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HOMOGENIZED ACOUSTIC EQUATIONS FOR A LAYERED MEDIUM CONSISTING OF A VISCOELASTIC MATERIAL AND A VISCOUS COMPRESSIBLE FLUID

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Abstract. We consider homogenized acoustic equations for a two-phase layered medium with periodic microstructure. The first phase of the medium is an isotropic viscoelastic material and the second one is a viscous compressible fluid. In addition, we assume that all layers are parallel to one of the coordinate planes. By means of solutions of auxiliary cell problems, we show that coefficients and convolution kernels of the homogenized equations depend on the volume fraction of the fluid phase inside the periodicity cell and do not depend on the number of layers and their geometrical position.

Keywords: homogenization, cell problems, layered media.

1. Introduction

The paper deals with the homogenization of an initial-boundary value problem for a system of partial differential and integro-differential equations whose coefficients are periodic and rapidly varying in one direction. The stated problem describes the joint motion of alternating layers of viscoelastic materials and viscous compressible fluids.

Homogenized acoustic equations for heterogeneous media composed of periodically repeating solid and fluid phases have been extensively studied by many authors (for example, see [1–8]). It was shown that the homogenized systems consist of integro-differential equations even if the corresponding original systems

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consist only of differential equations. The homogenized coefficients and kernels are determined by solutions of several auxiliary problems on the periodicity cell. However, their explicit calculation is available only for the simplest models of solid-fluid media, namely for layered media consisting of plane isotropic solid and fluid layers. This makes it possible to comprehensively study the dynamical behaviour of such media, which is of great importance in practical applications.

In this paper, we consider a layered medium whose periodicity cell contains $M+1$ layers of isotropic viscoelastic material and M layers of viscous compressible fluid, where $M \geq 1$ is a constant. Relying on the techniques outlined in [7–9], we write out the homogenized acoustic equations and find explicit formulas for their coefficients and kernels by direct solving auxiliary cell problems. The analysis of these formulas allows us to discover that the homogenized coefficients and kernels are independent of the number of fluid layers M and their locations inside the periodicity cell.

2. Statement of the original and homogenized problems

Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$ occupied by a two-phase layered medium with periodic microstructure. For the periodicity cell we take the cube $Y_\varepsilon = \varepsilon Y$, where $Y = (0,1)^3$ is the unite cube and ε is a small positive parameter. We suppose that all layers are parallel to the Ox_2x_3-plane and the cell Y_ε contains $M+1$ layers of isotropic viscoelastic material and M layers of viscous compressible fluid ($M \geq 1$). We define

$$Y_1 = \bigcup_{m=0}^{M} (q_{2m}, q_{2m+1}) \times (0,1)^2, \quad Y_2 = \bigcup_{m=1}^{M} (q_{2m-1}, q_{2m}) \times (0,1)^2,$$

$$\Gamma = \bigcup_{m=1}^{2M} \{q_m\} \times (0,1)^2, \quad 0 = q_0 < q_1 < q_2 < ... < q_{2M} < q_{2M+1} = 1$$

and assume that the sets $Y_{1\varepsilon} = \varepsilon Y_1$ and $Y_{2\varepsilon} = \varepsilon Y_2$ represent, respectively, the "viscoelastic" and "fluid" parts of the cell Y_ε.

Denote by q the total volume fraction of the fluid in Y_ε. Then

$$q = \frac{|Y_{2\varepsilon}|}{|Y_\varepsilon|} = \frac{|Y_2|}{|Y|} = \sum_{m=1}^{M} (q_{2m} - q_{2m-1}). \tag{1}$$

The subsets of Ω occupied by the viscoelastic and fluid layers are denoted by $\Omega_{1\varepsilon}$ and $\Omega_{2\varepsilon}$, respectively. Note that $\Omega_{s\varepsilon} = \Omega \cap \varepsilon E_s$, where E_s is the Y-periodic continuation of Y_s, i.e.,

$$E_s = \bigcup_{k\in\mathbb{Z}^3} \left(Y_s \cup (\partial Y_s \cap \partial Y) + k\right), \quad s = 1,2.$$

The initial-boundary value problem describing the joint motion of viscoelastic and fluid layers in Ω has the form

$$\begin{gathered} \rho_s \frac{\partial^2 u_i^\varepsilon}{\partial t^2} = \frac{\partial \sigma_{ij}^\varepsilon}{\partial x_j} + f_i(x,t) \ \text{ in } \ \Omega_{s\varepsilon} \times (0,T), \quad s = 1,2, \\ [u^\varepsilon]|_{S_\varepsilon} = 0, \quad [\sigma_{i1}^\varepsilon]|_{S_\varepsilon} = 0, \quad S_\varepsilon = \partial\Omega_{1\varepsilon} \cap \partial\Omega_{2\varepsilon}, \\ u^\varepsilon(x,t)|_{\partial\Omega} = 0, \quad u^\varepsilon(x,0) = \frac{\partial u^\varepsilon}{\partial t}(x,0) = 0, \end{gathered} \tag{2}$$

where $\rho_s = \text{const} > 0$ is the density in $\Omega_{s\varepsilon}$, $u^\varepsilon(x,t)$ is the displacement vector, σ^ε is the stress tensor, $f(x,t) \in H^2(0,T;(L^2(\Omega))^3)$ is the external force vector, and the square brackets $[\cdot]|_{S_\varepsilon}$ mean the jump in the enclosed quantity across the boundary S_ε. Note that in (2) and everywhere below the summation convention over repeated subscripts is employed.

The components of the stress tensor σ^ε are defined by

$$\sigma^\varepsilon_{ij} = a_{ijkh} e_{kh}(u^\varepsilon) + b^{(1)}_{ijkh} e_{kh}\left(\frac{\partial u^\varepsilon}{\partial t}\right) - d_{ijkh}(t) * e_{kh}(u^\varepsilon), \quad x \in \Omega_{1\varepsilon},$$

$$\sigma^\varepsilon_{ij} = -\delta_{ij} p^\varepsilon + b^{(2)}_{ijkh} e_{kh}\left(\frac{\partial u^\varepsilon}{\partial t}\right), \quad p^\varepsilon = -\gamma \operatorname{div} u^\varepsilon, \quad x \in \Omega_{2\varepsilon},$$

where $e_{kh}(u^\varepsilon) = e^x_{kh}(u^\varepsilon) = (\partial u^\varepsilon_k/\partial x_h + \partial u^\varepsilon_h/\partial x_k)/2$ are the components of the strain tensor, δ_{ij} is the Kronecker symbol, $p^\varepsilon(x,t)$ is the fluid pressure, γ is the bulk modulus of elasticity, a and $b^{(s)}$ are the positive-definite tensors of elasticity and viscosity coefficients, respectively, $d(t)$ is the tensor of the regular parts of relaxation kernels, and the symbol $*$ denotes the convolution operation with respect to t,

$$g_1(t) * g_2(t) = \int_0^t g_1(t-s) g_2(s)\, ds.$$

The components of the tensors a, $b^{(s)}$, and $d(t)$ are given by

$$\begin{aligned} a_{ijkh} &= \lambda \delta_{ij}\delta_{kh} + \mu(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}), \\ b^{(s)}_{ijkh} &= \zeta_s \delta_{ij}\delta_{kh} + \eta_s(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}), \quad s = 1,2, \\ d_{ijkh}(t) &= \left(G_1(t) - \frac{1}{3}G(t)\right)\delta_{ij}\delta_{kh} + \frac{1}{2}G(t)(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}), \end{aligned}$$

where λ and μ are the Lamé parameters, ζ_s and η_s are the viscosity coefficients in $\Omega_{s\varepsilon}$, and $G(t)$ and $G_1(t)$ are the regular parts of the bulk and shear relaxation kernels. In what follows, we will assume that $G(t)$ and $G_1(t)$ satisfy the following conditions:

$$G_1(t) = k_1 G(t), \quad G(t) = \sum_{n=1}^N v_n e^{-\gamma_n t}, \quad k_1 \geq 0, \quad \sum_{n=1}^N \frac{v_n}{\gamma_n} \leq K,$$

where $v_n, \gamma_n \in \mathbb{R}^+$ $(n = 1,...,N)$, $K = \min\{(3\lambda + 2\mu)/(3k_1), 2\mu\}$ for $k_1 > 0$ and $K = 2\mu$ for $k_1 = 0$.

The asymptotic behaviour as $\varepsilon \to 0$ of solutions of problems like (2) was widely studied in [7–9]. Based on the results of these studies, one can easily deduce that

$$u^\varepsilon(x,t) \to u(x,t) \ \text{ in } \ (L^2(\Omega))^3 \ \text{ for every } \ t \in [0,T],$$

where $u(x,t)$ is given as the solution of the corresponding homogenized problem, which is the initial boundary-value problem for a system of integro-differential equations of convolution type. Their coefficients and convolutions kernels are explicitly expressed in terms of Y-periodic solutions of auxiliary cell problems. Let us write down the homogenized and cell problems for our layered medium [7, 8].

Firstly, we define $Z^{kh}(y) \in (H^1_{per}(Y))^3$ as the solutions of the problems

$$(3) \qquad \frac{\partial}{\partial y_j}\left(\sigma^{(1)}_{ij}(Z^{kh})\right) = 0 \ \text{ in } \ Y, \quad \int_Y Z^{kh} dy = 0, \quad [\sigma^{(1)}_{i1}(Z^{kh})]|_\Gamma = 0,$$

where $H^1_{per}(Y)$ is the Sobolev space of Y-periodic functions and

$$\sigma^{(1)}_{ij}(Z^{kh}) = b^{(s)}_{ijkh} + b^{(s)}_{ijlm} e^y_{lm}(Z^{kh}), \quad y \in Y_s, \quad s = 1, 2.$$

Next, we define $D^{kh}(y) \in (H^1_{per}(Y))^3$ as the solutions of the problems

$$\frac{\partial}{\partial y_j}\left(\sigma^{(2)}_{ij}(D^{kh})\right) = 0 \text{ in } Y, \quad \int\limits_Y D^{kh} dy = 0, \quad [\sigma^{(2)}_{i1}(D^{kh})]|_\Gamma = 0, \tag{4}$$

where

$$\sigma^{(2)}_{ij}(D^{kh}) = a_{ijkh} + a_{ijlm} e^y_{lm}(Z^{kh}) + b^{(1)}_{ijlm} e^y_{lm}(D^{kh}), \quad y \in Y_1,$$
$$\sigma^{(2)}_{ij}(D^{kh}) = \gamma \delta_{ij}(\delta_{kh} + \operatorname{div}_y Z^{kh}) + b^{(2)}_{ijlm} e^y_{lm}(D^{kh}), \quad y \in Y_2.$$

Finally, we determine $W^{kh}(y,t) \in L^\infty(0,T;(H^1_{per}(Y))^3)$ as the solutions of the evolutionary problems

$$\begin{gathered} \frac{\partial}{\partial y_j}\left(\sigma^{(3)}_{ij}(W^{kh})\right) = 0 \text{ in } Y \times (0,T), \quad \int\limits_Y W^{kh} dy = 0, \\ W^{kh}(y,0) = D^{kh}(y) \text{ in } Y, \quad [\sigma^{(3)}_{i1}(W^{kh})]|_\Gamma = 0, \end{gathered} \tag{5}$$

where

$$\sigma^{(3)}_{ij}(W^{kh}) = a_{ijlm} e^y_{lm}(W^{kh}) + b^{(1)}_{ijlm} e^y_{lm}\left(\frac{\partial W^{kh}}{\partial t}\right) -$$
$$- d_{ijlm}(t) * e^y_{lm}(W^{kh}) - d_{ijlm}(t) e^y_{lm}(Z^{kh}) - d_{ijkh}(t), \quad y \in Y_1,$$
$$\sigma^{(3)}_{ij}(W^{kh}) = \gamma \delta_{ij} \operatorname{div}_y W^{kh} + b^{(2)}_{ijlm} e^y_{lm}\left(\frac{\partial W^{kh}}{\partial t}\right), \quad y \in Y_2.$$

Using the solutions of the above stationary and evolutionary cell problems, we introduce the tensors α, β, and $g(t)$ as follows:

$$\begin{aligned} \alpha_{ijkh} = {} & (1-q) a_{ijkh} + \gamma q \delta_{ij} \delta_{kh} + \int\limits_{Y_1} a_{ijlm} e^y_{lm}(Z^{kh}) dy + \\ & + \gamma \delta_{ij} \int\limits_{Y_2} \operatorname{div}_y Z^{kh} dy + \sum_{s=1}^2 \int\limits_{Y_s} b^{(s)}_{ijlm} e^y_{lm}(D^{kh}) dy, \end{aligned} \tag{6}$$

$$\beta_{ijkh} = (1-q) b^{(1)}_{ijkh} + q b^{(2)}_{ijkh} + \sum_{s=1}^2 \int\limits_{Y_s} b^{(s)}_{ijlm} e^y_{lm}(Z^{kh}) dy, \tag{7}$$

$$\begin{aligned} g_{ijkh}(t) = {} & (1-q) d_{ijkh}(t) - \int\limits_{Y_1} a_{ijlm} e^y_{lm}(W^{kh}) dy + \\ & + \int\limits_{Y_1} \left(d_{ijlm}(t) * e^y_{lm}(W^{kh}) + d_{ijlm}(t) e^y_{lm}(Z^{kh})\right) dy - \\ & - \gamma \delta_{ij} \int\limits_{Y_2} \operatorname{div}_y W^{kh} dy - \sum_{s=1}^2 \int\limits_{Y_s} b^{(s)}_{ijlm} e^y_{lm}\left(\frac{\partial W^{kh}}{\partial t}\right) dy. \end{aligned} \tag{8}$$

Then, the homogenized problem corresponding to (2) takes the form

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j} + f_i(x,t) \quad \text{in} \quad \Omega \times (0,T), \tag{9}$$

$$u(x,t)|_{\partial\Omega} = 0, \quad u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0,$$

where $\rho = \rho_1(1-q) + \rho_2 q$ and

$$\sigma_{ij} = \alpha_{ijkh} e_{kh}(u) + \beta_{ijkh} e_{kh}\left(\frac{\partial u}{\partial t}\right) - g_{ijkh}(t) * e_{kh}(u).$$

3. Solutions of the cell problems

We start this section by solving the stationary cell problem (3) for $k = h = 1$. We seek its solution in the form

$$Z^{11}(y) = (z_0(y_1), 0, 0),$$

where $z_0(y_1)$ is a piecewise linear function, which is defined by

$$z_0(y_1) = A_m y_1 + B_m, \quad y_1 \in (q_{m-1}, q_m), \quad m = 1, ..., 2M+1.$$

It follows from (3) that $4M+2$ coefficients A_1, ..., A_{2M+1}, B_1,..., B_{2M+1} must be chosen in such a way that the following conditions hold:

$$z_0(0) = z_0(1), \quad \int_0^1 z_0(y_1) dy_1 = 0, \quad [z_0]|_{y_1 = q_m} = 0, \quad m = 1, ..., 2M,$$

$$\left(b^{(1)}_{i111} + b^{(1)}_{i111} \frac{dz_0}{dy_1}\right)\Bigg|_{q_{2n-1}-0} = \left(b^{(2)}_{i111} + b^{(2)}_{i111} \frac{dz_0}{dy_1}\right)\Bigg|_{q_{2n-1}+0}, \quad n = 1, ..., M,$$

$$\left(b^{(1)}_{i111} + b^{(1)}_{i111} \frac{dz_0}{dy_1}\right)\Bigg|_{q_{2n}+0} = \left(b^{(2)}_{i111} + b^{(2)}_{i111} \frac{dz_0}{dy_1}\right)\Bigg|_{q_{2n}-0}, \quad n = 1, ..., M.$$

These conditions lead to the following linear system of $4M+2$ equations:

$$B_1 = A_{2M+1} + B_{2M+1}, \quad A_m q_m + B_m = A_{m+1} q_m + B_{m+1}, \tag{10}$$

$$A_{2n-1} b_1 + b_1 = A_{2n} b_2 + b_2, \quad A_{2n} b_2 + b_2 = A_{2n+1} b_1 + b_1, \tag{11}$$

$$\sum_{m=1}^{2M+1} \left(A_m (q_m^2 - q_{m-1}^2) + 2B_m (q_m - q_{m-1})\right) = 0, \tag{12}$$

$$m = 1, 2, ..., 2M, \quad n = 1, ..., M,$$

where $b_s = b^{(s)}_{1111} = \zeta_s + 2\eta_s$, $s = 1, 2$. It follows immediately from (11) that $A_1 = A_3 = ... = A_{2M+1}$ and $A_2 = A_4 = ... = A_{2M}$. Thus, equations (10), (11) become

$$A_1 - B_1 + B_{2M+1} = 0, \quad A_1 b_1 - A_2 b_2 = b_2 - b_1, \tag{13}$$

$$(A_1 - A_2) q_{2m-1} + B_{2m-1} - B_{2m} = 0, \quad m = 1, ..., M, \tag{14}$$

$$(A_2 - A_1) q_{2m} + B_{2m} - B_{2m+1} = 0, \quad m = 1, ..., M. \tag{15}$$

Summing (14) and (15) leads to

$$(A_2 - A_1) \sum_{m=1}^{M} (q_{2m} - q_{2m-1}) + B_1 - B_{2M+1} = 0.$$

Using (1) and the first equation in (13), we obtain

$$A_1(1-q)+A_2q=0,$$

which together with the second equation in (13) gives

$$A_1=-\frac{q}{b_{12}}(b_1-b_2),\quad A_2=\frac{1-q}{b_{12}}(b_1-b_2),$$

where $b_{12}=b_1q+b_2(1-q)$. Substituting these values of A_1 and A_2 into (14) and (15), we get

$$B_m=B_{m+1}-\frac{(-1)^m q_m}{b_{12}}(b_1-b_2),\quad m=1,...,2M$$

and consequently

$$B_m=B_{2M+1}-\frac{b_1-b_2}{b_{12}}\sum_{k=m}^{2M}(-1)^k q_k=$$

(16) $$=B_1+\frac{b_1-b_2}{b_{12}}\left(q-\sum_{k=m}^{2M}(-1)^k q_k\right),\quad m=1,...,2M.$$

Further, substituting the expressions

$$\sum_{m=1}^{2M+1}(A_m(q_m^2-q_{m-1}^2)=\frac{b_1-b_2}{b_{12}}\left(-q+\sum_{m=1}^{2M}(-1)^m q_m^2\right),$$

$$\sum_{m=1}^{2M+1}B_m(q_m-q_{m-1})=B_1+\frac{b_1-b_2}{b_{12}}\left(q-\sum_{m=1}^{2M}(q_m-q_{m-1})\sum_{k=m}^{2M}(-1)^k q_k\right)=$$

$$=B_1+\frac{b_1-b_2}{b_{12}}\left(q-\sum_{m=1}^{2M}(-1)^m q_m^2\right)$$

into (12), we get

$$B_1=\frac{b_1-b_2}{2b_{12}}\left(-q+\sum_{m=1}^{2M}(-1)^m q_m^2\right).$$

Finally, it remains to use (13) and (16) to find the constants B_2,..., B_{2M+1}. As a result, we obtain

$$B_m=\frac{b_1-b_2}{2b_{12}}\left(q+\sum_{k=1}^{2M}(-1)^k q_k^2-2\sum_{k=m}^{2M}(-1)^k q_k\right),\quad m=2,...,M,$$

$$B_{2M+1}=\frac{b_1-b_2}{2b_{12}}\left(q+\sum_{m=1}^{2M}(-1)^m q_m^2\right).$$

The solutions of the remaining stationary cell problems (3) and (4) are found similarly to what we have just done to find $Z^{11}(y)$. In order to write the solutions of all cell problems in a short way, it is convenient to introduce the notation

$$z(y_1)=\frac{b_{12}z_0(y_1)}{(1-q)(b_2-b_1)},\quad a_1=a_{1111}=\lambda+2\mu,$$

$$a_{12}=a_1q+\gamma(1-q),\quad \eta_{12}=\eta_1q+\eta_2(1-q).$$

Note that the coefficients of the piecewise linear function $z(y_1)$ depend only on q, q_1, ..., and q_{2M}. More precisely,

$$z(y_1) = \begin{cases} -y_1 - C_{2m}, & y_1 \in (q_{2m-1}, q_{2m}), \quad m = 1, ..., M, \\ \dfrac{qy_1}{1-q} - C_{2m+1}, & y_1 \in (q_{2m}, q_{2m+1}), \quad m = 0, ..., M, \end{cases}$$

where

$$C_1 = \frac{1}{2(1-q)} \left(-q + \sum_{m=1}^{2M} (-1)^m q_m^2 \right),$$

$$C_m = \frac{1}{2(1-q)} \left(q + \sum_{k=1}^{2M} (-1)^k q_k^2 - 2 \sum_{k=m}^{2M} (-1)^k q_k \right), \quad m = 2, ..., M,$$

$$C_{2M+1} = \frac{1}{2(1-q)} \left(q + \sum_{m=1}^{2M} (-1)^m q_m^2 \right).$$

The next assertion describes the solutions of the stationary cell problems.

Lemma 1. *Problems* (3) *and* (4) *have the following solutions:*

$$Z^{11}(y) = (c_{11} z(y_1), 0, 0), \quad Z^{22}(y) = Z^{33}(y) = (c_{12} z(y_1), 0, 0),$$
$$D^{11}(y) = (c_{21} z(y_1), 0, 0), \quad D^{22}(y) = D^{33}(y) = (c_{22} z(y_1), 0, 0),$$
$$Z^{12}(y) = Z^{21}(y) = (0, c_{13} z(y_1), 0), \quad Z^{13}(y) = Z^{31}(y) = (0, 0, c_{13} z(y_1)),$$
$$D^{12}(y) = D^{21}(y) = (0, c_{23} z(y_1), 0), \quad D^{13}(y) = D^{31}(y) = (0, 0, c_{23} z(y_1)),$$
$$Z^{23}(y) = Z^{32}(y) = D^{23}(y) = D^{32}(y) = (0, 0, 0),$$

where

$$c_{11} = \frac{1-q}{b_{12}} (b_2 - b_1), \quad c_{12} = \frac{1-q}{b_{12}} (\zeta_2 - \zeta_1),$$
$$c_{13} = \frac{1-q}{\eta_{12}} (\eta_2 - \eta_1), \quad c_{21} = \frac{1-q}{b_{12}^2} (\gamma b_1 - a_1 b_2),$$
$$c_{22} = \frac{1-q}{b_{12}^2} ((\gamma - \lambda) b_{12} - (\zeta_2 - \zeta_1) a_{12}), \quad c_{23} = -\frac{(1-q)\mu\eta_2}{\eta_{12}^2}.$$

Now we turn to the evolutionary cell problems (5). First, we consider problem (5) for $k = h = 1$ and seek its solution in the form

$$W^{11}(y, t) = (z(y_1) p(t), 0, 0), \quad p(0) = c_{21},$$

where $z(y_1)$ is the piecewise linear function introduced above. It follows from (5) that the unknown function $p(t)$ must be chosen in such a way that the condition

$$[\sigma_{i1}^{(3)}(Z^{11}, W^{11})]|_{y_1 = q_m} = 0, \quad m = 1, ..., 2M$$

is satisfied. It is easily checked that this leads to the integro-differential equation

$$b_{12} \frac{dp}{dt} - q d_1(t) * p(t) + a_{12} p(t) = \frac{b_2}{b_{12}} (1-q) d_1(t), \tag{17}$$

where

$$d_1(t) = d_{1111}(t) = \left(k_1 + \frac{2}{3} \right) \sum_{n=1}^{N} v_n e^{-\gamma_n t}.$$

We claim that

$$p(t) = \sum_{k=1}^{N+1} p_{1k} e^{-\xi_k t}, \tag{18}$$

where p_{1k} and ξ_k are uniquely determined by some linear system and rational equation, respectively. Indeed, substituting (18) into (17) yields

$$\sum_{k=1}^{N+1} A_k p_{1k} e^{-\xi_k t} + q\left(k_1 + \frac{2}{3}\right) \sum_{n=1}^{N} B_n v_n e^{-\gamma_n t} = 0, \tag{19}$$

where

$$A_k = b_{12}\xi_k - a_{12} - q\left(k_1 + \frac{2}{3}\right) \sum_{n=1}^{N} \frac{v_n}{\xi_k - \gamma_n},$$

$$B_n = \sum_{k=1}^{N+1} \frac{p_{1k}}{\xi_k - \gamma_n} + \frac{b_2(1-q)}{b_{12}q}.$$

It is clear that equality (19) is satisfied if and only if $A_k = 0$ and $B_n = 0$ for all $k = 1, ..., N+1$ and $n = 1, ..., N$. This means that ξ_1,..., ξ_{N+1} are the roots of the equation

$$b_{12}\xi - a_{12} = q\left(k_1 + \frac{2}{3}\right) \sum_{n=1}^{N} \frac{v_n}{\xi - \gamma_n}, \tag{20}$$

whereas p_{11}, ..., $p_{1(N+1)}$ is the solution of the linear system

$$\begin{cases} \displaystyle\sum_{k=1}^{N+1} \frac{p_{1k}}{\xi_k - \gamma_n} + \frac{b_2(1-q)}{b_{12}q} = 0, \quad n = 1, ..., N, \\ \displaystyle\sum_{k=1}^{N+1} p_{1k} = \frac{1-q}{b_{12}^2}(\gamma b_1 - a_1 b_2). \end{cases} \tag{21}$$

By similar arguments as above we solve the evolutionary cell problems (5) for other values of k and h. In order to formulate the obtained results, we consider the equation

$$\eta_{12}\tau - \mu q = \frac{q}{2} \sum_{n=1}^{N} \frac{v_n}{\tau - \gamma_n}, \tag{22}$$

whose roots we denote by τ_1, ..., τ_{N+1}. In addition, denote by p_{21}, ..., $p_{2(N+1)}$ and p_{31},..., $p_{3(N+1)}$ the solutions of the linear systems

$$\begin{cases} \displaystyle\sum_{k=1}^{N+1} \frac{p_{2k}}{\xi_k - \gamma_n} + \frac{1-q}{b_{12}}(\zeta_2 - \zeta_1) + \frac{(1-q)(3k_1 - 1)}{q(3k_1 + 2)} = 0, \quad n = 1, ..., N, \\ \displaystyle\sum_{k=1}^{N+1} p_{2k} = \frac{1-q}{b_{12}^2}((\gamma - \lambda)b_{12} - (\zeta_2 - \zeta_1)a_{12}), \end{cases} \tag{23}$$

and

$$\begin{cases} \sum_{k=1}^{N+1} \frac{p_{3k}}{\tau_k - \gamma_n} + \frac{\eta_2(1-q)}{\eta_{12} q} = 0, \quad n = 1, ..., N+1, \\ \sum_{k=1}^{N+1} p_{3k} = -\frac{(1-q)\mu\eta_2}{\eta_{12}^2}, \end{cases} \tag{24}$$

respectively. Note that before solving systems (21), (23), and (24), we need to find the roots of equations (20) and (22).

A simple geometric construction shows that the roots of (20) and (22) are different positive numbers. Thus, we can assume without loss of generality that

$$0 < \xi_1 < ... < \xi_{N+1}, \quad 0 < \tau_1 < ... < \tau_{N+1}.$$

The next assertion describes the solutions of the evolutionary cell problems.

Lemma 2. *Problems* (5) *have the following solutions:*

$$W^{11}(y,t) = \left(z(y_1) \sum_{k=1}^{N+1} p_{1k} e^{-\xi_k t}, 0, 0 \right), \quad W^{23}(y,t) = W^{32}(y,t) = 0,$$

$$W^{22}(y,t) = W^{33}(y,t) = \left(z(y_1) \sum_{k=1}^{N+1} p_{2k} e^{-\xi_k t}, 0, 0 \right),$$

$$W^{12}(y,t) = W^{21}(y,t) = \left(0, z(y_1) \sum_{k=1}^{N+1} p_{3k} e^{-\tau_k t}, 0 \right),$$

$$W^{13}(y,t) = W^{31}(y,t) = \left(0, 0, z(y_1) \sum_{k=1}^{N+1} p_{3k} e^{-\tau_k t} \right).$$

4. Calculation of the homogenized tensor components

Now we are able to find explicit expressions for the components of the homogenized tensors α, β, and $g(t)$. Before stating it let us note that these tensors satisfy the usual symmetry conditions, and it can be readily seen that $\alpha_{ijkh} = 0$, $\beta_{ijkh} = 0$, and $g_{ijkh}(t) = 0$ whenever $\delta_{ij}\delta_{kh} + \delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk} = 0$. Besides, it immediately follows from Lemmas 1 and 2 that

$$\alpha_{2323} = \mu(1-q), \quad \beta_{2323} = \eta_1(1-q) + \eta_2 q,$$

$$g_{2323}(t) = \frac{1-q}{2} \sum_{n=1}^{N} v_n e^{-\gamma_n t}.$$

In addition, it can easily be checked that

$$\alpha_{2222} = \alpha_{3333}, \quad \alpha_{1122} = \alpha_{1133}, \quad \alpha_{1212} = \alpha_{1313},$$

$$\alpha_{2222} - \alpha_{2233} = \int_{Y_1} (a_{2222} - a_{2233}) dy = 2\mu(1-q) = 2\alpha_{2323}$$

and quite similarly for the tensors β and $g(t)$. This means that for our purpose it is sufficiently to deduce explicit expressions for the components with subscripts $\{1111\}$, $\{2222\}$, $\{1122\}$, and $\{1212\}$.

Setting $i = j = k = h = 1$ in (6)–(8) and then using Lemmas 1 and 2, we get

$$\alpha_{1111} = a_1(1-q) + \gamma q + \sum_{m=0}^{M} \int\limits_{q_{2m}}^{q_{2m+1}} \frac{q}{1-q}(a_1 c_{11} + b_1 c_{21})dy_1 -$$

$$- \sum_{m=1}^{M} \int\limits_{q_{2m-1}}^{q_{2m}} (\gamma c_{11} + b_2 c_{21})dy_1 = \frac{\gamma b_1^2 q + a_1 b_2^2(1-q)}{b_{12}^2},$$

$$\beta_{1111} = b_1(1-q) + b_2 q + \sum_{m=0}^{M} \int\limits_{q_{2m}}^{q_{2m+1}} \frac{b_1 c_{11} q}{1-q} dy_1 - \sum_{m=1}^{M} \int\limits_{q_{2m-1}}^{q_{2m}} b_2 c_{11} dy_1 = \frac{b_1 b_2}{b_{12}},$$

$$g_{1111}(t) = (1-q)d_1(t) + \sum_{m=1}^{M} \int\limits_{q_{2m-1}}^{q_{2m}} \left(\gamma p(t) + b_2 \frac{dp}{dt}\right) dy_1 +$$

$$+ \sum_{m=0}^{M} \int\limits_{q_{2m}}^{q_{2m+1}} \frac{q}{1-q} \left(c_{11} d_1(t) + d_1(t) * p(t) - a_1 p(t) - b_1 \frac{dp}{dt}\right) dy_1 =$$

$$= \sum_{k=1}^{N+1} q c_{1k} p_{1k} e^{-\xi_k t},$$

where

$$c_{1k} = \gamma - a_1 + (b_1 - b_2)\xi_k - \left(k_1 + \frac{2}{3}\right) \sum_{n=1}^{N} \frac{v_n}{\xi_k - \gamma_n}.$$

Expressions for the components with subscripts {2222}, {1122}, and {1212} are obtained similarly. Omitting the detailed calculation, we write them in the following final form:

$$\alpha_{1122} = \frac{1}{b_{12}}(b_1 \gamma q + b_2 \lambda (1-q)) + \frac{q}{b_{12}^2}(1-q)(\zeta_1 - \zeta_2)(\gamma b_1 - a_1 b_2),$$

$$\alpha_{2222} = a_1(1-q) + \gamma q + \frac{q}{b_{12}}(1-q)(\zeta_1 - \zeta_2)\left(2\gamma - 2\lambda + \frac{a_{12}}{b_{12}}(\zeta_1 - \zeta_2)\right),$$

$$\alpha_{1212} = \frac{\mu \eta_2^2 (1-q)}{\eta_{12}^2}, \quad \beta_{1122} = \frac{1}{b_{12}}(b_1 \zeta_2 q + b_2 \zeta_1 (1-q)),$$

$$\beta_{2222} = b_1(1-q) + b_2 q - \frac{q}{b_{12}}(1-q)(\zeta_1 - \zeta_2)^2, \quad \beta_{1212} = \frac{\eta_1 \eta_2}{\eta_{12}},$$

$$g_{2222}(t) = \sum_{k=1}^{N+1} q c_{2k} p_{2k} e^{-\xi_k t} + \frac{(1-q)(6k_1+1)}{3k_1+2} \sum_{n=1}^{N} v_n e^{-\gamma_n t},$$

$$g_{1122}(t) = \sum_{k=1}^{N+1} q c_{1k} p_{2k} e^{-\xi_k t}, \quad g_{1212}(t) = \sum_{k=1}^{N+1} q c_{3k} p_{3k} e^{-\tau_k t},$$

where we have used the notation

$$c_{2k} = \gamma - \lambda + (\zeta_1 - \zeta_2)\xi_k - \left(k_1 - \frac{1}{3}\right) \sum_{n=1}^{N} \frac{v_n}{\xi_k - \gamma_n},$$

$$c_{3k} = -\mu + (\eta_1 - \eta_2)\tau_k - \frac{1}{2}\sum_{n=1}^{N} \frac{v_n}{\tau_k - \gamma_n}.$$

It is essential to note that the solutions of the cell problems depend on the number of layers and their geometrical positions $(q_k, q_{k+1}) \times (0,1)^2$ inside the unit cube Y. However, substituting these solutions into (6)-(8), we obtain sums of ordinary integrals of various constants over the intervals (q_k, q_{k+1}). As a result of the direct calculations, we found out that the homogenized tensor components depend on q, but not on M and q_k. Thus we arrive at the main result of this paper.

Theorem 1. *If the viscoelastic and fluid layers are parallel to one of the coordinate planes, then the coefficients and convolution kernels of the homogenized equations* (9) *depend on the volume fraction of the fluid inside the periodicity cell Y_ε and do not depend on the number of layers and their geometrical position inside Y_ε.*

5. Homogenized tensors for other types of solid phase

Up until now we have been dealing with the case when both tensors $b^{(1)}$ and $d(t)$ are nonzero. However, using the similar arguments as before, we can deduce the required explicit formulas if at least one of these two tensors is zero. We now briefly review the cell problems and the homogenized tensor components for three cases described below.

Case 1: $b^{(1)} \neq 0$ and $d(t) = 0$. This case corresponds to a medium whose solid part is a viscoelastic Kelvin-Voigt material. The cell problems (3) and (4) as well as the homogenized tensors α and β are exactly the same as above. Further, the solutions of the cell problems (5), where we set $d_{ijkh}(t) = 0$ for all possible indices, are as follows:

$$W^{11}(y,t) = \left(c_{21} z(y_1) \exp\left(-\frac{a_{12}t}{b_{12}}\right), 0, 0\right), \quad W^{23}(y,t) = W^{32}(y,t) = 0,$$

$$W^{22}(y,t) = W^{33}(y,t) = \left(c_{22} z(y_1) \exp\left(-\frac{a_{12}t}{b_{12}}\right), 0, 0\right),$$

$$W^{12}(y,t) = W^{21}(y,t) = \left(0, c_{23} z(y_1) \exp\left(-\frac{\mu q t}{\eta_{12}}\right), 0\right),$$

$$W^{13}(y,t) = W^{31}(y,t) = \left(0, 0, c_{23} z(y_1) \exp\left(-\frac{\mu q t}{\eta_{12}}\right)\right).$$

Substituting $Z^{kh}(y)$ and $W^{kh}(y,t)$ into (8), we get

$$g_{1111}(t) = \frac{q(1-q)}{b_{12}^3}(\gamma b_1 - a_1 b_2)^2 \exp\left(-\frac{a_{12}t}{b_{12}}\right),$$

$$g_{2222}(t) = \frac{q(1-q)}{b_{12}^3}((\gamma - \lambda)b_{12} - (\zeta_2 - \zeta_1)a_{12})^2 \exp\left(-\frac{a_{12}t}{b_{12}}\right),$$

$$g_{1122}(t) = \frac{q(1-q)}{b_{12}^3}(\gamma b_1 - a_1 b_2)((\gamma - \lambda)b_{12} - (\zeta_2 - \zeta_1)a_{12}) \exp\left(-\frac{a_{12}t}{b_{12}}\right),$$

$$g_{1212}(t) = \frac{q(1-q)\mu^2\eta_2^2}{\eta_{12}^3} \exp\left(-\frac{\mu q t}{\eta_{12}}\right).$$

Case 2: $b^{(1)} = 0$ and $d(t) \neq 0$. In this case the cell problems (3) and (4) are replaced by

$$\frac{\partial}{\partial y_j}\left(\sigma_{ij}^{(s)}(Z^{kh})\right) = 0 \quad \text{in} \quad Y_s \quad (s = 1, 2),$$
$$\int\limits_Y Z^{kh} dy = 0, \quad \sigma_{i1}^{(2)}(Z^{kh}) = 0 \quad \text{on} \quad \Gamma,$$

and

$$\frac{\partial}{\partial y_j}\left(\sigma_{ij}^{(3)}(D^{kh})\right) = 0 \quad \text{in} \quad Y_2,$$
$$\int\limits_{Y_2} D^{kh} dy = 0, \quad \sigma_{i1}^{(3)}(D^{kh}) = \sigma_{i1}^{(1)}(Z^{kh}) \quad \text{on} \quad \Gamma,$$

respectively, where

$$\sigma_{ij}^{(1)}(Z^{kh}) = a_{ijkh} + a_{ijlm} e_{lm}^{y}(Z^{kh}), \quad y \in Y_1,$$
$$\sigma_{ij}^{(2)}(Z^{kh}) = b_{ijkh}^{(2)} + b_{ijlm}^{(2)} e_{lm}^{y}(Z^{kh}), \quad y \in Y_2,$$
$$\sigma_{ij}^{(3)}(D^{kh}) = \gamma \delta_{ij}(\delta_{kh} + \operatorname{div}_y Z^{kh}) + b_{ijlm}^{(2)} e_{lm}^{y}(D^{kh}), \quad y \in Y_2.$$

Further, the initial conditions in the cell problems (5), where we set $b_{ijkh}^{(1)} = 0$ for all possible indices, must be replaced by the same conditions in Y_2.

Repeating the above reasoning (with natural modifications), we find that the components of α, β, and $g(t)$ are the same as in Section 4, if we put everywhere

$$b_1 = \zeta_1 = \eta_1 = 0, \quad b_{12} = b_2(1-q).$$

Case 3: $b^{(1)} = 0$ and $d(t) = 0$. This case corresponds to a medium whose solid part is an elastic material. The cell problems and the tensors α and β are the same as in Case 2, while the components of $g(t)$ are found by the formulas

$$g_{1111}(t) = \frac{a_1^2 q}{b_2(1-q)^2} \exp\left(-\frac{a_{12} t}{b_2(1-q)}\right),$$
$$g_{2222}(t) = \frac{q}{b_2}\left(\lambda - \gamma + \frac{a_{12}\zeta_2}{b_2(1-q)}\right)^2 \exp\left(-\frac{a_{12} t}{b_2(1-q)}\right),$$
$$g_{1122}(t) = \frac{a_1 q}{b_2(1-q)}\left(\lambda - \gamma + \frac{a_{12}\zeta_2}{b_2(1-q)}\right) \exp\left(-\frac{a_{12} t}{b_2(1-q)}\right),$$
$$g_{1212}(t) = \frac{\mu^2 q}{\eta_2(1-q)^2} \exp\left(-\frac{\mu q t}{\eta_2(1-q)}\right).$$

Note that Theorem 1 remains true in all three cases (it is clear that in Case 3 we must write “elastic” instead of “viscoelastic”).

References

[1] E. Sanchez-Palencia, *Non-homogeneous media and vibration theory*, Lecture Notes in Physics, **127**, Springer, Berlin, Heidelberg, 1980.

[2] J. Sanchez-Hubert, *Asymptotic study of the macroscopic behavior of a solid-liquid mixture*, Math. Methods Appl. Sci., **2** (1980), 1–11. Zbl 0427.73025

[3] G. Nguetseng, *Asimptotic analysis for a stiff variational problem arising in mechanics*, SIAM J. Math. Anal., **21**:6 (1990), 1396–1414. Zbl 0723.73011

[4] R.P. Gilbert, A. Mikelić, *Homogenizing the acoustic properties of the seabed. I*, Nonlinear Anal., **40**:1-8 (2000), 185–212. Zbl 0958.35108

[5] Th. Clopeau, J.L. Ferrin, R.P. Gilbert, A. Mikelić, *Homogenizing the acoustic properties of the seabed. II*, Math. Comput. Modelling, **33**:8-9 (2001), 821–841. Zbl 1050.76562

[6] A. Meirmanov, *A description of seismic acoustic wave propagation in porous media via homogenization*, SIAM J. Math. Anal., **40**:3 (2008), 1272–1289. Zbl 1180.35079

[7] A.S. Shamaev, V.V. Shumilova, *Homogenization of acoustic equations for a partially perforated viscoelastic solid with viscous liquid*, Doklady Physics, **56**:1 (2011), 43–46.

[8] A.S. Shamaev, V.V. Shumilova, *Homogenization of the acoustic equations for a porous long-memory viscoelastic material filled with a viscous fluid*, Differ. Equ., **48**:8 (2012), 1161–1173. Zbl 1261.35015

[9] A.S. Shamaev, V.V. Shumilova, *Spectrum of one-dimensional oscillations in the combined stratified medium consisting of a viscoelastic material and a viscous compressible fluid*, Fluid Dyn., **48**:1 (2013), 14–22. Zbl 1262.74016

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