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PRODUCTS OF MULTIDIMENSIONAL MATRICES, STOCHASTIC MATRICES, AND PERMANENTS

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Abstract: The paper is devoted to four basic multidimensional matrix operations (outer product, Kronecker product, contraction, and projection) and two derivative operations (dot and circle products). It is studied the interrelations between these operations, some of their algebraic properties, and their action on k-stochastic matrices. Also, it is proved several relations on the permanents of products of multidimensional matrices. In particular, it is shown that the permanent of the dot product of nonnegative multidimensional matrices is not less than the product of their permanents. Another result of the paper is that inequalities on the Kronecker product of nonnegative 2-dimensional matrices cannot be extended to the multidimensional case.

Keywords: outer product, Kronecker product, contraction, dot product, stochastic matrix, permanent.

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1 Introduction and basic definitions

Let $n, d \in \mathbb{N}$ and $I_n^d = \{(\alpha_1, \ldots, \alpha_d) | \alpha_i \in \{1, \ldots, n\}\}$ be the *index set*. For indices $\alpha \in I_n^{d_1}$, $\beta \in I_n^{d_2}$, let a concatenation $\alpha\beta$ be the index $\gamma \in I_n^{d_1+d_2}$ such that $\gamma = (\gamma_1, \ldots, \gamma_{d_1+d_2}) = (\alpha_1, \ldots, \alpha_{d_1}, \beta_1, \ldots, \beta_{d_2})$.

A *d-dimensional matrix A of order n* is an array $(a_\alpha)_{\alpha \in I_n^d}$, $a_\alpha \in \mathbb{R}$. Sometimes d-dimensional matrices of order n are considered as tensors of dimension n and order d.

Let $k \in \{0, \ldots, d\}$. A *k-dimensional plane* Γ in A is a submatrix of A obtained by fixing $d - k$ components of indices and letting the other k components vary from 1 to n. A 1-dimensional plane is said to be a *line*, and a $(d-1)$-dimensional plane is a *hyperplane*. Let the *direction* of a plane Γ in the matrix A be a $(0,1)$-vector whose i-th component is equal to 1 whenever the i-th component in Γ is fixed.

Let a *transpose* of a matrix A be a permutation of directions of its hyperplanes (permutation of components of all indices). We will say that matrices A and B of the same order and dimension are *equivalent* if one can be turned into the other by transposes and permutations of hyperplanes of the same direction.

In this paper we focus on the properties of four operations: outer and Kronecker products, projection, and contraction. These operations and their properties appear in many papers, but to our knowledge they have never been collected and compared like this before.

One can find many other multidimensional matrix operations and products of tensors in the literature (see, for example, papers [2, 6, 11, 17] or Chapter 15 "Tensors and hypermatrices" in [5]). We do not aim to observe their variety here, but we assert that the above four operations are enough to express any reasonable multidimensional matrix transformation. As an illustration, we consider the dot product and circle product of multidimensional matrices and present them as an appropriate composition of outer products and contractions.

Another aim of the present paper is to study the action of matrix products on a set of stochastic matrices and their connections with the multidimensional permanent.

If all $a_\alpha \geq 0$, a multidimensional matrix A is said to be *nonnegative*. A nonnegative d-dimensional matrix A of order n is called *k-stochastic* if the sum of entries in each k-dimensional plane equals 1. It is easy to see that if A is a k-stochastic matrix of order n and $k < d$, then $\frac{1}{n}A$ is a $(k+1)$-stochastic matrix. Two-dimensional 1-stochastic matrices are known as *doubly stochastic*, multidimensional 1-stochastic matrices are called *polystochastic*, and polystochastic $(0,1)$-matrices are said to be *multidimensional permutations*. We also denote by J_n^d the d-dimensional polystochastic matrix of order n, all of whose entries are equal to $1/n$.

Multidimensional stochastic matrices are closely related to latin hypercubes and orthogonal arrays. A *d-dimensional latin hypercube Q* of order n

is a d-dimensional matrix of order n such that its entries q_α take values from the set $\{1,\dots,n\}$ and in each line of Q all n symbols occur. Two-dimensional latin hypercubes are known as *latin squares*. A $t-(n,k,\lambda)$ *orthogonal array* is a rectangular $\lambda n^t \times k$ array R whose entries are chosen from a set $I_n = \{1,\dots,n\}$ such that in every subset of t columns of the array, every t-tuple of elements of I_n appears in exactly λ rows.

The correspondence between a d-dimensional latin hypercube Q and a $(d+1)$-dimensional permutation $M(Q)$ is given by the following rule: an entry $q_{\alpha_1,\dots,\alpha_d}$ of a latin hypercube Q equals α_{d+1} if and only if an entry $m_{\alpha_1,\dots,\alpha_{d+1}}$ of the permutation matrix $M(Q)$ equals 1. There is also a correspondence between a $t-(n,k,\lambda)$ orthogonal array R and a k-dimensional t-stochastic matrix $\frac{1}{\lambda}M$ of order n: an entry $m_{\alpha_1,\dots,\alpha_k}$ of the matrix M is equal to the number of appearances of the row $(\alpha_1,\dots,\alpha_k)$ in the array R.

To define the permanent of a multidimensional matrix, we need to define the set of its diagonals. A *diagonal* D in a d-dimensional matrix of order n is a collection of indices $\{\alpha^1,\dots,\alpha^n\}$ such that $\alpha^i_k \neq \alpha^j_k$ for all $k \in \{1,\dots,d\}$ and all $i,j \in \{1,\dots,n\}$, $i \neq j$. The diagonal

$$\{(1,\dots,1),\dots,(n,\dots,n)\}$$

is said to be *main*. For a nonnegative matrix A, we will say that a diagonal D is *positive* if $a_\alpha > 0$ for all $\alpha \in D$.

Let $\mathcal{D}(A)$ denote the set of all diagonals in the matrix A. The *permanent* of a d-dimensional matrix A of order n is

$$\operatorname{per} A = \sum_{D \in \mathcal{D}(A)} \prod_{\alpha \in D} a_\alpha.$$

A diagonal D in a latin hypercube Q is said to be a *transversal* if all q_α, $\alpha \in D$ are different. It is easy to see that every transversal in a latin hypercube Q is a positive diagonal in the $(d+1)$-dimensional permutation matrix $M(Q)$, so the number of transversals in Q is equal to the permanent of $M(Q)$.

The well-known Birkhoff theorem states that the permanent of every doubly stochastic matrix is positive. For dimensions d greater than 2, there exist d-dimensional polystochastic matrices with zero permanent. The simplest examples of such matrices are multidimensional permutations of odd dimensions d which correspond to iterated groups $\mathbb{Z}_n$ of even order n [16]. Since for other values of d and n we still do not have examples of d-dimensional polystochastic matrices of order n with zero permanent, in [12] it was conjectured that the permanent of every polystochastic matrix of odd order or even dimension is greater than zero. The study of stochastic matrices and different generalizations of permanents was continued in [15].

Thanks to the correspondence between the permanent of a multidimensional permutation matrix and the number of transversals in a latin hypercube, this conjecture generalizes the well-known Ryser's conjecture on existence of transversals in latin squares of odd order [10] and the conjecture on existence of transversals in latin hypercubes of odd dimension or odd order by Wanless [16].

For more information on the properties and applications of stochastic matrices and multidimensional permanents to combinatorial problems see [12, 13].

The rest of the paper is organized as follows. In Section 2, we introduce four basic multidimensional matrix operations (outer and Kronecker products, projection, and contraction) and two derived operations (dot and circle products). We also provide many basic algebraic properties and relations for compositions of these operations. Moreover, in Section 2.1 we discuss the application of the dot, Kronecker, and circle products to the composition and direct product of the multiary quasigroups, eigenvalues and coverings of multidimensional matrices.

Section 3 is devoted to the products of stochastic matrices. For given stochastic matrices A and B we establish the degree of stochastivity of their outer, Kronecker, dot, and circle products, as well as their contractions and projections. In particular, we prove that the Kronecker, dot, and circle product of polystochastic matrices is a polystochastic matrix and fix an error in the theorem on the dot product of stochastic matrices from [3].

At last, in Section 4 we study the permanent of products of multidimensional matrices. We prove that the permanent of the outer product of matrices A and B of order n is $n!$ times the product of the permanents of A and B. Although for the permanent of contractions there are no good estimations, we prove that (as in the 2-dimensional case) the permanent of the dot product of nonnegative matrices is not less than the product of their permanents. Finally, we consider the permanent of the Kronecker product of nonnegative matrices and show that the upper and lower bounds for 2-dimensional matrices from [1] cannot be extended to the multidimensional case.

2 Operations on multidimensional matrices

We start with four basic operations for multidimensional matrices and their properties. Most of these properties (maybe stated in other terms) are scattered among many papers and seem to be folklore. We give proofs only for several statements because the others can be deduced from definitions by straightforward calculation.

We assume everywhere that matrices have orders and dimensions such that all operations are well defined.

Outer product. Let A be a d_1-dimensional matrix of order n and B be a d_2-dimensional matrix of order n. Then the *outer product* $A \times B$ of matrices A and B is the (d_1+d_2)-dimensional matrix C of the same order n with entries $c_{\alpha\beta} = a_\alpha b_\beta$ for all $\alpha \in I_n^{d_1}$, $\beta \in I_n^{d_2}$.

Proposition 1 (Properties of the outer product)**.**

(1) $(A \times B) \times C = A \times (B \times C)$.

(2) *$A \times B$ is obtained from $B \times A$ by transposes of the first and last directions of hyperplanes.*
(3) $(A + B) \times C = A \times C + B \times C$.
(4) $(\lambda A) \times B = \lambda(A \times B) = A \times (\lambda B)$ *for every* $\lambda \in \mathbb{R}$.

Kronecker product. Let A be a d-dimensional matrix of order n_1 and B be a d-dimensional matrix of order n_2. Then the *Kronecker product* $A \otimes B$ of matrices A and B is the d-dimensional matrix C of order $n_1 n_2$ with entries $c_\gamma = a_\alpha b_\beta$, where $\gamma_i = (\alpha_i - 1)n_2 + \beta_i$ for each $i = 1, \ldots, d$. Since the Kronecker product of multidimensional matrices is a natural generalization from the 2-dimensional case, it has similar properties.

Proposition 2 (Properties of the Kronecker product)**.**
(1) $(A \otimes B) \otimes C = A \otimes (B \otimes C)$.
(2) *$A \otimes B$ can be obtained from $B \otimes A$ by permutations of hyperplanes of the same direction.*
(3) $(A + B) \otimes C = A \otimes C + B \otimes C$.
(4) $(\lambda A) \otimes B = \lambda(A \otimes B) = A \otimes (\lambda B)$ *for every* $\lambda \in \mathbb{R}$.

Contraction. For a d-dimensional matrix A of order n and a set $S = \{i_1, \ldots, i_\ell\}$, $i_j \in \{1, \ldots, d\}$, let the *S-contraction* A_S of the matrix A be the $(d - \ell)$-dimensional matrix B of the same order with entries

$$b_\beta = \sum_{i=1}^{n} a_{\beta_1, \ldots, i, \ldots, i, \ldots, \beta_{d-\ell}},$$

where components i are located exactly at positions $i_j \in S$. In other words, the matrix A_S is obtained from A by a summation of entries over the main diagonal in all ℓ-dimensional planes such that their varying components of indices are given by the set S. If $S = \{i, j\}$, we will write *(i, j)-contraction* instead of S-contraction, and if $S = \{i\}$, we will write *i-contraction*.

For 2-dimensional matrices, the $(1, 2)$-contraction is known as the trace of the matrix. The tensor contraction is an (i, j)-contraction for appropriate i and j.

It is possible to further generalize the definition of the contraction and take a summation over an arbitrary (not necessarily main) diagonal in ℓ-dimensional planes. But we will not study it here because it is the S-contraction of a matrix equivalent to the given one and so it has similar properties.

Sometimes it is helpful to apply more than one contraction to the same matrix. Assume that $S_1, \ldots, S_m$ is a collection of pairwise disjoint subsets of $\{1, \ldots, d\}$. Let $A_{S_1; \ldots; S_m}$ denote consecutive contraction $(\cdots (A_{S_1})_{S_2} \cdots)_{S_m}$, in which we renumber the components of indices of matrices only after the last S_m-contraction.

Proposition 3 (Properties of the contraction)**.**
(1) *If $S \cap T = \emptyset$, then $A_{S;T} = A_{T;S}$.*
(2) $(A + B)_S = A_S + B_S$.

(3) $(\lambda A)_S = \lambda A_S$ *for every* $\lambda \in \mathbb{R}$.

Projection. For a d-dimensional matrix A of order n and a set $S = \{i_1, \dots, i_\ell\}$, $i_k \in \{1, \dots, d\}$, define the *S-projection* $P_S(A)$ of a matrix A to be the $(d-\ell)$-dimensional matrix B of the same order with entries

$$b_\beta = \sum_{j_1,\dots,j_\ell=1}^{n} a_{\beta_1,\dots,j_1,\dots,j_\ell,\dots,\beta_{d-\ell}},$$

where components j_k are located exactly at positions $i_k \in S$. In other words, the matrix $P_S(A)$ is obtained from A by a summation of all entries of ℓ-dimensional planes in which varying components of indices are given by the set S. If $S = \{i\}$, we will write *i-projection* instead of $\{i\}$-projection. Note that i-contraction and i-projection are the same operation.

Given pairwise disjoint sets $S_1, \dots, S_m$, let $P_{S_1;\dots;S_m}(A)$ be the consecutive projection of the form $P_{S_m}(\cdots P_{S_2}(P_{S_1}(A))\cdots)$, in which we renumber the components of indices only after the last S_m-projection.

Proposition 4 (Properties of the S-projection)**.**
(1) *If* $S \cap T = \emptyset$, *then* $P_{S;T}(A) = P_{T;S}(A)$.
(2) $P_S(A+B) = P_S(A) + P_S(B)$.
(3) $P_S(\lambda A) = \lambda P_S(A)$ *for every* $\lambda \in \mathbb{R}$.

The S-contraction and S-projection can be considered as reverse operations to the outer product in the following sense.

Proposition 5. *Let A be a d-dimensional matrix of order n.*
(1) *If* $S = \{1, \dots, \ell\}$, *then* $(J_n^\ell \times A)_S = A$.
(2) *If* $S = \{1, \dots, \ell\}$, *then* $P_S(J_n^\ell \times A) = n^{\ell-1} A$.

Let us turn to other multidimensional matrix operations that can be obtained as a composition of the above. We start with a pair of trivial operations.

Suppose that a d-dimensional matrix A is composed of k-dimensional matrices B^i in such a way that B^i are all k-dimensional planes in A of a given direction. Then the matrix A can be presented as the sum of the outer products of B^i and appropriate multidimensional $(0,1)$-matrices.

On the other hand, for a given multidimensional matrix A one can take its k-dimensional plane Γ as a new k-dimensional matrix. This operation can be expressed as the outer product of A by an appropriate $(0,1)$-matrix and the consecutive contractions of the result.

Let us consider two more multidimensional matrix products.

Dot product. Let A be a d_1-dimensional matrix of order n and B be a d_2-dimensional matrix of order n. We define the *(i,j)-dot product* $A \cdot_{i,j} B$ of matrices A and B to be the (d_1+d_2-2)-dimensional matrix obtained as the (i, d_1+j)-contraction of the outer product $A \times B$. For example, if C is a $(d_1,1)$-dot product of matrices A and B, then $c_\gamma = \sum\limits_{i=1}^{n} a_{\alpha i} b_{i\beta}$, where

$\gamma = \alpha\beta$ is the concatenation of indices α and β. For shortness, we denote the $(d_1, 1)$-dot product of matrices A and B as AB or $A \cdot B$ and call it *dot product*. Note that the standard dot product AB of 2-dimensional matrices coincides with our dot product because it is the $(2, 1)$-dot product, where the first component of a 2-dimensional index is used for rows and the second component labels columns.

This notion of the dot product can be generalized to a larger collection of matrices. Suppose that $A^1, \ldots, A^\ell$ are matrices of the same order n and dimensions $d_1, \ldots, d_\ell$, respectively. Let $S = (i_1, \ldots, i_\ell)$ be a tuple such that $i_j \in \{1, \ldots, d_j\}$. Define the *$S$-dot product* $[A^1_{i_1}, \ldots, A^\ell_{i_\ell}]$ to be the $(d_1 + \cdots + d_\ell - \ell)$-dimensional matrix obtained as the $\{i_1, d_1 + i_2, \ldots, d_1 + \cdots + d_{\ell-1} + i_\ell\}$-contraction of their outer product $A^1 \times \cdots \times A^\ell$. In particular, the standard dot product AB of 2-dimensional matrices is the matrix $[A_2, B_1]$.

Let us provide some properties of the dot product. Most of them can be easily derived from the ones for the outer product and contraction. They remain essentially the same for a general (i, j)-dot product or for the S-dot product of more than two matrices.

Proposition 6 (Properties of the dot product)**.**

1. *$(AB)C = A(BC)$.*
2. *$(A + B)C = AC + BC$ and $A(B + C) = AB + BC$.*
3. *$(\lambda A)B = \lambda(AB) = A(\lambda B)$ for every $\lambda \in \mathbb{R}$.*
4. *If I is the 2-dimensional identity matrix, then $AI = IA = A$.*

As in the 2-dimensional case, $AB \neq BA$ in general, and BA cannot be obtained from AB by equivalent transformations.

Circle product. Let A be a d_1-dimensional matrix of order n and B be a d_2-dimensional matrix of order n. Define the *circle product* $A \circ B$ to be the $((d_1 - 1)(d_2 - 1) + 1)$-dimensional matrix of order n equal to the following $(d_1 - 1)$-times dot product of A and B:

$$A \circ B = (\cdots ((A \cdot_{d_1,1} B) \cdot_{d_1-1,1} B) \cdots) \cdot_{2,1} B.$$

Equivalently, if $C = A \circ B$, then for entries c_γ we have

$$c_\gamma = \sum_{j_2, \ldots, j_{d_1} = 1}^{n} a_{i, j_2, \ldots, j_{d_1}} b_{j_2 \beta_2} \cdots b_{j_{d_1} \beta_{d_1}},$$

where $\gamma = i\beta_2 \cdots \beta_{d_1}$, $i \in \{1, \ldots, n\}$, $\beta_j \in I_n^{d_2 - 1}$.

The following properties of the circle product can be found in [11] or derived from the definition and properties of the dot product.

Proposition 7 (Properties of the circle product)**.**

1. *$(A \circ B) \circ C = A \circ (B \circ C)$.*
2. *$(A + B) \circ C = A \circ C + B \circ C$.*
3. *If A is a 2-dimensional matrix, then $A \circ (B + C) = A \circ B + A \circ C$.*
4. *$(\lambda A) \circ B = \lambda(A \circ B)$ for every $\lambda \in \mathbb{R}$.*

(5) $A \circ (\lambda B) = \lambda^{d_1 - 1}(A \circ B)$ *for every* $\lambda \in \mathbb{R}$, *where* d_1 *is the dimension of the matrix* A.

We also note that in general $A \circ B \neq B \circ A$.

At the end of this section we provide several relations between the defined multidimensional matrix operations and prove some of them.

Proposition 8 (Properties of multidimensional operations)**.**

(1) $(A \times B) \otimes (C \times D) = (A \otimes C) \times (B \otimes D)$.
(2) $A_S \times B = (A \times B)_S$.
(3) $P_S(A) \times B = P_S(A \times B)$.
(4) $(A \cdot_{i,j} B) \times C = (A \times C) \cdot_{i,j} (B \times C)$.
(5) $A_S \otimes B_S = (A \otimes B)_S$.
(6) $P_S(A \otimes B) = P_S(A) \otimes P_S(B)$.
(7) $(A \otimes B) \cdot (C \otimes D) = (A \cdot C) \otimes (B \cdot D)$.
(8) *If* $S \cap T = \emptyset$, *then* $P_T(A_S) = (P_T(A))_S$.
(9) *If* A *is a* d*-dimensional matrix and* $d \notin S$, *then* $A_S \cdot B = (A \cdot B)_S$.
(10) *If* A *is a* d*-dimensional matrix and* $d \notin S$, *then* $P_S(A) \cdot B = P_S(A \cdot B)$.

Proof. 1. In order for all operations to be well defined, we require that A be a d_1-dimensional matrix of order n_1, B be a d_2-dimensional matrix of order n_1, C be a d_1-dimensional matrix of order n_2, and D be a d_2-dimensional matrix of order n_2.

Note that an entry with index μ on both sides of the equality is equal to $a_\alpha b_\beta c_\gamma d_\delta$, where components $\mu_i = (\alpha_i - 1)n_2 + \gamma_i$ if $1 \le i \le d_1$ and $\mu_i = (\beta_i - 1)n_2 + \delta_i$ if $d_1 + 1 \le i \le d_1 + d_2$.

5. Let A be a d-dimensional matrix of order n_1 and B be a d-dimensional matrix of order n_2. Without loss of generality, assume that $S = \{1, \dots, \ell\}$, $1 \le \ell \le d$. Then an entry c_γ of matrices on both sides of the equation is equal to

$$\left(\sum_{i=1}^{n_1} a_{i,\dots,i,\alpha} \right) \left(\sum_{j=1}^{n_2} b_{j,\dots,j,\beta} \right) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} a_{i,\dots,i,\alpha} b_{j,\dots,j,\beta},$$

where $\gamma_k = (\alpha_{\ell+k} - 1)n_2 + \beta_{\ell+k}$ for every $k = 1, \dots, d - \ell$.

7. In order for all operations to be well defined, we require that A be a d_1-dimensional matrix of order n_1, B be a d_1-dimensional matrix of order n_2, C be a d_2-dimensional matrix of order n_1, and D be a d_2-dimensional matrix of order n_2. The proof of this equation is essentially the same as for 2-dimensional matrices. □

2.1. Applications to eigenvalues, coverings and multiary quasigroups.

The notion of the circle product is motivated by eigenvalues and eigenvectors of multidimensional matrices and hypergraphs. Their study was initiated in [7, 9] and has been continued in many subsequent papers.

For a d-dimensional matrix A of order n, a number $\lambda \in \mathbb{C}$ is called an *eigenvalue* and a vector $v \in \mathbb{C}^n$ is the corresponding *eigenvector* if $A \circ v =$

$\lambda(\mathbb{I} \circ v)$. Here $\mathbb{I}$ is the d-dimensional identity $(0,1)$-matrix of order n that has ones only at the main diagonal.

For example, let us show that $(d-1)$-stochastic d-dimensional matrices have eigenvalue 1.

Proposition 9. *Let A be a d-dimensional $(d-1)$-stochastic matrix of order n. Then 1 is an eigenvalue of the matrix A corresponding to the eigenvector $e = (1, \dots, 1)$.*

Proof. By the definition of circle product, the vector $v = A \circ e$ has components v_i equal to the sums of entries in hyperplanes Γ_i of the first direction. Since A is $(d-1)$-stochastic, we have $A \circ e = (1, \dots, 1) = e$. On the other hand, it holds $\mathbb{I} \circ e = e$. So $A \circ e = \mathbb{I} \circ e$, and 1 is an eigenvalue corresponding to the eigenvector e. $\square$

The circle product also proves to be useful in the multidimensional matrix equation for perfect colorings of hypergraphs [14]. With the help of the circle product we also define the covering of matrices that can be considered as the inverse operation of the Kronecker product.

Let A be a d-dimensional matrix of order n_1 and B be a d-dimensional matrix of order n_2, where $n_2 \leq n_1$. We will say that the matrix A *covers* the matrix B (A is a *covering* of B) if there exists a rectangular $(0,1)$-matrix P of size $n_1 \times n_2$ such that $A \circ P = P \circ B$ and every row of P contains exactly one 1.

Proposition 10. *Let A be a d-dimensional matrix of order n_1. Then for every $n_2 \in \mathbb{N}$ matrices $\frac{1}{n_2^{d-1}}(A \otimes J_{n_2}^d)$ and $\frac{1}{n_2}(A \otimes \mathbb{I})$ cover the matrix A. Here $\mathbb{I}$ is the d-dimensional identity $(0,1)$-matrix of order n that has ones only at the main diagonal.*

Proof. To obtain a covering for both cases it is sufficient to take the $(0,1)$-matrix P of size $(n_1 n_2) \times n_2$ such that $p_{i,j} = 1$ if and only if $j = \lceil \frac{i}{n_2} \rceil$. $\square$

Let us turn to the connection between products of matrices and multiary quasigroups.

A *d-ary quasigroup* f of order n is a function $f : I_n^d \to I_n$ such that the equation $x_0 = f(x_1, \dots, x_d)$ has a unique solution for any one variable if all the other d variables are specified arbitrarily. The Cayley table of a d-ary quasigroup of order n is a d-dimensional latin hypercube of order n, and vice versa, every d-dimensional latin hypercube can be considered as the Cayley table of some d-ary quasigroup. The graph

$$\{(x_0, x_1, \dots, x_d) |\ x_0 = f(x_1, \dots, x_d)\}$$

of a quasigroup f is the set of ones of the $(d+1)$-dimensional permutation M^f of order n.

For d-ary quasigroups we define *transversals* so that they coincide with those in the corresponding latin hypercubes. Then every transversal in a d-ary quasigroup f is a positive diagonal in the $(d+1)$-dimensional permutation matrix M^f, so the number of transversals in f is equal to the permanent of M^f.

Let f be a d_1-ary quasigroup of order n and g be a d_2-ary quasigroup of order n. The *composition* $f \cdot g$ of quasigroups f and g is the (d_1+d_2-1)-ary quasigroup of order n defined as

$$(f \cdot g)(x_1, \dots, x_{d_1+d_2-1}) = x_0 \Leftrightarrow f(x_1, \dots, x_{d_1-1}, g(x_{d_1}, \dots, x_{d_1+d_2-1})) = x_0.$$

Proposition 11. *Let f be a d_1-ary quasigroup of order n and g be a d_2-ary quasigroup of order n. Suppose that M^f and M^g are the multidimensional permutations corresponding to quasigroups f and g. Then $M^f M^g$ is the multidimensional permutation corresponding to the composition $f \cdot g$.*

Proof. By the definition, an entry c_γ of the matrix $C = M^f M^g$ is equal to $c_\gamma = \sum\limits_{i=1}^{n} m^f_{\alpha i} m^g_{i\beta}$, where $\gamma = \alpha\beta$, $\alpha = (x_0, x_1, \dots, x_{d_1-1})$, and $\beta = (x_{d_1}, \dots, x_{d_1+d_2-1})$.

Since every line of matrices M^f and M^g contains exactly one 1, an entry $c_\gamma = 1$ if $m^f_{\alpha k} = m^g_{k\beta} = 1$ for some $k \in \{1, \dots, n\}$, and $c_\gamma = 0$ otherwise. Equivalently, we have that $c_\gamma = 1$ if and only if $f(x_1, \dots, x_{d_1-1}, k) = x_0$ and $g(x_{d_1}, \dots, x_{d_1+d_2-1}) = k$ for some $k \in \{1, \dots, n\}$. It remains to note that the matrix C is exactly the graph of the composition $f \cdot g$. □

The proposition also implies that the graph of a consecutive composition of a series of quasigroups can be obtained as an appropriate dot product of their multidimensional permutations. Moreover, the following result from [12, Proposition 12] is a direct corollary of the inequality on the dot product of nonnegative matrices that will be stated as Theorem 11 below.

Proposition 12 ([12])**.** *Let f be a d_1-ary quasigroup of order n and g be a d_2-ary quasigroup of order n. Assume that f and g have $T(f)$ and $T(g)$ transversals, respectively. Then the number of transversals in the composition $f \cdot g$ is at least $T(f)T(g)$.*

Let f be a d-ary quasigroup of order n_1 and g be a d-ary quasigroup of order n_2. Define the *direct product* $f \times g$ of quasigroups f and g as the d-ary quasigroup of order $n_1 n_2$ over the set $I_{n_1 n_2} = I_{n_1} \times I_{n_2}$ as

$$(f \times g)((x_1, y_1), \dots, (x_d, y_d)) = (x_0, y_0) \Leftrightarrow$$
$$f(x_1, \dots, x_d) = x_0 \text{ and } g(y_1, \dots, y_d) = y_0.$$

Proposition 13. *Let f be a d-ary quasigroup of order n_1 and g be a d-ary quasigroup of order n_2. Suppose that M^f and M^g are the multidimensional permutations corresponding to f and g. Then $M^f \otimes M^g$ is the multidimensional permutation corresponding to the direct product $f \times g$.*

Proof. By the definition, an entry c_γ of the matrix $C = M^f \otimes M^g$ is equal to $c_\gamma = m^f_\alpha m^g_\beta$, where $\gamma_i = (\alpha_i - 1)n_2 + \beta_i$ for each $i = 1, \dots, d$, $\alpha = (x_0, x_1, \dots, x_d)$, and $\beta = (y_0, y_1, \dots, y_d)$.

Note that an entry $c_\gamma = 1$ if $m^f_\alpha = m^g_\beta = 1$, and $c_\gamma = 0$ otherwise. Equivalently, $c_\gamma = 1$ if and only if $f(x_1, \ldots, x_d) = x_0$ and $g(y_1, \ldots, y_d) = y_0$. It remains to note that the matrix C is exactly the graph of $f \times g$. □

3 Products of stochastic matrices

In this section, we study the products of stochastic matrices. We start with the outer product.

Theorem 1. *Let A be a d_1-dimensional k_1-stochastic matrix of order n and B be a d_2-dimensional k_2-stochastic matrix of order n. Then $n^{k_1+k_2-r}(A\times B)$ is a (d_1+d_2)-dimensional r-stochastic matrix, where $r = \max\{d_1+k_2, d_2+k_1\}$.*

Proof. Let us choose an r-dimensional plane Γ in the matrix $C = A \times B$. Without loss of generality, we assume that the first r_1 components and the last r_2 components of indices vary in Γ, $r_1 + r_2 = r$. Since $r = \max\{d_1 + k_2, d_2 + k_1\}$, we have $r_1 \geq k_1$ and $r_2 \geq k_2$. Then

$$\sum_{\gamma\in\Gamma} c_\gamma = \sum_{i_1,\ldots,i_{r_1}=1}^{n} \sum_{j_{d_2-r_2+1},\ldots,j_{d_2}=1}^{n} a_{i_1,\ldots,i_{r_1},\alpha} b_{\beta,j_{d_2-r_2+1},\ldots,j_{d_2}} =$$

$$\sum_{i_1,\ldots,i_{r_1}=1}^{n} a_{i_1,\ldots,i_{r_1},\alpha} \sum_{j_{d_2-r_2+1},\ldots,j_{d_2}=1}^{n} b_{\beta,j_{d_2-r_2+1},\ldots,j_{d_2}} = n^{r_1-k_1}n^{r_2-k_2} = n^{r-k_1-k_2},$$

where index $\gamma = (i_1, \ldots, i_{r_1}, \alpha, \beta, j_{d_2-r_2+1}, \ldots, j_{d_2})$.

Thus, the matrix $n^{k_1+k_2-r}C$ is r-stochastic. □

Next, we consider the Kronecker product of stochastic matrices. Note that one of the basic constructions of orthogonal arrays from [8] can be considered as the Kronecker product of the corresponding stochastic matrices.

Theorem 2. *Let A be a d-dimensional k_1-stochastic matrix of order n_1 and B be a d-dimensional k_2-stochastic matrix of order n_2. Then $n^{k_1+k_2-2r}(A \otimes B)$ is a d-dimensional r-stochastic matrix, where $r = \max\{k_1, k_2\}$. In particular, the Kronecker product of polystochastic matrices is polystochastic.*

Proof. Let us choose an r-dimensional plane Γ in the matrix $C = A \otimes B$. Without loss of generality, we assume that the first r components of the indices vary in Γ. Then

$$\sum_{\gamma\in\Gamma} c_\gamma = \sum_{i_1,\ldots,i_r=1}^{n} \sum_{j_1,\ldots,j_r=1}^{n} a_{i_1,\ldots,i_r,\alpha} b_{j_1,\ldots,j_r,\beta} =$$

$$\sum_{i_1,\ldots,i_r=1}^{n} a_{i_1,\ldots,i_r,\alpha} \sum_{j_1,\ldots,j_r=1}^{n} b_{j_1,\ldots,j_r,\beta} = n^{r-k_1}n^{r-k_2} = n^{2r-k_1-k_2},$$

where the index γ is such that $\gamma_\ell = (i_\ell - 1)n_2 + j_\ell$ for $\ell \in \{1, \ldots, r\}$ and $\gamma_\ell = (\alpha_\ell - 1)n_2 + \beta_\ell$ for $\ell \in \{r + 1, \ldots, d\}$. Thus the matrix $n^{k_1+k_2-2r}C$ is r-stochastic. □

The contraction and projection preserve or strengthen the degree of stochastivity.

Theorem 3. *Let A be a d-dimensional k-stochastic matrix of order n and $S \subseteq \{1, \dots, d\}$, $|S| = \ell$. If $k + \ell \leq d$, then the contraction $\frac{1}{n}A_S$ is a $(d-\ell)$-dimensional k-stochastic matrix of order n.*

Proof. Let us choose a k-dimensional plane Γ in the matrix $C = A_S$. Without loss of generality, we assume that $S = \{d - \ell + 1, \dots, d\}$ and that the first k components of the indices vary in the plane Γ. Then

$$\sum_{\gamma \in \Gamma} c_\gamma = \sum_{i_1,\dots,i_k=1}^{n} \sum_{j=1}^{n} a_{i_1,\dots,i_k,\alpha,j,\dots,j} = \sum_{j=1}^{n} \sum_{i_1,\dots,i_k=1}^{n} a_{i_1,\dots,i_k,\alpha,j,\dots,j} = n,$$

where the index $\gamma = (i_1, \dots, i_k, \alpha)$ and $\alpha \in I_n^{d-k-\ell}$. Thus, the matrix $\frac{1}{n}C$ is k-stochastic. □

Theorem 4. *Let A be a d-dimensional k-stochastic matrix of order n and $S \subseteq \{1, \dots, d\}$, $|S| = \ell$. Then the projection $P_S(A)$ is a $(d-\ell)$-dimensional $(k-\ell)$-stochastic matrix of order n if $k > \ell$, and $P_S(A) = n^{\ell-k+1}J_n^{d-\ell}$ if $k \leq \ell$.*

Proof. Assume that $k > \ell$. Let us choose a $(k-\ell)$-dimensional plane Γ in the matrix $C = P_S(A)$. Without loss of generality, we assume that $S = \{d - \ell + 1, \dots, d\}$ and that the first $k - \ell$ components of the indices vary in Γ. Then

$$\sum_{\gamma \in \Gamma} c_\gamma = \sum_{i_1,\dots,i_{k-\ell}=1}^{n} \sum_{j_1,\dots,j_\ell=1}^{n} a_{i_1,\dots,i_{k-\ell},\alpha,j_1,\dots,j_\ell} = 1,$$

where the index $\gamma = (i_1, \dots, i_{k-\ell}, \alpha)$. Thus, the matrix C is $(k-\ell)$-stochastic.

If $k \leq \ell$, then every element of the matrix $C = P_S(A)$ is

$$c_\gamma = \sum_{j_1,\dots,j_\ell=1}^{n} a_{\gamma,j_1,\dots,j_\ell} = n^{\ell-k}.$$

Thus, $P_S(A)$ is equal to $n^{\ell-k+1}J_n^{d-\ell}$. □

Let us now turn to more complex matrix operations. The dot product of stochastic matrices has already been studied in paper [3]. Meanwhile, Lemma 3.4 from [3] contains a mistake that we correct here.

Theorem 5. *Let A be a d_1-dimensional k_1-stochastic matrix of order n, B be a d_2-dimensional k_2-stochastic matrix of order n, and $r = \max\{k_1 + d_2, k_2 + d_1\}$. If $r \leq d_1 + d_2 - 2$, then $n^{k_1+k_2-r-1}(AB)$ is a $(d_1 + d_2 - 2)$-dimensional r-stochastic matrix of order n. Moreover, if A and B are polystochastic matrices, then AB is a polystochastic matrix.*

Proof. Let $k_1 > 1$ or $k_2 > 1$. By the definition, AB is the $(d_1, 1)$-contraction of $A \times B$. By Theorem 1, $n^{k_1+k_2-r}(A \times B)$ is an r-stochastic matrix with

$r = \max\{d_1 + k_2, d_2 + k_1\}$. Using Theorem 3 for the $(d_1, 1)$-contraction, we obtain that $n^{k_1+k_2-r-1}(AB)$ is a $(d_1 + d_2 - 2)$-dimensional r-stochastic matrix of order n.

Assume now that $k_1 = k_2 = 1$. Without loss of generality, consider the 1-dimensional plane Γ in which the first component of indices vary. Then for the matrix $C = AB$ we have

$$\sum_{\gamma\in\Gamma} c_\gamma = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,\alpha,j} b_{j,\beta} = \sum_{j=1}^{n} b_{j,\beta} \sum_{i=1}^{n} a_{i,\alpha,j} = 1,$$

where index $\gamma = (i, \alpha, \beta)$, $\alpha \in I_n^{d_1-2}$, $\beta \in I_n^{d_2-1}$. Thus, the matrix C is polystochastic. □

Lemma 3.4 from [3] claimed that the dot product of k_1- and k_2-stochastic matrices A and B is an r-stochastic matrix with $r = \max\{k_1+d_2, k_2+d_1\}-2$. It is not hard to find a pair of 2-dimensional 2-stochastic matrices A and B (in which the sums of all entries are equal to 1) satisfying $AB = 0$ that will be a counterexample to this statement.

The following result on the dot product of a polystochastic matrix and the uniform matrix can be proved by a direct calculation.

Proposition 14. *If A is a d-dimensional polystochastic matrix of order n, then for every integer $t \geq 2$ it holds*

$$AJ_n^t = J_n^{d+t-2} \text{ and } J_n^t A = J_n^{d+t-2}.$$

As a corollary from Theorem 5, we have the following property of the circle product.

Theorem 6. *Let A be a d_1-dimensional polystochastic matrix of order n and B be a d_2-dimensional polystochastic matrix of order n. Then $A \circ B$ is a polystochastic matrix.*

Proof. By the definition, $A \circ B$ is the $(d_1 - 1)$-times dot product:

$$A \circ B = (\cdots((A \cdot_{d_1,1} B) \cdot_{d_1-1,1} B) \cdots) \cdot_{2,1} B.$$

By Theorem 5, if A and B are polystochastic matrices, then the dot product of A and B is a polystochastic matrix. So in this case, the matrix $A \circ B$ is also polystochastic. □

If A is a k_1-stochastic matrix and B is a k_2-stochastic matrix, where k_1 or k_2 is greater than 1, then $A \circ B$ is not necessarily a k-stochastic matrix. By Theorem 5, for $d_1 \geq 2$ each of the $d_1 - 1$ dot products in the circle product $A \circ B$ decreases the difference between the degree of stochastivity r and the dimension d of the matrices until the former exceeds the latter.

Since $A \circ B$ is an appropriate dot product of matrices A and B, the following is also straightforward from Proposition 14.

Proposition 15. *If A is a d-dimensional polystochastic matrix of order n, then for every integer $t \geq 2$ it holds*

$$A \circ J_n^t = J_n^{(d-1)(t-1)+1} \text{ and } J_n^t \circ A = J_n^{(d-1)(t-1)+1}.$$

4 Permanents of products of multidimensional matrices

In this section we estimate the permanent of the above products of matrices by means of the permanent of the factors. We start with the outer product, for which we have the exact equality.

Theorem 7. *Let A be a d_1-dimensional matrix of order n and B be a d_2-dimensional matrix of order n. Then*

$$\operatorname{per}(A \times B) = n!(\operatorname{per} A \cdot \operatorname{per} B).$$

Proof. Let $D_A = \{\alpha^1, \dots, \alpha^n\}$ be a diagonal in the matrix A and $D_B = \{\beta^1, \dots, \beta^n\}$ be a diagonal in the matrix B. For every permutation $\sigma \in S_n$, consider a collection of indices $D = \{\gamma^1, \dots, \gamma^n\}$ such that $\gamma^i = \alpha^i \beta^{\sigma(i)}$ for all $i \in \{1, \dots, n\}$. From the definition of outer product, we see that D is a diagonal in the matrix $A \times B$. Moreover, every diagonal of $A \times B$ is covered by this construction. Therefore, for the matrix $C = A \times B$ we have

$$\operatorname{per} C = \sum_{D \in \mathcal{D}(A \times B)} \prod_{\gamma \in D} c_\gamma = n! \sum_{D_A \in \mathcal{D}(A), D_B \in \mathcal{D}(B)} \prod_{\alpha \in D_A} a_\alpha \prod_{\beta \in D_B} b_\beta$$

$$= n! \left(\sum_{D_A \in \mathcal{D}(A)} \prod_{\alpha \in D_A} a_\alpha \right) \left(\sum_{D_B \in \mathcal{D}(B)} \prod_{\beta \in D_B} b_\beta \right) = n!(\operatorname{per} A \cdot \operatorname{per} B).$$

□

As a corollary, we have that for every pair of matrices A and B of the same order and dimensions d_1 and d_2, there exists a matrix of dimension $d_1 + d_2$ whose permanent is equal to $\operatorname{per} A \cdot \operatorname{per} B$. As it was shown [4], one can find a matrix of one less dimension satisfying this property. For the sake of completeness, we provide its proof here.

Proposition 16 ([4])**.** *For a d_1-dimensional matrix A of order n and a d_2-dimensional matrix B of order n, there exists a $(d_1 + d_2 - 1)$-dimensional matrix C such that* $\operatorname{per} C = \operatorname{per} A \cdot \operatorname{per} B$.

Proof. Let us fix some permutation $\sigma \in S_n$ and define the matrix C with entries c_γ, $\gamma = i\alpha\beta$, as

$$c_{i\alpha\beta} = a_{i\alpha} b_{\sigma(i)\beta}.$$

In other words, the matrix C is composed by n different $(d_1 + d_2 - 2)$-dimensional planes of the matrix $A \times B$ of the same direction such that no two of these planes were in the same hyperplane of $A \times B$.

Note that each pair of diagonals D_A and D_B, where a diagonal $D_A = \{(1, \alpha^1), \dots, (n, \alpha^n)\}$ is from the matrix A and $D_B = \{(1, \beta^1), \dots, (n, \beta^n)\}$ is from the matrix B, one-to-one corresponds to a diagonal $D = \{\gamma^1, \dots, \gamma^n\}$ of the matrix C, where $\gamma^i = i\alpha^i\beta^{\sigma(i)}$.

Repeating the calculations from the proof of Theorem 7, we obtain $\operatorname{per} C = \operatorname{per} A \cdot \operatorname{per} B$. □

Next, we study the permanent of the Kronecker product of multidimensional matrices. In [1], Brualdi proved the following inequalities on the permanent of the Kronecker product of nonnegative 2-dimensional matrices.

Theorem 8 ([1]). *Let A and B be nonnegative 2-dimensional matrices of orders n_1 and n_2, respectively. Then there exists a constant K, $K \geq 1$, depending only on n_1 and n_2 such that*

$$(\operatorname{per} A)^{n_2} (\operatorname{per} B)^{n_1} \leq \operatorname{per}(A \otimes B) \leq K (\operatorname{per} A)^{n_2} (\operatorname{per} B)^{n_1}.$$

Unfortunately, both of these inequalities do not hold for multidimensional matrices.

Remark 1. *There exists a nonnegative 3-dimensional matrix A of order n such that* $\operatorname{per}(A \otimes A) < (\operatorname{per} A)^{2n}$.

Proof. Consider the following 3-dimensional matrix A of order $n = 3$

$$A = \left(\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array} \right).$$

It is easy to see that the permanent of A is equal to 2, so $(\operatorname{per} A)^{2n} = (\operatorname{per} A)^6 = 64$. On the other hand, it can be checked by direct calculations that the permanent of $A \otimes A$ is 40. □

Remark 2. *There exist nonnegative 3-dimensional matrices A and B such that* $\operatorname{per} B = 0$ *but* $\operatorname{per}(A \otimes B) > 0$.

Proof. Consider the following 3-dimensional matrices of order 2:

$$A = \left(\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array} \right); \quad B = \left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array} \right).$$

Note that $\operatorname{per} A = 4$ and $\operatorname{per} B = 0$. Then

$$A \otimes B = \left(\begin{array}{cccc|cccc|cccc|cccc} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array} \right)$$

and it can be checked that $\operatorname{per}(A \otimes B) = 64 > 0$. □

Meanwhile, the following lower bound on the Kronecker product of nonnegative matrices remains true for the multidimensional case.

Theorem 9. *Let A and B be nonnegative d-dimensional matrices of orders n_1 and n_2, respectively. Then*

$$\operatorname{per}(A \otimes B) \geq \operatorname{per} A (\operatorname{per} B)^{n_1} + (\operatorname{per} A)^{n_2} \operatorname{per} B - \operatorname{per} A \operatorname{per} B.$$

Proof. By the definition, the Kronecker product $C = A \otimes B$ has entries $c_\gamma = a_\alpha b_\beta$, where $\gamma_i = (\alpha_i - 1)n_2 + \beta_i$ for each $i = 1, \ldots, d$.

Let us fix a positive diagonal $D_A = \{\alpha^1, \ldots, \alpha^{n_1}\}$ in the matrix A. To every collection of n_1 (not necessarily different) positive diagonals $D_B^1 = \{\beta^{1,1}, \ldots, \beta^{n_2,1}\}, \ldots, D_B^{n_1} = \{\beta^{1,n_1}, \ldots, \beta^{n_2,n_1}\}$ of the matrix B, we put into a correspondence a set of indices $D = \{\gamma^1, \ldots, \gamma^{n_1 n_2}\}$ in the matrix $A \otimes B$, where for each $m \in \{1, \ldots, n_1 n_2\}$, $m = (t-1)n_2 + r$, $1 \leq r \leq n_2$, and $i \in \{1, \ldots, d\}$, we take $\gamma_i^m = (\alpha_i^t - 1)n_2 + \beta_i^{r,t}$.

It can be checked that the set D is a diagonal in the Kronecker product C. Since $c_{\gamma^m} = a_{\alpha^t} b_{\beta^{r,t}}$ and all a_{α^t} and $b_{\beta^{r,t}}$ are nonzero, a diagonal D is also positive. So we proved that $\operatorname{per}(A \otimes B) \geq \operatorname{per} A (\operatorname{per} B)^{n_1}$.

Acting similarly, one can deduce that $\operatorname{per}(A \otimes B) \geq (\operatorname{per} A)^{n_2} \operatorname{per} B$. Note that these two constructions produce different diagonals except of diagonals of the form $D = \{\gamma^1, \ldots, \gamma^{n_1 n_2}\}$ such that $\gamma_i^m = (\alpha_i^t - 1)n_2 + \beta_i^r$ for some positive diagonals $D_A = \{\alpha^1, \ldots, \alpha^{n_1}\}$ and $D_B = \{\beta^1, \ldots, \beta^{n_2}\}$. Such diagonals D arise exactly once in both constructions. Therefore,

$$\operatorname{per}(A \otimes B) \geq \operatorname{per} A (\operatorname{per} B)^{n_1} + (\operatorname{per} A)^{n_2} \operatorname{per} B - \operatorname{per} A \operatorname{per} B.$$

□

For the permanent of the projection of a nonnegative matrix we also have a simple lower bound.

Theorem 10. *Let A be a nonnegative d-dimensional matrix of order n and $S \subseteq \{1, \ldots, d\}$. Then* $\operatorname{per} P_S(A) \geq \operatorname{per} A$.

Proof. Without loss of generality, we assume that $S = \{1, \ldots, \ell\}$. Denote $C = P_S(A)$. By the definitions of the permanent and the projection, we have

$$\operatorname{per} C = \sum_{D_C \in \mathcal{D}(C)} \prod_{\gamma \in D_C} c_\gamma = \sum_{D_C \in \mathcal{D}(C)} \prod_{\gamma \in D_C} \left(\sum_{i_1, \ldots, i_\ell = 1}^{n} a_{i_1, \ldots, i_\ell, \gamma} \right).$$

To each diagonal $D_A \in \mathcal{D}(A)$, $D_A = \{(i_1^1, \ldots, i_\ell^1, \gamma^1), \ldots, (i_1^n, \ldots, i_\ell^n, \gamma^n)\}$ we put into a correspondence a diagonal $D_C = \{\gamma^1, \ldots, \gamma^n\}$. So in $\operatorname{per} C$ every $\prod\limits_{\alpha \in D_A} a_\alpha$ is contained as a summand. Since the matrix A is nonnegative, all other summands in $\operatorname{per} C$ are also nonnegative.

Therefore, $\operatorname{per} C = \operatorname{per} P_S(A) \geq \operatorname{per} A$. □

Note that the permanent of a projection cannot be estimated from the above by the permanent of a matrix.

Remark 3. *There exist nonnegative d-dimensional matrices A of order n such that* $\operatorname{per} A = 0$ *but* $\operatorname{per} P_S(A) > 0$.

Proof. It is sufficient to consider a d-dimensional $(0,1)$-matrix A of order n such that $a_\alpha = 1$ if and only if α belongs to a fixed hyperplane Γ of the first direction (direction $(1, 0, \ldots, 0)$). Then the 1-projection of A is the matrix nJ_n^{d-1}, whose permanent is equal to $(n!)^{d-2}$. □

For the contraction of matrices (if it does not coincide with the projection), it is also not possible to bound the permanent by means of the permanent of the initial matrix.

Remark 4.

(1) *There exists a d-dimensional nonnegative matrix A of order n such that* $\operatorname{per} A = 0$ *but* $\operatorname{per} A_S > 0$ *for some* $S \subseteq \{1, \dots, d\}$, $|S| \geq 2$.
(2) *There exists a d-dimensional nonnegative matrix A of order n such that* $\operatorname{per} A > 0$ *but* $\operatorname{per} A_S = 0$ *for some* $S \subseteq \{1, \dots, d\}$, $|S| \geq 2$.

Proof. 1. Consider a d-dimensional $(0,1)$-matrix A of order n such that $a_\alpha = 1$ if and only if α belongs to a fixed hyperplane Γ of direction $(1, 0, \dots, 0)$. Then for every subset S such that $|S| = \ell \geq 2$ and $1 \in S$, the S-contraction of A is the matrix $nJ_n^{d-\ell}$, whose permanent is equal to $(n!)^{d-\ell-1}$.

2. Consider a d-dimensional $(0,1)$-matrix A, all of whose ones are located at the diagonal

$$D = \{(1, 2, \dots, 2), (2, 3, \dots, 3), \dots, (n, 1, \dots, 1)\}.$$

Obviously, $\operatorname{per} A = 1$. On the other hand, for every subset S such that $|S| = \ell \geq 2$ and $1 \in S$, the S-contraction of A is the $(d-\ell)$-dimensional zero matrix, whose permanent is 0. $\square$

At last, let us estimate the permanent of the dot product of multidimensional matrices. For 2-dimensional matrices, a similar result was proved in [1].

Theorem 11. *Let A be a nonnegative d_1-dimensional matrix of order n and B be a nonnegative d_2-dimensional matrix of order n. Then*

$$\operatorname{per}(AB) \geq \operatorname{per} A \cdot \operatorname{per} B.$$

Proof. By the definition of the permanent,

$$\operatorname{per} A \cdot \operatorname{per} B = \left(\sum_{D_A \in \mathcal{D}(A)} \prod_{\alpha \in D_A} a_\alpha \right) \left(\sum_{D_B \in \mathcal{D}(B)} \prod_{\beta \in D_B} b_\beta \right) = \sum_{D_A \in \mathcal{D}(A)} \sum_{D_B \in \mathcal{D}(B)} \prod_{\alpha \in D_A} \prod_{\beta \in D_B} a_\alpha b_\beta.$$

On the other hand, from the definition of the dot product, for the matrix $C = AB$ we have

$$\operatorname{per}(AB) = \sum_{D_C \in \mathcal{D}(C)} \prod_{\gamma \in D_C} c_\gamma = \sum_{D_C \in \mathcal{D}(C)} \prod_{\gamma \in D_C} \left(\sum_{i=1}^{n} a_{\overline{\alpha}, i} b_{i, \overline{\beta}} \right),$$

where $\gamma = (\overline{\alpha}, \overline{\beta})$, $\overline{\alpha} \in I_n^{d_1 - 1}$, and $\overline{\beta} \in I_n^{d_2 - 1}$.

Let $D_A = \{\alpha^1, \dots, \alpha^n\}$ and $D_B = \{\beta^1, \dots, \beta^n\}$ be a pair of diagonals in A and B, respectively. For every $j \in \{1, \dots, n\}$, we present α^j as $(\overline{\alpha^j}, j)$ and β^j as $(j, \overline{\beta^j})$. Note that for each pair of diagonals D_A and D_B the summand $\prod\limits_{\alpha \in D_A} \prod\limits_{\beta \in D_B} a_\alpha b_\beta$ in $\operatorname{per} A \cdot \operatorname{per} B$ appears in the expansion of $\prod\limits_{\gamma \in D_C} \left(\sum\limits_{i=1}^{n} a_{\overline{\alpha}, i} b_{i, \overline{\beta}} \right)$

in $\mathrm{per}(AB)$, where the diagonal $D_C = \{\gamma^1, \dots, \gamma^d\}$ is such that $\gamma^j = (\overline{\alpha^j}, \overline{\beta^j})$ for all $j \in \{1, \dots, n\}$. Since the matrices A and B are nonnegative, all other summands in $\mathrm{per}(AB)$ are also nonnegative. As a result,

$$\mathrm{per}(AB) \geq \mathrm{per}\, A \cdot \mathrm{per}\, B.$$

□

As a corollary, we have the following inequality on the circle product of nonnegative matrices.

Corollary 1. *Let A be a nonnegative d_1-dimensional matrix of order n and B be a nonnegative d_2-dimensional matrix of order n. Then*

$$\mathrm{per}(A \circ B) \geq \mathrm{per}\, A \cdot (\mathrm{per}\, B)^{d_1 - 1}.$$

At last, we note that it is not possible to bound $\mathrm{per}(AB)$ by $\mathrm{per}\, A$ and $\mathrm{per}\, B$ from above.

Remark 5. *There exist nonnegative 2-dimensional matrices A and B of order n such that* $\mathrm{per}\, A = \mathrm{per}\, B = 0$ *but* $\mathrm{per}(AB) > 0$.

Proof. Consider

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

It is obvious that $\mathrm{per}\, A = \mathrm{per}\, B = 0$. On the other hand, AB is the matrix nJ_n^2, whose permanent is equal to $n!$. □

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