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DESCRIBING EDGES IN NORMAL PLANE MAPS HAVING NO ADJACENT 3-FACES

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Abstract: The weight $w(e)$ of an edge e in a normal plane map (NPM) is the degree-sum of its end-vertices. An edge $e = uv$ is an (i, j)-edge if $d(u) \leq i$ and $d(v) \leq j$. In 1940, Lebesgue proved that every NPM has a $(3, 11)$-edge, or $(4, 7)$-edge, or $(5, 6)$-edge, where 7 and 6 are best possible. In 1955, Kotzig proved that every 3-polytope has an edge e with $w(e) \leq 13$, which bound is sharp. Borodin (1987), answering Erdős' question, proved that every NPM has such an edge. Moreover, Borodin (1991) refined this by proving that there is either a $(3, 10)$-edge, or $(4, 7)$-edge, or $(5, 6)$-edge.

Given an NPM, we observe some upper bounds on the minimum weight of all its edges, denoted by w, of those incident with a 3-face, w^*, and those incident with two 3-faces, w^{**}. In particular, Borodin (1996) proved that if $w^{**} = \infty$, that is if an NPM has no edges incident with two 3-faces, then either $w^* \leq 9$ or $w \leq 8$, where both bounds are sharp.

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The purpose of our note is to refine this result by proving that in fact $w^{**} = \infty$ implies either a $(3,6)$- or $(4,4)$-edge incident with a 3-face, or a $(3,5)$-edge, which description is tight.

Keywords: planar graph, plane map, structure properties, 3-polytope, weight.

1 Introduction

A normal plane map (NPM) is a plane pseudograph in which loops and multiple edges are allowed, but the degree of each vertex and face is at least three. 3-polytopes are in 1–1 correspondence with 3-connected plane graphs, as proved by Wernicke [23] back in 1904.

The degree of a vertex or a face x, that is the number of edges incident with x (loops and cut-edges are counted twice in the degree of vertex and face, respectively) is denoted by $d(x)$. A *k-vertex* is a vertex v with $d(v) = k$. A *k-face* f satisfies $d(f) = k$. By k^+ or k^- we denote any integer not smaller or not greater than k, respectively. Hence, a k^+-vertex v satisfies $d(v) \geq k$, etc.

An edge uv is an *(i,j)-edge* if $d(u) \leq i$ and $d(v) \leq j$. The *weight* $w(e)$ of an edge e in an NPM is the degree-sum of its end-vertices. By $\delta(G)$ and $w(G)$ we denote the minimum vertex degree and the minimum weight of edges of a graph G, respectively. We will drop the argument when it is clear from context.

Already in 1904, Wernicke [23] proved that every NPM with $\delta = 5$ satisfies $w \leq 11$. In 1940, Lebesgue [22] proved that every NPM has a $(3,11)$-edge, or $(4,7)$-edge, or $(5,6)$-edge, where 7 and 6 are best possible. In 1955, Kotzig [21] proved that every 3-polytope satisfies $w \leq 13$, which bound is sharp.

In 1972, Erdős (see [18]) conjectured that Kotzig's bound $w \leq 13$ holds for all planar graphs with $\delta \geq 3$. The first proof of Erdős' conjecture is due to Borodin [3], and is, moreover, given for all NPMs. Borodin [6,7] refined this result by proving that every NPM contains a $(3,10)$-, or $(4,7)$-, or $(5,6)$-edge (as easy corollaries of some stronger structural facts having applications to coloring of plane graphs).

In some coloring applications, it is important to find a light edge incident with one or two 3-faces. Given an NPM, the minimum weight of all its edges is denoted by w, of those incident with a 3-face, by w^*, and those incident with two 3-faces, by w^{**}.

Borodin [5] proved that any NPM has either $w^{**} \leq 13$, or $w^* \leq 10$ or else $w \leq 8$, where all bounds are best possible. Some other related results, as well as conjectures and references can be found in surveys Borodin, Ivanova [10], Jendrol', Voss [20]) and papers [1–23]).

In particular, in results on the entire coloring we often deal with NPMs having $w^{**} = \infty$. In Borodin [8], it is proved that such NPMs satisfy the sharp bound $w \leq 9$, which was strengthened in Borodin [9] to either $w^* \leq 9$ or $w \leq 8$, where both bounds are sharp.

The purpose of this note is to refine the latter result as follows.

Theorem 1. *Every normal plane map without adjacent* 3*-faces has either a* $(3,6)$*- or* $(4,4)$*-edge incident with a* 3*-face, or a* $(3,5)$*-edge, which description is tight.*

2 Proving Theorem 1

In Borodin [9], there is obtained a plane graph (with $w^{**} = \infty$, as also assumed throughout the proof below) with vertices of degree 3 and 6 only, in which every edge is *semi-weak*, that is incident with a 3-face, and joins a 3^+-vertex to a 6-vertex. This confirms that the first option in Theorem 1 is necessary and best possible.

The second option is due to by the well-known (3,4,3,4)-Archimedean solid, in which every edge joins two 4-vertices and is incident with a 3-face and 4-face. A simple way to obtain this construction is to join the middles of all edges of the cube inside the corresponding faces, followed by deleting the edges and vertices of the initial cube.

The third option is confirmed by the dual of the $(3,5,3,5)$-Archimedean solid, in which every edge joins a 3-vertex with a 5-vertex and is incident with two 4-faces.

2.1. Discharging and its consequences. By M denote a counterexample to Theorem 1. Let V, E, and F be the sets of vertices, edges and faces of M, respectively. Euler's formula $|V| - |E| + |F| = 2$ for M may be rewritten as

$$\sum_{x \in V \cup F} (d(x) - 4) = -8. \tag{1}$$

Every vertex and face x contributes the *charge* $\mu(x) = d(x) - 4$ to (1), so only the charges of 3-vertices and 3-faces are negative. Using the properties of M as a counterexample, we define a local redistribution of μ's, preserving their sum, such that the *new charge* $\mu'(x)$ is non-negative for all $x \in V \cup F$. This will contradict the fact that the sum of the new charges is, by (1), equal to -8.

In what follows, we denote the vertices adjacent to a vertex v in a cyclic order by $v_1, \ldots, v_{d(v)}$. An edge is *strong* if it is not incident with a 3-face.

We apply the following rules of discharging.

R1. *Every* 3*-face receives* $\frac{1}{3}$ *from each incident vertex.*

R2. *Every* 3*-vertex receives* $\frac{1}{2}$ *along each semi-weak edge from a* 7^+*-vertex.*

R3. *Every* 3*-vertex receives* $\frac{1}{3}$ *along each strong edge from a* 6^+*-vertex.*

R4. *Every* 4*-vertex receives* $\frac{1}{6}$ *along each semi-weak edge from a* 5^+*-vertex.*

2.2. Checking that $\mu'(x) \geq 0$ whenever $x \in V \cup F$.

CASE 1. $f \in F$. If $d(f) = 3$ then $\mu'(f) = 3-4+3\times\frac{1}{3} = 0$ by R1. If $d(f) \geq 4$ then f does not participate in R1–R4, so $\mu'(f) = \mu(f) = d(f) - 4 \geq 0$.

CASE 2. $v \in V$.

Subcase 2.1. $d(v) = 3$. If v is incident with a 3-face $f = v_1vv_2$ (precisely one, since $w^{**} = \infty$), then v receives $2 \times \frac{1}{2}$ from the two incident 7^+-vertices v_1, v_2 by R2 and $\frac{1}{3}$ from the 6^+-vertex v_3 by R3 due to the absence of $(3,5)$-edge and semi-weak $(3,6)$-edge in M. Also, v gives $\frac{1}{3}$ to f by R1, so $\mu'(v) = 3 - 4 + 2 \times \frac{1}{2} + \frac{1}{3} - \frac{1}{3} = 0$.

If v is not incident with 3-faces, then $\mu'(v) = 3 - 4 + 3 \times \frac{1}{3} = 0$ by R3 due to the absence of $(3,5)$-edge in M.

Subcase 2.2. $d(v) = 4$. Note that each 3-face $f = v_1vv_2$ at v results in giving $\frac{1}{3}$ to f by R1 and receiving $2 \times \frac{1}{6}$ from the incident 5^+-vertices v_1, v_2 by R4 due to the absence of semi-weak $(4,4)$-edge in M, so in fact f costs nothing to v. Therefore, $\mu'(v) = 0$, no matter whether v is incident with two, one or no 3-faces.

Subcase 2.3. $d(v) = 5$. Now v is incident with at most two 3-faces, and each 3-face $f = v_1vv_2$ receives $\frac{1}{3}$ from v by R1, while at most one of v_1, v_2 is a 4-vertex due to the absence of semi-weak $(4,4)$-edges in M, receiving $\frac{1}{6}$ from v by R4. So the total expenditure of v caused by f is at most $\frac{1}{2}$, which means that $\mu'(v) \geq 5 - 4 - 2 \times \frac{1}{2} = 0$.

Subcase 2.4. $d(v) = 6$. Due to the argument in Subcase 2.3 combined with the absence of semi-weak $(3,6)$-edges in M, each 3-face at v costs v at most $\frac{1}{3} + \frac{1}{6}$ by R1 and R4. Let T be the number of 3-faces at v; clearly $0 \leq T \leq 3$. Now it follows from R1, R3 and R4 that $\mu'(v) \geq 6-4-T\times\frac{1}{2}-(6-2T)\times\frac{1}{3} = \frac{T}{6} \geq 0$.

Subcase 2.5. $d(v) \geq 7$. Now a 3-face $f = v_1vv_2$ at v collects at most $\frac{1}{3} + \frac{1}{2} = \frac{5}{6}$ from v by R1, R2 and R4. Since v is incident with at most $\lfloor\frac{d(v)}{2}\rfloor$ 3-faces, we have $\mu'(v) \geq d(v)-4-T\times\frac{5}{6}-(d(v)-2T)\times\frac{1}{3} = \frac{2d(v)}{3}-4-T\times\frac{1}{6} \geq \frac{2d(v)}{3} - 4 - \frac{d(v)}{2} \times \frac{1}{6} = \frac{7d(v)-48}{12} > 0$, as desired.

Thus we have proved that $\mu'(x) \geq 0$ for all $x \in V \cup F$, which contradicts (1) and thus completes the proof of Theorem 1.

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