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FINITE GROUPS WITH MODULAR AND SUBMODULAR SUBGROUPS

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Abstract: A subgroup H of a group G is modular in G if H is a modular element of subgroup lattice of G, and is submodular in G if there is a subgroup chain $H = H_0 \leq \ldots \leq H_i \leq H_{i+1} \leq \ldots \leq H_n = G$ such that H_i is modular in H_{i+1} for every i. We prove that if every Sylow subgroup of a group G is modular in G, then G is supersolvable and $G/F(G)$ is a cyclic group of square-free order. We also obtain new signs of supersolvabilty of groups with some submodular subgroups (normalizers of Sylow subgroups, Hall subgroups, maximal subgroups). For a such group G, $G/\Phi(G)$ is a supersolvable group of square-free exponent. Moreover, we describe the structure of groups with modular (submodular) or self-normalizing primary subgroups.

Keywords: finite group, modular subgroup, submodular subgroup, self-normalizing subgroup.

1 Introduction

All groups in this paper are finite.

In many respects, the group structure depends on the way of embedding of its subgroups. In particular, normal subgroups play an important role in the

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structure of non-simple groups. Modularity is an extension of the concept of normality.

Definition 1 ([2, p. 43]). *A subgroup H of a group G is modular in G if H is a modular element of subgroup lattice of G, i. e. the following statements hold:*

(1) $\langle A, H\rangle \cap B = \langle A, H \cap B\rangle$ *for all* $A, B \leq G$ *such that* $A \leq B$;
(2) $\langle A, H\rangle \cap B = \langle H, A \cap B\rangle$ *for all* $A, B \leq G$ *such that* $H \leq B$.

A subgroup H of a group G is quasinormal in G if H permutes with every subgroup of G. Clearly, every quasinormal subgroup, in particular, normal subgroup, is modular. The converse is not true in general. For example, in the non-abelian group of order 6, a subgroup of order 2 is modular and not quasinormal. Groups in which every subnormal subgroup is modular have been considered in [1]. In [2], the properties of modular subgroups were described in detail. Like normality, modularity is not a transitive relation. In [3], there were obtained a description of finite groups in which modularity is transitive.

Submodularity is an extension of modularity.

Definition 2 ([4, p. 546]). *A subgroup H of a group G is submodular in G if there is a subgroup chain*

$$H = H_0 \leq \ldots \leq H_i \leq H_{i+1} \leq \ldots \leq H_n = G \tag{1}$$

such that H_i is modular in H_{i+1} for every i.

It is clear that every modular subgroup and every subnormal subgroup is submodular. The converse is not true in general. Unlike normality and modularity, the relation of submodularity is transitive. Groups with submodular Sylow subgroups were studied in [4, 5, 6]. We denote the class of all such groups by $\mathfrak{Z}$. In [6], there were introduced and described the class $\mathfrak{C}$ of all group in which every cyclic subgroup is submodular. Since a subnormal subgroup is submodular, $\mathfrak{Z} \subset \mathfrak{C}$, and $C_7^3 \rtimes Q \in \mathfrak{C} \setminus \mathfrak{Z}$, where Q is a non-abelian subgroup of order 3^3 and exponent 3 that acts irreducibly on the elementary abelian group C_7^3 of order 7^3, [6, Example 3].

In this paper we prove that a group G in which every Sylow subgroup is modular is supersolvable, and $G/F(G)$ is a cyclic group of square-free order (see Proposition 1). If every Sylow subgroup of a group G is submodular in G, then G can be non-supersolvable (see Example 3). We obtain new signs of supersolvabilty of groups with some submodular subgroups (normalizers of Sylow subgroups, Hall subgroups, maximal subgroups). For a such group G, $G/\Phi(G)$ is a supersolvable group of square-free exponent. Here $F(G)$ and $\Phi(G)$ are the Fitting subgroup and the Frattini subgroup, respectively.

Sometimes, when we impose restrictions on subgroups, we have a choice between several possible properties of subgroups. We say that such restrictions are alternative. Restrictions can be strictly alternative when one of the possible subgroup properties excludes all other one, or partially alternative, otherwise. For instance, a normal subgroup can not be self-normalizing. But there are groups that contain self-normalizing modular subgroup, for example, the non-abelian group of order 6. Groups with partially alternative restrictions on subgroups were studied in papers of many authors, see [7, 8, 9, 10] and their references. We describe the structure of groups with modular (submodular) or self-normalizing primary subgroups (see Theorem 1 and Theorem 2).

2 Preliminaries

Let $\mathbb{P}$ be the set of all primes. We use $\pi(G)$ to denote the set of all prime divisors of $|G|$. If X is a subgroup (proper subgroup, normal subgroup, maximal subgroup) of a group Y, then we write $X \leq Y$ ($X < Y$, $X \lhd Y$, $X \lessdot Y$, respectively).

Let $\mathfrak{F}$ be a formation. The $\mathfrak{F}$-residual of a group G is the intersection of all normal subgroups of G with quotients in $\mathfrak{F}$. It is denoted by $G^{\mathfrak{F}}$. We use $\mathfrak{N}$ and $\mathfrak{U}$ to denote the formation of all nilpotent groups and the formation of all supersolvable groups, respectively. If $\mathfrak{X}$ is a class of groups, then $\mathfrak{X}_1$ is a class of all groups from $\mathfrak{X}$ with square-free exponents. In particular, $\mathfrak{U}_1$ is the formation of all supersolvable groups of square-free exponent.

Recall a Carter subgroup of a group G is a nilpotent self-normalizing subgroup of G [14, VI.12]. In solvable groups, Carter subgroups exist and are conjugate. An insolvable group may have no Carter subgroups, but by a result of E. P. Vdovin [13], which uses the classification of finite simple groups, Carter subgroups are conjugate whenever they exist.

A group G is a P-group [2, p. 49] if G is either an elementary abelian group of order r^{n+1} for a prime r or the semidirect product of an elementary abelian normal subgroup R of order r^n and a subgroup Q of order q, $q \neq r$, which induces a nontrivial power automorphism on R. The properties of such groups are described in [2, Lemma 2.2.2]. In particular, every Sylow subgroup of a non-abelian P-group is modular.

The following lemmas contains the properties of modular and submodular subgroups that we use.

Lemma 1 ([2, p. 201], [4, Lemma 1])**.** *Let G be a group, let H be a modular (submodular) subgroup of G, let K be a subgroup of G, and let N be a normal subgroup of G. Then*

(1) *H^g is modular (submodular) in G for any $g \in G$;*

(2) *$H \cap K$ is modular (submodular) in K, in particular, if $H \leq K$, then H is modular (submodular) in K;*

(3) *HN/N is modular (submodular) in G/N;*

(4) *if $N \leq K$ and K/N is modular (submodular) in G/N, then K is modular (submodular) in G.*

Lemma 2. (1) *A subgroup H of a group G is quasinormal in G if and only if H is modular and subnormal in G.*

(2) *Let Q be a Sylow q-subgroup of G for a prime $q \in \pi(G)$. The subgroup Q is modular in G if and only if either Q is normal in G or Q^G/Q_G is a normal Hall subgroup, which is a non-abelian P-group of order $r^n q$ for primes $r > q$ and has a normal supplement.*

(3) *Let A be a proper subgroup of a Sylow subgroup of a group G. The subgroup A is modular in G if and only if A is quasinormal in G. In particular, if a primary modular subgroup of G is not a Sylow subgroup of G, then it is subnormal in G.*

Proof. (1) This statement is true in view of [2, Theorem 5.1.1].

(2) Assume that Q is a Sylow q-subgroup of group G for a prime $q \in \pi(G)$ and Q is modular in G. Let Q be not normal in G. In view of Statement (1), Q is not quasinormal in G. Therefore $G/Q_G = Q^G/Q_G \times K/Q_G$, where Q^G/Q_G is a non-abelian P-group of order $r^n q$ for primes $r > q$, Q^G/Q_G and K/Q_G of coprimes order, by [2, Lemma 5.1.9]. Hence Q^G/Q_G is a normal Hall subgroup and has a normal supplement.

Conversely, if Q is normal in G, then Q is modular in G by definition. Assume that Q^G/Q_G is a normal Hall subgroup, which is a non-abelian P-group of order $r^n q$ for primes $r > q$ and has a normal supplement. In that case, Q is modular in G by [2, Lemma 5.1.9].

(3) Assume that A is a proper subgroup of a Sylow subgroup of G. If A is modular in G, then A is quasinormal in G by [2, Lemma 5.1.9]. In particular, A is subnormal in G by Statement (1). Conversely, if A is quasinormal in G, then A is modular in G by Statement (1). □

Notice that a subgroup M of a group G is a maximal modular subgroup of G if M satisfies the following conditions: M is a proper subgroup of G; M is a modular subgroup of G; if $M < K < G$, then K is not modular in G.

Lemma 3. *A subgroup M of a group G is a maximal modular subgroup of G if and only if either M is normal in G and G/M is a simple group or M is a maximal subgroup of G and G/M_G is a non-abelian group of order pq for primes p and q.*

Proof. Let M be a maximal modular subgroup of a group G. Suppose that M is normal in G and G/M is not simple. In that case, there is a non-trivial normal subgroup H/M of G/M. Therefore H is normal in G, and H is modular in G. Since $M < H$, we obtain a contradiction with the choice of M. Thus, G/M is a simple group. Now assume that M is not normal in G. In that case, G/M_G is a non-abelian group of order pq for primes p and q by [2, Lemma 5.1.2], and M is a maximal subgroup of G.

Conversely, let M be a normal subgroup of G such that G/M is a simple group. In that case, M is modular in G. Suppose that there is a maximal modular subgroup H of G such that $M \leq H$. Since G/M is simple, we deduce that H is not normal in G. Therefore G/H_G is a non-abelian group of order pq for primes p and q in view of [2, Lemma 5.1.2]. By the choice of M, we have $M = H_G$ and G/H_G is a simple group, a contradiction. Now assume that M is a maximal subgroup of G and G/M_G is a non-abelian group of order pq for primes p and q. In that case, M/M_G is a maximal subgroup of G/M_G. Since $|G/M_G| = pq$, we deduce that M/M_G is modular in G/M_G. Hence M is modular in G by Lemma 1 (4). Since M is a maximal subgroup of G, M is a maximal modular subgroup of G. □

Lemma 4. *Let R be a submodular r-subgroup of a group G for a prime $r \in \pi(G)$.*

(1) *If $r = \max \pi(G)$, then R is subnormal in G* [6, Lemma 2.3].

(2) *If R is a Sylow r-subgroup of G, then G is r-solvable* [6, Proposition 2.4].

Lemma 5 ([6, Theorem 1.3])**.** *Let G be a group. The following statements are equivalent.*

(1) *Every Sylow subgroup of G is submodular in G, i. e. $G \in \mathfrak{Z}$.*

(2) $G/\Phi(G) \in \mathfrak{Z}_1$.

(3) $A/\Phi(A) \in \mathfrak{U}_1$ *for any metanilpotent subgroup A of G.*

(4) $B/\Phi(B) \in \mathfrak{U}_1$ *for any biprimary subgroup B of G.*

Lemma 6 ([6, Theorem 1.4])**.** *Let G be a group. The following statements are equivalent.*

(1) *Every cyclic primary subgroup of G is submodular in G, i. e. $G \in \mathfrak{C}$.*

(2) $G/\Phi(G) \in \mathfrak{C}_1$.

(3) $A/\Phi(A) \in \mathfrak{U}_1$ *for any subgroup A with the nilpotent derived subgroup.*

(4) $B/\Phi(B) \in \mathfrak{U}_1$ *for any biprimary subgroup B with a cyclic Sylow subgroup.*

A group G is primitive if G contains a maximal subgroup M such that $M_G = 1$. It is easy to check that the following lemma is true.

Lemma 7. *Let $\mathfrak{F}$ be a saturated formation, and let G be a solvable group. If $G \notin \mathfrak{F}$ and $G/N \in \mathfrak{F}$ for every non-trivial normal subgroup N of G, then G is a primitive group.*

The following lemma contains the properties of primitive groups that we use.

Lemma 8 ([11, Theorem 1.1.7,1.1.10])**.** *Let G be a solvable primitive group and let M be a maximal subgroup of G with $M_G = 1$. The following statements hold.*

(1) $\Phi(G) = 1$.

(2) *G contains a unique minimal normal subgroup N such that $N = C_G(N) = F(G) = O_p(G)$ for a prime p.*

(3) *$G = F(G) \rtimes M$ and $O_p(M) = 1$.*

3 Groups with modular primary subgroups

Proposition 1. *Let every Sylow subgroup of a group G be modular in G. The following statements hold.*

(1) *Q^G/Q_G is a normal Hall subgroup, which is a P-group of order r^nq for primes $r > q$ and has normal supplement for every non-normal Sylow q-subgroup Q of G.*

(2) *G is supersolvable.*

(3) *$G/F(G)$ is a cyclic group of square-free order.*

Conversely, if G satisfies Statement (1)*, then every Sylow subgroup of G is modular in G.*

Proof. Assume that every Sylow subgroup of a group G is modular in G.

(1) If Q is a non-normal Sylow subgroup of G, then by Lemma 2 (2), Q^G/Q_G is a normal Hall subgroup, which is a non-abelian P-group of order r^nq for primes $r > q$ and has a normal supplement.

(2) We use induction on $|G|$ to prove that G is supersolvable. Let N be a non-trivial normal subgroup of G and let $\overline{P}$ be a Sylow p-subgroup of $\overline{G} = G/N$ for a prime $p \in \pi(G/N)$. In that case, there is a Sylow p-subgroup P of G such that $\overline{P} = PN/N$. By hypothesis, P is modular in G. Therefore $\overline{P}$ is modular in $\overline{G}$ by Lemma 1 (3). Consequently, G/N is supersolvable by induction. Let R be a Sylow r-subgroup of G for $r = \max \pi(G)$. By Lemma 4 (1), R is normal in G, and G/R is supersolvable by induction. Therefore G is solvable. In view of Lemma 7, G is primitive, and by Lemma 8, $G = F(G) \rtimes M$, where $F(G)$ is a unique minimal normal subgroup of G, M is a maximal subgroup of G with $M_G = 1$. Since R is normal in G, we have $F(G) = R$. Let Q be a Sylow q-subgroup of M. In that case, Q is a Sylow q-subgroup of G and $q < r$. By hypothesis, Q is modular in G. From $Q_G = 1$ and Lemma 2 (2), it follows that $G = Q^G$ is a non-abelian P-group of order r^nq and $M = Q$. Thus, Q is a non-normal maximal modular subgroup of G. Consequently, by Lemma 3, we have $|G| = rq$ and G is supersolvable.

(3) By Statement (2), G is supersolvable. Therefore $\overline{G} = G/F(G)$ is abelian. Let $\overline{Q}$ be a Sylow q-subgroup of $\overline{G}$. In that case, there is a Sylow q-subgroup Q of G such that $\overline{Q} = QF(G)/F(G) \cong Q/Q \cap F(G) = Q/Q_G$. Since Q is modular in G, by Lemma 2 (2), $Q^G/Q_G = TQ^G/Q_G \rtimes Q/Q_G$ is a non-abelian P-group of order t^nq for a prime $t > q$. Hence $|\overline{Q}| = |Q/Q_G| = q$, and so every Sylow subgroup of $G/F(G)$ is of prime order and $G/F(G)$ is cyclic.

Conversely, let G satisfy Statement (1) and let Q be a Sylow q-subgroup of G for a prime $q \in \pi(G)$. If Q is normal in G, then Q is modular in G. Suppose that Q is not normal in G. In that case, Q^G/Q_G is a normal Hall subgroup, which is a non-abelian P-group of order r^nq for primes $r > q$ and has a normal supplement. Consequently, Q is modular in G by Lemma 2 (2). □

Example 1. The group $G = C_3 \times S_3$ [12, SmallGroup(18,3)] is supersolvable, $F(G) = C_3^2$ and $G/F(G) \cong C_2$. At the same time, a Sylow 2-subgroup C_2 is not modular in G. Thus, a group satisfying only Statement (2) or Statement (3) of Proposition 1 may contain a non-modular Sylow subgroup.

Corollary 1. *Every Sylow subgroup of a biprimary group G is modular in G if and only if either G is nilpotent or G is supersolvable and G/Q_G is* *a non-abelian* P*-group of order* $r^n q$*,* $r > q$*, where* Q *is a Sylow* q*-subgroup of* G*.*

Proof. Assume that G is a biprimary group, Q is a Sylow q-subgroup of G, R is a Sylow r-subgroup of G, $r > q$. By Lemma 4 (1), R is normal in G. If Q is normal in G, then G is nilpotent. Suppose that Q is not normal in G. By Proposition 1, G is supersolvable and G/Q_G is a non-abelian P-group of order $r^n q$.

Conversely, if G is nilpotent, then every Sylow subgroup of G is normal in G. Hence every Sylow subgroup of G is modular in G. Let G is supersolvable and G/Q_G is a non-abelian P-group of order $r^n q$, $r > q$. In that case, every Sylow subgroup of G is modular in G by Proposition 1. □

Remark 1. Further, we use $\mathfrak{U}_m$ to denote the class of all supersolvable groups satisfying Statements (1)–(3) of Proposition 1.

Theorem 1. *Every Sylow subgroup of a group* G *is modular or self-normalizing in* G *if and only if either* $G \in \mathfrak{U}_m$ *or* $G = G^{\mathfrak{N}} \rtimes R$*, where* R *is a Sylow* r*-subgroup and a Carter subgroup of* G *for a prime* $r \in \pi(G)$*,* $G^{\mathfrak{N}} \in \mathfrak{N}$*.*

Proof. Assume that every Sylow subgroup of a group G is modular or self-normalizing in G. If every Sylow subgroup of G is modular in G, then $G \in \mathfrak{U}_m$ by Proposition 1. Now, we can assume that $G \notin \mathfrak{U}_m$. In that case, G contains a Sylow r-subgroup R that is not modular in G. By hypothesis, $R = N_G(R)$, and R is a Carter subgroup of G. Let Q be a Sylow q-subgroup of G for a prime $q \neq r$. If Q is self-normalizing in G, then Q is a Carter subgroup of G. By [13], Carter subgroups are conjugate whenever they exist. Hence R and Q are conjugate, a contradiction. Therefore all Sylow r'-subgroups of G are modular in G. If Q is not normal in G, then by Lemma 2 (2), $G/Q_G = Q^G/Q_G \times K/Q_G$ and $(|Q^G/Q_G|, |K/Q_G|) = 1$. Since $r \neq q$, we have $RQ_G/Q_G \leq Q^G/Q_G$ or $RQ_G/Q_G \leq K/Q_G$. In view of [13, Theorem 9.3], RQ_G/Q_G is a Carter subgroup of G/Q_G. Hence $RQ_G/Q_G = Q^G/Q_G$ or $K/Q_G = G/Q_G$, a contradiction, since $Q \neq Q_G$. Thus, all Sylow r'-subgroups of G are normal in G. Therefore a Hall r'-subgroup $G_{r'}$ of G is nilpotent and normal in G, G is solvable and $G = G_{r'} \rtimes R$. Since $G/G_{r'} \cong R \in \mathfrak{N}$, we get $G^{\mathfrak{N}} \leq G_{r'}$. Suppose that $G^{\mathfrak{N}} \rtimes R \leq M \lessdot G$. In that case, $M/G^{\mathfrak{N}}$ is a maximal subgroup of the nilpotent group $G/G^{\mathfrak{N}}$. Consequently, $M/G^{\mathfrak{N}}$ is normal in $G/G^{\mathfrak{N}}$, and M is normal in G. Since $R = N_G(R)$, we get $M = N_G(M)$, a contradiction. Hence $G = G^{\mathfrak{N}} \rtimes R$ and $G^{\mathfrak{N}} = G_{r'} \in \mathfrak{N}$.

Conversely, if $G \in \mathfrak{U}_m$, then every Sylow subgroup of G is modular in G by Proposition 1. Let $G = G^{\mathfrak{N}} \rtimes R$, where R is a Sylow r-subgroup and a Carter subgroup of G for a prime $r \in \pi(G)$, $G^{\mathfrak{N}} \in \mathfrak{N}$. Choose an arbitrary Sylow q-subgroup Q of G. If $q = r$, then $Q = R^g$ for some element $q \in G$ and Q is self-normalizing in G. Let $q \neq r$. In that case, $Q \leq G^{\mathfrak{N}} \in \mathfrak{N}$, and Q is normal in $G^{\mathfrak{N}}$. Consequently, Q is normal in G, and Q is modular in G. □

A subgroup H of a group G is abnormal in G if $g \in \langle H, H^g \rangle$ for every $g \in G$. Since a primary subgroup is abnormal if and only if it is self-normalizing, we obtain the following corollary.

Corollary 1. *If every Sylow subgroup of a group G is modular or self-normalizing in G, then every non-abnormal Sylow subgroup of G is normal in G.*

Corollary 2. *Let G be a biprimary group, let Q be a Sylow q-subgroup of G, and let R be a Sylow r-subgroup of G, $r > q$. Every Sylow subgroup of G is modular or self-normalizing in G if and only if G is a group of one of the following types:*

(1) *G is nilpotent;*

(2) *G is supersolvable and G/Q_G is a non-abelian* P*-group of order $r^n q$;*

(3) *$G = R \rtimes Q$, $G^{\mathfrak{N}} = R$ and Q is a Carter subgroup of G.*

Example 2. In $G = C_3^3 \rtimes A_4$ [12, SmallGroup(324,160)], a Sylow 3-subgroup $C_3 \wr C_3$ is maximal and self-normalizing in G, a Sylow 2-subgroup C_2^2 is contained in a normal maximal subgroup $C_3^3 \rtimes C_2^2$. Since $C_3^3 \rtimes C_2^2 \in \mathfrak{U}_1$, we conclude that C_2^2 is submodular in $C_3^3 \rtimes C_2^2$. Consequently, C_2^2 is submodular in G. At the same time, C_2^2 is not normal in G. Therefore in Theorem 1, the condition of modularity of Sylow subgroups can not be weakened to submodularity.

Corollary 3. *Every primary subgroup of a group G is modular or self-normalizing in G if and only if G is a group of one of the following types:*

(1) *$G \in \mathfrak{U}_m$ and every primary subgroup, which is not a Sylow subgroup of G, is quasinormal in G;*

(2) *$G = G^{\mathfrak{N}} \rtimes R$, where R is a Sylow r-subgroup and a Carter subgroup of G for a prime $r \in \pi(G)$, $G^{\mathfrak{N}} \times X \in \mathfrak{N}$ for any proper subgroup X of R and all primary subgroups except for Carter subgroups are quasinormal in G.*

Proof. Assume that every primary subgroup of G is modular or self-normalizing in G. By Theorem 1, either $G \in \mathfrak{U}_m$ or $G = G^{\mathfrak{N}} \rtimes R$, where R is a Sylow r-subgroup and a Carter subgroup of G for a prime $r \in \pi(G)$, $G^{\mathfrak{N}} \in \mathfrak{N}$. Let $G \in \mathfrak{U}_m$ and let X be a primary subgroup of G, which is not a Sylow subgroup of G. In that case, X is modular in G, and in view of Lemma 2 (3), X is quasinormal in G. Now assume that $G = G^{\mathfrak{N}} \rtimes R$, where R is a Sylow r-subgroup and a Carter subgroup of G for a prime $r \in \pi(G)$, $G^{\mathfrak{N}} \in \mathfrak{N}$. Let X be a proper subgroup of R. If X is self-normalizing in G, then X is a Carter subgroup of G. Hence X and R are conjugate, a contradiction. Therefore X is modular in G. Since $X < R$, we deduce that X is quasinormal in G by Lemma 2 (3), in particular, X is subnormal in G. Hence X is normal in $G^{\mathfrak{N}}$ and $G^{\mathfrak{N}}X = G^{\mathfrak{N}} \rtimes X \in \mathfrak{N}$. Let A be a q-subgroup of G for a prime $q \neq r$. If A is a Sylow subgroup of G, then $A \leq G^{\mathfrak{N}} \in \mathfrak{N}$. Consequently, A is normal in $G^{\mathfrak{N}}$, and A is normal in G, in particular, A is quasinormal in G. Suppose that A is not a Sylow subgroup of G. In that case, A is modular in G. Indeed, if A is not modular in G, then A is self-normalizing in G by hypothesis, and A

is a Carter subgroup. Consequently, A and R are conjugate, a contradiction. Therefore A is quasinormal in G by Lemma 2 (3).

Conversely, let $G \in \mathfrak{U}_m$ and let every primary subgroup, which is not a Sylow subgroup of G, be quasinormal in G. In that case, in view of Proposition 1, every Sylow subgroup of G is modular in G, and by Lemma 2 (1), every primary subgroup, which is not a Sylow subgroup of G, is modular in G. Thus, every primary subgroup of G is modular in G. Now, assume that $G = G^{\mathfrak{N}} \rtimes R$, where R is a Sylow r-subgroup and a Carter subgroup of G for a prime $r \in \pi(G)$, $G^{\mathfrak{N}} \rtimes X \in \mathfrak{N}$ for any proper subgroup X of R and every primary subgroup except for Carter subgroups is quasinormal in G. By Lemma 2 (1), every primary subgroup except for Carter subgroups is modular in G. All Carter subgroups are self-normalizing in G. □

Remark 2. In view of [2, Lemma 2.3.2], a primary group has modular subgroup lattice if and only if any two of its subgroups permute. Therefore if a group G satisfies Statement (2) of Corollary 3 and A is a q-subgroup of G that is not a Carter subgroup, then A has modular subgroup lattice. By [2, Theorem 2.3.1], A is a group of one of the following types:

(a) $A = Q_8 \times C_2^n$ for some n;

(b) A contains an abelian normal subgroup N such that A/N is cyclic; further there exists an element $x \in A$, $A = N\langle x\rangle$, and a positive integer s such that $x^{-1}yx = y^{1-q^s}$ for all $y \in N$, with $s \geq 2$ in case $q = 2$.

Corollary 4. *Assume that every primary subgroup of a group G is modular or self-normalizing in G, K is a Carter subgroup of G and A is a primary subgroup of G. If $|K|$ divides $|A|$, then A is abnormal in G. If $|K|$ does not divide $|A|$, then A is quasinormal in G.*

4 Groups with submodular subgroups

Example 3. In $G = C_7^2 \rtimes S_3$ [12, SmallGroup(294,7)], a Sylow 7-subgroup $R = C_7^2$ is normal. Hence R is modular in G.

For a Sylow 3-subgroup $Q = C_3$, there is a subgroup chain

$$Q = C_3 \lessdot Q_1 = C_7 \rtimes C_3 \lessdot Q_2 = C_7^2 \rtimes C_3 \lessdot G.$$

Since $Q_1 = C_7 \rtimes C_3$, we deduce that every subgroup of Q_1 is modular in Q_1, in particular, Q is modular in Q_1. We have $(Q_1)_{Q_2} \cong C_7$ and $Q_2/(Q_1)_{Q_2} \cong C_7 \rtimes C_3$. Therefore Q_1 is modular in Q_2. Since Q_2 is normal in G, we get Q_2 is modular in G. Thus, Q is submodular in G.

For a Sylow 2-subgroup $P = C_2$, there is a subgroup chain

$$P = C_2 \lessdot P_1 = C_{14} \lessdot P_2 = C_7 \rtimes D_{14} \lessdot G.$$

Since P is normal in P_1, we get P is modular in P_1. We have $(P_1)_{P_2} \cong C_7$ and $P_2/(P_1)_{P_2} \cong D_{14}$. Hence P_1 is modular in P_2. From $(P_2)_G \cong C_7^2$ and $G/(P_2)_G \cong S_3$, it follows that P_2 is modular in G. Thus, P is submodular in G and $G \in \mathfrak{Z}$. But G is not supersolvable. Therefore $\mathfrak{Z} \not\subseteq \mathfrak{U}$.

Proposition 2. *Let G be a group. Then $G/\Phi(G) \in \mathfrak{U}_1$ in every of the following cases.*

(1) *The normalizer of every Sylow subgroup of G is submodular in G.*
(2) *Every Hall subgroup of G is submodular in G.*
(3) *Every maximal subgroup of G is submodular in G.*

Proof. (1) Since every subgroup is normal in own normalizer, every Sylow subgroup of G is submodular in G. Consequently, $G \in \mathfrak{Z}$, in particular, G is solvable by Lemma 4 (2) and $G/\Phi(G) \in \mathfrak{Z}_1$ by Lemma 5.

Now we prove that $G \in \mathfrak{U}$. Let N be a non-trivial normal subgroup of G and let $\overline{P}$ be a Sylow p-subgroup of $\overline{G} = G/N$ for a prime $p \in \pi(G/N)$. In that case, there is a Sylow p-subgroup P of G such that $\overline{P} = PN/N$, and $N_{\overline{G}}(\overline{P}) = N_{G/N}(PN/N) = N_G(P)N/N$. Since $N_G(P)$ is submodular in G, we deduce that $N_G(P)N/N$ is submodular in G/N by Lemma 1 (3). Consequently, by induction, $G/N \in \mathfrak{U}$ for every non-trivial normal subgroup N. Hence G is primitive by Lemma 7, and in view of Lemma 8, $G = F(G) \rtimes M$, where M is a maximal subgroup of G with $M_G = 1$. In view of Lemma 4 (2), a Sylow r-subgroup R of G is normal in G for $r = \max \pi(G)$. Therefore $R = F(G)$ and $G = R \rtimes M$. By induction, $G/R \cong M \in \mathfrak{U}$. Consequently, for $q = \max \pi(M)$, a Sylow q-subgroup Q of M is normal in M, and so $N_M(Q) = M \leq N_G(Q) \leq G$ and $M = N_G(Q)$. Since M is a Hall subgroup of G, we deduce that Q is a Sylow subgroup of G. Hence M is submodular in G, and $|R| = |G : M| \in \mathbb{P}$ in view of Lemma 3. Consequently, G is supersolvable, a contradiction.

Thus, $G/\Phi(G) \in \mathfrak{Z}_1 \cap \mathfrak{U} \subseteq \mathfrak{U}_1$.

(2) It is clear that $G \in \mathfrak{Z}$, in particular, G is solvable by Lemma 4 (2) and $G/\Phi(G) \in \mathfrak{Z}_1$ by Lemma 5.

Now we prove that $G \in \mathfrak{U}$. Let N be a non-trivial normal subgroup of G and let $\overline{H}$ be a Hall subgroup of $\overline{G} = G/N$. In that case, there is a Hall subgroup H of G such that $\overline{H} = HN/N$. Since H is submodular in G, we get HN/N is submodular in G/N by Lemma 1 (3). By induction, $G/N \in \mathfrak{U}$ for every non-trivial normal subgroup N. Therefore G is primitive by Lemma 7, and $G = F(G) \rtimes M$, where M is a maximal subgroup of G with $M_G = 1$, by Lemma 8. In view of Lemma 4 (1), a Sylow r-subgroup R is normal in G for $r = \max \pi(G)$. Hence $R = F(G)$ and $G = R \rtimes M$. By induction, $G/R \cong M \in \mathfrak{U}$. Since M is a Hall subgroup of G, we have that M is submodular in G. It follows that $|R| = |G : M| \in \mathbb{P}$ by Lemma 3, and G is supersolvable, a contradiction.

Thus, $G/\Phi(G) \in \mathfrak{Z}_1 \cap \mathfrak{U} \subseteq \mathfrak{U}_1$.

(3) Let M be a maximal subgroup of G. By hypothesis, M is modular in G. If M is normal in G, then $M = M_G$ and $|G/M_G| \in \mathbb{P}$. Hence $G/M_G \in \mathfrak{U}_1$. If M is not normal in G, then $G/M_G \in \mathfrak{U}_1$ by Lemma 3. Thus, $G/M_G \in \mathfrak{U}_1$ for every maximal subgroup M of G. Therefore $G/\Phi(G) = G/\bigcap_{M \lessdot G} M_G = \bigcap_{M \lessdot G} G/M_G \in \mathfrak{U}_1$. □

Example 4. The Frobenius group $F_7 = C_7 \rtimes C_6 \in \mathfrak{U}_1$, $\Phi(G) = 1$, but a maximal subgroup C_6 is not submodular in F_7. Therefore the statements of Proposition 2 are not converse.

Example 5. In the cyclic group C_{12} of order 12, every subgroup is normal, and so is submodular, but $\exp(C_{12}) = 12$ and $G \notin \mathfrak{U}_1$. Therefore we can not replace "$G/\Phi(G) \in \mathfrak{U}_1$" by "$G \in \mathfrak{U}_1$" in Proposition 2.

Theorem 2. *Every primary subgroup of a group G is submodular or self-normalizing in G if and only if G is a group of one of the following types:*

(1) *$G \in \mathfrak{Z}$;*

(2) *$G \notin \mathfrak{C}$ and $G = G' \rtimes \langle x\rangle$, where $\langle x\rangle$ is a Sylow r-subgroup and a Carter subgroup of G, $r \in \pi(G)$, $G' \rtimes \langle x^r\rangle \in \mathfrak{Z}$;*

(3) *$G \in \mathfrak{C} \setminus \mathfrak{Z}$ and $G = G^{\mathfrak{N}} \rtimes R$, where R is a non-submodular non-cyclic Sylow r-subgroup and a Carter subgroup of G, $r = \min \pi(G)$, $G^{\mathfrak{N}} \rtimes X \in \mathfrak{Z}$ for every $X < R$.*

Proof. Let every primary subgroup of a group G be submodular or self-normalizing in G. If every primary subgroup of G is submodular in G, then $G \in \mathfrak{Z}$. Further, we can assume that $G \notin \mathfrak{Z}$. In that case, G contains a primary subgroup R that is not submodular in G. By hypothesis, $R = N_G(R)$, and so R is a Sylow r-subgroup and a Carter subgroup of G for a prime $r \in \pi(G)$. Let Q be a Sylow q-subgroup of G for a prime $q \neq r$. If Q is self-normalizing in G, then Q a Carter subgroup of G. Since Carter subgroups are conjugate whenever they exist by [13], R and Q are conjugate, a contradiction. Therefore all Sylow r'-subgroups of G are modular in G. By Lemma 4 (2), G is q-solvable for every $q \in \pi = \pi(G) \setminus \{r\}$. Hence G is π-solvable. Since $|\pi(G) \setminus \pi| = |\{r\}| = 1$, G is solvable. Consequently, $G = G^{\mathfrak{N}}R$ and $G_{r'} \leq G^{\mathfrak{N}}$.

Let X be a proper subgroup of R and let Y be a primary subgroup of $G^{\mathfrak{N}}X$. If Y is self-normalizing in G, then Y is a Carter subgroup of G, and so Y and R are conjugate, a contradiction. Consequently, every primary subgroup of $G^{\mathfrak{N}}X$ is submodular in $G^{\mathfrak{N}}X$ by Lemma 1 (2) and $G^{\mathfrak{N}}X \in \mathfrak{Z}$.

Assume that $G \in \mathfrak{C}$. In that case, R is not cyclic. By Lemma 4, G has a Sylow tower of supersolvable type. Hence $r = \min \pi(G)$ and there is a normal Hall r'-subgroup $G_{r'}$ of G. Since $G/G_{r'} \cong R \in \mathfrak{N}$, we have $G^{\mathfrak{N}} \leq G_{r'}$, and so $G^{\mathfrak{N}} = G_{r'}$ and $G = G^{\mathfrak{N}} \rtimes R$.

Now assume that $G \notin \mathfrak{C}$. In that case, $R = \langle x\rangle$ is cyclic. By [14, IV.2.6], there is a normal Hall r'-subgroup $G_{r'}$ of G. Since $G/G_{r'} \cong R \in \mathfrak{A}$, we have $G^{\mathfrak{N}} \leq G' \leq G_{r'}$. Hence $G' = G^{\mathfrak{N}} = G_{r'}$ and $G = G' \rtimes \langle x\rangle$. Since $G^{\mathfrak{N}} \rtimes X \in \mathfrak{Z}$ for every $X < R$, we deduce that $G' \rtimes \langle x^r\rangle \in \mathfrak{Z}$.

Conversely, if $G \in \mathfrak{Z}$, then every primary subgroup of G is submodular in G. Now, we can assume that $G \notin \mathfrak{Z}$. Let $G \notin \mathfrak{C}$ and let $G = G' \rtimes \langle x\rangle$, where $\langle x\rangle$ is a Sylow r-subgroup and a Carter subgroup of G, $r \in \pi(G)$, $G' \rtimes \langle x^r\rangle \in \mathfrak{Z}$. Choose an arbitrary q-subgroup A of G for a prime $q \in \pi(G)$. If $q \neq r$, then $A \leq G'$. From $G' \leq G' \rtimes \langle x^r\rangle \in \mathfrak{Z}$, it follows that A is submodular in G'. Since G' is submodular in G, Q is submodular in G. Let $q = r$. In that case, $Q^g \leq \langle x \rangle$ for $g \in G$. If $Q^g = \langle x \rangle$, then Q is self-normalizing. If $Q^g < \langle x \rangle$, then $Q^g \leq \langle x^r \rangle \leq G' \rtimes \langle x^r \rangle \in \mathfrak{Z}$. Consequently, Q^g is submodular in $G' \rtimes \langle x^r \rangle$, and $G' \rtimes \langle x^r \rangle$ is submodular in G. Thus, Q is submodular in G by Lemma 1 (1). Now, let $G \in \mathfrak{C} \setminus \mathfrak{Z}$ and let $G = G^{\mathfrak{N}} \rtimes R$, where R is a non-submodular non-cyclic Sylow r-subgroup and a Carter subgroup of G, $r = \min \pi(G)$, $G^{\mathfrak{N}} \rtimes X \in \mathfrak{Z}$ for every $X < R$. As above we can prove that every primary subgroup of G is submodular or self-normalizing in G. □

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