S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

http://semr.math.nsc.ru

Том 20, №2, стр. 797–813 (2023) УДК 519.86
DOI 10.33048/semi.2023.20.047 MSC 46N10

MATHEMATICAL MODEL OF ECONOMIC DYNAMICS IN AN EPIDEMIC

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Abstract. The paper proposes a model of economic growth in an epidemic. It takes into account the dependence of the labor force on the parameters of the epidemic and the contacts restrictions, built on the base of the stable equilibrium in the corresponding SIR model, which evolves in a faster time compared to the main model. The model is formalized as an optimal control problem on an infinite horizon. The verification theorem is proved and the turnpike for the growth model without the epidemic is found. The study of a non-trivial stationary regime in a growth model during an epidemic makes it possible to analyze the dependence of the main macroeconomic indicators on the model parameters. Examples of calculations are presented that confirm the adequacy of the developed model.

Keywords: optimal control problem, Hamilton-Jacobi-Bellman equation, SIR model, economic growth model, epidemic, lockdown.

1. Introduction

The impact of the epidemic on the economic performance of different countries was clearly manifested during the Covid-19 pandemic. Quarantine measures introduced to limit the spread of the disease have led to large economic losses. During this period, an active discussion began on the need for lockdowns with varying degrees of spread of the disease and the consistency of the degree of these restrictions with the goals of economic development.

Boranbayev, A. Obrosova, N., Shananin, A., Mathematical model of economic dynamics in an epidemic.

This research is funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP09261118).

Received June, 26, 2023, published September, 25, 2023.

An adequate tool for studying such problems is mathematical models that allow substantiating quarantine measures consistent with the epidemiological situation, which allow minimizing economic losses in an epidemic.

Against the background of the Covid-19 pandemic, a huge number of scientific papers have been published over the last few years in which the authors analyze the spread of the pandemic and its impact on the economies of different countries using various types of mathematical models (for ex., [1]- [3]).

A whole series of scientific papers examines the relationship between the results of modeling the spread of the epidemic using SIR models with disease related mortality and changes in economic indicators ([4]-[6]).

In our paper we introduce a SIR epidemiological model into a neoclassical economic growth model that is formalized in terms of optimal control problem on infinite-time horizon. We suggest the framework that allows to study the impact of the spread of the disease and quarantine measures on macroeconomic indicators, such as fixed capital and final consumption. We take into account in the model the need to invest in the development of effective anti-epidemic capacities and analyze the redistribution of capital between investment in medicine and consumption, depending on the model parameters, which characterize the intensity of the disease and quarantine measures.

The most close to our field are the papers [7], [8]. The paper [7] studies an optimal growth model where there is an infectious disease with SIR dynamics. The main purpose of authors is to study implications of two different ways to model the disease related mortality – early and late in infection mortality – on the equilibrium health and economic outcomes.

The paper [8] in terms of SIR-model and neoclassical production economy discusses the economic effectiveness of reallocation of the final consumption structure in face of epidemiological restrictions.

In contrast to these papers, we ignore the impact of additional mortality due to the epidemic, suggesting that it does not have a significant impact on economic performance, and focus more on the economic consequences of the disease itself and quarantine measures. These factors most strongly affect the economically active population and lead to an increase in economic costs in conditions of Covid-19 type pandemic.

The plan of the paper is as follows.

In the section 2 we discuss the non-trivial Lyapunov-stable equilibrium state in a simple SIR-model without mortality. We introduce in the model the dependence of labor force from the spread of disease and quarantine measures at the stable point of the SIR-model. In terms of our framework we suggest that the dynamic of the spread of the disease has reached the steady proportions in a fast time scale and the characteristic time of change in economic indicators is much slower than the pandemic stabilization time.

In the section 3 we consider the economic growth model without epidemic. We formalize the model in terms of the optimal control problem with infinite horizon. The difficulties with the studying of infinite-horizon optimal control problem are connected with the transversality condition constructing ([9]). We find out the singular regime of the model with optimal non-trivial proportions between investments and consumption. We construct the Hamilton-Jacobi-Bellman equation for the model and prove the verification theorem. The results implies that the singular regime is the turnpike in the model.

In the section 4 we introduce the epidemic factor in the economic growth model through the influence of labor force (obtained from the SIR-model) on the economic dynamic. The model is formalized in terms of the infinite-horizon optimal control problem which solution implies an optimal distribution of capital between consumption, fixed capital investments and pharma investments for the development of capacity to fight the epidemic. We evaluate the non-trivial equilibrium state of the model and calculate the main macroeconomic characteristics at this state.

In the section 5 we present several examples of evaluations results which confirm the possibility of applying the model to analyze the economic consequences of the epidemic.

2. The SIR Epidemic Model

In this section we consider the simplest SIR model of spread of an epidemic. The total population N is divided into three groups:

S - the susceptible (healthy and susceptible to the disease),

I -the infective (infected and capable of transmitting the disease),

R - the recovered (recovered with subsequent temporary immunity to the disease).

Thus, the following equality holds

$$N = S + I + R.$$

We assume that the mortality rate connected with the disease is small and does not determine the spread of the epidemic. We assume that that the natural mortality rate approximately equal to the birth rate, i.e.

$$\frac{\mathrm{dN}}{\mathrm{dt}} = 0.$$

Let $\beta > 0$ be the contact rate that means the average number of contacts of a person to catch the disease per unit time. This rate may be changed by contact restrictions and lockdowns $0 \leq L \leq 1$ imposed by the government with the intensity $0 \leq \theta \leq 1$. These measures lead to a reduction in contacts between the all types of individuals. Then, the number of new cases per unit of time is

$$\beta \left(1 - \theta L\right)^2 \frac{I\left(t\right)}{N} S\left(t\right),$$

depending on the fraction of the infected.

The recovery of individuals is governed by the parameter $\gamma > 0$ and the total number of individuals who recover from the disease at each time period is $\gamma I(t)$.

Let $\lambda > 0$ be the rate of transition of individuals from immune to susceptible (average period to preserve immunity to the disease). Then the considered SIR model takes the form

$$\frac{\mathrm{dS}(t)}{\mathrm{dt}} = -\beta \left(1 - \theta L\right)^2 \frac{S(t)I(t)}{N} + \lambda R\left(t\right)$$

$$\frac{\mathrm{dI}(t)}{\mathrm{dt}} = \beta \left(1 - \theta L\right)^2 \frac{S(t)I(t)}{N} - \gamma I\left(t\right)$$

$$\frac{\mathrm{dR}(t)}{\mathrm{dt}} = \gamma I\left(t\right) - \lambda R\left(t\right).$$

The change of variables

$$s = \frac{S}{N}, \quad j = \frac{I}{N}, \quad r = \frac{R}{N}, \quad s + j + r = 1$$

leads to the following form of the SIR model

$$\frac{ds(t)}{t} = -\beta (1 - \theta L)^2 s(t) j(t) + \lambda r(t)$$

$$\frac{dj(t)}{dt} = \beta (1 - \theta L)^2 s(t) j(t) - \gamma j(t)$$

$$\frac{dr(t)}{dt} = \gamma j(t) - \lambda r(t).$$

We denote

$$\omega = \beta (1 - \theta L)^2$$

and eliminate the variable r accordingly to the equality $r = 1 - s - j$. Then we rewrite the SIR system as follows

$$\frac{ds(t)}{dt} = -\omega s(t) j(t) + \lambda (1 - s(t) - j(t))$$

$$\frac{dj(t)}{dt} = \omega s(t) j(t) - \gamma j(t).$$

Note, that for $\omega < \gamma$ the domain

$$\Gamma = \{(s(t), j(t)) \,|s(t) \geq 0, \quad j(t) \geq 0, \quad s(t) + j(t) \leq 1 \ \}$$

is a trap, i.e. if $(s(t), j(t)) \in \Gamma$, then $(s(\tau), j(\tau)) \in \Gamma$ for $\tau > t$. Therefore by $t \to +\infty$ the trajectory $(s(t), j(t))$ tends to an attractor belonging to Γ. The system has a trivial Lyapunov stable equilibrium $s = 1, j = 0$ if and only if $\omega < \gamma$. The equilibrium state $s = 1, j = 0$ corresponds to a disease like seasonal flu, which ends due to natural causes.

If $\omega > \gamma$, then as a bifurcation result the equilibrium state $s = 1, j = 0$ loses stability and new Lyapunov stable equilibrium appears

$$\hat{s} = \frac{\gamma}{\omega}, \; \hat{j} = \frac{\lambda (\omega - \gamma)}{\omega (\lambda + \gamma)},$$

that corresponds to an ongoing epidemic of the new reality type.

Quarantine restrictions $L, \; 0 \leq L \leq 1$ on contacts between people reduces the parameter

$$\omega = \beta (1 - \theta L)^2,$$

i.e. reduces the rate of spread of diseases. At the same time, quarantine restrictions reduce the labor force

$$N (1 - j) (1 - \theta_1 L)$$

that leads to economic losses. The parameter $0 \leq \theta_1 \leq 1$ reflects the response of the labor force to the epidemic restrictions.

We assume that the characteristic time of the spread of an epidemic is much less than the characteristic time of change in economic indicators. Therefore, in order to simulate estimates of the impact of an epidemic and strategies of epidemic control on economic dynamics, we consider that

- the dynamics of the spread of the disease has reached the steady proportions $\left(\hat{s}, \hat{j}\right)$,

- the total labor force, depending on the characteristics of the epidemic and quarantine restrictions, is given by the value

$$N\left(1-\hat{j}\right)(1-\theta_1 L).$$

For evaluations, we use the parameter

$$(1-i)=\left(1-\hat{j}\right)(1-\theta_1 L)$$

as the characteristic of the labor force in the context of disease and the government measures against the pandemic.

3. The Economic Growth Model

In this section we analyze the simple neoclassical economic growth model. We assume that the representative household's preferences are given as

$$\int_0^{+\infty} e^{-\delta t} u\left(C\left(t\right)\right) dt,$$

where δ is the rate of discount and $C\left(t\right)$ is the consumption of households and the utility function is trivial

$$u\left(C\left(t\right)\right)=C(t).$$

Assume that the law of evolution of capital stock is as follows

$$\frac{dk\left(t\right)}{dt}=\hat{F}\left(k\left(t\right),N\right)-C\left(t\right)-\mu k\left(t\right),$$

where $0<\mu<1$ is the fixed capital retirement rate. The production function

$$\hat{F}\left(k\left(t\right),N\right)=Bk^{\alpha}N^{1-\alpha},\ \ 0<\alpha<1,\ \ B>0$$

is assumed to be a Cobb-Douglas function. We denote

$$A=\mathrm{BN}^{1-\alpha}.$$

Thus, the production is defined by the function

$$F\left(k\left(t\right)\right)=Ak(t)^{\alpha}.$$

Assume that the income $F\left(k\left(t\right)\right)$ is distributed between the final consumption and capital by the control variable $0\leq u(t)\leq 1$, i.e.

$$C\left(t\right)=u\left(t\right)Ak\left(t\right)^{\alpha}.$$

Accordingly to the assumptions made the economic growth model takes the following form

$$\begin{aligned}
&\textstyle\int_0^{+\infty} e^{-\delta t}u\left(t\right)Ak\left(t\right)^{\alpha}dt\to\max\\
&\tfrac{\mathrm{dk}}{\mathrm{dt}}=\left(1-u\right)Ak^{\alpha}-\mu k,\ \ k\left(0\right)=k_0>0, \qquad (1)\\
&0\leq u\leq 1.
\end{aligned}$$

The Hamilton-Pontryagin function of the problem (1) has the form

$$H\left(k,u,p,t\right)=uAk^{\alpha}e^{-\delta t}+e^{-\delta t}p\left(\left(1-u\right)Ak^{\alpha}-\mu k\right).$$

The adjoint system has the form

$$\frac{dp}{dt}=\left(\delta+\mu\right)p-\alpha uAk^{\alpha-1}-\alpha\left(1-u\right)Ak^{\alpha-1}p.$$

By the Pontryagin maximum principle the optimal control depends on the value of the Lagrange multiplier p:

$$u = \begin{cases} 1, \text{ if } p > 1 \\ [0,1], \text{ if } p = 1 \\ 0, \text{ if } p < 1. \end{cases}$$

In the domain $D_1 = \{(k,p)\,|k > 0,\ 1 > p \geq 0\ \}$ we obtain the following system of equations

$$\tfrac{dk}{dt} = -\mu k$$

$$\tfrac{dp}{dt} = (\delta + \mu)\, p - \alpha A k^{\alpha - 1}.$$

The trajectory of the system which starts (in the moment τ) at the point $(k(\tau), p(\tau)) \in D_1$ has the form

$$(2) \qquad \begin{cases} lk(t) = k(\tau)\, e^{-\mu(t-\tau)} \\ \\ p(t) = p(\tau)\, e^{(\delta+\mu)(t-\tau)} + \frac{\alpha A k(\tau)^{\alpha-1}}{\delta+\alpha\mu}\left(e^{(1-\alpha)\mu(t-\tau)} - e^{(\delta+\mu)(t-\tau)}\right) \end{cases}$$

In the domain $D_2 = \{(k,p)\,|k > 0,\ 1 < p\}$ we have the system of equations

$$\tfrac{dk}{dt} = Ak^{\alpha} - \mu k$$

$$\tfrac{dp}{dt} = (\delta + \mu)\, p - \alpha A k^{\alpha - 1} p.$$

The fixed point

$$p^* = 1,\ \ k^* = \left(\frac{\alpha A}{\delta + \mu}\right)^{\frac{1}{1-\alpha}}$$

of the system of differential equations corresponds to the singular regime with the optimal control value

$$u^* = \frac{\delta + (1-\alpha)\,\mu}{\delta + \mu}.$$

Let's consider the auxiliary problem with the finite-time horizon

$$\int_0^T e^{-\delta t} u(t)\, Ak(t)^{\alpha} dt \to \max$$

$$\tfrac{dk}{dt} = (1-u)\, Ak^{\alpha} - \mu k,\ \ k(0) = k_0 > 0,$$

$$0 \leq u \leq 1.$$

The transversality condition for the finite-time horizon problem is as $p(T) = 0$. Thus the trajectory has to finish in the domain D_1 .

Proposition 1. *If* $(k(\tau - 0), p(\tau - 0)) \in D_2$, $(k(\tau + 0), p(\tau + 0)) \in D_1$, $\frac{dp(\tau)}{dt} < 0$ *then*

$$p(\tau) = 1,\ \ k(\tau) < k^*,\ \ p(t) < 1,\ \ k(t) < k^* \ \textit{for}\ t > \tau$$

Besides that, for the value $p(T) = 0$ *and* $p(t) > 0$ *if and only if* $t < T$.

Proof. Note that for $p(t) = 1$ the inequality $\frac{dp(t)}{dt} < 0$ holds if and only if $k(t) < k^*$. Then the equalities (2) implies the Proposition 1. □

Proposition 2. *If*

$$k_0 > k^* = \left(\frac{\alpha A}{\delta+\mu}\right)^{\frac{1}{1-\alpha}},$$

then the optimal solution of the optimal control problem(1) is as follows

$$u(t) = \begin{cases} 0, \ if \ t < \frac{1}{\mu}\ln\frac{k_0}{k^*} \\ \frac{\delta+(1-\alpha)\mu}{\delta+\mu}, \ if \ t \geq \frac{1}{\mu}\ln\frac{k_0}{k^*}, \end{cases}$$

$$k(t) = \begin{cases} k_0 e^{-\mu t}, \ if \ t < \frac{1}{\mu}\ln\frac{k_0}{k^*} \\ k^*, \ if \ t \geq \frac{1}{\mu}\ln\frac{k_0}{k^*}. \end{cases}$$

Proposition 3. *If* $k_0 \leq k^* = \left(\frac{\alpha A}{\delta+\mu}\right)^{\frac{1}{1-\alpha}}$, *then on the solution of the optimal control problem (1) the following equalities hold*

$$\frac{dk}{dt} = Ak^\alpha - \mu k$$

$$u(t) = \begin{cases} 0, \ if \ \ t < \int_{k_0}^{k^*} \frac{dk}{Ak^\alpha - \mu k} \\ \frac{\delta+(1-\alpha)\mu}{\delta+\mu}, \ if \ \ t \geq \int_{k_0}^{k^*} \frac{dk}{Ak^\alpha - \mu k}. \end{cases}$$

Proposition 4. *The cost function of the optimal control problem (1) has the form*

$$V(k) = \begin{cases} \frac{Ak^\alpha}{\delta+\mu} + A\frac{(1-\alpha)\mu}{\delta(\delta+\mu)}\left(\frac{\alpha A}{\delta+\mu}\right)^{\frac{\delta+\alpha\mu}{\mu(1-\alpha)}}\left(\frac{1}{k}\right)^{\frac{\delta}{\mu}}, \quad k \geq \left(\frac{\alpha A}{\delta+\mu}\right)^{\frac{1}{1-\alpha}}, \\ \frac{A}{\delta}\left(\frac{\delta+(1-\alpha)\mu}{\delta+\mu}\right)\left(\frac{\alpha A}{\delta+\mu}\right)^{\frac{\alpha}{1-\alpha}} \times \\ \qquad \exp\left(-\delta\int_k^{\left(\frac{\alpha A}{\delta+\mu}\right)^{\frac{1}{1-\alpha}}} \frac{dy}{Ay^\alpha - \mu y}\right), \quad k < \left(\frac{\alpha A}{\delta+\mu}\right)^{\frac{1}{1-\alpha}}. \end{cases} \tag{3}$$

Note that the cost function $V(k)$ is a continuous function on the interval $(0, +\infty)$.

According to [10] (chapter 1, part 3) we can write the Hamilton-Jacobi-Bellman equation that corresponds to the optimal control problem (1):

$$\delta V(k) = A\left(1 - \frac{\mathrm{dV}(k)}{\mathrm{dk}}\right)_+ k^\alpha + (Ak^\alpha - \mu k)\frac{\mathrm{dV}(k)}{\mathrm{dk}} \tag{4}$$

Proposition 5. *(The verification theorem). The cost function of the problem (1) is a solution to the Hamilton-Jacobi-Bellman equation (4).*

Proof. Let's consider the case of $k < k^* = \left(\frac{\alpha A}{\delta+\mu}\right)^{\frac{1}{1-\alpha}}$. In this case $V(k)$ is a continuously differentiable function and the following equality holds

$$\frac{dV(k)}{dk} = \frac{\delta V(k)}{Ak^\alpha - \mu k}.$$

Note, that in the considered case of $k \leq \ k^* = \left(\frac{\alpha A}{\delta+\mu}\right)^{\frac{1}{1-\alpha}}$ the inequality $\frac{dV(k)}{dk} \geq 1$ holds if and only if the following inequality is true

$$A\left(\frac{\delta+(1-\alpha)\,\mu}{\delta+\mu}\right)\left(\frac{\alpha A}{\delta+\mu}\right)^{\frac{\alpha}{1-\alpha}} \geq \left(Ak^{\alpha}-\mu k\right)\exp\left(\delta\int_{k}^{\left(\frac{\alpha A}{\delta+\mu}\right)^{\frac{1}{1-\alpha}}}\frac{dy}{Ay^{\alpha}-\mu y}\right).$$

The inequality $k \leq k^* = \left(\frac{\alpha A}{\delta+\mu}\right)^{\frac{1}{1-\alpha}}$ implies that

$$\frac{d}{dk}\left\{\left(Ak^{\alpha}-\mu k\right)\exp\left(\delta\int_{k}^{\left(\frac{\alpha A}{\delta+\mu}\right)^{\frac{1}{1-\alpha}}}\frac{dy}{Ay^{\alpha}-\mu y}\right)\right\} =$$

$$\left(\alpha Ak^{\alpha-1}-\mu-\delta\right)\exp\left(\delta\int_{k}^{\left(\frac{\alpha A}{\delta+\mu}\right)^{\frac{1}{1-\alpha}}}\frac{dy}{Ay^{\alpha}-\mu y}\right) \geq 0$$

Therefore,

$$\max_{k\leq k^*}\left\{\left(Ak^{\alpha}-\mu k\right)\exp\left(\delta\int_{k}^{\left(\frac{\alpha A}{\delta+\mu}\right)^{\frac{1}{1-\alpha}}}\frac{dy}{Ay^{\alpha}-\mu y}\right)\right\} =$$

$$A\left(\frac{\alpha A}{\delta+\mu}\right)^{\frac{\alpha}{1-\alpha}}-\mu\left(\frac{\alpha A}{\delta+\mu}\right)^{\frac{1}{1-\alpha}} = A\left(\frac{\alpha A}{\delta+\mu}\right)^{\frac{\alpha}{1-\alpha}}\left(\frac{\delta+(1-\alpha)\mu}{\delta+\mu}\right).$$

These equalities imply that $\frac{dV(k)}{dk} \geq 1$ and

$$A\left(1-\frac{dV(k)}{dk}\right)_{+}k^{\alpha}+\left(Ak^{\alpha}-\mu k\right)\frac{dV(k)}{dk} = \left(Ak^{\alpha}-\mu k\right)\frac{dV(k)}{dk} =$$

$$= \left(Ak^{\alpha}-\mu k\right)\frac{\delta V(k)}{Ak^{\alpha}-\mu k} = \delta V\left(k\right),$$

i.e. the function $V(k)$ satisfies to the Hamilton-Jacobi-Bellman equation (4) of the problem (1).

Let's consider the case of $k \geq \ k^* = \left(\frac{\alpha A}{\delta+\mu}\right)^{\frac{1}{1-\alpha}}$. In this case $V(k)$ is a continuously differentiable function and the following equality holds

$$\frac{dV(k)}{dk} = \frac{\alpha Ak^{\alpha-1}}{\delta+\mu} - A\frac{(1-\alpha)}{\delta+\mu}\left(\frac{\alpha A}{\delta+\mu}\right)^{\frac{\delta+\alpha\mu}{\mu(1-\alpha)}}k^{-\left(\frac{\delta+\mu}{\mu}\right)}.$$

Note, that in this case

$$\frac{dV\left(k^*+0\right)}{dk} = \frac{A\left(k^*\right)^{\alpha}-\delta V\left(k^*\right)}{\mu k^*}$$

and

$$\frac{dV(k)}{dk} = \left(\frac{k}{k^*}\right)^{\alpha-1} - \frac{(1-\alpha)}{\alpha}\left(\frac{k}{k^*}\right)^{-\left(\frac{\delta+\mu}{\mu}\right)} =$$

$$= \left(\frac{k}{k^*}\right)^{\alpha-1}\left(1-\frac{(1-\alpha)}{\alpha}\left(\frac{k}{k^*}\right)^{-\left(\frac{\delta}{\mu}+\alpha\right)}\right) \leq 1.$$

The inequality $k \geq k^* = \left(\frac{\alpha A}{\delta+\mu}\right)^{\frac{1}{1-\alpha}}$ implies that

$$A\left(1-\frac{dV(k)}{dk}\right)_+ k^\alpha + (Ak^\alpha - \mu k)\frac{dV(k)}{dk} = Ak^\alpha - \mu k\frac{dV(k)}{dk} =$$

$$= Ak^\alpha - \frac{\alpha\mu Ak^\alpha}{\delta+\alpha\mu} + A\frac{(1-\alpha)\mu}{\delta+\mu}\left(\frac{\alpha A}{\delta+\mu}\right)^{\frac{\delta+\alpha\mu}{\mu(1-\alpha)}} k^{-\frac{\delta}{\mu}} =$$

$$= \frac{\delta Ak^\alpha}{\delta+\alpha\mu} + A\frac{(1-\alpha)\mu}{\delta+\mu}\left(\frac{\alpha A}{\delta+\mu}\right)^{\frac{\delta+\alpha\mu}{\mu(1-\alpha)}} k^{-\frac{\delta}{\mu}} = \delta V(k).$$

Thus, if $k \geq k^* = \left(\frac{\alpha A}{\delta+\mu}\right)^{\frac{1}{1-\alpha}}$ then the function $V(k)$ satisfies to the Hamilton-Jacobi-Bellman equation (4) of the problem (1).

Note, that

$$\frac{dV(k^*+0)}{dk} - \frac{dV(k^*-0)}{d} = \frac{A(k^*)^\alpha - \delta V(k^*)}{\mu k^*} - \frac{\delta V(k^*)}{A(k^*)^\alpha - \mu k^*} =$$

$$= \frac{A(k^*)^\alpha (A(k^*)^\alpha - \mu k^* - \delta V(k^*))}{\mu k^*(A(k^*)^\alpha - \mu k^*)} = \frac{A(k^*)^\alpha((1-u^*)A(k^*)^\alpha - \mu k^*)}{\mu k^*(A(k^*)^\alpha - \mu k^*)} = 0.$$

The Proposition 5 is proved. □

4. Economic Dynamics in a Pandemic Conditions

In this section we modify the economic growth model to analyze the economic effects from both the epidemic and restrictions measures with simultaneous investments to pharma industry. Recall the results of the section 2, in the case of $\omega > \gamma$ the SIR model has a not trivial stable equilibrium solution

$$\hat{s} = \frac{\gamma}{\omega}, \hat{j} = \frac{\lambda(\omega-\gamma)}{\omega(\lambda+\gamma)},$$

that corresponds to an ongoing epidemic of the new reality type. The epidemic control methods may include both lockdowns and the capital investment in the development of means of protection against the epidemic (vaccines, medical centers, etc.). These pharma investments are connected to the additional funding. We consider the economic growth model in pandemic conditions, that allows to compare and measure the economic effects from the pandemic decrease of the labor force $N(1-i)$ as a result of quarantine measures L, $0 \leq L \leq 1$ and from pharma investments (see the section 2).

We assume that the total income of the economy splits on the final consumption of households and pharma investments by the shares

$$u_1(t) \geq 0,\ u_2(t) \geq 0,\ u_1(t) + u_2(t) \leq 1.$$

Then in terms of the previous section the dynamics of capital stock $k(t) \geq 0$ and pharma capital stock $P(t) \geq 0$ are given as follows

$$\frac{dk}{dt} = (1-u_1-u_2)Ak^\alpha(1-i)^{1-\alpha} - \mu k,\ \ k(0) = k_0$$

$$\frac{dP}{dt} = u_2 Ak^\alpha(1-i)^{1-\alpha},\ \ P(0) = P_0.$$

We assume that the epidemic ends in a limited period of time T. After the end of the epidemic the economic dynamic is described by the economic growth model from the section 3.

We introduce the parameter $\sigma \geq 0$ of the health loss into the economic growth model with epidemic, i.e. we assume that households spend the sum of σi to restore the health after disease. Then the economic growth model during an epidemic period has the form

$$(5)\quad \begin{aligned} & E\left\{\int_0^T e^{-\delta t}\left(u_1 A k^\alpha (1-i)^{1-\alpha} - \sigma i\right)dt + e^{-\delta T} V(k(T))\right\} \to \max \\ & \tfrac{dk}{dt} = (1-u_1-u_2) A k^\alpha (1-i)^{1-\alpha} - \mu k, \;\; k(0) = k_0 \\ & \tfrac{dP}{dt} = u_2 A k^\alpha (1-i)^{1-\alpha}, \;\; P(0) = P_0 \\ & \tfrac{d\Lambda}{dt} = -\delta - \zeta P, \;\; \Lambda(0) = 0 \\ & u_1 \geq 0, \;\; u_2 \geq 0, \;\; u_1 + u_2 \leq 1 \end{aligned}$$

where the objective functional corresponds to the expected value of the discounted consumption utility and $V(\cdot)$ is the cost function (3).

We assume that the random epidemic duration T has a distribution

$$F_T(t) = 1 - \exp\left\{-\int_0^t H(P(s))\,ds\right\}.$$

Lemma 1. *The following equality holds*

$$E\left(\int_0^T e^{-\delta t}(C(t) - \sigma i)\,dt + e^{-\delta T} V(k(T))\right) =$$

$$\int_0^{+\infty} e^{-\int_0^\tau (\delta + H(P(s)))ds} \left(H(P(\tau)) V(k(\tau)) + C(\tau) - \sigma i\right) d\tau$$

Proof. Obviously, we have

$$E\left(\int_0^T e^{-\delta t}(C(t) - \sigma i)\,dt + e^{-\delta T} V(k(T))\right) =$$

$$= \int_0^{+\infty}\left(\int_0^\tau e^{-\delta t}(C(t) - \sigma i)\,dt + e^{-\delta\tau} V(k(\tau))\right) dF_T(\tau) =$$

$$= \int_0^{+\infty}\left(\int_0^\tau e^{-\delta t}(C(t) - \sigma i)\,dt\right) dF_T(\tau) +$$

$$+ \int_0^{+\infty} e^{-\int_0^\tau (\delta + H(P(s)))ds} H(P(\tau)) V(K(\tau))\,d\tau =$$

$$F_T(\tau)\int_0^\tau e^{-\delta t}(C(t) - \sigma i)\,dt\Big|_{\tau=0}^{\tau=\infty} - \int_0^{+\infty} e^{-\delta t}(C(t) - \sigma i) F_T(t)\,dt +$$

$$\int_0^{+\infty} e^{-\int_0^\tau (\delta + H(P(s)))ds} H(P(\tau)) V(K(\tau))\,d\tau =$$

$$\int_0^{+\infty} e^{-\delta t}(1 - F_T(t))(C(t) - \sigma i)\,dt +$$

$$+ \int_0^{+\infty} e^{-\int_0^\tau (\delta + H(P(s)))ds} H(P(\tau)) V(K(\tau))\,d\tau =$$

$$\int_0^{+\infty} e^{-\int_0^\tau (\delta + H(P(s)))ds} \left(H(P(\tau)) V(K(\tau)) + (C(t) - \sigma i)\right) d\tau.$$

Lemma 1 is proved. □

We assume that $H(P) = \zeta P$ – the linear function with the parameter $\zeta > 0$. We denote

$$\Lambda(t) = -\int_0^t (\delta + H(P(s)))\, ds.$$

Then the economic growth model (5) transforms to the following infinite-horizon optimal control problem

$$\begin{aligned}
&\textstyle\int_0^{+\infty} e^{\Lambda}\left(\zeta P V(k) + u_1 A k^{\alpha}(1-i)^{1-\alpha} - \sigma i\right) dt \to \max \\
&\tfrac{dk}{dt} = (1 - u_1 - u_2) A k^{\alpha}(1-i)^{1-\alpha} - \mu k,\ k(0) = k_0 \\
&\tfrac{dP}{dt} = u_2 A k^{\alpha}(1-i)^{1-\alpha},\ \ P(0) = P_0 \\
&\tfrac{d\Lambda}{dt} = -\delta - \zeta P,\ \ \Lambda(0) = 0, \\
&u_1 \geq 0,\ u_2 \geq 0,\ u_1 + u_2 \leq 1.
\end{aligned} \tag{6}$$

The difficulties of studying an infinite-horizon optimal control problem are connected with the transversality condition formulation (for ex., see [9]. The Hamilton-Pontryagin function of the problem (6) has the form

$$\begin{aligned}
&H(k, P, \Lambda, u_1, u_2, \varphi_k, \varphi_P, \varphi_\Lambda) = e^{\Lambda}\left(\zeta P V(k) + u_1 A k^{\alpha}(1-i)^{1-\alpha} - \sigma i\right) + \\
&e^{\Lambda}\varphi_k\left((1 - u_1 - u_2) A k^{\alpha}(1-i)^{1-\alpha} - \mu k\right) + e^{\Lambda}\varphi_P\left(u_2 A k^{\alpha}(1-i)^{1-\alpha}\right) - \\
&e^{\Lambda}\varphi_\Lambda(\delta + \zeta P).
\end{aligned}$$

Denote the cost function of the problem (6) as $W(k, P)$. In accordance to [10] (chapter 1, part 3) we can write the Hamilton-Jacobi-Bellman equation for the problem (6):

$$\begin{aligned}
&(\delta + \zeta P) W(k, P) = \\
&A\left(1 - \tfrac{\partial W(k,P)}{\partial k}\right)_+ k^{\alpha}(1-i)^{1-\alpha} + \left(A k^{\alpha}(1-i)^{1-\alpha} - \mu k\right)\tfrac{\partial W(k,P)}{\partial k} - \sigma i + \\
&\zeta P V(k) + A\left(\tfrac{\partial W(k,P)}{\partial P} - \tfrac{\partial W(k,P)}{\partial k}\right)_+ k^{\alpha}(1-i)^{1-\alpha}.
\end{aligned} \tag{7}$$

The adjoint system has the form

$$\begin{aligned}
&\tfrac{d\varphi_k}{dt} = (\delta + \zeta P + \mu)\varphi_k - \alpha\left(u_1 + (1 - u_1 - u_2)\varphi_k + u_2\varphi_P\right) A k^{\alpha-1}(1-i)^{1-\alpha} - \\
&\zeta P \tfrac{dV(k)}{dk} \\
&\tfrac{d\varphi_P}{dt} = (\delta + \zeta P)\varphi_P - \zeta V(k) + \zeta\varphi_\Lambda \\
&\tfrac{d\varphi_\Lambda}{dt} = (\delta + \zeta P)\varphi_\Lambda - \left(\zeta P V(k) + u_1 A k^{\alpha}(1-i)^{1-\alpha} - \sigma i\right).
\end{aligned}$$

The Pontryagin maximum principle implies that

- if $\max(\varphi_k, \varphi_P) < 1$ then $u_1 = 1, u_2 = 0$;
- if $\max(1, \varphi_P) < \varphi_k$ then $u_1 = 0, u_2 = 0$;

- if $\max(1,\varphi_k)<\varphi_P$ then $u_1=0, u_2=1$.

In case of $\max(1,\varphi_k)<\varphi_P$ we have

$$\frac{dk}{dt}=-\mu k$$

$$\frac{dP}{dt}=Ak^{\alpha}(1-i)^{1-\alpha}$$

$$\frac{d\varphi_k}{dt}=(\delta+\zeta P+\mu)\varphi_k-\alpha\varphi_P Ak^{\alpha-1}(1-i)^{1-\alpha}-\zeta P\frac{dV(k)}{dk}$$

$$\frac{d\varphi_P}{dt}=(\delta+\zeta P)\varphi_P-\zeta V(k)+\zeta\varphi_\Lambda$$

$$\frac{d\varphi_\Lambda}{dt}=(\delta+\zeta P)\varphi_\Lambda-(\zeta PV(k)-\sigma i).$$

In case of $\max(\varphi_k,\varphi_P)<1$ we have

$$\frac{dk}{dt}=-\mu k$$

$$\frac{dP}{dt}=0$$

$$\frac{d\varphi_k}{dt}=(\delta+\zeta P+\mu)\varphi_k-\alpha Ak^{\alpha-1}(1-i)^{1-\alpha}-\zeta P\frac{dV(k)}{dk}$$

$$\frac{d\varphi_P}{dt}=(\delta+\zeta P)\varphi_P-\zeta V(k)+\zeta\varphi_\Lambda$$

$$\frac{d\varphi_\Lambda}{dt}=(\delta+\zeta P)\varphi_\Lambda-\left(\zeta PV(k)+Ak^{\alpha}(1-i)^{1-\alpha}-\sigma i\right).$$

In case of $\max(1,\varphi_P)<\varphi_k$ we have

$$\frac{dk}{dt}=Ak^{\alpha}(1-i)^{1-\alpha}-\mu k$$

$$\frac{dP}{dt}=Ak^{\alpha}(1-i)^{1-\alpha}$$

$$\frac{d\varphi_k}{dt}=(\delta+\zeta P+\mu)\varphi_k-\alpha\varphi_k Ak^{\alpha-1}(1-i)^{1-\alpha}-\zeta P\frac{dV(k)}{dk}$$

$$\frac{d\varphi_P}{dt}=(\delta+\zeta P)\varphi_P-\zeta V(k)+\zeta\varphi_\Lambda$$

$$\frac{d\varphi_\Lambda}{dt}=(\delta+\zeta P)\varphi_\Lambda-(\zeta PV(k)-\sigma i).$$

In contrast to trivial regimes, the special regime $\varphi_k=1,\varphi_P=1$ is of interest for further research. In this regime the following equalities hold:

$$(1-u_1-u_2)Ak^{\alpha}(1-i)^{1-\alpha}=\mu k$$

$$u_2Ak^{\alpha}(1-i)^{1-\alpha}=0$$

$$(\delta+\zeta P+\mu)=\alpha Ak^{\alpha-1}(1-i)^{1-\alpha}+\zeta P\frac{dV(k)}{dk}$$

$$(\delta+\zeta P)=\zeta V(k)-\zeta\varphi_\Lambda$$

$$(\delta+\zeta P)\varphi_\Lambda=\left(\zeta PV(k)+u_1Ak^{\alpha}(1-i)^{1-\alpha}-\sigma i\right)$$

where the function $V(k)$ is defined by (3).

Thus, we have

$$u_1 = 1 - \frac{\mu k}{Ak^\alpha (1-i)^{1-\alpha}},\ Ak^\alpha (1-i)^{1-\alpha} \geq \mu k,$$

$$u_2 = 0$$

$$(\delta + \zeta P)\left(1 - \frac{dV(k)}{dk}\right) = \alpha Ak^{\alpha-1}(1-i)^{1-\alpha} - \delta \frac{dV(k)}{dk} - \mu$$

$$(\delta + \zeta P)^2 = \zeta\left(\delta V(k) - Ak^\alpha (1-i)^{1-\alpha} + \mu k + \sigma i\right)$$

$$(\delta + \zeta P)\varphi_\Lambda = \left(\zeta P V(k) + Ak^\alpha (1-i)^{1-\alpha} - \mu k - \sigma i\right).$$

Therefore, if the solution of the equation

$$\begin{aligned}&\left(\alpha Ak^{\alpha-1}(1-i)^{1-\alpha} - \mu - \delta\frac{dV(k)}{dk}\right)^2 = \\ &\zeta\left(\delta V(k) - Ak^\alpha(1-i)^{1-\alpha} + \mu k + \sigma i\right)\left(1 - \frac{dV(k)}{dk}\right)^2\end{aligned} \tag{8}$$

exists, which satisfies to the inequalities

$$Ak^\alpha (1-i)^{1-\alpha} \geq \mu k \frac{\alpha Ak^{\alpha-1}(1-i)^{1-\alpha} - \delta - \mu}{1 - \frac{dV(k)}{dk}} \geq 0,$$

then the equilibrium state (the special regime) exists and is defined by the following equalities:

$$P = \frac{1}{\zeta}\left(\frac{\alpha Ak^{\alpha-1}(1-i)^{1-\alpha} - \delta - \mu}{1 - \frac{dV(k)}{dk}}\right),\ \ \varphi_k = 1,\ \ \varphi_P = 1,$$

$$u_1 = 1 - \frac{\mu k}{Ak^\alpha(1-i)^{1-\alpha}},\ \ u_2 = 0,$$

$$\left(\frac{\alpha Ak^{\alpha-1}(1-i)^{1-\alpha} - \mu - \delta\frac{dV(k)}{dk}}{1 - \frac{dV(k)}{dk}}\right)^2 = \zeta\left(\delta V(k) - Ak^\alpha(1-i)^{1-\alpha} + \mu k + \sigma i\right).$$

5. Model Evaluations

In this section we evaluate some examples by the help of the model which shows the economic effects of an epidemic. In accordance to the results of the section 4, we can evaluate the following characteristics for the special equilibrium regime of the model:

- k – fixed capital stock, which is the solution of the equation (8),
- $S = u_1 Ak^\alpha (1-i)^{1-\alpha}$ – final consumption of households,
- $P = \frac{1}{\zeta}\left(\frac{\alpha Ak^{\alpha-1}(1-i)^{1-\alpha} - \delta - \mu}{1 - \frac{\mathrm{d}V(k)}{\mathrm{d}k}}\right)$ – pharma capital stock.

The goal of the state administration is to choose quarantine measures in such a way as to increase the number of workers. That leads to the maximization by $0 \leq L \leq 1$ of the value

$$(1-i) = \left(1 - \hat{j}\right)(1 - \theta_1 L),$$

lam/beta = 0.0075		Theta1										
		0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
	0	0	0	0	0	0	0	0	0	0	0	0
	0.1	1	0	0	0	0	0	0	0	0	0	0
	0.2	1	0	0	0	0	0	0	0	0	0	0
	0.3	1	0	0	0	0	0	0	0	0	0	0
	0.4	1	0	0	0	0	0	0	0	0	0	0
Theta	0.5	1	0	0	0	0	0	0	0	0	0	0
	0.6	1	0	0	0	0	0	0	0	0	0	0
	0.7	1	0	0	0	0	0	0	0	0	0	0
	0.8	1	1	0	0	0	0	0	0	0	0	0
	0.9	1	1	1	1	1	0	0	0	0	0	0
	1	1	1	1	1	1	1	1	1	1	1	0

lam/beta = 0.025		Theta1										
		0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
	0	0	0	0	0	0	0	0	0	0	0	0
	0.1	1	0	0	0	0	0	0	0	0	0	0
	0.2	1	0	0	0	0	0	0	0	0	0	0
	0.3	1	0	0	0	0	0	0	0	0	0	0
	0.4	1	0	0	0	0	0	0	0	0	0	0
Theta	0.5	1	0	0	0	0	0	0	0	0	0	0
	0.6	1	0	0	0	0	0	0	0	0	0	0
	0.7	1	0	0	0	0	0	0	0	0	0	0
	0.8	1	1	0	0	0	0	0	0	0	0	0
	0.9	1	1	1	1	1	0	0	0	0	0	0
	1	1	1	1	1	1	1	1	1	1	1	0

lam/beta = 0.075		Theta1										
		0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
	0	0	0	0	0	0	0	0	0	0	0	0
	0.1	1	0	0	0	0	0	0	0	0	0	0
	0.2	1	0	0	0	0	0	0	0	0	0	0
	0.3	1	0	0	0	0	0	0	0	0	0	0
	0.4	1	1	0	0	0	0	0	0	0	0	0
Theta	0.5	1	1	0	0	0	0	0	0	0	0	0
	0.6	1	1	1	0	0	0	0	0	0	0	0
	0.7	1	1	1	1	1	0	0	0	0	0	0
	0.8	1	1	1	1	1	1	1	0	0	0	0
	0.9	1	1	1	1	1	1	1	1	1	0	0
	1	1	1	1	1	1	1	1	1	1	1	0

FIG. 1. The quarantine restrictions $L(\theta, \theta_1)$

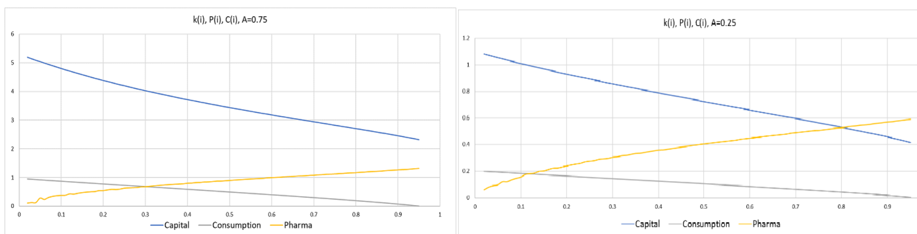

FIG. 2. Model calculations of macroeconomic characteristics during epidemic with quarantine measures

where (see the section 2)

$$\hat{j} = \frac{\lambda(\omega - \gamma)}{\omega(\lambda + \gamma)}, \quad \omega = \beta(1 - \theta L)^2.$$

The evaluation shows that the maximum value of $(1 - i)$ is reached by $L = 1$ (the total lockdown) or $L = 0$ (no restrictions) in dependence from the parameters θ (Theta) and θ_1 (Theta1) by the fixed set of other parameters of the model. For qualitative calculations we focus on data panel of parameters from [7]. On the Fig.1 we demonstrate the qualitative dependence of quarantine restrictions $L(\theta, \theta_1)$ for several values of parameter λ/β (lam/beta). We can see that the higher rate of disappearance of immunity (λ/β is higher) leads to the higher effectiveness of quarantine measures.

On the Fig.2 we show the result of model simulations of the dependence of macroeconomic characteristics from capital k, consumption C and pharma investments P from the characteristic i of the share of the population, which do not participate in production activities due to the disease or quarantine measures. We fix the following values of the model parameters for test evaluations: $\alpha = 0.3$, $\delta = 0.02$, $\zeta = 0.5$, $\mu = 0.05$, $\sigma = 0$.

Note, that the parameter $A > 0$ can be interpreted as the level of development of the economy. Model calculations shows that the spread of an epidemic leads to the decrease in the capital and the consumption of households while investment in pharma increases. The level of industrial development A has a positive effect on the scale of investment and final consumption. These results seems to be natural and confirm the validation of the developed model.

In Fig.4,5 dependencies of macroeconomic indicators $k(A)$, $C(A)$, $P(A)$ from A by the fixed optimal value of L are given, which show how the strategy of the epidemic control measures change for different technological levels of the economy.

θ_1	θ	β	λ	γ
0.1	0.7	0.4	0.01	0.03

⇨ $L = 1$

θ_1	θ	β	λ	γ
0.4	0.7	0.4	0.01	0.03

⇨ $L = 0$

FIG. 3. The test values of parameters of the SIR model

For analyzing the dependencies, we choose the two sets of SIR-model parameters that correspond to the total lockdown measures $L = 1$ and no restrictions case $L = 0$. The panels of corresponding parameters are shown on the Fig.3.

Besides that, we take into account that the rate of households discount δ (it is denoted as "delta" on Fig.4,5) is higher for the for the less developed economies. Therefore, in our test calculations the increase of the parameter A on the interval $[0.25, 0.8]$ with the step 0.025 is accompanied by a corresponding decrease of the parameter δ from the value 0.1 to the value 0.02 with a uniform step.

On the Fig.4 we demonstrate results of test evaluations with the developed model in the case of $L = 1$ for several values of parameter σ (it is denoted as "sigma" on Fig. 4,5) which characterizes the expenses of the population for the restoration of health after disease.

On the Fig.5 we demonstrate the results of similar test evaluations in the case of $L = 0$ for several values of the parameter σ.

In the case of $L = 1$ (Fig.4) the model calculations show that the final consumption of households and the total capital increase by the level of economic development. That is the natural results that validate the model. At the same time, we can conclude that the developed economies choose the higher level of investments in pharma during the epidemic compared to less developed economies. Note, that the growth of restoring health cost leads to the increase of the pharma investments. In the case of $L = 0$ (Fig.5) the model calculations show the similar economic dynamic but with a higher level of the pharma investments.

6. Conclusion

In the paper, we discuss the possible economic consequences of the epidemic in terms of the new mathematical model of economic growth, which parameters are customized on the epidemic regime of the corresponding SIR model. The new economic growth model takes into account the spread of epidemic (for ex., Covid-19 pandemic), that leads to a reduction in labor forces. We take into account the impact on economic performance of the costs, which arise from government quarantine measures and additional investment in pharma capital to develop effective tools for epidemic control. The developed model represents actually the two Hamilton-Jacobi-Bellman equations:

- the Hamilton-Jacobi-Bellman equation (4) describes economic dynamics in the absence of an epidemic,

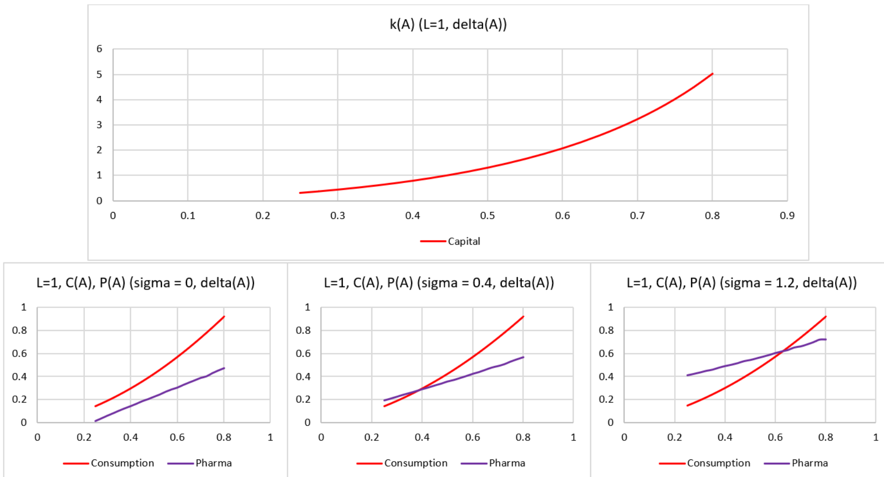

FIG. 4. The dependence of distribution of capital from the level of economic development. The case $L = 1$

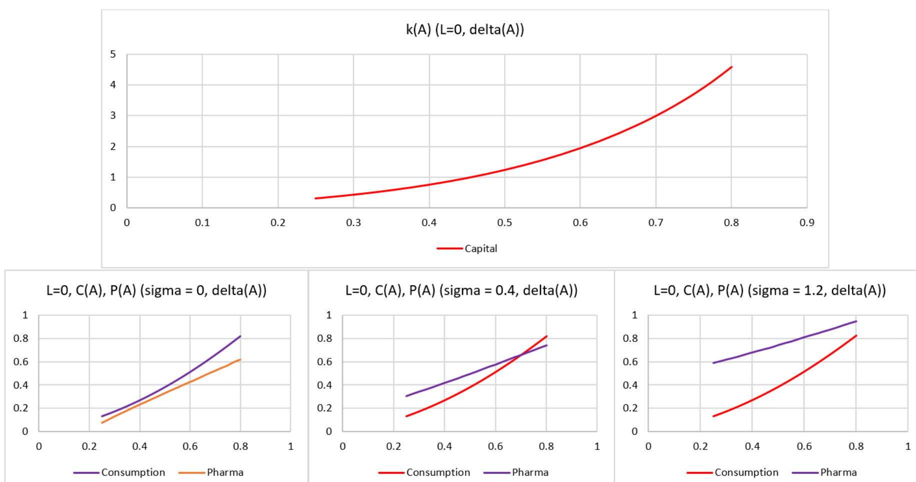

FIG. 5. The dependence of distribution of capital from the level of economic development. The case $L = 0$

- the Hamilton-Jacobi-Bellman equation (7) describes the economic dynamics in an epidemic and takes into account the economic losses associated with the disease.

Similar problems arise, for example, in the analysis of shock effects in climate models [11]. The study and analysis of the turnpike in such a problem is an open question. This question may be field of the future research.

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