S©MR

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ
Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru
ISSN 1813-3304

Том 21, № 1, стр. 271–276 (2024) УДК 510.67, 512.57
https://doi.org/10.33048/semi.2024.21.020 MSC 03C

PSEUDOFINITE S-ACTS

A.A. STEPANOVA, E.L. EFREMOV, S.G. CHEKANOV

Communicated by S.V. SUDOPLATOV

Abstract: The work has begun to study the structure of pseudofinite acts over a monoid. A theorem on the finiteness of an arbitrary cyclic subacts of S-act is proved under the condition that this S-act is pseudofinite and the number of types of isomorphisms of finite cyclic S-acts is finite. It is shown that a coproduct of finite S-acts is pseudofinite. As a consequence, it is shown that any S-act, where S is a finite group, is pseudofinite.

Keywords: pseudofinite act, pseudofinite theory, coproduct, act over monoid.

Introduction

Recently, the model theory of pseudofinite structures is an actively developing area of mathematics. The concept of pseudofiniteness was first introduced by J. Ax to show the solvability of the theory of all finite fields [1]. Subsequently, the theory of pseudofinite fields received great development (see, for example, [2], [3]). A large review of the theory of models of finite and pseudofinite groups is presented in [4]. In [5] the characterization of

STEPANOVA, A.A., EFREMOV, E.L., CHEKANOV, S.G., PSEUDOFINITE S-ACTS.

Supported by RF Ministry of Education and Science (Suppl. Agreement No. 075-02-2024-1440 of 28.02.2024).
Received November, 10, 2023, Published April, 8, 2024.

pseudofinite groups is given in terms of Szmielew's invariants. The results related to the model theory of finite and pseudofinite rings can be found in [6]. In [7] the structure of pseudofinite acyclic graphs is studied. In [8] the authors of this work considered the issues of pseudofiniteness of connected unars without cycles.

An acts over monoids are a generalization of unars. In this work we study pseudofinite S-acts. A theorem on the finiteness of an arbitrary cyclic subacts of S-act is proved under the condition that this S-act is pseudofinite and the number of types of isomorphisms of finite cyclic S-acts is finite. It is shown that a coproduct of finite S-acts is pseudofinite. As a consequence, it is shown that any S-act, where S is a finite group, is pseudofinite.

1 Preliminaries

Let us recall some definitions and facts from act theory and model theory (see [9, 10, 11]). Throughout this paper S will denote a monoid with identity 1. An algebraic system $\langle A; s\rangle_{s\in S}$ of the language $L_S = \{s \mid s \in S\}$ consisting of unary operation symbols is a (left) *S-act* if $s_1(s_2a) = (s_1s_2)a$ and $1a = a$ for all $s_1, s_2 \in S$ and $a \in A$. An S-act $\langle A; s\rangle_{s\in S}$ is denoted by ${}_SA$. An S-act ${}_SA$ is called *cyclic* if there exists $a \in A$ such that $A = \{sa \mid s \in S\}$. The cyclic act that generated by a is denoted by ${}_SSa$. Elements x, y of an S-act ${}_SA$ are called *connected* (denoted by $x \sim y$) if there exist $n \in \omega$, $a_0, \ldots, a_n \in A$, $s_1, \ldots, s_n \in S$ such that $x = a_0$, $y = a_n$, and $a_i = s_ia_{i-1}$ or $a_{i-1} = s_ia_i$. An S-act ${}_SA$ is called *connected* if we have $x \sim y$ for any $x, y \in {}_SA$. It is easy to check that $\sim$ is a congruence relation on the S-act ${}_SA$. The classes of this relation are called *connected components* of the S-act ${}_SA$. *A coproduct* of S-acts ${}_SA_i$ is a disjunctive union of this S-acts. The coproduct of S-acts ${}_SA_i$ is denoted by $\coprod_{i\in I} {}_SA_i$. It is known (see [9]) that every S-act ${}_SA$ can be uniquely represented as a coproduct of connected components.

A structure $\mathfrak{M}$ of the language L is *pseudofinite* if any sentence of language L true in $\mathfrak{M}$ is true in some finite of language L.

If ${}_SA$ is an S-act, then writing $\bar{a} \in A$ means that $a_1, \ldots, a_n \in A$, where $\bar{a} = \langle a_1, \ldots, a_n\rangle$. If $\bar{u} = \langle u_1, \ldots, u_n\rangle$, then $l(\bar{u})$ will denote the length of the tuple $\bar{u}$ and the notation $v \in \bar{u}$ means that v is an element of the tuple $\bar{u}$.

2 Pseudofiniteness of a coproduct of finite S-acts

Lemma 1. *Let ${}_SA = \coprod_{\alpha<\kappa} {}_SA_\alpha$, κ be an infinite ordinal, ${}_SA_\alpha$ be the connected components of ${}_SA$,*

$$ {}_SD_\beta = \coprod_{\alpha<\beta} {}_SA_\alpha \quad (\beta < \kappa), $$

$$ {}_SD = \prod_{\beta<\kappa} {}_SD_\beta / F, $$

where F is an ultrafilter on κ containing a Fréchet filter, i.e. filter $\{\alpha \in \kappa \mid |\kappa \setminus \alpha| < \kappa\}$. For any $a \in A$, by a' we denote an element of $\prod_{\beta\in\kappa} D_\beta$ such that $a'(\alpha) = a$ if $\alpha \geq \alpha_0$, and $a'(\alpha)$ is an arbitrary element of D_α, if $\alpha < \alpha_0$, for some $\alpha_0 < \kappa$; by $\bar{A}$ we denote a set $\{a'/F \mid a \in A\}$. Then ${}_SA \cong {}_S\bar{A}$ and ${}_S\bar{A} \preccurlyeq {}_SD$.

Proof. The map $\theta : A \to \bar{A}$ such that $\theta(a) = a'/F$ is an isomorphism from ${}_SA$ to ${}_S\bar{A}$, so ${}_SA \cong {}_S\bar{A}$. We show that ${}_S\bar{A} \preccurlyeq {}_SD$. Let $n \in \omega$, $\bar{x} = \langle x_1, \ldots, x_n\rangle$, $\bar{a} = \langle a_1, \ldots, a_n\rangle \in A$, $\bar{a}' = \langle a'_1/F, \ldots, a'_n/F\rangle \in \bar{A}$, $\Phi(\bar{x})$ be a formula of language L_S, $L_{S'}$ be a language of formula $\Phi(\bar{x})$, ${}_S\bar{A} \models \Phi(\bar{a}')$. It is necessary to prove that ${}_SD \models \Phi(\bar{a}')$.

For an arbitrary formula $\Psi(\bar{x})$ of a language L we will use standard notation: $(\Psi(\bar{x}))^1 \leftrightharpoons \Psi(\bar{x})$, $(\Psi(\bar{x}))^0 \leftrightharpoons \neg\Psi(\bar{x})$. For $m \in \omega$, let $E^{(m)}$ denote the set $\{\varepsilon \mid \varepsilon : (S')^2 \times \{1, \ldots, m\}^2 \to \{0, 1\}\}$; for $\varepsilon \in E^{(m)}$ let $\Psi_\varepsilon(x_1, \ldots, x_m)$ denote the formula $\bigwedge_{(i,j)\in\{1,\ldots,m\}^2} \bigwedge_{(s,t)\in(S')^2} (sx_i = tx_j)^{\varepsilon((s,t),(i,j))}$.

Since θ^{-1} is an isomorphism from ${}_S\bar{A}$ to ${}_SA$, then ${}_SA \models \Phi(\bar{a})$. In addition, $\Phi(\bar{x}) \equiv \Phi'(\bar{x})$, where

$$\Phi'(\bar{x}) \leftrightharpoons \forall y_1 \exists z_1 \ldots \forall y_k \exists z_k \bigvee_{\varepsilon\in E} \Psi_\varepsilon(\bar{x}, \bar{y}, \bar{z}),$$

$\bar{y} = \langle y_1, \ldots, y_k\rangle$, $\bar{z} = \langle z_1, \ldots, z_k\rangle$, $E \subseteq E^{(n+2k)}$. We can assume that $\Phi(\bar{x}) \leftrightharpoons \Phi'(\bar{x})$. Since ${}_SA \models \Phi(\bar{a})$, then there are functions $f_i : A^i \to A$ $(1 \leq i \leq k)$ such that

$${}_SA \models \forall y_1 \ldots y_k \bigvee_{\varepsilon\in E} \Psi_\varepsilon(\bar{a}, y_1, \ldots, y_k, f_1(y_1), \ldots, f_k(y_1, \ldots, y_k)).$$

For an arbitrary $\bar{b} = \langle b_1, \ldots, b_k\rangle \in A$ there is unique $\varepsilon_{\bar{b}} \in E^{(n+2k)}$ such that ${}_SA \models \Psi_{\varepsilon_{\bar{b}}}(\bar{a}, b_1, \ldots, b_k, f_1(b_1), \ldots, f_k(b_1, \ldots, b_k))$. Since the set $E^{(n+2k)}$ is finite, then there is a finite number of different $\varepsilon_{\bar{b}_1}, \ldots, \varepsilon_{\bar{b}_m}$ such that for any $\bar{c} \in A$ there is $\bar{b}_i \in \{\bar{b}_1, \ldots, \bar{b}_m\}$, satisfying the equality $\varepsilon_{\bar{c}} = \varepsilon_{\bar{b}_i}$. We denote the set $\{\bar{b}_1, \ldots \bar{b}_m\}$ by M. For $|l|$-element set $J = \{j_1, \ldots, j_l\} \subseteq \{1, \ldots, k\}$, $j \in \{1, \ldots, k\}$ and arbitrary $\bar{c} = \langle c_1, \ldots, c_k\rangle \in A$, we introduce some more notations:

$$\bar{c}^j = \langle c_1, \ldots c_j\rangle,$$

$$K^J_{\bar{b}} = \{(\alpha_{j_1}, \ldots, \alpha_{j_l}) \in \kappa^l \mid \exists \bar{c} \in A\ \forall j \in J\ (f_j(\bar{c}^j) \in A_{\alpha_j},\ \varepsilon_{\bar{b}} = \varepsilon_{\bar{c}})\}.$$

Consider an S-act ${}_SD_{\alpha_0}$, where $\alpha_0 \in \kappa$, such that $a_1, \ldots, a_n \in D_{\alpha_0}$ and for any $\bar{b} \in M$, $J = \{j_1, \ldots, j_l\} \subseteq \{1, \ldots, k\}$ $(l = |J|)$ the following conditions are satisfied:

(1) if $|K^J_{\bar{b}}| > p$, where $p = l \cdot (n + 2k - l)$, then $({}_SA_{\alpha^1_{j_1}}, \ldots, {}_SA_{\alpha^1_{j_l}})$, …, $({}_SA_{\alpha^p_{j_1}}, \ldots, {}_SA_{\alpha^p_{j_l}})$ are l-tuples of the connected components of S-act ${}_SD_{\alpha_0}$ for some different $(\alpha^1_{j_1}, \ldots, \alpha^1_{j_l}), \ldots, (\alpha^p_{j_1}, \ldots, \alpha^p_{j_l}) \in K^J_{\bar{b}}$;

(2) if $|K^J_{\bar{b}}| \leq p$ and $K^J_{\bar{b}} = \{(\alpha^1_{j_1}, \ldots, \alpha^1_{j_l}), \ldots, (\alpha^p_{j_1}, \ldots, \alpha^p_{j_l})\}$, then

$$({}_SA_{\alpha^1_{j_1}}, \ldots, {}_SA_{\alpha^1_{j_l}}), \ldots, ({}_SA_{\alpha^p_{j_1}}, \ldots, {}_SA_{\alpha^p_{j_l}})$$

are l-tuples of the connected components of S-act ${}_SD_{\alpha_0}$.

Let $\alpha \geq \alpha_0$. Then $D_{\alpha_0} \subseteq D_\alpha$. We show that ${}_SD_\alpha \models \Phi(\bar{a})$. For this we construct functions $g_i : D_\alpha^i \to D_\alpha$ $(1 \leq i \leq k)$ such that

$$ {}_SD_\alpha \models \forall y_1 \dots y_k \bigvee_{\varepsilon \in E} \Psi_\varepsilon(\bar{a}, y_1, \dots, y_k, g_1(y_1), \dots, g_k(y_1, \dots, y_k)). $$

Let $\bar{c} = \langle c_1, \dots, c_k \rangle \in D_\alpha$. By the definition of $\varepsilon_{\bar{c}}$, we have

$$ {}_SA \models \Psi_{\varepsilon_{\bar{c}}}(\bar{a}, c_1, \dots, c_k, f_1(c_1), \dots, f_k(c_1, \dots, c_k)). $$

By the definition of the set M, there exists $\bar{b} \in M$ such that $\varepsilon_{\bar{b}} = \varepsilon_{\bar{c}}$. Let $J = \{j \in \{1, \dots, k\} \mid f_j(\bar{c}^j) \notin D_\alpha\}$. If $J = \varnothing$, then

$$ {}_SD_\alpha \models \Psi_{\varepsilon_{\bar{c}}}(\bar{a}, c_1, \dots, c_k, f_1(c_1), \dots, f_k(c_1, \dots, c_k)) $$

and we set $g_i(\bar{c}^i) = f_i(\bar{c}^i)$ for all i, $1 \leq i \leq k$.

Suppose that $J = \{j_1, \dots, j_l\}$ and $J \neq \varnothing$. Then in the formula $\Psi_{\varepsilon_{\bar{b}}}(\bar{x}, \bar{y}, \bar{z})$ there is no subformulas of the form $(sz_j = tu)^1$, where $s, t \in S'$, $j \in J$, and $u \in \{x_i \mid i \in \{1, \dots, n\}\} \cup \{y_i \mid i \in \{1, \dots, k\}\} \cup \{z_i \mid i \in \{1, \dots, k\} \setminus J\}$.

If $|K_{\bar{b}}^J| \leq p$ and $K_{\bar{b}}^J = \{(\alpha_{j_1}^1, \dots, \alpha_{j_l}^1), \dots, (\alpha_{j_1}^p, \dots, \alpha_{j_l}^p)\}$, then by (2), $({}_SA_{\alpha_{j_1}^1}, \dots, {}_SA_{\alpha_{j_l}^1}), \dots, ({}_SA_{\alpha_{j_1}^p}, \dots, {}_SA_{\alpha_{j_l}^p})$ are l-tuples of the connected components of S-act ${}_SD_{\alpha_0}$, hence there is $(\alpha_{j_1}^i, \dots, \alpha_{j_l}^i) \in K_{\bar{b}}^J$ such that $f_j(\bar{c}^j) \in A_{\alpha_j^i} \subseteq D_\alpha$ for all $j \in J$, contradiction.

Therefore $|K_{\bar{b}}^J| > p$, where $p = l \cdot (n + 2k - l)$. By (1), $({}_SA_{\alpha_{j_1}^1}, \dots, {}_SA_{\alpha_{j_l}^1})$, $\dots$, $({}_SA_{\alpha_{j_1}^p}, \dots, {}_SA_{\alpha_{j_l}^p})$ are l-tuples of the connected components of S-act ${}_SD_\alpha$, where $(\alpha_{j_1}^1, \dots, \alpha_{j_l}^1), \dots, (\alpha_{j_1}^p, \dots, \alpha_{j_l}^p)$ are the different elements of $K_{\bar{b}}^J$. Then there is $(\alpha_{j_1}, \dots, \alpha_{j_l}) \in \{(\alpha_{j_1}^1, \dots, \alpha_{j_l}^1), \dots, (\alpha_{j_1}^p, \dots, \alpha_{j_l}^p)\}$ such that ${}_SA_{\alpha_{j_1}} \cup \dots \cup {}_SA_{\alpha_{j_l}}$ does not contain elements from the set

$$ \{a_i \mid i \in \{1, \dots, n\}\} \cup \{c_i \mid i \in \{1, \dots, k\}\} \cup \{f_i(\bar{c}^i) \mid i \in \{1, \dots, k\} \setminus J\}. $$

By definition of $K_{\bar{b}}^J$, l-tuple $(\alpha_{j_1}, \dots, \alpha_{j_l})$ corresponds to $\bar{d} \in A$ such that $\varepsilon_{\bar{d}} = \varepsilon_{\bar{b}}$ and $f_j(\bar{d}^j) \in A_{\alpha_j}$ for all $j \in J$. Since in the formula $\Psi_{\varepsilon_{\bar{b}}}(\bar{x}, \bar{y}, \bar{z})$ there is no subformulas of the form $(sz_j = tu)^1$, where $s, t \in S'$, $j \in J$,

$$ u \in \{x_i \mid i \in \{1, \dots, n\}\} \cup \{y_i \mid i \in \{1, \dots, k\}\} \cup \{z_i \mid i \in \{1, \dots, k\} \setminus J\} $$

and

$$ {}_SA \models \Psi_{\varepsilon_{\bar{d}}}(\bar{a}, d_1, \dots, d_k, g_1(d_1), \dots, g_k(d_1, \dots, d_k)), $$

then

$$ {}_SD_\alpha \models \Psi_{\varepsilon_{\bar{b}}}(\bar{a}, c_1, \dots, c_k, g_1(c_1), \dots, g_k(c_1, \dots, c_k)), $$

where $g_i(\bar{c}^i) = f_i(\bar{c}^i)$ if $i \notin J$, and $g_i(\bar{c}^i) = f_i(\bar{d}^i)$ if $i \in J$.

Thus, we construct functions $g_i : D_\alpha^i \to D_\alpha$ $(1 \leq i \leq k)$ such that

$$ {}_SD_\alpha \models \forall y_1 \dots y_k \bigvee_{\varepsilon \in E} \Psi_\varepsilon(\bar{a}, y_1, \dots, y_k, g_1(y_1), \dots, g_k(y_1, \dots, y_k)). $$

So ${}_SD_\alpha \models \Phi(\bar{a})$ for any $\alpha \geq \alpha_0$. Since the ultrafilter F contains Fréchet filter, then by Łoś's Theorem ${}_SD \models \Phi(\bar{a}')$. □

Corollary 1. *Let* ${}_SA = \coprod_{\alpha<\kappa} {}_SA_\alpha$*,* κ *be an infinite ordinal,* ${}_SA_\alpha$ *be the connected components of* ${}_SA$*,* Φ *be a sentence of language* L_S*, and* ${}_SA \models \Phi$*. Then there is a finite set* $\{{}_SB_1, \ldots, {}_SB_m\} \subseteq \{{}_SA_\alpha \mid \alpha < \kappa\}$ *such that* ${}_SB_1 \sqcup \ldots \sqcup {}_SB_m \models \Phi$*.*

Proof. Let the conditions of Corollary be satisfied. We will use the notation introduced in the proof of Lemma 1. As in Lemma 1, we can assume that

$$\Phi \leftrightharpoons \forall y_1 \exists z_1 \ldots \forall y_k \exists z_k \bigvee_{\varepsilon \in E} \Psi_\varepsilon(\bar{y}, \bar{z}),$$

where $\bar{y} = \langle y_1, \ldots, y_k \rangle$, $\bar{z} = \langle z_1, \ldots, z_k \rangle$, $E \subseteq E^{(2k)}$. As the desired S-act ${}_SB_1 \sqcup \ldots \sqcup {}_SB_m$ we take an S-act ${}_SD_{i_0}$, constructed in the proof of Lemma 1 for $n = 0$. □

Theorem 1 follows directly from Corollary 1.

Theorem 1. *Any coproduct of finite* S*-acts is a pseudofinite* S*-act.*

Since any S-act over a group is a cyclic S-act, we get

Corollary 2. *Any* S*-act, where* S *is a finite group, is a pseudofinite* S*-act.*

In [12], there are the examples of connected infinite pseudofinite and non-pseudofinite S-acts over a finite monoid S.

References

[1] J. Ax, *The elementary theory of finite fields*, Ann. Math., **88**:2 (1968), 239–271. Zbl 0195.05701

[2] J.-L. Duret, *Les corps pseudo-finis ont la propriété d'independance*, C.R. Acad. Sei. Paris Sér. A, **290** (1980), 981–983. Zbl 0469.03020

[3] Z. Chatzidakis, *Notes on the model theory of finite and pseudo-finite fields*, CNRS-Université Paris 7, 2009.

[4] D. Macpherson, *Model theory of finite and pseudofinite groups*, Arch. Math. Logic, **57**:1-2 (2018), 159–184. Zbl 1388.03037

[5] I.I. Pavlyuk, S.V. Sudoplatov, *Approximations for theories of abelian groups*, Mathematics and Statistics, **8**:2 (2020), 220–224.

[6] R. Bello-Aguirre, *Model theory of finite and pseudofinite rings*, PhD thesis, University of Leeds, 2016.

[7] N.D. Markhabatov, *Approximations of acyclic graphs*, Izv. Irkutsk. Gos. Univ., Ser. Mat., **40** (2022), 104–111. Zbl 1495.03058

[8] E.L. Efremov, A.A. Stepanova, S.G. Chekanov, *Pseudofinite unars*, Algebra Logic (in press).

[9] M. Kilp, U. Knauer, A.V. Mikhalev, *Monoids, acts and categories. With applications to wreath products and graphs. A handbook for students and researchers*, Walter de Gruyter, Berlin, 2000. Zbl 0945.20036

[10] I.B. Kozhukhov, A.V. Mikhalev, *Acts over semigroups*, J. Math. Sci., New York, **269**:3 (2023), 362–401. Zbl 1517.20092

[11] C.C. Chang, H.J. Keisler, *Model theory*, North-Holland Pub. Co., Amsterdam, American Elsevier, New York, 1973. Zbl 0276.02032

[12] E.L. Efremov, A.A. Stepanova, S.G. Chekanov, *On pseudofinite acts over finite monoids*, in A.G. Pinus, E.N. Poroshenko, S.V. Sudoplatov eds., *Algebra and Model theory*, **14**, NSTU, Novosibirsk, 2023, 138–142.

Alena Andreevna Stepanova
Far Eastern Federal University,
10 Ajax Bay, Russky Island,
690922, Vladivostok, Russia
Email address: stepltd@mail.ru

Evgenii Leonidovich Efremov
Far Eastern Federal University,
10 Ajax Bay, Russky Island,
690922, Vladivostok, Russia
Email address: efremov-el@mail.ru

Sergei Gennadevich Chekanov
Far Eastern Federal University,
10 Ajax Bay, Russky Island,
690922, Vladivostok, Russia
Email address: chekanov.sg@dvfu.ru