S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 20, №2, стр. 833–846 (2023) УДК 512.554
DOI 10.33048/semi.2023.20.050 MSC 17A60

QUASI-DEFINITE PRIMITIVE AXIAL ALGEBRAS OF JORDAN TYPE HALF

I.B. GORSHKOV AND V.YU. GUBAREV

Abstract. Axial algebras are commutative nonassociative algebras generated by a finite set of primitive idempotents whose action on an algebra is semisimple, and the fusion laws on the products between eigenvectors for these idempotents are fulfilled. We find the sufficient conditions in terms of the Frobenius form and the properties of idempotents under which an axial algebra of Jordan type half is unital.

Keywords: axial algebra of Jordan type, Frobenius form, semisimplicity.

1. Introduction

The notion of an axial algebra appeared in the article of J. Hall, F. Rehren, and S. Shpectorov in 2015 [4] as a natural extension of Majorana theory earlier developed by A.A. Ivanov [6]. Axial algebras generalize associative algebras, Jordan algebras, and the Griess algebra whose automorphism group is the Monster group. The main idea which lies behind the theory of axial algebras is to realize finite simple groups as automorphism (sub)groups of some finite-dimensional commutative nonassociative algebras.

Primitive axial algebras of Jordan type are the most well-studied objects among all axial ones, their fusion laws represent the properties of the Peirce decomposition fulfilled for all Jordan algebras [1]. It is known that every primitive axial algebra A of Jordan type has an invariant normalized bilinear Frobenius form $(\cdot,\cdot)$ [5]. The radical of the form coincides with the maximal ideal of A not containing axes from A. An axial algebra is called semisimple if the radical of its Frobenius form is

Gorshkov, I.B., Gubarev, V.Yu. Quasi-definite primitive axial algebras of Jordan type half.

The work is supported by Mathematical Center in Akademgorodok under agreement No. 075-15-2022-281 with the Ministry of Science and Higher Education of the Russian Federation.

Received April, 20, 2023, published October, 20, 2023.

zero. Despite of the obtained results, the question when an axial algebra of Jordan type 1/2 is finite-dimensional or contains a unit is still open. We find the sufficient conditions under which an axial algebra of Jordan type 1/2 containing a unit is semisimple and finite-dimensional.

A primitive axial algebra A of Jordan type 1/2 is called quasi-definite if the equality $(a,b)=1$ for axes $a,b\in A$ implies $a=b$. Simple Jordan algebra $J(f)$ over $\mathbb{R}$ with positive-definite form f and the Jordan algebra $H_n(\mathbb{R})$ of symmetric algebras of order n give examples of quasi-definite primitive axial algebras of Jordan type 1/2 (see §3). A generalization of quasi-definite primitive axial algebras of Jordan type 1/2 is provided by primitive axial algebras which have a quasi-definite linear basis X of axes, i. e. $(x,y)\neq 1$ for all pairwise distinct $x,y\in X$. We show that the matrix algebra $M_n(F)$ over an infinite field F has a quasi-definite basis consisting of axes. However, $M_n(F)$ for $n\geq 2$ is not quasi-definite. Also, every Matsuo algebra $M_{1/2}(G,D)$ has a quasi-definite basis of axes (see §3).

A primitive axial algebra A of Jordan type 1/2 is called strongly axial if every semisimple primitive idempotent $e\in A$ such that $\mathrm{Spec}(\mathrm{ad}_e)\subset\{0,1/2,1\}$, where ad_e denotes the operator of the multiplication on e, is a primitive axis in A. Thus, in a strongly axial algebra, the fusion laws for all semisimple primitive idempotents (not necessarily axes) have to be fulfilled. Actually, every axial algebra A of Jordan type 1/2 that is also a Jordan algebra is strongly axial [1, Theorem 6]. Since there are no known non-Jordan primitive axial algebras of Jordan type 1/2, we may not provide an example of a primitive axial algebra which is not strongly axial.

We concentrate on the study of the properties of finitely generated quasi-definite primitive strongly axial algebras of Jordan type 1/2. Let A be such an algebra. We state that given an axis $a\in A$, a subalgebra $A_0(a)$ is a primitive axial algebra of Jordan type 1/2 which has a quasi-definite basis of axes.

The main results of the work are the following. Suppose that A is a finitely generated quasi-definite primitive strongly axial algebra of Jordan type 1/2. If A is additionally finite-dimensional and semisimple, then A is unital (Theorem 1). We say that a unital primitive axial algebra A has a finite capacity k, if its unit e is represented as a sum of k pairwise orthogonal axes and such k is minimal. We prove that finitely generated quasi-definite primitive strongly axial algebras of Jordan type 1/2 has a finite capacity (Theorem 2).

Let us give a brief outline of the work. In §2, the required preliminaries are stated. In §3, definite and quasi-definite primitive axial algebras of Jordan type 1/2 are defined and different examples of them are given.

In §4, we state that the element $x_a(b)=\dfrac{2ab-(a,b)a-b}{(a,b)-1}$ constructed by two distinct axes a,b of a quasi-definite axial algebra A of Jordan type 1/2 is a primitive semisimple idempotent in A (Lemma 9). In §5, we show that if A is also strongly axial, then $x_a(b)$ is an axis in both A and $A_0(a)$.

In §6, we prove the first of the two main results of the work: any semisimple finite-dimensional quasi-definite primitive strongly axial algebra of Jordan type 1/2 is unital (Theorem 1).

In §7, we prove that a finitely generated quasi-definite primitive strongly axial algebras of Jordan type 1/2 has a finite capacity (Theorem 2).

By a subalgebra of axial algebra we mean only a subspace closed under the product.

2. Preliminaries

We consider only commutative (but not necessarily associative or unital) algebras over a ground field F of characteristic not two. Given an element $a \in A$ and $\lambda \in F$, we introduce the subspace $A_\lambda(a) = \{x \in A \mid ax = \lambda x\}$. Define the map $\mathrm{ad}_a \colon A \to A$ as follows, $\mathrm{ad}_a(x) = ax$. If $\lambda \in \mathrm{Spec}(\mathrm{ad}_a)$, then $A_\lambda(a) \neq (0)$, otherwise $A_\lambda(a) = (0)$. An idempotent $e \in A$ is called semisimple if ad_a is diagonalizable. An idempotent $e \in A$ is called primitive if $A_1(e) = \mathrm{Span}\{e\}$.

A (primitive) *axial algebra of Jordan type* $\eta \neq 0, 1$, is, by the definition, a commutative algebra generated by (primitive) semisimple idempotents a_i, $i \in I$, such that $\mathrm{Spec}(\mathrm{ad}_{a_i}) \subset \{0, \eta, 1\}$, and the following fusion rules are fulfilled,

$$\begin{gathered} A_0(a_i)^2 \subseteq A_0(a_i), \quad A_\eta(a_i)^2 \subseteq A_0(a_i) + A_1(a_i), \\ (A_0(a_i) + A_1(a_i))A_\eta(a_i) \subseteq A_\eta(a_i), \quad A_0(a_i)A_1(a_i) = (0). \end{gathered} \tag{1}$$

In particular, it implies that $(A_0(a_i) \oplus A_1(a_i)) \oplus A_\eta(a)$ is a $\mathbb{Z}_2$-graded algebra.

Given a primitive axial algebra A of Jordan type $1/2$ and an axis $a \in A$, the map $\tau_a \colon A \to A$ which acts as follows, $\tau_a(x) = (-1)^{2\lambda} x$ for $x \in A_\lambda(a)$, is an involution of A called Miyamoto.

Primitive axial algebras of Jordan type are known to have some significant properties. Every primitive axial algebra of Jordan type is spanned as a vector space by axes [4]. Moreover, any primitive axial algebra A of Jordan type admits a unique Frobenius form, a nonzero bilinear symmetric form $(\cdot,\cdot) \colon A \times A \to F$ which is invariant, i. e., $(ab, c) = (a, bc)$ for all $a, b, c \in A$, and which satisfies the property that $(a, a) = 1$ for every axis $a \in A$ [5].

Given a primitive axial algebra of Jordan type A, A_μ and A_λ are orthogonal with respect to the Frobenius form when $\lambda \neq \mu$. The radical of the Frobenius form $A^\perp := \{x \in A \mid (x, v) = 0 \text{ for all } v \in A\}$ coincides with the unique largest ideal $R(A)$ of A containing no axes from A [4]. A primitive axial algebra A of Jordan type is called semisimple if $R(A) = (0)$.

Given an algebra A, by $\langle X \rangle_{\mathrm{alg}}$ we denote the subalgebra of A generated by the set X. The set of all words in an alphabet X we denote as X^*.

Lemma 1. *(Seress Lemma,* [4, Lemma 4.3]*) Given a primitive axial algebra A of Jordan type $1/2$ and an axis $a \in A$, we have $a(xz) = (ax)z$ for all $x \in A$ and $z \in A_0(a) \oplus A_1(a)$.*

Proposition 1 ([4, §4])**.** *Let $A = \langle a, b \rangle_{\mathrm{alg}}$ be a primitive axial algebra of Jordan type $1/2$ generated by two distinct axes a, b. Denote $\sigma = ab - (a+b)/2$ and $\alpha = (a, b)$.*

a) A is 2-dimensional precisely in the following cases:

(1) $ab = 0$, then $A = Fa \oplus Fb$ and $\alpha = 0$;

(2) $\sigma = 0$ and $\alpha = 1$.

b) A is 3-dimensional precisely when $\sigma, ab \neq 0$ and $\sigma v = \pi v$ for $v \in \{a, b, \sigma\}$, where $\pi = (\alpha - 1)/2$. Moreover, A is unital if and only if $\alpha \neq 1$, in which case the unit equals σ/π.

Corollary 1. *Let $A = \langle a, b \rangle_{\mathrm{alg}}$ be a primitive axial algebra of Jordan type $1/2$ generated by two distinct axes a, b. Denote $\alpha = (a, b)$. Then we have*

$$(ab)b = \frac{1}{2}(\alpha b + ab), \quad (ab)a = \frac{1}{2}(\alpha a + ab), \quad (ab)(ab) = \frac{\alpha}{4}(a + b + 2ab). \tag{2}$$

Given axes $a, b \in A$, we write

$$b = a_0(b) + a_{1/2}(b) + \alpha(b)a,$$

where $a_0(b) \in A_0(a)$, $a_{1/2}(b) \in A_{1/2}(a)$, and $\alpha(b) = (a, b) \in F$.

Lemma 2. *Let A be a primitive axial algebra of Jordan type $1/2$, let $a, b \in A$ be axes, $\alpha = (a, b)$. Then we have the following equalities,*

$$a_0(b)^2 = (1-\alpha)a_0(b), \tag{3}$$

$$a_{1/2}(b)^2 = \alpha a_0(b) + (\alpha - \alpha^2)a, \tag{4}$$

$$a_0(b)a_{1/2}(b) = \frac{1}{2}(1-\alpha)a_{1/2}(b). \tag{5}$$

Proof. Denote $a_0 = a_0(b)$ and $a_{1/2} = a_{1/2}(b)$. We have $b = a_0 + a_{1/2} + \alpha a$. Thus,

$$a_0 + a_{1/2} + \alpha a = b = b^2 = (a_0 + a_{1/2} + \alpha a)^2 = a_0^2 + a_{1/2}^2 + \alpha^2 a + 2a_0 a_{1/2} + \alpha a_{1/2}. \tag{6}$$

The equality of components on $A_{1/2}(a)$ gives (5).

Also, we have $a_0 = b - 2ab + \alpha a$. So,

$$a_0^2 = b + 4(ab)(ab) + \alpha^2 a - 4(ab)b + 2\alpha ab - 4\alpha(ab)a.$$

Applying the equalities (2), we derive

$$a_0^2 = (1-\alpha)b - 2(1-\alpha)ab + \alpha(1-\alpha)a = (1-\alpha)a_0,$$

it is (3). Substituting (3) in (6) and looking at summands in $A_0(a) \oplus A_1(a)$, we obtain (4). $\square$

Proposition 2 ([3, Theorem 1])**.** *Let $A = \langle a, b, c\rangle_{\mathrm{alg}}$ be a primitive axial algebra of Jordan type $1/2$ generated by axes a, b, c. Then A is a Jordan algebra and* $\dim A \leq 9$.

Lemma 3. *Let A be a primitive axial algebra of Jordan type $1/2$ with unit e. Then $(e, a) = 1$ for every axis a.*

Proof. It follows from the properties of the Frobenius form,

$$(e, a) = (e, a^2) = (ea, a) = (a, a) = 1. \quad \square$$

Lemma 4. *Let A be a primitive axial algebra of Jordan type $1/2$, and let a be an axis in A. Suppose that $A_0(a)$ is a primitive axial algebra of Jordan type $1/2$, then $R(A_0(a)) = R(A) \cap A_0(a)$. In particular, $R(A_0(a)) = (0)$ when A is semisimple.*

Proof. Since the Frobenius form is defined on an axial algebra uniquely, the restriction of the Frobenius form from A on $A_0(a)$ coincides with the one defined on $A_0(a)$. The inclusion $R(A) \cap A_0(a) \subseteq R(A_0(a))$ is trivial. Suppose that $x \in R(A_0(a)) = (A_0(a))^\perp$, then $x \in R(A)$, since both $A_{1/2}(a)$ and $A_1(a)$ are orthogonal to $A_0(a)$ in A. $\square$

3. Definite and quasi-definite axial algebras

Definition 1. *Given a primitive axial algebra A of Jordan type and a Frobenius form $(\cdot,\cdot)$ on it, we call A anisotropic if $(x, x) = 0$ implies $x = 0$.*

Thus, an anisotropic primitive axial algebra is semisimple.

Definition 2. *We call an anisotropic primitive axial algebra of Jordan type $1/2$* **definite**.

Definition 3. *We call a primitive axial algebra A of Jordan type $1/2$* **quasi-definite***, if we have $(a, b) \neq 1$ for every pair of distinct axes $a, b \in A$.*

Definition 4. *Let X be a basis of a primitive axial algebra A of Jordan type $1/2$ consisting of axes. We say that X is a quasi-definite basis of A if $(x_i, x_j) \neq 1$ for distinct $x_i, x_j \in X$.*

The following lemma provides a sufficient condition, under which an axial algebra is quasi-definite.

Lemma 5. *Every definite primitive axial algebra of Jordan type $1/2$ is quasi-definite.*

Proof. Denote by A a definite primitive axial algebra of Jordan type $1/2$ and consider its distinct axes a and b. Suppose that $(a, b) = 1$. Then by Lemma 2, $a_{1/2}^2 = a_0$. Thus, $(a_{1/2}, a_{1/2}) = 2(a_{1/2}, a_{1/2}a) = 2(a_{1/2}^2, a) = 2(a_0, a) = 0$. Since A is anisotropic, $a_{1/2} = 0$ and so, $a_0 = 0$. Hence, $a = b$, a contradiction. □

Further, we provide an example of quasi-definite but not definite primitive axial algebra (see [4]).

Let us show that a unital 3-dimensional 2-generated primitive axial algebra A of Jordan type $1/2$ over a quadratically closed field F (it means that the equation $x^2 - a = 0$ has solutions for all $a \in F$) of characteristic not two is quasi-definite. It is known that A is isomorphic to the simple Jordan algebra of a symmetric bilinear non-degenerate form f defined on a two-dimensional vector space V, i. e., $A = F1 \oplus V$ with the product

$$(\alpha 1 + x)(\beta 1 + y) = (\alpha\beta + f(x, y))1 + \alpha y + \beta x, \tag{7}$$

where $\alpha, \beta \in F$ and $x, y \in V$.

Let us show that over F the 3-dimensional 2-generated axial algebra A is unique (up to isomorphism). One may diagonalize a quadratic form over any field. Thus, there exist $e_1, e_2 \in V$ such that $f(e_i, e_i) = d_i$ and $f(e_1, e_2) = 0$. Since f is non-degenerate, we have $d_1, d_2 \neq 0$. Define $u = e_1/\sqrt{d_1}$ and $v = e_2/\sqrt{d_2}$. Then, in the basis $1, u, v$, we have the following multiplication table:

$$1^2 = 1, \quad 1 \cdot u = u, \quad 1 \cdot v = v, \quad u^2 = v^2 = 1, \quad uv = 0.$$

Therefore, we get the unique algebraic structure. Moreover, we have a Frobenius form

$$(1, 1) = (u, u) = (v, v) = 2, \quad (1, u) = (1, v) = (u, v) = 0.$$

Firstly, we find all idempotents in A. Let $e = \alpha 1 + \beta u + \gamma v$, then $e^2 = e$ is equivalent to the relations

$$\alpha^2 + \beta^2 + \gamma^2 = \alpha, \quad 2\alpha\beta = \beta, \quad 2\alpha\gamma = \gamma.$$

If $\alpha = 0$, then $\beta = \gamma = 0$ and $e = 0$. Otherwise, either $e = 1$ or $\alpha = 1/2$ and $\beta^2 + \gamma^2 = 1/4$.

Let us check that a non-unital idempotent e is a primitive one. Indeed, $A_0(e) = \mathrm{Span}\{1/2 - (\beta u + \gamma v)\}$, $A_{1/2}(e) = \mathrm{Span}\{-\gamma u + \beta v\}$, and $A_1(e) = \mathrm{Span}\{e\}$.

Note that the definition of the Frobenius form is correct, since for every primitive idempotent e, we have

$$(e, e) = (1/2 + \beta u + \gamma v, 1/2 + \beta u + \gamma v) = 1/2 + 2(\beta^2 + \gamma^2) = 1/2 + 1/2 = 1.$$

Suppose that $(e, e') = 1$, where $e = 1/2 + \beta u + \gamma v$ and $e' = 1/2 + \delta u + \varepsilon v$ are distinct axes. Then we check that

$$(e, e') = (1/2 + \beta u + \gamma v, 1/2 + \delta u + \varepsilon v) = 1/2 + 2(\beta\delta + \gamma\varepsilon) = 1,$$

so, $\beta\delta+\gamma\varepsilon = 1/4$. Thus, we have $\begin{pmatrix} \beta & \gamma \\ \delta & \varepsilon \end{pmatrix} w = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, where $w = (\beta-\delta, \gamma-\varepsilon)^T$ in the basis u, v. If $e \neq e'$, then the matrix is degenerate. Suppose that $\delta u+\varepsilon v = k(\beta u+\gamma v)$ for some $k \in F$. Therefore,

$$1/4 = \beta\delta + \gamma\varepsilon = k(\beta^2 + \gamma^2) = k/4.$$

It means that $k = 1$ and $e = e'$.

However, A is not definite, e. g., $(1 + iu, 1 + iu) = 0$. Here i is a solution of the equation $x^2 + 1 = 0$. Since F is quadratically closed, $i \in F$.

Now, consider the case of $(n + 1)$-dimensional Jordan algebra $J = F1 \oplus V$ of Clifford type (also known as a spin factor) with the product defined by (7). Let us restrict the conditions as follows, we assume that $F = \mathbb{R}$ and the form f is positive-definite. Then we may find a linear basis $v_1, \ldots, v_n$ of V such that the Frobenius form satisfies $(1, 1) = (v_1, v_1) = \ldots = (v_n, v_n) = 2$ and $(1, v_i) = (v_i, v_j) = 0$ for all $i, j \in \{1, \ldots, n\}$ and $i \neq j$. Thus, $a_i = (1 + v_i)/2$, $i = 1, \ldots, n$, are axes generating J and satisfying $(a_i, a_i) = 1$. For every $x \in J$, we have a representation $x = \alpha 1 + \sum\limits_{i=1}^{n} \beta_i v_i$. Then $(x, x)/2 = \alpha^2 + \sum\limits_{i=1}^{n} \beta_i^2$. If $(x, x) = 0$, then $\alpha = \beta_1 = \ldots = \beta_n = 0$ and $x = 0$. Hence, J is a definite axial algebra of Jordan type $1/2$. Note that there is no contradiction to the previous example, since the field $\mathbb{R}$ is not quadratically closed.

Denote by $M_n^{(+)}(F)$ the matrix algebra $M_n(F)$ under the product $A \circ B = (AB + BA)/2$. It is well-known that $M_n^{(+)}(F)$ is a simple Jordan algebra. Thus, it is an axial algebra of Jordan type $1/2$. Moreover, its Frobenius form coincides with the trace one, i. e. $(X, Y) = \mathrm{tr}(X \circ Y) = \mathrm{tr}(XY)$ for all $X, Y \in M_n^{(+)}(F)$.

The matrix unity is denoted by e_{ij}, $1 \leq i, j \leq n$.

Lemma 6. *A matrix $X \in M_n^{(+)}(F)$ is a primitive axis if and only if $X^2 = X$ and* $\mathrm{rank}\, X = 1$.

Proof. Let X be an idempotent matrix from $A = M_n^{(+)}(F)$. Then X is conjugate to the matrix $e_{11} + \ldots + e_{rr}$, where $r = \mathrm{rank}\, X$. Note that $A_1(X) = \{S \in M_n^{(+)}(F) \mid X \circ S = S\} = \mathrm{Span}\{e_{11}, \ldots, e_{rr}\}$. So, X is primitive if and only if $r = 1$. □

For $n \geq 2$, $M_n^{(+)}(F)$ is not quasi-definite. Indeed, consider primitive axes $X = e_{11}$ and $Y = e_{11}+e_{12}$, then $(X, Y) = 1$. On the other hand, $M_n^{(+)}(F)$ has a quasi-definite basis of axes.

Given $a = (a_1, \ldots, a_n) \in F_n$ and $b = (b_1, \ldots, b_n) \in F_n$, introduce $\langle a, b\rangle = \sum\limits_{j=1}^{n} a_j b_j$.

Proposition 3. *Let F be an infinite field. Then $M_n^{(+)}(F)$ is a primitive axial algebra of Jordan type $1/2$ which has a quasi-definite basis of axes.*

Proof. We prove the statement by induction on n. For $n = 1$, we take the matrix e_{11} as the required basis.

Suppose that we have found quasi-definite basis for $M_n^{(+)}(F)$ consisting of matrices $H_1, \ldots, H_{n^2}$. By Lemma 6, since $\operatorname{rank} H_i = 1$, we may present $H_i = l_i^T r_i$, where $l_i = (l_{i1}, \ldots, l_{in})$ and $r_i = (r_{i1}, \ldots, r_{in})$ are vectors from F_n. Then $H_1, \ldots, H_{n^2}$ form a quasi-definite basis of axes if and only if

$$\langle l_i, r_i \rangle = 1,\ i = 1, \ldots, n^2, \quad \langle l_i, r_j \rangle \langle l_j, r_i \rangle \neq 1,\ i \neq j,\ i, j \in \{1, \ldots, n^2\}.$$

Now consider the algebra $A = M_{n+1}^{(+)}(F)$. We identify matrices H_i with their images in A under the embedding $\psi\colon M_n^{(+)}(F) \to M_{n+1}^{(+)}(F)$ defined as follows, $\psi(e_{ij}) = e_{ij}$, $i, j = 1, \ldots, n$.

Introduce matrices

$$F_i(b_i, d_i) = (1 - d_i)e_{ii} + b_i e_{i\,n+1} + \frac{d_i(1 - d_i)}{b_i} e_{n+1\,i} + d_i e_{n+1\,n+1}, \quad i = 1, \ldots, n.$$

By Lemma 6, they are all primitive axes.

We want to find $d_i, b_i, c_i \in F$ for $i = 1, \ldots, n$ such that

$$S = \{H_1, \ldots, H_{n^2}, F_1(b_1, d_1), F_1(c_1, d_1), \ldots, F_n(b_n, d_n), F_n(c_n, d_n), e_{n+1\,n+1}\}$$

is a required quasi-definite basis of A.

For each $i = 1, \ldots, n$, we take nonzero b_i, c_i such that $b_i \neq \pm c_i$ and $d_i \neq 0, 1$. The inductive hypothesis implies that the set S of primitive idempotents forms a basis of A.

It remains to check that $(X, Y) \neq 1$ for all $X, Y \in S$, $X \neq Y$. By the inductive hypothesis $(H_i, H_j) \neq 1$ when $i \neq j$. Note that $(H_i, e_{n+1\,n+1}) = 0$. We have $(F_i(b_i, d_i), e_{n+1\,n+1}) = (F_i(c_i, d_i), e_{n+1\,n+1}) = d_i \neq 1$.

Further, we have

$$(H_i, F_j(x, d_j)) = (1 - d_j) l_{ij} r_{ij},$$

where $x = b_i$ or $x = c_i$, for all $i = 1, \ldots, n^2$ and $j = 1, \ldots, n$. Thus, we take d_j such that $1/(1 - d_j) \notin K_j = \{l_{ij} r_{ij} \mid i = 1, \ldots, n^2\}$.

Now, we compute

$$(F_i(x, d_i), F_j(y, d_j)) = d_i d_j, \quad i \neq j,$$

where $x \in \{b_i, c_i\}$ and $y \in \{b_j, c_j\}$. Thus, we take d_k in such a way that $d_i d_j \neq 1$ for all $i, j = 1, \ldots, n$.

Finally, we get for $i = 1, \ldots, n$,

$$\begin{aligned}(F_i(b_i, d_i), F_i(c_i, d_i)) &= \operatorname{tr}\left(\begin{pmatrix} 1 - d_i & b_i \\ \frac{d_i(1-d_i)}{b_i} & d_i \end{pmatrix} \begin{pmatrix} 1 - d_i & c_i \\ \frac{d_i(1-d_i)}{c_i} & d_i \end{pmatrix} \right) \\ &= (1 - d_i)^2 + d_i^2 + d_i(1 - d_i)\left(\frac{b_i}{c_i} + \frac{c_i}{b_i}\right) = 1 + d_i(1 - d_i)\left(\frac{b_i}{c_i} + \frac{c_i}{b_i} - 2\right) \neq 1\end{aligned}$$

when $b_i \neq c_i$.

Summarizing, we choose nonzero b_i, c_i, $i = 1, \ldots, n$, such that $b_i \neq \pm c_i$. Also, we choose $d_i \notin \{0, 1\}$ and $1/(1 - d_i) \notin K_i$ and, moreover, $d_i d_j \neq 1$ for all $i, j = 1, \ldots, n$. We may do it, since F is infinite. □

Let $n \geq 2$, denote by $H_n(F)$ the set of all symmetric matrices of order n over F. The space $H_n(F)$ under the product $A \circ B = (AB + BA)/2$ is also a simple Jordan algebra. By [7, Theorem 3.4], its Jordan subalgebra $H_n'(F)$ consisting of all symmetric matrices with zero row sum has a quasi-definite basis, since $H_n'(F)$ is isomorphic to the Matsuo algebra $M_{1/2}(G,D)$ for corresponding group G and set of involution D.

Let us recall the definition of a Matsuo algebra. Given a group G generated by a set D of involutions, the Matsuo algebra $M_\eta(G,D)$, where $\eta \neq 0,1$, is a vector space $\mathrm{Span}\{D\}$ with the product

$$c \cdot d = \begin{cases} c, & |cd| = 1, \\ 0, & |cd| = 2, \\ \frac{\eta}{2}(c + d - c^d), & |cd| = 3. \end{cases}$$

The Frobenius form on $M_\eta(G,D)$ is defined as follows,

$$(c,d) = \begin{cases} 1, & c = d, \\ 0, & |cd| = 2, \\ \frac{\eta}{2}, & |cd| = 3. \end{cases} \tag{8}$$

If $\eta = 1/2$, then $M_{1/2}(G,D)$ is a primitive axial algebra of Jordan type $1/2$. By (8), $M_{1/2}(G,D)$ has a quasi-definite basis. If the automorphism group of $M_{1/2}(G,D)$ coincides with the Miyamoto group, then $M_{1/2}(G,D)$ is quasi-definite.

For the algebra $H_n(F)$ over the field of real numbers, we may say much more.

Proposition 4. *Algebra $H_n(\mathbb{R})$ is a definite primitive axial algebra of Jordan type $1/2$.*

Proof. For the Frobenius form $(X,Y) = \mathrm{tr}(XY^T)$, we have the Cauchy—Bunyakovsky—Schwarz inequality $\mathrm{tr}(XY^T)^2 \leq \mathrm{tr}(XX^T)\mathrm{tr}(YY^T)$. Applying it for pairwise distinct axes A and B, we get $(A,B)^2 \leq (A,A)(B,B) = 1$, and we have the equality if and only if A and B are linearly dependent. We conclude that $A = B$, a contradiction. □

Remark 1. *For $n = 2$, the statement holds over every field, since $H_2(F)$ coincides with 3-dimensional simple Jordan algebra of Clifford type considered above.*

4. Idempotent $x_a(b)$ in quasi-definite axial algebra

Given a primitive axial algebra A of Jordan type $1/2$ and distinct axes $a, b \in A$, we introduce the element

$$x_a(b) = \frac{2ab - (a,b)a - b}{(a,b) - 1} = \frac{a_0(b)}{1 - (a,b)}. \tag{9}$$

It is clear that element $x_a(b)$ is defined only if $(a,b) \neq 1$. By Lemma 2, $x_a(b)$ is nonzero.

Lemma 7. *Let A be a primitive axial algebra of Jordan type $1/2$, and let $a, b \in A$ be axes such that $(a,b) \neq 1$. Then $x_a(b)$ is an idempotent, $a + x_a(b)$ is a unit in $\langle a,b\rangle_{\mathrm{alg}}$, and $(x_a(b), x_a(b)) = 1$.*

Proof. If $\dim(\langle a,b\rangle_{\mathrm{alg}}) = 3$, then by Proposition 1b, σ/π is a unit of $\langle a,b\rangle_{\mathrm{alg}}$. Thus, $x_a(b) = \sigma/\pi - a$ is an idempotent. If $\dim(\langle a,b\rangle_{\mathrm{alg}}) = 2$, then $\langle a,b\rangle_{\mathrm{alg}} = Fa \oplus Fb$ by Proposition 1a. Thus, $x_a(b) = b$.

Denote $\alpha = (a,b)$ and calculate by Lemma 2:

$$(x_a(b), x_a(b)) = \frac{1}{(1-\alpha)^2}(a_0(b), a_0(b))$$

$$= \frac{1}{(1-\alpha)^2}((a_0(b)+a_{1/2}(b)+\alpha a, a_0(b)+a_{1/2}(b)+\alpha a)-(a_{1/2}(b), a_{1/2}(b))-\alpha^2(a,a))$$

$$= \frac{1}{(1-\alpha)^2}(1-\alpha^2-2(a_{1/2}, a_{1/2}a)) = \frac{1}{(1-\alpha)^2}(1-\alpha^2-2(a_{1/2}^2, a)) = 1. \quad \square$$

Lemma 8. *Let A be a primitive axial algebra of Jordan type $1/2$. Let a,b,c be axes such that $(a,b) \neq 1$ and $C = \langle a,b,c\rangle_{\mathrm{alg}}$. Then $x_a(b)$ is a primitive axis in C.*

Proof. By Lemma 7, $x_a(b)$ is an idempotent in C. By Proposition 2, $x_a(b)$ is a semisimple idempotent in C with the fusion rules (1) fulfilled. So, it remains to check that $x_a(b)$ is a primitive idempotent.

When C is the universal 3-generated 9-dimensional axial algebra U_3 of Jordan type $1/2$, it follows by [2, Code#1]. The fact that $x_a(b)$ is a primitive axis in every quotient of U_3 by the ideal I not containing $x_a(b)$ follows from the property that I contains all homogeneous components of its elements. Indeed, let $y \in U_3$ be such that $x_a(b)y - y \in I$, i.e., $x_a(b)y = y$ in U_3/I. Since $y = y_0 + y_{1/2} + \alpha x_a(b)$ with respect to the idempotent $x_a(b)$, we get

$$x_a(b)y - y = -(y_0 + y_{1/2}/2) \in I.$$

Thus, $y_0, y_{1/2} \in I$ and $y \in \mathrm{Span}\{x_a(b)\} + I$, it means that $x_a(b)$ is primitive in U_3/I. $\square$

Lemma 9. *Let A be a primitive axial algebra A of Jordan type $1/2$. Let a,b be two distinct axes such that $(a,b) \neq 1$. Then $c = x_a(b)$ is a primitive semisimple idempotent in A, and eigenvalues of ad_c lie in the set $\{0, 1/2, 1\}$.*

Proof. By Lemma 7, $c = x_a(b)$ is an idempotent.

Let B be a linear basis of A consisting of axes. We may assume that $a,b \in B$. Let us state that

$$A = Y_0(c) + Y_{1/2}(c) + \mathrm{Span}\{c\}, \tag{10}$$

where $Y_\varepsilon(c) = \{z \in A \mid cz = \varepsilon z\}$. By Lemma 8, for every $r \in B$ we have $r = r_0(c) + r_{1/2}(c) + \beta_r x_a(b)$. Here $r_0(c) \in Y_0(c)$, $r_{1/2}(c) \in Y_{1/2}(c)$, and $\beta_r \in F$. Since each element of the basis has a decomposition on the eigensubspaces of $\mathrm{ad}_{x_a(b)}$, the statement follows. $\square$

Suppose that B is a basis of an axial algebra A of Jordan type $1/2$ consisting of axes such that $(a,y) \neq 1$ for a fixed $a \in B$ and every $y \in B \setminus \{a\}$. Define $B_0 = \{x_a(y) \mid y \in B \setminus \{a\}\}$.

Lemma 10. *Let A be a primitive axial algebra A of Jordan type $1/2$, and let A have a quasi-definite basis B. For $a \in B$, we have $\mathrm{Span}\{B_0\} = A_0(a)$.*

Proof. Let $z \in A_0(a)$, we may present it as a linear combination of elements from B,

$$z = \sum_{j\in J} \varkappa_j b_j = \sum_{j\in J_1} \varkappa_j b_j + \sum_{j\in J_2} \varkappa_j b_j,$$

where $J = J_1 \dot\cup J_2$ and $j \in J_2$ if and only if $a_0(b_j) = 0$. Also each $b_j \in B$ we write as $b_j = a_0(b_j) + a_{1/2}(b_j) + (a,b_j)a$.

For $j \in J_1$, we have $a_0(b_j) = (1-(a,b_j))x_a(b_j)$. Since A is a direct vector-space sum of its subspaces $A_0(a)$, $A_{1/2}(a)$, and $\mathrm{Span}\{a\}$, z lies in $\mathrm{Span}\{B_0\}$. ☐

Let a,b,c be axes in an axial algebras of Jordan type $1/2$ such that $(a,b),(a,c),(b,c) \neq 1$. The following example shows that it may happen $(x_a(b),x_a(c)) = 1$ when $x_a(b) \neq x_a(c)$.

Example 1. *Let $A = A(\alpha,\beta,\gamma,\phi)$ be the universal 3-generated axial algebra of Jordan type $1/2$. Assume that $\alpha = (a,b)$, $\beta = (b,c)$ and $\gamma = (a,c)$ are not equal to 1. For the axes $x_a(b)$ and $x_a(c)$, we have by* [2, Code#2]

$$(x_a(b),x_a(c)) = \frac{-\alpha\gamma - \beta + 2\phi}{-\alpha\gamma + \alpha + \gamma - 1},$$

which is equal to 1 if and only if $\alpha+\beta+\gamma-2\phi-1 = 0$. However, $x_a(b) \neq x_a(c)$.

Lemma 11. *Let A be a primitive axial algebra of Jordan type $1/2$. Let a,b be axes of A such that $(a,b) \neq 0,1$. Then $A_0(a) \cap A_{1/2}(b) = (0)$.*

Proof. Put $\alpha = (a,b)$. For $z \in A_0(a) \cap A_{1/2}(b)$, we compute

$$z/2 = zb = z(a_0(b) + a_{1/2}(b) + \alpha a) = za_0(b) + za_{1/2}(b).$$

Since $za_0(b) \in A_0(a)$, $za_{1/2}(b) \in A_{1/2}(a)$, we get $za_{1/2}(b) = 0$ and $zx_a(b) = z/(2-2\alpha)$. By Lemma 9, $\mathrm{ad}_{x_a(b)}$ has eigenvalues only from the set $\{0,1/2,1\}$. Since $1/(2-2\alpha) \neq 0$ and $1/(2-2\alpha) \neq 1/2$ when $\alpha \neq 0$, we have the only case $\alpha = 1/2$ and $z \in A_1(x_a(b))$. Since $x_a(b)$ is primitive, we derive that $z = kx_a(b)$ for some $k \in F$. If $k \neq 0$, then $(x_a(b))^2 = x_a(b)$ lies in $A_{1/2}(b)$ and in $A_{1/2}(b)^2 \subset A_0(b) + \mathrm{Span}(b)$ at the same time. Thus, $z = 0$. ☐

5. Subalgebra $A_0(a)$ in quasi-definite strongly axial case

Definition 5. *We call a primitive axial algebra A of Jordan type $1/2$* **strongly axial**, *if every primitive semisimple idempotent with eigenvalues $\{0,1/2,1\}$ in A is a primitive axis.*

Definition 5 says that in a strongly axial algebra, the fusion rules (1) hold for every primitive semisimple idempotent. Every primitive axial algebra of Jordan type $1/2$ which is also a Jordan one is strongly axial [1].

Lemma 12. *Let A be a primitive axial algebra of Jordan type, let B be a subalgebra of A, and let $a \in B$ be an axis in A. Then a is an axis in B.*

Proof. The equality $B = B_0(a) \oplus B_{1/2}(a) \oplus \mathrm{Span}\{a\}$ holds due to the fact that ad_a is a semisimple operator on A with the eigenvalues $\{0,1/2,1\}$. Since $B_0(a) = B \cap A_0(a)$, $B_{1/2}(a) = B \cap A_{1/2}(a)$, the fusion rules (1) are automatically fulfilled in B. ☐

Proposition 5. *Let A be a quasi-definite primitive strongly axial algebra of Jordan type $1/2$ and let B be a subalgebra of A. Let $a,b \in B$ be distinct axes in A, then $x_a(b)$ is a primitive axis in A, B, and in $A_0(a) \cap B$.*

Proof. By Lemma 9, $x_a(b)$ is a primitive semisimple idempotent with eigenvalues $\{0,1/2,1\}$ in A. Since A is strongly axial, then $x_a(b)$ is a primitive axis in A.

By Lemma 12, $x_a(b)$ is an axis in B and $A_0(a) \cap B$ too. ☐

Corollary 2. *Let A be a quasi-definite primitive strongly axial algebra of Jordan type $1/2$, and let a be an axis in A, and $\dim A > 1$. Then $A_0(a)$ is an axial algebra of Jordan type $1/2$ which contains a quasi-definite basis.*

Proof. Let X be a basis of A consisting of axes. From Lemma 10 it follows that $X_0 = \{x_a(x_i) \mid x_i \in X\}$ includes a basis of $A_0(a)$ which by Proposition 5 consists of axes in $A_0(a)$. Since all $x_a(x_i)$ are axes in A, this basis is quasi-definite. □

Lemma 13. *Let A be a quasi-definite primitive strongly axial algebra, and let a and b be axes. We have $(a, b) = 0$ if and only if $b \in A_0(a)$.*

Proof. If $b \in A_0(a)$, then $(a, b) = 0$ by definition.

Assume that $b \notin A_0(a)$. Since $(a, b) = 0$, we have $x_a(b) = a_0(b)$. By Proposition 5, $a_0(b)$ is an axis of A. Therefore, applying orthogonality of distinct eigenspaces due to the Frobenius form, we compute

$$(a_0(b), b) = (a_0(b), a_0(b) + a_{1/2}(b) + \alpha a) = (a_0(b), a_0(b)) = 1.$$

Hence, we get a contradiction with the definition of quasi-definite axial algebra. □

Lemma 14. *Let A be a quasi-definite primitive strongly axial algebra of Jordan type $1/2$ with a generating set X of axes. Let $w \in X^* \setminus \{1\}$. Then there exists a linear combination s of words of the length less than $|w|$ such that $q = w + s$ is (up to a nonzero scalar) an axis of A.*

Proof. Let us prove by induction on $|w|$. If $|w| = 1$, then the statement is trivial. When $w = ab$ for $a, b \in X$, it follows by the formula (9), where we take $q = x_a(b)$.

Suppose that $w = w_1 a$ for some $w_1 \in X^*$ and $a \in X$. The induction hypothesis states that there exist a linear combination s_1 of words of the length less than $|w_1|$ and a nonzero scalar α such that $q_1 = \alpha(w_1 + s_1)$ is an axis of A. Thus, the following equality $w + s_1 a = (w_1 + s_1)a = (1/\alpha)q_1 a$ reduces the general case to the already studied case when $|w| = 2$. □

6. Sufficient conditions to contain unit

Lemma 15. *Let A be a primitive axial algebra of Jordan type $1/2$, and let a be an axis in A such that $A_0(a)$ contains a unit $e_0(a)$. Then we have $(e_0(a) + a, b) = 1$ for every axis $b \in A$ satisfying the condition that $(a, b) \neq 1$.*

Proof. Denote $\alpha = (a, b)$, $a_0 = a_0(b)$, $a_{1/2} = a_{1/2}(b)$. Consider $e = e_0(a) + a$ and compute

$$eb = (e_0 + a)(a_0 + a_{1/2} + \alpha a) = a_0 + e_0 a_{1/2} + (1/2)a_{1/2} + \alpha a = b - (1/2)a_{1/2} + e_0 a_{1/2}.$$

Further, using pairwise orthogonality of $A_0(a)$, $A_{1/2}(a)$, and $A_1(a)$, we deduce

$$\begin{aligned}(11)\quad (e, b) = (eb, b) &= (b, b - (1/2)a_{1/2} + e_0 a_{1/2}) \\ &= 1 - (1/2)(a_{1/2}, a_{1/2}) + (a_{1/2}, e_0 a_{1/2}).\end{aligned}$$

We apply Lemmas 2 and 7 and the condition that $(a, b) \neq 1$ to compute

$$\begin{aligned}&(a_{1/2}, e_0 a_{1/2}) = (a_{1/2}^2, e_0) = (\alpha a_0 + \alpha(1 - \alpha)a, e_0) = \alpha(a_0, e_0) \\ &= \alpha(1-\alpha)(x_a(b), e_0) = \alpha(1-\alpha)(x_a(b)^2, e_0) = \alpha(1-\alpha)(x_a(b), x_a(b)) = \alpha(1-\alpha), \\ &(a_{1/2}, a_{1/2}) = 2(aa_{1/2}, a_{1/2}) = 2(a, a_{1/2}^2) = 2(a, \alpha a_0 + \alpha(1 - \alpha)a) = 2\alpha(1 - \alpha).\end{aligned}$$

Hence, $(e, b) = 1$. □

Lemma 16. *Let A be a semisimple primitive axial algebra of Jordan type $1/2$ with a quasi-definite basis X. Suppose that $A_0(x_i)$ contains a unit $e_0(x_i)$ for each $x_i \in X$. Then $e = e_0(x_1) + x_1$ is a unit of A.*

Proof. Let $a, b \in X$ be distinct. Denote $e = e_0(a) + a$ and $\tilde{e} = e_0(b) + b$. From Lemma 15 it follows that $(e, x_i) = (\tilde{e}, x_i) = 1$ for each $x_i \in X$.

Let $r = e - \tilde{e}$. For each $x_i \in X$, we have $(r, x_i) = 0$.

Since X is a linear basis (maybe, infinite) consisting of axes, $e - \tilde{e}$ lies in the radical $A^\perp$ of the Frobenius form. By $A^\perp = (0)$, we get $e = \tilde{e}$. Hence,

$$eb = \tilde{e}b = (e_0(b) + b)b = b^2 = b.$$

It holds for every axis $b \in X$, thus, e is a unit of A. □

Theorem 1. *Let A be a semisimple finite-dimensional quasi-definite primitive strongly axial algebra of Jordan type $1/2$. Then A is unital.*

Proof. Denote $N = \dim A$. If $N = 1$, then $A \cong F$ and A is unital. Let C_0 be a basis of A consisting of axes. Fix $a_0 \in C_0$, by Corollary 2, one may find a quasi-definite basis $C_1 \subset \{x_{a_0}(y) \mid y \in C_0 \setminus \{a_0\}\}$ of an axial algebra $A_0(a_0)$, which consist of axes of A by Proposition 5. Let us indicate that $A_0(a_0)$ may be not necessarily quasi-definite and strongly axial algebra. However, we do not need such conditions! Throughout the proof, we consider only elements of the form $x_y(z)$, where y, z are axes in A, thus, Proposition 5 implies that all such elements are axes in A as well as in all required axial subalgebras of A. Note that $A_0(a_0)$ is semisimple by Lemma 4.

Now, we choose $a_1 \in C_1$, by Corollary 2 one may find a quasi-definite basis $C_2 \subset \{x_{a_1}(y) \mid y \in C_1 \setminus \{a_1\}\}$ of an axial algebra $A_0(a_0)$, which consist of axes of A again by Proposition 5. We denote $(A_0(a_0))_0(a_1)$ as $A_0(a_0, a_1)$.

Continue on, since the choice of $a_0, \ldots, a_n, \ldots$ is not unique, we get a finite set of algebras of the form $A_0(a_0, \ldots, a_n)$, where $n < N$ and $A_0(a_0, \ldots, a_n)$ is defined by induction as $(A_0(a_0, \ldots, a_{n-1}))_0(a_n)$. Now, we prove by induction on the dimension that all of them are unital. If $\dim A_0(a_0, \ldots, a_n) = 1$, then $A_0(a_0, \ldots, a_n) \cong F$ and hence is unital.

Suppose it is proved that $A_0(a_0, \ldots, a_n)$ is unital if $\dim A_0(a_0, \ldots, a_n) < k$. Consider the case when $\dim Y = k$, where $Y = A_0(a_0, \ldots, a_n)$. Applying Lemma 16 and the induction hypothesis, we conclude that Y is unital. Therefore, by induction, $A_0(a)$ is unital for every $a \in C_0$. Then A is unital by Lemma 16. □

7. Finite capacity of unital algebras

Definition 6. *Let e be a unit of a primitive axial algebra A of Jordan type $1/2$. Suppose that e equals a sum of n pairwise orthogonal axes and n is minimal satisfying such property. We say that e has the* **capacity** *n and write it as $c(e) = n$.*

For the simple Jordan algebra $A = F1 + Fu + Fv$ of a nondegenerate form defined above, we have $e = (1/2 + (u+v)/\sqrt{2}) + (1/2 - (u+v)/\sqrt{2})$, it is a decomposition of e into a sum of two axes. Thus, $c(e) = 2$, and all decompositions of e into a finite sum of pairwise orthogonal axes consist of only two summands.

Something similar we have with the algebra $M_n^{(+)}$, now $c(e) = n$, and there is a natural decomposition $e = e_{11} + \ldots + e_{nn}$. Note that the decomposition is not unique, e. g., $e = (e_{11} + e_{12}) + (e_{22} - e_{12})$ for $n = 2$.

Lemma 17. *Let A be a quasi-definite primitive strongly axial algebra of Jordan type $1/2$. Let q, a, b be axes in A. Suppose that $qa = qx_a(b) = 0$. Then $qb = 0$.*

Proof. Denote $\alpha = (a, b)$. By Lemma 13, we prove the statement,

$$0 = (\alpha-1)(q, x_a(b)) = (q, 2ab-\alpha a-b) = 2(q, ab)-(q, b) = 2(qa, b)-(q, b) = -(q, b).$$

□

Theorem 2. *Let A be a quasi-definite primitive strongly axial algebra of Jordan type $1/2$ and let $G = \{a_1, \ldots, a_n\}$ be a generating set of A consisting of axes. Let e be a unit of A. Then e has a finite capacity $k \leq n$.*

Proof. Given a subalgebra B of A and $x \in B$, denote the eigensubspace of the operator $\mathrm{ad}_x \colon B \to B$ corresponding to the eigenvalue 0 by $A_0(x, B)$. Note that $e_2 = e - a_1$ is a unit of the algebra $A_2 = A_0(a_1, A)$. Denote

$$G_2(a_1) = \{b_1 = x_{a_1}(a_2), \ldots, b_{n-1} = x_{a_1}(a_n)\}. \tag{12}$$

By Corollary 2, the set G_2 contains a subset of the quasi-definite basis of A_2. On the next step, we fix b_1 and project the remaining axes on the subalgebra $A_3 = A_0(b_1, A_2)$:

$$G_3(a_1, b_1) = \{x_{b_1}(b_2), \ldots, x_{b_1}(b_{n-1})\}, \tag{13}$$

and $e_3 = e_2 - b_1 = e - a_1 - b_1$ is a unit of A_3. By induction, on the last step, we get the set $G_k(a_1, b_1, \ldots, c_1) = \{d_1 = x_{c_1}(c_2)\}$.

Define $e_{k+1} = e_k - d_1 = e - (a_1 + b_1 + \ldots + c_1 + d_1)$. If $e_{k+1} = 0$, then e can be represented in the required form.

Suppose that $e_{k+1} \neq 0$. Then by Corollary 2, $A_{k+1} = A_0(d_1, A_k)$ is an axial algebra with a basis of axes from A. Take q such an axis. Then, by the definition, $qd_1 = 0$ and $qc_1 = 0$. By Lemma 17, $qc_2 = 0$.

Suppose that $c_1 = x_{f_1}(f_2)$ and $c_2 = x_{f_1}(f_3)$. By the definition, $qf_1 = 0$. Since $qc_2 = 0$, then again by Lemma 17, $qf_2 = 0$. Continuing the procedure, we get $qa_i = 0$ for every $1 \leq i \leq n$.

If $qA = 0$, then $q \cdot q = q = 0$, a contradiction.

Suppose that $qA \neq 0$. Let w be a word in the alphabet G of a minimal length such that $qw \neq 0$. Let $w = w_1w_2$. By the assumption, $qw_1 = qw_2 = 0$. Let s be a linear combination of words of the length less than $|w_1|$ and $\alpha \in F \setminus \{0\}$ such that $\alpha(w_1 + s) = r$ for some axis r (see Lemma 14). By the assumption, we have $qr = q(sw_2) = 0$. By Seress Lemma,

$$\alpha q(w_1w_2) = \alpha q(w_1w_2) \pm \alpha q(sw_2) = q(rw_2) = r(qw_2) = 0,$$

a contradiction. □

Let A be a finitely generated quasi-definite primitive strongly axial algebra of Jordan type $1/2$ and let $G = \{a_1, \ldots, a_n\}$ be a generating set of A consisting of axes. Moreover, let $y_1, \ldots, y_s \in A$ be axes such that $y_1 \in G$, $y_2 \in G_2(y_1)$, $\ldots$, $y_s \in G_s(y_1, \ldots, y_{s-1})$, where the sets $G_i(y_1, \ldots, y_{i-1})$ are defined by (12), (13) etc. Denote the subalgebra $A_0(y_1) \cap A_0(y_2) \cap \ldots \cap A_0(y_s)$ by $A_0(y_1, \ldots, y_s)$ and let us call it as a *special subalgebra* of A.

Given a special subalgebra $B = A_0(y_1, \ldots, y_s)$, let us call $y_{s+1} \in G_{s+1}(y_1, \ldots, y_s)$ as a special axis in B. Actually, y_{s+1} is an axis in A too.

Corollary 3. *Let A be a quasi-definite primitive strongly axial algebra of Jordan type $1/2$ and $G = \{a_1, \dots, a_n\}$ be a generating set of A consisting of axes. Let e be a unit of A. Then there exists a sequence of special subalgebras for $k \leq n$,*

$$A \supset A_0(y_1) \supset A_0(y_1, y_2) \dots \supset A_0(y_1, \dots, y_k) \supset A_0(y_1, \dots, y_{k+1}) = (0).$$

Proof. It follows from the proof of Lemma 2. □

Acknowledgment

We are grateful to the anonymous reviewer for the helpful remarks.

References

[1] A.A. Albert, *A structure theory for Jordan algebras*, Ann. Math. (2), **48** (1947), 546–567. Zbl 0029.01003

[2] https://github.com/GubarevV/Quasi-definite-axial-algebras.

[3] I. Gorshkov, A. Staroletov, *On primitive 3-generated axial algebras of Jordan type*, J. Algebra, **563** (2020), 74–99. Zbl 1460.17004

[4] J.I. Hall, F. Rehren, S. Shpectorov, *Primitive axial algebras of Jordan type*, J. Algebra, **437** (2015), 79–115. Zbl 1339.17019

[5] J.I. Hall, Y. Segev, S. Shpectorov, *On primitive axial algebras of Jordan type,* Bull. Inst. Math. Acad. Sin. (N.S.), **13**:4 (2018), 397–409. Zbl 1460.17005

[6] A.A. Ivanov, *The monster group and majorana involutions*, Cambridge University Press, Cambridge, 2009.

[7] T. De Medts, F. Rehren, *Jordan algebras and 3-transposition groups*, J. Algebra, **478** (2017), 318–340. Zbl 1386.17028

Ilya Borisovich Gorshkov
Sobolev Institute of Mathematics,
pr. Koptyuga, 4,
630090, Novosibirsk, Russia
Email address: ilygor8@gmail.com

Vsevolod Yurevich Gubarev
Sobolev Institute of Mathematics,
pr. Koptyuga, 4,
630090, Novosibirsk, Russia
Email address: wsewolod89@gmail.com