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GROUPS WITH SYMMETRIC NON-COMMUTING GRAPHS

A. ABDOLLAHI

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Abstract: In this paper we characterize non-abelian finite 2-generator groups G whose non-commuting graphs are $\mathrm{Aut}(G)$-symmetric. We also find some general results on these groups. These partially answer Problem 31 posed in Peter Cameron's home page, old problems.

Keywords: non-commuting graph; automorphism group; symmetric graphs.

1 Introduction

Problem 31 of [3] is the following:

Question 1. *Which finite groups have the property that the automorphism group acts transitively on the set of ordered pairs of non-commuting elements?*

Every abelian group trivially satisfies the property stated in Question 1. Let G be a finite non-abelian group satisfying the property stated in Question 1. This property means that the non-commuting graph Γ_G is $\mathrm{Aut}(G)$-symmetric, where $\mathrm{Aut}(G)$ is the automorphism group of G, Γ_G is the graph whose vertex set is $G\backslash Z(G)$ ($Z(G)$ is the center of G) and the edge set is the set of all non-commuting pairs of elements of G (see [1] or [2]) and recall that a graph Γ is called K-symmetric for a subgroup K of $\mathrm{Aut}(\Gamma)$ if K acts

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transitively on the set of ordered pairs of adjacent vertices of Γ.
Pablo Spiga has pointed out that the group G must be a p-group of Frattini class 2, for some prime p (see [3]). We generalize this result by proving that the group G is nilpotent of class 2 and $G/Z(G)$ is elementary abelian (see Lemma 1, below). We also prove some general properties of these groups (see Lemma 1, below): for example, it is proved that every two non-abelian 2-generator subgroups of G are isomorphic and every non-abelian 2-generator subgroup H of G has the same property as G.

2 Proofs

Some properties of the groups in question are as follows:

Lemma 1. *Let G be a finite non-abelian group having the property that the automorphism group of G acts transitively on the set of ordered pairs of non-commuting elements. Then*

(1) $\mathrm{Aut}(G)$ *acts transitively on* $G\backslash Z(G)$.
(2) *for all* $x, y \in G\backslash Z(G)$, *we have* $|x| = |y|$.
(3) *for all* $x, y \in G\backslash Z(G)$, *we have* $C_G(x) \cong C_G(y)$.
(4) *every two non-abelian* 2*-generator subgroups of* G *are isomorphic and every non-abelian* 2*-generator subgroup* H *of* G *has the same property as* G, *i.e., the automorphism group of* H *acts transitively on the set of ordered pairs of non-commuting elements of* H.
(5) G *is a* p*-group for some prime* p.
(6) $\Phi(G) \leq Z(G)$ ($\Phi(G)$ *denotes the Frattini subgroup of* G). *In particular,* G *is nilpotent of class* 2 *and* $G/Z(G)$ *is elementary abelian.*

Proof. (1) Let $x, y \in G\backslash Z(G)$. Then there exist elements x' and y' such that $xx' \neq x'x$ and $yy' \neq y'y$. Thus by hypothesis, there exists $\alpha \in \mathrm{Aut}(G)$ such that $(x, x')^\alpha = (y, y')$. It follows that $x^\alpha = y$. This completes the proof of (1).
(2) This easily follows from (1).
(3) By (1), there exists $\alpha \in \mathrm{Aut}(G)$ such that $x^\alpha = y$. Now it is easy to see that $C_G(y) = \big(C_G(x)\big)^\alpha$. On the other hand, clearly we have $C_G(x) \cong \big(C_G(x)\big)^\alpha$. This completes the proof of (3).
(4) Let H_1 and H_2 be two non-abelian 2-generator subgroups of G. Then $H_1 = \langle x, x'\rangle$ and $H_2 = \langle y, y'\rangle$ for some elements $x, x', y, y' \in G$. Thus by hypothesis, there exists $\alpha \in \mathrm{Aut}(G)$ such that $(x, x')^\alpha = (y, y')$. It follows that $x^\alpha = y$ and $x'^\alpha = y'$. Therefore $H_1^\alpha = H_2$ and $H_1 \cong H_2$. Now let (x_1, x_2) and (y_1, y_2) be two non-commuting pairs with entries from H. By hypothesis, there exists $\beta \in \mathrm{Aut}(G)$ such that $(x_1, x_2)^\beta = (y_1, y_2)$. By the first part of (4), $\langle x_1, x_2\rangle^\beta = \langle y_1, y_2\rangle \cong H$ and since $\langle x_1, x_2\rangle$ and $\langle y_1, y_2\rangle$ are subgroups of H and these groups are all finite, it follows that $\langle x_1, x_2\rangle = \langle y_1, y_2\rangle = H$. Hence the restriction of β to H, is an automorphism of H. This completes the proof of (4).
(5) Since G is non-abelian, there exists a non-central element x in G. Then

$xy \neq yx$ for some $y \in G$. Since x is of finite order, there are commuting elements $x_1, \ldots, x_n \in G$ of prime power orders such $x = x_1 \cdots x_n$. It follows that $x_i y \neq y x_i$ for some i. This implies that G contains a non-central element of p-power order for some prime p. It follows from (2) that every element in $G \backslash Z(G)$ has the same p-power order. Now let $z \in Z(G)$ and $t \in G \backslash Z(G)$, where $t^{p^s} = 1$. Then $tz \in G \backslash Z(G)$ and $(tz)^{p^s} = 1$. Since $z \in Z(G)$, $1 = t^{p^s} z^{p^s} = z^{p^s}$. This completes the proof of (5).
(6) Suppose, for a contradiction, that there exists $x \in \Phi(G) \backslash Z(G)$. Let $X = \{x_1, \ldots, x_d\}$ be a minimal generating set for G, that is no proper subset of X generates G. Since $\Phi(G)$ is the set of non-generators of G, $\Phi(G) \cap X = \varnothing$. Since G is not abelian, there exists $i \in \{1, \ldots, d\}$ such that $x_i \in G \backslash Z(G)$. By (1), $x^\alpha = x_i$ for some $\alpha \in \mathrm{Aut}(G)$. Since $\Phi(G)$ is a characteristic subgroup of G, $x_i \in \Phi(G)$ which is a contradiction. Therefore $\Phi(G) \leq Z(G)$.
By part (5), G is a finite p-group for some prime p. Thus $\Phi(G) = G^p G'$ is a subgroup of $Z(G)$. This implies that G is nilpotent of class 2 and $G/Z(G)$ is elementary abelian. □

Note that if G is a nilpotent p-group of class 2 (p a prime number), then its commutator subgroup G' is non-trivial and contained in the center of G. We shall make frequent use without reference of well-known relations such as

$$[x, yz] = [x, y][x, z],\ [x, y^r] = [x^r, y] = [x, y]^r,\ (xy)^n = x^n y^n [y, x]^{\frac{n(n-1)}{2}}$$

for all $x, y, z \in G$, $n \in \mathbb{N}$ and $r \in \mathbb{Z}$.
For a prime number p, an integer $n > 0$ such that $p^n > 2$ and an arbitrary integer $d > 1$, we denote by $\mathcal{G}_d(p^n)$ the free group of rank d in the variety $\mathcal{V}(p^n)$ of groups of class at most 2 satisfying the laws $x^{p^n} = [x, y]^p = 1$. Lemma 1(6) show that any finite group G whose non-commuting graph is $\mathrm{Aut}(G)$-symmetric is in the variety $\mathcal{V}(p^n)$ for some prime p and integer $n \geq 1$. Thus G is isomorphic to a quotient of $\mathcal{G}_d(p^n)$. The following result shows that free groups in the latter variety is a good source for the groups in question.

Theorem 1. *The non-commuting graph of $\mathcal{G}_d(p^n)$ is $\mathrm{Aut}(\mathcal{G}_d(p^n))$-symmetric.*

Proof. Let $\mathcal{G}_d = \mathcal{G}_d(p^n)$ and suppose that $x_1, \ldots, x_d$ are free generators of $\mathcal{G}_d$. Suppose that $X = x_1^{i_1} \cdots x_d^{i_d} c_1$ and $Y = x_1^{j_1} \cdots x_d^{j_d} c_2$ ($c_1, c_2 \in \Phi(\mathcal{G}_d)$ and $i_1, \ldots, i_d, j_1, \ldots, j_d$ in $\mathbb{Z}$) are two non-commuting elements of $\mathcal{G}_d$. Now since $\mathcal{G}_d'$ is of exponent p and

$$1 \neq [X, Y] = \prod_{k<\ell} [x_k, x_\ell]^{i_k j_\ell - i_\ell j_k},$$

there exist k and ℓ, $k < \ell$ such that p does not divide $K = i_k j_\ell - i_\ell j_k$. For any $x \in \mathcal{G}_d$, let $\overline{x}$ denote $x\Phi(\mathcal{G}_d)$. We may write $\overline{X}$ and $\overline{Y}$ additively in the vector space $\mathcal{G}_d/\Phi(\mathcal{G}_d)$ over $\mathbb{Z}_p$:

$$\overline{X} = i_1 \overline{x_1} + \cdots + i_d \overline{x_d} \quad \text{and} \quad \overline{Y} = j_1 \overline{x_1} + \cdots + j_d \overline{x_d}.$$

It follows that

$$j_\ell\overline{X} - i_\ell\overline{Y} = K\overline{x_k} + \overline{x} \text{ and } i_k\overline{Y} - j_k\overline{X} = K\overline{x_\ell} + \overline{y},$$

for some $x, y \in \langle x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{\ell-1}, x_{\ell+1}, \ldots, x_d\rangle$. As $\gcd(K, p) = 1$, it follows that

$$\overline{x_k}, \overline{x_\ell} \in \langle \overline{x_1}, \ldots, \overline{x_{k-1}}, \overline{X}, \overline{x_{k+1}}, \ldots, \overline{x_{\ell-1}}, \overline{Y}, \overline{x_{\ell+1}}, \ldots, \overline{x_d}\rangle.$$

Hence

$$\mathcal{G}_d = \langle x_1, \ldots, x_{k-1}, X, x_{k+1}, \ldots, x_{\ell-1}, Y, x_{\ell+1}, \ldots, x_d\rangle.$$

Therefore, as $\mathcal{G}_d$ is free in the variety, there exists an automorphism $\Psi_{(X,Y)}$ of $\mathcal{G}_d$ which maps x_1 to X, x_2 to Y and x_i to x_i for all $i > 2$. Now let X' and Y' be another two non-commuting elements of $\mathcal{G}_d$. Then $\Psi_{(X',Y')}\Psi^{-1}_{(X,Y)} \in \mathrm{Aut}(\mathcal{G}_d)$ is sending the pair (X, Y) to (X', Y'). This completes the proof. □

Lemma 2. *Let G be a finite non-abelian p-group ($p > 2$) having the property that the automorphism group acts transitively on the set of ordered pairs of non-commuting elements. Suppose that $x_1, \ldots, x_\ell \in G$ are such that $x_1Z(G), \ldots, x_\ell Z(G)$ form a basis for $G/Z(G)$. Then $\langle x_1^p, \ldots, x_\ell^p\rangle = \langle x_1^p\rangle \times \cdots \times \langle x_\ell^p\rangle$.*

Proof. Let $x_1^{pi_1} \cdots x_\ell^{pi_\ell} = 1$ for some integers $i_1, \ldots, i_\ell$. Suppose that $i_k = p^{\alpha_k} j_k$, where $\gcd(j_k, p) = 1$ $(k = 1, \ldots, \ell)$. Then, since G' is of exponent $p > 2$ and G is nilpotent of class 2, we can write

$$1 = x_1^{pi_1} \cdots x_\ell^{pi_\ell} = \left(x_1^{s_1} \cdots x_\ell^{s_\ell}\right)^{p^{1+\alpha_i}},$$

where $\alpha_i = \min\{\alpha_k \mid k = 1, \ldots, \ell\}$ and $s_k = p^{\alpha_k - \alpha_i} j_k$ $(k = 1, \ldots, \ell)$. Since $\gcd(s_i, p) = 1$ and $x_1Z(G), \ldots, x_\ell Z(G)$ form a basis for $G/Z(G)$, we have that $x = x_1^{s_1} \cdots x_\ell^{s_\ell} \notin Z(G)$. Now it follows from Lemma 1(2) that $|x_k| = |x| = p^n$ (for some integer $n \geq 1$) for each $k = 1, \ldots, \ell$. Thus p^n divides p^{α_i+1} and so $n - 1 \leq \alpha_k$ for all $k = 1, \ldots, \ell$. This completes the proof. □

Theorem 2. *A finite non-abelian 2-generator group G has the property that the automorphism group acts transitively on the set of ordered pairs of non-commuting elements if and only if $G \cong \mathcal{G}_2(p^n)$ which is isomorphic to*

$$\mathcal{G} = \langle x, y \mid x^{p^n} = y^{p^n} = [x, y]^p = [x, y, y] = [x, y, x] = 1\rangle,$$

for some prime number p and integer $n > 0$ with $p^n > 2$ or $G \cong Q_8$, the quaternion group of order 8.

Proof. Throughout we denote $\mathcal{G}_2(p^n)$ by $\mathcal{G}$. We first prove the sufficiency. It follows from Theorem 1 that the non-commuting graph of $\mathcal{G}$ is $\mathrm{Aut}(\mathcal{G})$-symmetric.

Since every two non-commuting elements of Q_8 generate the group, it is easy by using the following presentation of Q_8

$$\langle x, y \mid x^4 = 1, x^2 = y^2, x^y = x^{-1}\rangle$$

to see that $\mathrm{Aut}(Q_8)$ acts transitively on the ordered pairs of non-commuting elements of Q_8.

Now we are going to show the necessity. Let $G = \langle a, b\rangle$. By Lemma 1(5), G is a finite p-group for some prime number p. Also it follows from Lemma 1(6), that $G/Z(G) \cong C_p \times C_p$, since G is not abelian. On the other hand, as G is 2-generator and non-abelian, $G/\Phi(G) \cong C_p \times C_p$. Thus by Lemma 1(6), $Z(G) = \Phi(G) = G'G^p$. This implies that $Z(G) = \langle a^p, b^p, [a, b]\rangle$. By Lemma 1(2), we have $|a| = |b| = p^n$ for some integer $n \geq 1$ (if $p = 2$, then since G is not abelian, it follows from Lemma 1(5) that $n \geq 2$). Note that, since $G^p \leq Z(G)$, $[a, b]^p = [a^p, b] = 1$ and so $|[a, b]| = p$, since G is not abelian.
We first prove that either $\langle a^p\rangle \cap \langle b^p\rangle = 1$ or $G \cong Q_8$. Suppose that $a^{pi}b^{pj} = 1$ for some $i, j \in \mathbb{Z}$. Let $pi = p^t i'$ and $pj = p^s j'$, where $i', j' \in \mathbb{Z}$ and $\gcd(i'j', p) = 1$. Assume $s \geq t$ and note that $t \geq 1$. Then

$$\left(a^{i'}b^{p^{s-t}j'}\right)^{p^t} = a^{p^t i'}b^{p^s j'}[b^{p^{s-t}j'}, a^{i'}]^{\frac{p^t(p^t-1)}{2}} = [b^{p^{s-t}j'}, a^{i'}]^{\frac{p^t(p^t-1)}{2}}. \qquad (\#)$$

Suppose that $p > 2$ or $s \geq 2$ or $s > t$. Then it follows from (#) that

$$[b^{p^{s-t}j'}, a^{i'}]^{\frac{p^t(p^t-1)}{2}} = [a, b]^{-j'i'\frac{p^s(p^t-1)}{2}} = 1.$$

It follows that $|a^{i'}b^{p^{s-t}j'}|$ divides p^t. Now since $\gcd(i', p) = 1$, $[a^{i'}b^{p^{s-t}j'}, b] \neq 1$. It follows that $a^{i'}b^{p^{s-t}j'} \in G\backslash Z(G)$ and so by Lemma 1(2), $|a^{i'}b^{p^{s-t}j'}| = |a| = p^n$. Therefore $n \leq t \leq s$ and so $a^{pi} = b^{pj} = 1$. Hence, in this case, we have that $\langle a^p\rangle \cap \langle b^p\rangle = 1$. Thus we may assume $p = 2$ and $s = t = 1$. It follows that $(a^i)^2(b^j)^2 = 1$ and i and j are odd. Without loss of generality we may assume that $i = j = 1$. Since (a, b), (ab, b) and (ab, a) are ordered pairs of non-commuting elements of G, there exists $\alpha, \beta \in \mathrm{Aut}(G)$ such that $(a, b)^\alpha = (ab, b)$ and $(a, b)^\beta = (ab, a)$. It follows that

$$1 = (a^2b^2)^\alpha = (ab)^2b^2 = a^2b^4[a, b] = b^2[a, b],$$
$$1 = (a^2b^2)^\beta = (ab)^2a^2 = a^4b^2[a, b] = a^2[a, b].$$

Thus $a^2 = b^2 = [a, b]$ and $a^4 = b^4 = 1$ and so $G \cong Q_8$.
From now on, suppose that

$$\langle a^p\rangle \cap \langle b^p\rangle = 1. \qquad (I)$$

We now show that

$$\langle [a, b]\rangle \cap \langle a^p, b^p\rangle = 1. \qquad (II)$$

It is enough to show that $[a, b] \notin \langle a^p, b^p\rangle$, since $|[a, b]|$ is prime. Suppose, for a contradiction, that

$$[a, b] = a^{pi}b^{pj} \text{ for some } i, j \in \mathbb{Z}. \qquad (*)$$

First assume that $p > 2$. Therefore a and b are of odd order and so (a, b), (a^2, b) and (a, b^2) are ordered non-commuting pairs of elements of G. Thus, by hypothesis, there exist $\alpha, \beta \in \mathrm{Aut}(G)$ such that $(a, b)^\alpha = (a, b^2)$ and $(a, b)^\beta = (a^2, b)$. It follows from (*) that $[a, b]^\alpha = (a^{pi}b^{pj})^\alpha$ and $[a, b]^\beta = (a^{pi}b^{pj})^\beta$. Therefore $[a, b]^2 = a^{pi}b^{2pj}$ and $[a, b]^2 = a^{2pi}b^{pj}$. On the other hand, (*)

implies that $[a,b]^2 = a^{2pi}b^{2pj}$ (note that $a^p, b^p \in Z(G)$). These relations yield that $a^{pi} = b^{pi} = 1$ and so $[a,b] = 1$, a contradiction. Hence $\langle [a,b]\rangle \cap \langle a^p, b^p\rangle = 1$, if $p > 2$.
Now suppose that $p = 2$. Since (a,b) and (b,a) are ordered non-commuting pairs of elements in G, there exists an automorphisms $\alpha \in \mathrm{Aut}(G)$ such that $(a,b)^\alpha = (b,a)$. It follows from $(*)$ that $[b,a] = b^{2i}a^{2j}$. Since $[a,b]^2 = 1$, $a^{2i}b^{2j} = a^{2j}b^{2i}$ and so $a^{2(i-j)}b^{2(j-i)} = 1$. It follows from (I), we have that $b^{2(j-i)} = 1$. Thus $b^{2i} = b^{2j}$ and so

$$[a,b] = a^{2i}b^{2i}. \qquad (III)$$

Now since (a,b), (ab,b) and (a,ab) are non-commuting pairs of elements of G, there exist automorphisms $\beta, \gamma \in \mathrm{Aut}(G)$ such that $(a,b)^\beta = (ab,a)$ and $(a,b)^\gamma = (a,ab)$. Now it follows from (III) that $[ab,b] = (ab)^{2i}b^{2i}$ and $[a,ab] = a^{2i}(ab)^{2i}$. Since $[a,b] = [ab,b] = [a,ab]$, we have that $b^{2i}[b,a]^{i(2i-1)} = a^{2i}[b,a]^{i(2i-1)} = 1$ and so $a^{2i}b^{2i} = [b,a]^{2i(2i-1)} = 1$. Hence $[a,b] = 1$, a contradiction.
Now it follows from (I) and (II), that

$$Z(G) = \langle a^p\rangle \times \langle b^p\rangle \times \langle [a,b]\rangle.$$

Therefore $|Z(G)| = p^{2n-1}$ and $|G| = p^{2n+1}$. Now by von Dyck's theorem, G is an epimorphic image of $\mathcal{G}$ and since $|G| = |\mathcal{G}|$, $G \cong \mathcal{G}$. This completes the proof. □

We can now produce a further number of examples.

Theorem 3. *Let G be a finite 2-generated non-abelian group whose non-commuting graph is $\mathrm{Aut}(G)$-symmetric. If G is of exponent q and A is any finite abelian group of exponent dividing q, and not equal to q whenever $G \cong Q_8$, then the non-commuting graph of $G \times A$ is also $\mathrm{Aut}(G \times A)$-symmetric.*

Proof. By Theorem 2, $q = p^n$ for some prime p and integer $n > 0$. Let $A = C_{p^{n_1}} \times \cdots \times C_{p^{n_k}}$. If $G \not\cong Q_8$, then $G \times A$ has the following presentation:

$$\langle x, y, a_1, \ldots, a_k \mid x^{p^n} = y^{p^n} = [x,y]^p = [x,y,x] = [x,y,y] = [a_i,a_j] =$$
$$= [a_i,x] = [a_i,y] = a_i^{p^{n_i}} = 1\ \forall i,j\rangle,$$

and if $G \cong Q_8$, then $G \times A$ has the following presentation

$$\langle x, y, a_1, \ldots, a_k \mid x^4 = 1, x^2 = y^2, x^y = x^{-1},$$
$$[a_i,a_j] = [a_i,x] = [a_i,y] = a_i^2 = 1\ ;\forall\ i,j\rangle.$$

It is now easy to see by von Dyck's theorem and Lemma 1(2) that for any two non-commuting elements $x_1, x_2 \in \langle x,y\rangle$ and any two elements $a, b \in \langle a_1, \ldots, a_k\rangle$, the map α which is defined by $x \mapsto x_1 a$, $y \mapsto x_2 b$, $a_i \mapsto a_i$ for all i, can be extended to an automorphism of $G \times A$. From this, it now follows that the non-commuting graph of $G \times A$ is $\mathrm{Aut}(G \times A)$-symmetric. □

Remark 1. For any two non-commuting elements x and y in Q_8, we have $x^2 = y^2 = [x,y]$.

We can give the classification of finite non-abelian 2-groups G having a subgroup isomorphic to Q_8 whose non-commuting graphs are $\mathrm{Aut}(G)$-symmetric.

Theorem 4. *Let G be a finite non-abelian 2-group having a subgroup isomorphic to Q_8. Then the non-commuting graph of G is $\mathrm{Aut}(G)$-symmetric if and only if $G \cong Q_8 \times E$ for some elementary abelian 2-group E.*

Proof. We prove that G is a Dedekind group. Let x and y be two non-commuting elements of G. Then it follows from Lemma 1(2) and Theorem 2 that $\langle x, y\rangle \cong Q_8$. Now by Remark 1 we have that $x^2 = [x, y]$ which implies that $x^y = x^3$. Hence $\langle x\rangle \trianglelefteq G$ for all $x \in G$ and so every subgroup of G is normal. Now by a famous result of Dedekind-Baer (see 5.3.7 of [4]) that $G \cong Q_8 \times E$ for some elementary abelian 2-group E.
The converse follows from Theorem 3. □

3 3-Generator groups G whose non-commuting graphs are $\mathrm{Aut}(G)$-symmetric

In this section we study groups with the property of the title of the section and we find some properties of them.

Theorem 5. *Let G be a finite non-abelian 3-generator group having the property that the automorphism group acts transitively on the set of ordered pairs of non-commuting elements. Then $C_G(x) = \langle x\rangle Z(G)$ for all $x \in G\backslash Z(G)$.*

Proof. Let $x \in G\backslash Z(G)$. By Lemma 1(5)-(6), G is a p-group (for some prime p) of class 2 and $x \notin \Phi(G)$. Then Burnside's basis theorem implies that $G = \langle x, y, z\rangle$ for some $y, z \in G$. Suppose, for a contradiction, that $C_G(x) \neq \langle x\rangle Z(G)$. Since $G/Z(G)$ is elementary abelian, it follows that $y^i z^j \in C_G(x)$ for some integers i and j such that $0 \le i \le j < p$ with $(i, j) \neq (0, 0)$. Thus either $G = \langle x, y^i z^j, z\rangle$ or $G = \langle x, y^i z^j, y\rangle$. Therefore, without loss of generality, we may assume that $[x, y] = 1$. Since x is not central, $[x, z] \neq 1$. If $C_G(z) = \langle z\rangle Z(G)$, then by Lemma 1(1) and (3), $C_G(x) = \langle x\rangle Z(G)$, a contradiction. Therefore $x^\ell y^k \in C_G(z)\backslash\langle z\rangle Z(G)$ for some integers ℓ and k such that $0 \le \ell \le k < p$ with $(\ell, k) \neq (0, 0)$. Since $[x, z] \neq 1$, $k \neq 0$ and so $G = \langle x, x^\ell y^k, z\rangle$ and $x^\ell y^k \in Z(G)$, a contradiction. This completes the proof. □

For a finite group G, we denote by $\mathrm{d}(G)$ the minimum number of elements of a generating set of G.

Lemma 3. *Let G be a non-abelian nilpotent group of class 2. If $\mathrm{d}(G) = 3$, then there exist pairwise non-commuting elements x, y and z such that $G = \langle x, y, z\rangle$.*

Proof. Let $G = \langle a, b, c\rangle$. Since G is not abelian, we may assume that $a \in G\backslash Z(G)$. Thus a does not commute with either b or c. Thus we may assume that $[a, b] \neq 1$. Suppose that $[a, c] = 1$. Then as G is nilpotent of class 2,

$[a, bc] = [a, b][a, c] = [a, b] \neq 1$. Since $G = \langle a, b, bc\rangle$, if $[b, bc] \neq 1$, we are done. Thus we may assume that $[b, bc] = 1$. Now we have $[ab, bc] = [a, bc] \neq 1$ and as $G = \langle a, ab, bc\rangle$, we are done. Therefore we may assume that $[a, c] \neq 1$. If $[b, c] \neq 1$, the proof is complete. If $[b, c] = 1$, since $G = \langle a, ab, c\rangle$ the proof completes. □

Lemma 4. *Let G be a 3-generator nilpotent group of class two whose derived subgroup is of prime exponent p. Then $G' = \{[x, y] \mid x, y \in G\}$.*

Proof. Let $G = \langle a, b, c\rangle$. Then, since G is nilpotent of class two, $G' = \langle [a, b], [a, c], [b, c]\rangle$. As $G' \leq Z(G)$, every element of G' can be written as $[a, b]^i[a, c]^j[b, c]^k$. It is enough to show that there are elements $x, y \in G$ such that $[x, y] = [a, b]^i[a, c]^j[b, c]^k$. If $p \mid j$, then we have

$$[a, b]^i[a, c]^j[b, c]^k = [a, b]^i[b, c]^k = [a^j c^{-k}, b]$$

and we are done. Now suppose $p \nmid j$ and so there is an integer j' such that $p \mid (jj' - 1)$. Thus we may write

$$[a, b]^i[a, c]^j[b, c]^k = [a, b^i c^j][b^k, c]^{jj'} = [a, b^i c^j][b^{kj'}, b^i c^j] = [ab^{kj'}, b^i c^j].$$

This completes the proof. □

Lemma 5. *Let G be a finite non-abelian p-group having the property that the automorphism group acts transitively on the set of ordered pairs of non-commuting elements. If $\mathrm{d}(G) = 3$ and N is a characteristic subgroup of G, then either $G' \leq N$ or $G' \cap N = 1$.*

Proof. Suppose that $G' \cap N \neq 1$. Then by Lemma 4, there exist $x, y \in G$ such that $1 \neq [x, y] \in N$. Now let a and b are two arbitrary elements of G such that $1 \neq [a, b]$. By hypothesis, there exists an automorphism α of G such that $[a, b] = [x, y]^\alpha$. Since $N^\alpha = N$, we have that $[a, b] \in N$. This completes the proof. □

Lemma 6. *Let G be a finite non-abelian group whose non-commuting graph is $\mathrm{Aut}(G)$-symmetric. If $\mathrm{d}(G) = 3$, then $\mathrm{d}(G') \neq 2$.*

Proof. By Lemma 3, there exist pairwise non-commuting elements x, y, z generating G. Suppose, for a contradiction, that $\mathrm{d}(G') = 2$. By Lemma 1, G is a p-group for some prime p and $G' \leq Z(G)$ is a $\mathbb{Z}_p$-vector space. It follows that there is a $\mathbb{Z}_p$-basis of size 2 for G' in $\{[x, y], [x, z], [y, z]\}$. Suppose without loss of generality that $\{[x, y], [y, z]\}$ is a basis for G'. Then $[x, y]^i[y, z]^j = [x, z]$ for some integers i and j. Then $[x^i z^{-j}, y] = [x, z]$. Since $[x, z] \neq 1$, either i or j is coprime to p. If $\gcd(i, p) = 1$, consider the equality $[x^i z^{-j}, y^i] = [x^i z^{-j}, z]$ and if $\gcd(j, p) = 1$, consider the equality $[x^i z^{-j}, y^{-j}] = [x, x^i z^{-j}]$. Thus if $p \nmid i$, (respectively, $p \nmid j$) $\{a = x^i z^{-j}, b = y^i, c = z\}$ (respectively, $\{a = x^i z^{-j}, b = x^{-1}, c = y^{-j}\}$) is a generating set for G consisting of pairwise non-commuting elements with the property that $[a, b] = [a, c]$ so that $\{[a, b], [b, c]\}$ is a basis for G'. Now by hypothesis there is an automorphism α of G sending (a, b) to (b, a). Therefore $\{[a, b]^\alpha, [b, c]^\alpha\}$ must be a basis for G'. We have

$[a,b]^\alpha = [a,b]^{-1}$ and $[b,c]^\alpha = [a,b]^{i+j}$, where $c^\alpha = a^\ell b^i c^j f$ for some $f \in \Phi(G)$. These imply that $G' = \langle [a,b] \rangle$, a contradiction. This completes the proof. □

Lemma 7. *Let G be a finite non-abelian group whose non-commuting graph is* $\mathrm{Aut}(G)$*-symmetric. If* $\mathrm{d}(G) = 3$ *and G' is non-cyclic then* $\Phi(G) = Z(G)$.

Proof. By Lemma 1(6), $\Phi(G) \leq Z(G)$. If $\Phi(G) \neq Z(G)$, then there is a generating set $\{x, y, z\}$ for G such that $z \in Z(G)$. Since $G' \leq Z(G)$, it follows that $G' = \langle [x,y], [y,z], [x,z] \rangle$. Thus $G' = \langle [x,y] \rangle$ as $[x,z] = [y,z] = 1$. Hence $\mathrm{d}(G') = 1$, a contradiction. This completes the proof. □

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Alireza Abdollahi
Department of Pure Mathematics,
Faculty of Mathematics and Statistics,
University of Isfahan,
Isfahan 81746-73441, Iran
Email address: a.abdollahi@math.ui.ac.ir