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TWO-DIMENSIONAL GAVRILOV FLOWS

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Abstract: A steady solution to the Euler equations is called a Gavrilov flow if the velocity vector is orthogonal to the pressure gradient at any point. Such flows can be localized that yields compactly supported solutions to the Euler equations. Gavrilov flows exist in dimentions 2 and 3. We present a complete description of two-dimensional Gavrilov flows.

Keywords: Euler equations, Gavrilov flow.

1 Introduction

In dimensions 2 and 3, the Euler equations

$$u\cdot\nabla u+\nabla p=0, \tag{1}$$

$$\nabla\cdot u=0 \tag{2}$$

describe steady flows of ideal incompressible fluid. The equations are also of some mathematical interest in an arbitrary dimension. Here

$$u=\big(u_1(x),\dots,u_n(x)\big)$$

is a vector field on an open set $U\subset\mathbb{R}^n$ (the fluid velocity) and p is a scalar function on U (the pressure). We use also the term "fluid speed" for $|u|$ and "fluid accelaration" for $u\cdot\nabla u$, integral curves of the vector field u are

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also called *particle trajectories*. We consider only smooth real solutions to the Euler equations, i.e., $u_i \in C^\infty(U)$ $(i = 1, \dots, n)$ and $p \in C^\infty(U)$ are assumed to be real functions (the term "smooth" is used as the synonym of "C^∞-smooth").

We say that a solution (u, p) to (1)–(2) is a *Gavrilov flow* if it satisfies

$$u \cdot \nabla p = 0, \tag{3}$$

i.e., the velocity is orthogonal to the pressure gradient at all points of U. We use also the abbriviation GF for "Gavrilov flow".

Equations (1)–(3) constitute the overdetermined system of first order differential equations: $n + 2$ equations in $n + 1$ unknown function. Therefore every GF is an exception in a certain sense. Nevertheless, such flows exist and deserve study.

Let a solution (u, p) to the Euler equations (1)–(2) be defined on an open set $U \subset \mathbb{R}^n$. We say that $x \in U$ is a *regular point* if $\nabla p(x) \neq 0$. The vector field u does not vanish at regular points as is seen from (1). The sets $M_{p_0} = \{x \in U \mid p(x) = p_0 = \text{const}\}$ will be called *isobaric hypersurfaces* (*isobaric surfaces* in the 3D case and *isobaric curves* in the 2D case). In the general case, an arbitrary closed subset of U can be an isobaric hypersurface M_p. But M_p is indeed a smooth hypersurface of $\mathbb{R}^n$ in a neighborhood of a regular point $x \in M_p$. We say that M_p is a *regular isobaric hypersurface* if it consists of regular points.

Gavrilov flows obey the following important property: Given a GF (u, p), a pair of functions $(\tilde{u}, \tilde{p})$ defined by

$$\tilde{u} = \varphi(p)u, \quad \nabla\tilde{p} = \varphi^2(p)\nabla p, \tag{4}$$

where $\varphi(p)$ is an arbitrary smooth function, is again a GF. This property underlies the following construction that will be called the *Gavrilov localization*. Given a GF (u, p) on a domain $U \subset \mathbb{R}^n$, let $M_{p_0} = \{x \in U \mid p(x) = p_0\}$ be a compact regular isobaric hypersurface. Then we can construct a compactly supported smooth solution to the Euler equations on the whole of $\mathbb{R}^n$ by choosing $\varphi(p)$ as a cutoff function supported in a small neighborhood of p_0. Indeed, the new velocity vector field $\tilde{u}$ and the gradient $\nabla\tilde{p}$ are supported in some compact neighborhood $\tilde{U} \subset U$ of the surface M_{p_0}, as is seen from (4), and we define $\tilde{u}$ as zero in $\mathbb{R}^n \setminus U$. Thus, the new pressure $\tilde{p}$ is constant on every connected component of $U \setminus \tilde{U}$. Since only the gradient ∇p participates in (1)–(3), we can assume without lost of generality that $\tilde{p} = 0$ on the "exterior component" of $U \setminus \tilde{U}$. It is now clear that $\tilde{p}$ can be extended to a compactly supported function $\tilde{p} \in C^\infty(\mathbb{R}^n)$.

For some neighborhood $O(\mathcal{C})$ of the circle $\mathcal{C} = \{(x_1, x_2, 0) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = 1\}$, Gavrilov [2] proved the existence of an axisymmetric solution

$$u \in C^\infty\big(O(\mathcal{C}) \setminus \mathcal{C}; \mathbb{R}^3\big), \quad p \in C^\infty\big(O(\mathcal{C}) \setminus \mathcal{C}\big)$$

to the equations (1)–(3) such that the isobaric surface $M_{p_0} \subset O(\mathcal{C}) \setminus \mathcal{C}$ is diffeomorphic to the 2-torus for some p_0. Using the localization procedure described above, Gavrilov proved the existence of a solution $\tilde{u} \in C^\infty(\mathbb{R}^3;\mathbb{R}^3)$, $\tilde{p} \in C^\infty(\mathbb{R}^3)$ to the Euler equations supported in a small neighborhood of M_{p_0}. Thus, Gavrilov gave the positive answer to the long standing question: Is there a smooth compactly supported solution to the Euler equations on the whole of $\mathbb{R}^3$ which is not identically equal to zero? Gavrilov's example is discussed in the subsequent articles [1] and [4]. In particular, a geometric illustration of Gavrilov's example is presented in [4, Section 8].

We say that two GFs (u,p) and $(\tilde{u},\tilde{p})$, defined on the same open set $U \subset \mathbb{R}^n$, are *equivalent* if (4) holds with a smooth non-vanishing function $\varphi(p)$. For example, (u,p) and $(-u,p)$ are equivalent GFs. Equivalent GFs have coincident isobaric hypersurfaces.

The following example of a two-dimensional GF is well known and is called a *vortex*. Let (x_1,x_2) be Cartesian coordinates on $\mathbb{R}^2$. Set

$$u_1(x_1,x_2) = -x_2, \quad u_2(x_1,x_2) = x_1, \quad p(x_1,x_2) = \frac{1}{2}(x_1^2 + x_2^2). \tag{5}$$

One easily checks that these functions satisfy equations (1)–(3). All $0 \neq x \in \mathbb{R}^2$ are regular points of the GF (5), isobaric curves are circles centered at the origin, and the fluid speed $|u|$ is constant on an isobaric curve. This example can be generalized to any even dimension [4].

In order to apply the Gavrilov localization to the flow (5), choose a compactly supported smooth function $\alpha : [0,\infty) \to \mathbb{R}$ such that $\alpha(r) = 0$ for $r \le \varepsilon$ with some $\varepsilon > 0$ and define the function $\beta : [0,\infty) \to \mathbb{R}$ by $\beta(r) = -\int_r^\infty s\alpha^2(s)\,ds$. Then

$$\tilde{u}(x) = \alpha(|x|)u(x), \quad \tilde{p}(x) = \beta(|x|)$$

is a smooth compactly supported GF on the whole of $\mathbb{R}^2$ satisfying $|\tilde{u}|^2 = \psi(\tilde{p})$ with a function ψ uniquely determined by α. In particular, if α is supported in $(r_0-\delta, r_0+\delta)$ for some $r_0 > \delta > 0$, then the velocity $\tilde{u}$ is supported in the annulus $\{x \in \mathbb{R}^2 \ :\ r_0 - \delta < |x| < r_0 + \delta\}$, and the pressure $\tilde{p}$ is supported in the disk $\{|x| < r_0 + \delta\}$ with $\tilde{p} = \mathrm{const}$ in the smaller disk $\{|x| \le r_0 - \delta\}$. Then we can take a linear combination of several such localized flows with disjoints supports. In particular, a periodic GF can be constructed in this way.

An axisymmetric GF on $\mathbb{R}^3$ can be obtained as a direct product of the flow (5) with a constant velocity flow. Namely,

$$u_1(x) = -x_2, \quad u_2(x) = x_1, \quad u_3(x) = a = \mathrm{const},$$
$$p(x) = \frac{1}{2}(x_1^2 + x_2^2) \quad (x = (x_1,x_2,x_3)).$$

Isobaric surfaces are cylinders $\{x_1^2 + x_2^2 = \mathrm{const}\}$, and particle trajectories are either circles (if $a = 0$) or helices (if $a \neq 0$).

The following theorem is the main result of the present article.

Theorem 1. *Let a Gavrilov flow (u,p) be defined on an open set $U \subset \mathbb{R}^2$. For every regular point $x \in U$, after shifting the coordinate origin to some point $c \neq x$, the following statement is valid.*

If $U' \subset U \cap (\mathbb{R}^2 \setminus \{0\})$ is an arbitrary connected neighborhood of the point x consisting of regular points, then the restriction of the flow (u, p) to U' is equivalent to the restriction of the flow (5) to U'.

Roughly speaking, Theorem 1 means that the vortex (5) is the unique two-dimensional GF up to natural ambiguities: shifting the origin, replacing a GF with an equivalent GF, and restricting to a subdomain.

The author considers Theorem 1 as a preliminary step in solving the difficult problem of classification of three-dimensional Gavrilov flows. Even the simplest question remains open: are there GFs on $\mathbb{R}^3$ which are not axisymmetric? A compact regular isobaric surface M_{p_0} of a three-dimensional GF (u, p) is diffeomorhic to the 2-torus $\mathbb{T}^2$ since it is furnished with the non-vanishing tangent vector field $u|_{M_{p_0}}$. Therefore the problem is closely related to classification of geodesic foliations of $\mathbb{T}^2 \subset \mathbb{R}^3$. See details in [4, Section 9].

A similar result (particle trajectories are circles) was recently obtained for a general steady solution (that do not need to be a GF) of the 2D Euler equations [5]. For general solutions, the proof is much more complicated and very different of our elementary geometrical approach.

2 Proof of Theorem 1

In the two-dimensional case, equations (1)–(3) are written in Cartesian coordinates as follows:

$$\begin{aligned} u_1\frac{\partial u_1}{\partial x_1} + u_2\frac{\partial u_1}{\partial x_2} + \frac{\partial p}{\partial x_1} = 0, \\ u_1\frac{\partial u_2}{\partial x_1} + u_2\frac{\partial u_2}{\partial x_2} + \frac{\partial p}{\partial x_2} = 0; \end{aligned} \tag{6}$$

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0; \tag{7}$$

$$u_1\frac{\partial p}{\partial x_1} + u_2\frac{\partial p}{\partial x_2} = 0. \tag{8}$$

The vector field u can be treated as the first order differential operator

$$u = u_1\frac{\partial}{\partial x_1} + u_2\frac{\partial}{\partial x_2}.$$

With the help of the operator, equations (6) and (8) are written in the shorter form:

$$uu_1 + \frac{\partial p}{\partial x_1} = 0, \quad uu_2 + \frac{\partial p}{\partial x_2} = 0; \tag{9}$$

$$up = 0. \tag{10}$$

In all propositions below, a Gavrilov flow $(u, p) = (u_1, u_2; p)$ is assumed to be defined on an open set $U \subset \mathbb{R}^2$.

Proposition 1. *The equality*

$$u|u|^2 = 0 \tag{11}$$

holds in the domain U. In other words, the fluid speed $|u|$ is constant along integral curves of the vector field u.

Proof. Indeed, we obtain on using (9)–(10)

$$\frac{1}{2}u|u|^2 = \frac{1}{2}u(u_1^2 + u_2^2) = u_1(uu_1) + u_2(uu_2) = -u_1\frac{\partial p}{\partial x} - u_2\frac{\partial p}{\partial y} = -up = 0.$$

□

Proposition 2. *The modulus $|\nabla p|$ is constant along integral curves of the vector field u, i.e.,*

$$u|\nabla p|^2 = 0. \tag{12}$$

Proof. We will demonstrate that the equation (12) is equivalent to the incompressibility equation (7). To this end we will write down the equation (12) in local coordinates adapted to the foliation of the domain U into isobaric curves. The equation (12) trivially holds at non-regular points. It suffices to prove the validity of (12) at a regular point.

Fix a regular point $(x_1^0, x_2^0) \in U$ and set $p_0 = p(x_1^0, x_2^0)$. For $p \in \mathbb{R}$ sufficiently close to p_0, the isobaric curve M_p is a regular curve near (x_1^0, x_2^0). We parameterize M_{p_0} by the arc length

$$\begin{aligned} &x_1 = r_1(s), \quad x_2 = r_2(s) \quad (-\varepsilon < s < \varepsilon); \quad {r_1'}^2 + {r_2'}^2 = 1,\\ &r_1(0) = x_1^0, \quad r_2(0) = x_2^0. \end{aligned} \tag{13}$$

We introduce curvilinear coordinates (s,t) in some neighborhood of the point (x_1^0, x_2^0) as follows. Define the vector field

$$\xi = (\xi_1, \xi_2) = \frac{\nabla p}{|\nabla p|^2}. \tag{14}$$

For a sufficiently small $|s|$, let

$$R(s,t) = (R_1(s,t), R_2(s,t)) \quad (p_0 - \delta < t < p_0 + \delta) \tag{15}$$

be the integral curve of the vector field ξ starting from the point $(r_1(s), r_2(s))$ at the initial time $t = p_0$. Thus, $(R_1(s,t), R_2(s,t))$ is the solution to the Cauchy problem

$$\begin{aligned} &\frac{\partial R_1}{\partial t}(s,t) = \xi_1\big(R_1(s,t), R_2(s,t)\big), \quad \frac{\partial R_2}{\partial t}(s,t) = \xi_2\big(R_1(s,t), R_2(s,t)\big);\\ &R_1(s,p_0) = r_1(s), \quad R_2(s,p_0) = r_2(s). \end{aligned} \tag{16}$$

Obviously R is a diffeomorphism between some neighborhoods of points $(0, p_0)$ и (x_1^0, x_2^0); therefore the variables (s,t) constitute a local coordinate system on the plane in some neighborhood of the point (x_1^0, x_2^0). By our construction, R satisfies the identity

$$p\big(R(s,t)\big) = t, \tag{17}$$

which means that the coordinate t coincides with the pressure p. Nevertheless, we use the different notation for the coordinate since t is considered as an independent variable while p is a function on U. By (17), the isobaric curve M_p coincides with the coordinate line $\{t = p\}$ in some neighborhood of (x_1^0, x_2^0) for every p sufficiently close to p_0.

Let

$$dx_1^2 + dx_2^2 = E\,ds^2 + 2F\,ds\,dt + G\,dt^2 \tag{18}$$

be the expression for the Euclidean metric in coordinates (s, t). Then

$$\begin{aligned} E &= \Big(\frac{\partial R_1}{\partial s}\Big)^2 + \Big(\frac{\partial R_2}{\partial s}\Big)^2, \quad F = \frac{\partial R_1}{\partial s}\frac{\partial R_1}{\partial t} + \frac{\partial R_2}{\partial s}\frac{\partial R_2}{\partial t},\\ G &= \Big(\frac{\partial R_1}{\partial t}\Big)^2 + \Big(\frac{\partial R_2}{\partial t}\Big)^2. \end{aligned} \tag{19}$$

On using (14) and (16), we find

$$F = \frac{\partial R_1}{\partial s}\,\xi_1 + \frac{\partial R_2}{\partial s}\,\xi_2 = \frac{\partial R}{\partial s}\cdot\frac{\nabla p}{|\nabla p|^2} = 0.$$

The last equality here holds since the vector $\frac{\partial R}{\partial s}(s, t)$ is tangent to the isobaric curve M_t at the point $R(s, t)$, while the vector $\nabla p(s, t)$ is orthogonal to M_t at the same point. In the same way we find

$$G = \Big|\frac{\partial R}{\partial t}\Big|^2 = |\nabla p|^{-2}.$$

The formula (18) is thus simplified to the following one:

$$dx_1^2 + dx_2^2 = E\,ds^2 + |\nabla p|^{-2}\,dt^2. \tag{20}$$

Besides this, initial conditions in (16) imply with the help of the equality ${r_1'}^2 + {r_2'}^2 = 1$ that

$$E(s, p_0) = 1. \tag{21}$$

We compute Christoffel symbols of the Euclidean metric (20) by standard formulas of differential geometry

$$\begin{aligned} &\Gamma^s_{ss} = \frac{E'_s}{2E}, \quad \Gamma^s_{st} = \frac{E'_t}{2E}, \quad \Gamma^s_{tt} = \frac{1}{|\nabla p|}\,\frac{\partial|\nabla p|}{\partial s},\\ &\Gamma^t_{ss} = -\frac{1}{2}|\nabla p|E'_t, \quad \Gamma^t_{st} = -\frac{1}{|\nabla p|}\,\frac{\partial|\nabla p|}{\partial s}, \quad \Gamma^t_{tt} = -\frac{1}{|\nabla p|}\,\frac{\partial|\nabla p|}{\partial t}. \end{aligned} \tag{22}$$

The vector field u can be written in coordinates (s, t) as

$$u(s, t) = u^s(s, t)\frac{\partial}{\partial s} + u^t(s, t)\frac{\partial}{\partial t} \tag{23}$$

and its divergence is expressed by the formula

$$\nabla\cdot u = \nabla_s u^s + \nabla_t u^t = \frac{\partial u^s}{\partial s} + \Gamma^s_{ss}u^s + \Gamma^s_{st}u^t + \frac{\partial u^t}{\partial t} + \Gamma^t_{st}u^s + \Gamma^t_{tt}u^t. \tag{24}$$

Actually $u^t(s,t) \equiv 0$ since the vector field u is tangent to isobaric curves $\{t = \mathrm{const}\}$. Formulas (23)–(24) are simplified to the following ones:

$$u(s,t) = u^s(s,t)\frac{\partial}{\partial s}, \tag{25}$$

$$\nabla \cdot u = \frac{\partial u^s}{\partial s} + (\Gamma^s_{ss} + \Gamma^t_{st})u^s. \tag{26}$$

By (26), the incompressibility equation (7) looks as follows in coordinates (s,t):

$$\frac{\partial u^s}{\partial s} + (\Gamma^s_{ss} + \Gamma^t_{st})u^s = 0. \tag{27}$$

Substituting values (22) for Christoffel symbols, we arrive to the equation

$$\frac{\partial u^s}{\partial s} + \frac{E'_s}{2E}u^s - \frac{u^s}{|\nabla p|}\,\frac{\partial|\nabla p|}{\partial s} = 0. \tag{28}$$

Let us now use the equation (11). We first observe that, by (20) and (25),

$$|u|^2 = E(u^s)^2$$

and

$$u|u|^2 = u^s\,\frac{\partial(E(u^s)^2)}{\partial s} = 2E(u^s)^2\,\frac{\partial u^s}{\partial s} + E'_s(u^s)^3.$$

Therefore the equation (11) gives $2E(u^s)^2\,\frac{\partial u^s}{\partial s} + E'_s(u^s)^3 = 0$. Since u^s does not vanish,

$$\frac{\partial u^s}{\partial s} + \frac{E'_s}{2E}\,u^s = 0. \tag{29}$$

With the help of the latter equality, (28) implies

$$u^s\,\frac{\partial|\nabla p|}{\partial s} = 0.$$

By (25), this can be written in the form $u|\nabla p| = 0$. □

Proposition 3. *Every connected interval of a regular isobaric curve is either a circle arc or whole circle.*

Proof. Let

$$\gamma(s) = (\gamma_1(s), \gamma_2(s)) \quad (s_0 < s < s_1)$$

be a connected interval of a regular isobaric curve parameterized by the arc length. Let $(\tau(s), \nu(s))$ be the Fresnet frame of the curve γ and $k(s)$, the curvature of γ. Recall Fresnet formulas for a planar curve:

$$\tau' = k\nu, \quad \nu' = -k\tau.$$

Observe that the curvature $k(s)$ is not equal to zero for any s. Indeed, by Proposition 1, the fluid speed $|u|$ is constant along γ. The fluid acceleration at the point $\gamma(s)$ is equal to $|u(\gamma(s))|^2 k(s)\nu(s)$. By the Euler equation

$$u \cdot \nabla u + \nabla p = 0,$$

the acceleration is equal to $-\nabla p(\gamma(s))$. Thus,

$$|u|^2 k(s)\nu(s) = -\nabla p(\gamma(s)).$$

Hence $k(s) \neq 0$ at regular points.

We have to demonstrate that $k(s)$ is a constant function, i.e., $\frac{dk}{ds} = 0$. By the Fresnet formulas, $k = \nu \cdot \frac{d\tau}{ds}$ and $\frac{dk}{ds} = \nu \cdot \frac{d^2\tau}{ds^2}$. Thus, we have to prove the equality

$$\nu(s) \cdot \frac{d^2\tau(s)}{ds^2} = 0. \tag{30}$$

By (11),

$$\tau(s) = au(\gamma(s)) \tag{31}$$

with some positive constant a. And by Proposition 2,

$$\nu(s) = b(\nabla p)(\gamma(s))$$

with some positive constant b. Therefore the equation (30) is equivalent to the following one:

$$(\nabla p)(\gamma(s)) \cdot \frac{d^2u(\gamma(s))}{ds^2} = 0,$$

or in coordinates

$$\frac{\partial p}{\partial x_1}(\gamma(s))\frac{d^2\big(u_1(\gamma(s))\big)}{ds^2} + \frac{\partial p}{\partial x_2}(\gamma(s))\frac{d^2\big(u_2(\gamma(s))\big)}{ds^2} = 0. \tag{32}$$

The equalities

$$\frac{d\big(f(\gamma(s))\big)}{ds} = a(uf)(\gamma(s)), \quad \frac{d^2\big(f(\gamma(s))\big)}{ds^2} = a^2(u^2f)(\gamma(s))$$

hold for any function $f(x_1, x_2)$, where $a > 0$ is the same constant as in (31). Therefore to prove (32) it suffices to demonstrate that

$$\frac{\partial p}{\partial x_1}\,(u^2u_1) + \frac{\partial p}{\partial x_2}\,(u^2u_2) = 0. \tag{33}$$

By (9),

$$u^2u_1 = u(uu_1) = -u\Big(\frac{\partial p}{\partial x_1}\Big), \quad u^2u_2 = u(uu_2) = -u\Big(\frac{\partial p}{\partial x_2}\Big).$$

With the help of these equalities, the equation (33) takes the form

$$\frac{\partial p}{\partial x_1}\,u\Big(\frac{\partial p}{\partial x_1}\Big) + \frac{\partial p}{\partial x_2}\,u\Big(\frac{\partial p}{\partial x_2}\Big) = 0$$

or

$$u|\nabla p|^2 = 0.$$

This is true by Proposition 2. □

Proposition 4. *Let $U' \subset U$ be a convex open set consisting of regular points. Integral curves of the vector field ∇p living in U' are straight-line intervals, i.e., every such integral curve coincides with the intersection $U' \cap L$ for some straight line L. The length of a segment of such integral curve between isolines $\{p = p_0\}$ and $\{p = p_1\}$ depends on p_0 and p_1 only.*

Proof. First of all we observe that both statements are of a local character, i.e., it suffices to prove the statements for a sufficiently small convex neighborhood $U' \subset U$ of an arbitrary regular point.

Fix a regular point $(x_1^0, x_2^0) \in U$ and set $p_0 = p(x_1^0, x_2^0)$. Introduce curvilinear coordinates (s, t) in some neighborhood of (x_1^0, x_2^0) in the same way as at the beginning of the proof of Proposition 2. The Euclidean metric is expresses in these coordinates by the formula (20). By (17), coordinate lines $t = \text{const}$ are isolines of the function p. By Proposition 3, the modulus $|\nabla p|$ is constant on coordinate lines $t = \text{const}$, i.e., $|\nabla p| = |\nabla p|(t)$. Now formulas (22) imply that

$$\Gamma^s_{tt} = 0, \quad \Gamma^t_{tt} = \Gamma^t_{tt}(t). \tag{34}$$

For a point (s_0, t_0) belonging to the domain of the chosen coordinate system, let us consider the geodesic $(s(\tau), t(\tau))$ starting at the point (s_0, t_0) orthogonally to the coordinate line $\{s = s_0\}$. The functions $s(\tau)$ and $t(\tau)$ solve the Cauchy problem

$$\begin{aligned} &s'' + \Gamma^s_{ss} s'^2 + 2\Gamma^s_{st} s' t' + \Gamma^s_{tt} t'^2 = 0, \quad t'' + \Gamma^t_{ss} s'^2 + 2\Gamma^t_{st} s' t' + \Gamma^t_{tt} t'^2 = 0;\\ &s(0) = s_0, \quad t(0) = t_0, \quad s'(0) = 0, \quad t'(0) = 1. \end{aligned}$$

By (34), the problem is specified as follows:

$$\begin{aligned} &s'' + \Gamma^s_{ss} s'^2 + 2\Gamma^s_{st} s' t' = 0, \quad t'' + \Gamma^t_{ss} s'^2 + 2\Gamma^t_{st} s' t' + \Gamma^t_{tt}(t) t'^2 = 0;\\ &s(0) = s_0, \quad t(0) = t_0, \quad s'(0) = 0, \quad t'(0) = 1. \end{aligned} \tag{35}$$

The solution of the problem is of the form $s = s_0, t = t(\tau)$, where $t = t(\tau)$ is the solution of the Cauchy problem

$$t'' + \Gamma^t_{tt}(t) t'^2 = 0; \quad t(0) = t_0, \quad t'(0) = 1.$$

Thus, our geodesic coincides with the coordinate line $\{s = s_0\}$ up to parametrization. In other words, coordinate lines $\{s = s_0\}$ are geodesics of the Euclidean metric, i.e., straight lines.

By (14)–(16), coordinate lines $\{s = \text{const}\}$ coincide with integral curves of the vector field ∇p up to parametrization. We have thus proved that integral curves of the vector field ∇p are straight-line. Moreover, these straight-line integral curves are orthogonal to isolines $\{t = \text{const}\}$ of the function p. Now, applying the first variation formula for the length of a geodesic [3, Section 4.1], we obtain the second statement: The length of a segment of such integral curve between isolines $\{p = p_0\}$ and $\{p = p_1\}$ depends on p_0 and p_1 only. □

Proposition 5. *Let $U' \subset U$ be a connected open set consisting of regular points. Every isoline of the function p living in U' coincides with the intersection $U' \cap C$, where C is a circle centered at a point $c \in \mathbb{R}^2 \setminus U'$ independent of the choice of the isoline.*

Proof. For a point $(x_1, x_2) \in U'$, let

$$\gamma(s) \quad (-\varepsilon < s < \varepsilon); \quad \gamma(0) = (x_1, x_2)$$

be the isoline of the function p through the point (x_1, x_2) parameterized by the arc length, where $\varepsilon > 0$ is sufficiently small. By Proposition 3, γ is a circle arc centered at some point $c(x_1, x_2)$. The map $U' \to \mathbb{R}^2, (x_1, x_2) \mapsto c(x_1, x_2)$ is continuous. It suffices to prove that it is a locally constant map.

We fix a point $(x_1^0, x_2^0) \in U'$, set $p_0 = p(x_1^0, x_2^0)$ and denote by

$$\gamma_0(s) \quad (-\varepsilon < s < \varepsilon); \quad \gamma_0(0) = (x_1^0, x_2^0)$$

the isoline of the function p passing through (x_1^0, x_2^0) and parameterized by the arc length. Proposition 4 implies that, for every p_1 sufficiently close to p_0, the intersection of the isoline $\{p = p_1\}$ with some neighborhood of the point (x_1^0, x_2^0) can be obtained by the following construction. Let $N(s)$ be the normal line of γ_0 through the point $\gamma_0(s)$. Choose the orientation of the normal $N(s)$ accordingly to the vector $\nabla p(\gamma_0(s))$. On each normal $N(s)$ we plot a segment of constant length (depending on (p_0, p_1)) from the point $\gamma_0(s)$. Ends of these segments constitute the isoline $\{p = p_1\}$. In other words, the isoline $\{p = p_1\}$ is an equidistant curve of the curve γ_0 near (x_1^0, x_2^0) for any p_1 sufficiently close to p_0. Therefore circle arcs $\{p = p_0\}$ and $\{p = p_1\}$ coincide, i.e., $c(x_1, x_2) = c(x_1^0, x_2^0)$ for all points $(x_1, x_2) \in U'$ sufficiently close to (x_1^0, x_2^0).

Thus, all isolines of the function p living in U' are circle arcs with a common center $c \in \mathbb{R}^2$. It can happen that $c \in U$. But in such a case c cannot be a regular point since different isolines of p pass through c. Therefore $c \notin U'$. □

Proposition 6. *Let $U' \subset U$ be a connected open set consisting of regular points. Assume that every isoline of the function p living in U' coincides with the intersection $U' \cap C$, where C is a circle centered at the origin $(0, 0)$. Set $r = r(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$. Then, in the domain U', the functions $|u|$ and $|\nabla p|$ depend on r only and satisfy*

$$|u|^2 = r|\nabla p|. \tag{36}$$

Proof. The statement is of a local character. It suffices to prove the statement for some neighborhood of an arbitrary regular point.

Fix a regular point $(x_1^0, x_2^0) \in U$. For every regular point $(x_1, x_2) \in U$ sufficiently close to (x_1^0, x_2^0), let $\gamma(s)$ $(-\varepsilon < s < \varepsilon)$ be an interval of the isoline of the function p through the point (x_1, x_2) parameterized by the arc length and lying near (x_1^0, x_2^0). By the assumption, γ is a circle arc centered at the origin. The radius of the circle arc is equal to $r = r(x_1, x_2)$. By Proposition 1, the fluid speed $|u|$ is constant on the circle arc γ. Hence $|u(x_1, x_2)| = |u|(r)$ at least for all (x_1, x_2) belonging to some neighborhood of (x_1^0, x_2^0). The fluid acceleration at the point $\gamma(s)$ is equal to $\frac{|u(r)|^2}{r}\nu(s)$, where $\nu(s)$ is the unit normal vector of the curve γ. On the other hand, by the Euler equation, the acceleration is equal to $-\nabla p(\gamma(s))$. Hence $|\nabla p| = |\nabla p|(r)$ and the equality (36) holds. □

Proof of Theorem 1. Let a Gavrilov flow (u, p) be defined on an open set $U \subset \mathbb{R}^2$. Fix a regular point $x \in U$. Let γ be a connected interval of the isobaric curve through the point x. By Proposition 3, γ is a circle arc centered at some point $c \neq x$. We shift the origin to the point c. Now γ is a circle arc centered at the origin.

Let $U' \subset U \cap (\mathbb{R}^2 \setminus \{0\})$ be an arbitrary connected neiborhood of the point x consisting of regular points. By Proposition 5, every isoline of the function p living in U' coincides with the intersection $U' \cap C$, where C is a circle centered at the origin. By Proposition 6, restrictions of the functions $|u|$ and $|\nabla p|$ to the domain U' depend on r only and the equality (36) holds.

Living in U' isolines of the functions $r(x_1, x_2)$ and $p(x_1, x_2)$ coincide (both are circle arcs centered at the origin). Hence gradients of the functions r and p are collinear to each other at any point of the domain U'. The gradient of r does not vanish in U' since $(0, 0) \notin U'$. The gradient of p does not vanish in U' since U' consists of regular points. Therefore the representation $r = r(p)$ is possible with a smooth right-hand side. With the help of the representation, we make sure that $|u|$ and $|\nabla p|$ can be also represented in U' as smooth functions of the pressure p, i.e., $|u| = |u|(p)$ and $|\nabla p| = |\nabla p|(p)$.

We introduce the positive function $\varphi(p)$ by

$$\varphi(p) = \frac{r(p)}{|u|(p)} \tag{37}$$

and define a new GF $(\tilde{u}, \tilde{p})$ on the domain U' by the formulas

$$\tilde{u} = \pm\varphi u, \quad \nabla\tilde{p} = \varphi^2\, \nabla p. \tag{38}$$

The GF $(\tilde{u}, \tilde{p})$ is equivalent to $(u|_{U'}, p|_{U'})$. The sign in (38) is chosen so that integral curves $(x_1(t), x_2(t))$ of the vector field $\tilde{u}$ (which are circle arcs centered at the origin) go around origin in the counter clockwise direction when t is increasing. Formulas (36)–(37) imply that

$$|\tilde{u}| = |\nabla\tilde{p}| = r. \tag{39}$$

The vector field $\tilde{u}$ is tangent to circle arcs centered at the origin. The same circle arcs are isolines of the function $\tilde{p}$. Together with (39) and the above remark on the sign choice, this gives

$$\begin{aligned} &\tilde{u}_1(x_1, x_2) = -x_2, \quad \tilde{u}_2(x_1, x_2) = x_1, \\ &p(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2) + \mathrm{const} \quad \big((x_1, x_2) \in U'\big). \end{aligned}$$

The constant can be equated to zero since only $\nabla\tilde{p}$ participates in the Euler equations but not the function $\tilde{p}$ itself. We see that the flow $(\tilde{u}, \tilde{p})$ coincides with the restriction of the flow (5) to the domain U'. $\square$

In particular, if a GF (u, p) is defined on $\mathbb{R}^2 \setminus \{0\}$ and all points of $\mathbb{R}^2 \setminus \{0\}$ are regular ones, then this flow is equivalent to the restriction of the flow (5) to $\mathbb{R}^2 \setminus \{0\}$. But unlike (5), the vector field u and function p can be singular at the origin.

References

[1] P. Constantin, J. La, V. Vicol, *Remarks on a paper by Gavrilov: Grad-Shafranov equations, sready solutions of the three dimensional incompressible Euler equations with compactly supported velocities, and applications*, Geom. Funct. Anal., **29**:6 (2019), 1773–1793. Zbl 1427.35189

[2] A.V. Gavrilov, *A steady Euler flow with compact support*, Geom. Funct. Anal., **29**:1 (2019), 190–197. Zbl 1414.35157

[3] D. Gromoll, W. Klingenberg, W. Meyer, *Riemannsche Geometrie im Großen*, Springer-Verlag, Berlin-Heidelberg-New York, 1968. Zbl 0155.30701

[4] V.Yu. Rovenski, V.A. Sharafutdinov, *Steady flows of ideal incompressible fluid with velocity pointwise orthogonal to the pressure gradient*, Arnold Math. J., (2023). See also arXiv:2209.14572.

[5] D. Ruiz, *Symmetry results for compactly supported sready solutions of the 2D Euler equations*, Arch. Ration. Mech. Anal., **247**:3 (2023), Paper No. 40. Zbl 1518.35391

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