even with large values of ε, the obtained solution does not deviate by a large amount, which can be seen in Fig. 4.

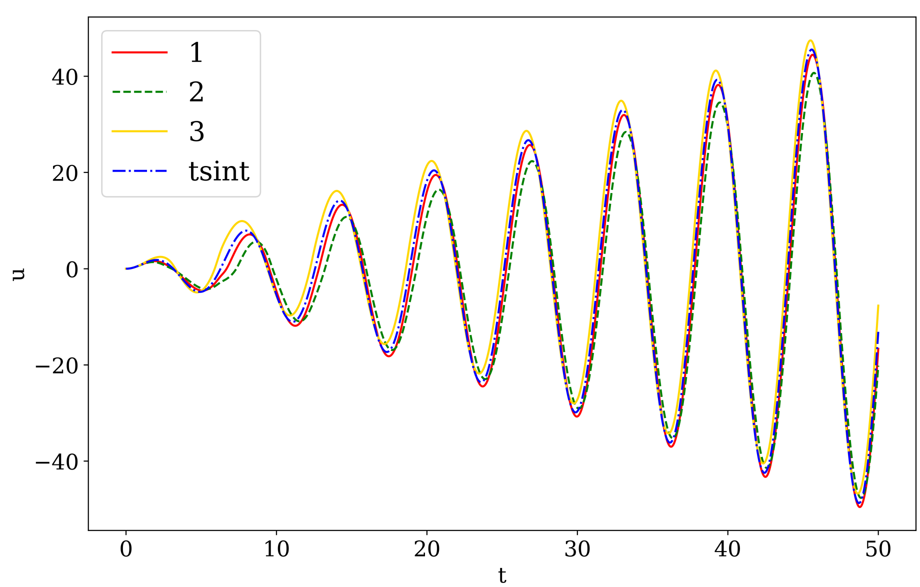


FIG. 4. Solutions of the (12) system with the parameters $a = 10$, $b = 1$ and deviation $f(t)$ by (1) $\varepsilon = -1$, (2) $\varepsilon = -2$, (3) $\varepsilon = 2$

With values $a = 40$ and $b = 1$, the solution becomes more sensitive to small deviations, even with the deviation of $\varepsilon = -0.05$, the solution deviates greatly from what we need (Fig. 5).

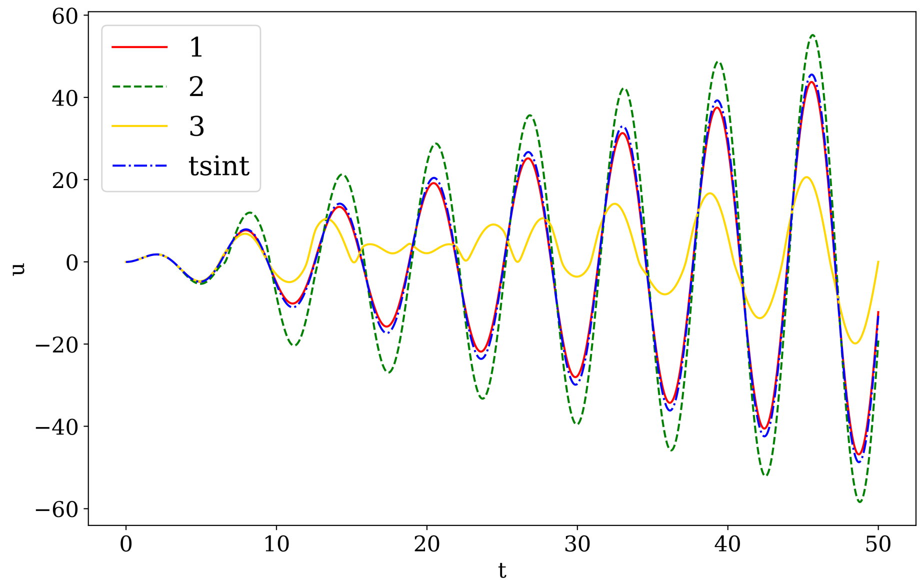


FIG. 5. Solutions of the (12) system with parameters $a = 40$, $b = 1$ and deviation $f(t)$ by (1) $\varepsilon = -0.01$, (2) $\varepsilon = 0.2$, (3) $\varepsilon = -0.05$

That is, either with the increase in the parameter a or with the decrease in b, the system (12) becomes more sensitive to small deviations ε. However, it is also worth noting that with an increase in the amplitude of the self-consistent external force, the solution changes slightly, and only the amplitude of the oscillations increases, whereas a decrease in the amplitude of the external force can cause an unpredictable sharp change in the solution, as can be seen in Fig. 5.

Next, let us consider the effect on the system (6) only by the cosine with the amplitude ε:

$$\frac{d^2u}{dt^2} + u + \frac{au^2}{1 + b^2u^2} = \varepsilon \cos t \tag{13}$$

Depending on the amplitude of the cosine effect on the oscillator, two different behaviors of the system can be distinguished. Since only at intermediate amplitudes the nonlinearity (in the left part of the equation (13)) has strong changes, the cosine with a small amplitude leads to oscillations with the limited amplitude, which (the oscillations) look like beats. However, a remarkable feature of this system is that if the cosine amplitude is sufficient to skip the intermediate oscillation amplitudes and reach those ones when the system is almost linear, we observe the resonance (Fig. 6).

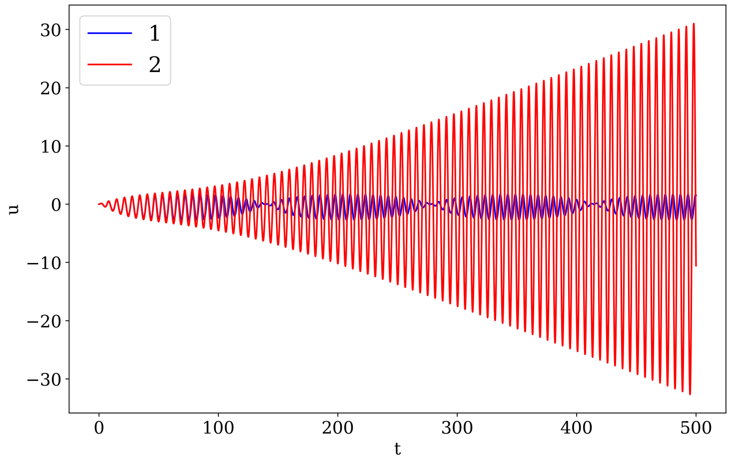


FIG. 6. Solutions of the (13) system with the parameters $a = b =$ $= 1$. Line 1 is a graph for the value $\varepsilon = 0.15$, line 2 is a graph for the value $\varepsilon = 0.16$

It is possible to disperse the system (6) and with an arbitrarily small amplitude ε of the cosine, if we add to it the self-consistent external force obtained in Section 2. Namely, it is necessary to look for a solution in the form of $u(t) = \frac{\varepsilon}{2} tsint$. Then, when the cosine of the unit frequency affects the system , as well as the force

$$f(t) = \frac{a(\frac{\varepsilon}{2} t\sin t)^2}{1 + (b\frac{\varepsilon}{2} t\sin t)^2} \tag{14}$$

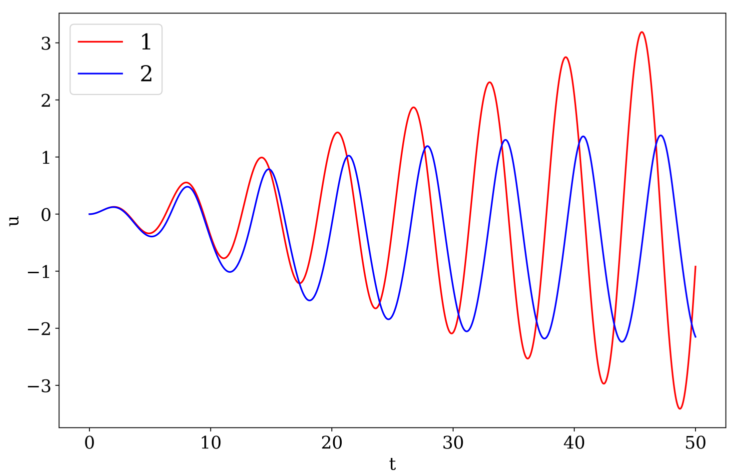


FIG. 7. Solutions of the (13) system with the parameters $a = b =$ $= 1, \varepsilon = 0.14$. Line 1 is a graph with the addition of a self-consistent external force, line 2 is a graph without its addition

it is possible to excite the resonance (Fig. 7), and after the system jumps the intermediate amplitudes, the self-consistent source $f(t)$ may be removed, so the system may be acted upon only with the cosine.

5. Conclusion

To sum up everything mentioned above, the given paper describes the method obtained to find the self-consistent external influences on an oscillatory system, whose oscillations are described by an ODE that is linear in the limit of small and large amplitudes and nonlinear at intermediate amplitudes, that excite resonant oscillations like in linear systems. In the analyzed example of the nonlinear oscillator with saturation, this approach is demonstrated; it is also shown that as the amplitude of the nonlinear saturation function increases, the system becomes more sensitive to changes in the amplitude of a self-consistent external force. At the same time, in our opinion, the search and study of resonance in nonlinear isochronous systems, which are not rare or exceptional examples of nonlinear systems, is an interesting subject for subsequent research.

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