S©MR

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ
Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru
ISSN 1813-3304

Vol. 21, No. 2, pp. 684–701 (2024)
https://doi.org/10.33048/semi.2024.21.047

УДК 517.218.244,517.518.86
MSC 26D10,26A51,26A33

NEW GENERALIZED WEIGHTED FRACTIONAL VARIANTS OF HERMITE-HADAMARD INEQUALITIES WITH APPLICATIONS

J. E. NÁPOLES, B. BAYRAKTAR, S. I. BUTT

Communicated by E.M. Rudoy

Abstract: Integral inequalities play a fundamental role in various mathematical fields, which have led to new methods and theoretical developments, both pure and applied. The need for searching for precise inequalities, in which the notion of convexity plays an important role, has a high impact on approximation theory calculus. In this paper, we first obtained a new version of the weighted fractional identity that led us to obtain new variants of weighted Hermite-Hadamard and Bullen type inequalities. We then presented several refinements of it in the framework of weighted integrals for the modified second type (h, m)–convex functions.

Keywords: Hermite–Hadamard integral inequality, Bullen type inequality, Hölder's inequality, $(h, m)-$convex modified functions, weighted fractional integrals operators.

1 Introduction

Let $\varsigma_1, \varsigma_2 \in \mathbb{R}$ with $\varsigma_1 < \varsigma_2$. A function $\psi : I = [\varsigma_1, \varsigma_2] \to \mathbb{R}$ is said to be convex if

$$\psi\left(\zeta\theta + (1-\zeta)\vartheta\right) \leq \zeta\psi(\theta) + (1-\zeta)\psi(\vartheta)$$

Napoles J. E., Bayraktar B., Butt S. I., New generalized weighted fractional variants of Hermite-Hadamard inequalities with applications.

Received Febriary, 11, 2023, published September, 30, 2024.

holds for all $\zeta \in [0,1]$ and $\theta, \vartheta \in I$. If the function is concave, then the inequalities will be in reverse order.

In the last few decades, convex functions have become the object of extensive research by scientists from different countries. Today, this concept in the literature has a lot of generalizations and extensions. In [22], the authors presented a fairly wide range of different classes convexity.

Of the theory of inequalities, the most important one is the Hermite–Hadamard inequality:

$$\psi\left(\frac{\varsigma_1+\varsigma_2}{2}\right) \leq \frac{1}{\varsigma_2-\varsigma_1}\int_{\varsigma_1}^{\varsigma_2}\psi(x)dx \leq \frac{\psi(\varsigma_1)+\psi(\varsigma_2)}{2}, \tag{1}$$

holds for convex functions on an interval.

The ground breaking inequality helps to predict estimations of the mean values of a convex function and it is pertinent mention that it also provides a refine estimation of Jensen gap.

Interesting results for integral inequalities by using fractional operators can be found, for example, [4–13, 30, 35] and the references therein.

For example, the authors in [4–7] obtained new results in terms of fractional integral operators of the Riemann–Liouville for (s,m), (α,m) and $(h,m)-$convex functions. In [8], Bermudo et al. presented some variants of the Hermite–Hadamard inequality general convex functions in the context of $q-$calculus. In [9], Butt and Pečarić obtained some generalized Hermite–Hadamard's integral inequalities for the monotone functions of the form f/h. In [11], the authors presented an article in which they obtained new Hermite–Hadamard inequalities of the Jensen–Mercer type for a harmonically convex function through fractional integrals. In [16], Guzmán et al. studied the most important properties of the generalized integral operators and obtained different inequalities of the Hermite–Hadamard type for the weighted symmetric convex and $h-$symmetrized convex functions. In [21], the authors have defined new integral operators to obtain new Hermite–Hadamard inequalities for $h-$convex functions.

Dragomir and Agarwal in [12], by using the convexity of the first derivatives of the function, obtained the following estimate for the trapezoid inequality:

Theorem 1. *Let $\varsigma_1, \varsigma_2 \in \mathbb{R}$ with $\varsigma_1 < \varsigma_2, \psi : I = [\varsigma_1, \varsigma_2] \to \mathbb{R}$ and $\psi \in C^1(\varsigma_1, \varsigma_2)$. If $|\psi'|$ is convex on $[\varsigma_1, \varsigma_2]$, then the inequality holds:*

$$\left|\frac{\psi(\varsigma_1)+\psi(\varsigma_2)}{2} - \frac{1}{\varsigma_2-\varsigma_1}\int_{\varsigma_1}^{\varsigma_2}\psi(\zeta)d\zeta\right| \leq \frac{\varsigma_2-\varsigma_1}{8}\left(|\psi'(\varsigma_1)|+|\psi'(\varsigma_2)|\right). \tag{2}$$

The definitions of sequentially expanding the concept of convex functions, which form the basis of our article, are given below [6, 7].

Definition 1. *Let $h : [0,1] \to (0,1]$, $\psi : [0,+\infty) \to [0,+\infty)$. If inequality*

$$\psi(\zeta\theta + m(1-\zeta)\vartheta) \leq h^s(\zeta)\psi(\theta) + m(1-h^s(\zeta))\psi(\vartheta)$$

is fulfilled for all $\zeta \in [0,1]$ and $\theta, \vartheta \in [0,+\infty)$, then the function ψ is called the $(h,m)-$convex modified of the first type on $[0,+\infty)$. Here $m \in [0,1]$ and $s \in [-1,1]$.

Definition 2. *et $h:[0,1]\to(0,1]$, $\psi:[0,+\infty)\to[0,+\infty)$. If inequality*

$$\psi\left(\zeta\theta+m(1-\zeta)\vartheta\right)\leq h^s(\zeta)\psi(\theta)+m(1-h(\zeta))^s\psi(\vartheta)$$

is fulfilled for all $\zeta\in[0,1]$ and $\theta,\vartheta\in[0,+\infty)$, then the function ψ is called the $(h,m)-$convex modified of the second type on $[0,+\infty)$. Here $m\in[0,1]$ and $s\in[-1,1]$.

We denote by $N^{s,1}_{h,m}[0,+\infty)$ and $N^{s,2}_{h,m}[0,+\infty)$ the class of all $(h,m)-$convex modified functions of the first and second type respectively on the $[0,+\infty)$.

Remark 1. *From Definitions 1 and 2, we have the following classifications of functions on $[0,+\infty)$.*

- (i01) *If $h(\zeta)=\zeta$, then ψ recaptures the class of m–convex functions, see [31].*
- (i02) *If $h(\zeta)=\zeta^\alpha$ with $a\in(0,1]$, then ψ recaptures the class of (α,s,m)–convex functions, see [37].*
- (i03) *If $s=1$, then ψ recaptures the class of (h,m)–convex function.*
- (i04) *If $h(\zeta)=\zeta, s\in(0,1]$ and $m=1$, then ψ recaptures the class of s–convex functions, see [17].*
- (i05) *If $h(\zeta)=\zeta, m=1$ and $s\in[-1,1]$, then ψ recaptures the extended class of s–convex functions, see [36].*
- (i06) *If $h(\zeta)=\zeta$, $m=s=1$, then ψ is a convex function.*

To begin with, we will give a definition of Riemann–Liouville fractional integrals, and then, for a better understanding of the topic, we will give a definition of weighted integral operators(with $0\leq\varsigma_1<\zeta<\varsigma_2\leq\infty$ and real number $\kappa\geq0$), which will be the basis for our article.

Definition 3. *Let $\psi\in L_1[\varsigma_1,\varsigma_2]$, $\alpha\in\mathbb{C}$ and $\Re(\alpha)>0$. Right and left fractional integrals defined below*

$$^{\alpha}I_{\varsigma_1^+}\psi(x)=\frac{1}{\Gamma(\alpha)}\int_{\varsigma_1}^{x}(x-\zeta)^{\alpha-1}\psi(\zeta)d\zeta,\quad x>\varsigma_1,$$

$$^{\alpha}I_{\varsigma_2^-}\psi(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{\varsigma_2}(\zeta-x)^{\alpha-1}\psi(\zeta)d\zeta,\quad x<\varsigma_2,$$

are called right and left Riemann–Liouville fractional integrals, respectively. Here $\Gamma(\alpha)$ is the gamma function.

Definition 4. *Let $\psi\in L_1(\varsigma_1,\varsigma_2)$, $w:[0,\infty)\to[0,\infty)$ and $w\in C[0,\infty)$ with derivatives piecewise continuous on $[0,\infty)$ up to the second order inclusive. Integral operators defined below*

$$J^w_{\varsigma_1^+}\psi(x)=\int_{\varsigma_1}^{x}w'\left(\frac{x-\zeta}{\frac{\varsigma_2-\varsigma_1}{1+\kappa}}\right)\psi(\zeta)d\zeta,\quad x>\varsigma_1,$$

$$J^w_{\varsigma_2^-}\psi(x)=\int_{x}^{\varsigma_2}w'\left(\frac{\zeta-x}{\frac{\varsigma_2-\varsigma_1}{1+\kappa}}\right)\psi(\zeta)d\zeta,\quad x<\varsigma_2.$$

called weighted fractional integrals (right and left, respectively).

Remark 2. *If we take* $w'(\zeta) = \frac{(\varsigma_2-\varsigma_1)\zeta^{\alpha-1}}{\Gamma(\alpha)}$ *and* $\kappa = 0$*, we get the Riemann–Liouville fractional integral, but if we take* $w'(\zeta) \equiv 1$ *it turns into a classical Riemann integral.*

In the present article, we presented some new unified variants of weighted Hadamard inequality (1) by employing generalized convexity for generalized fractional integral operators given in Definition 4. Several special cases are discussed as well.

2 Main results

The results are obtained with the help of the following lemma.

Lemma 1. *Let* $\psi : [\varsigma_1, \varsigma_2] \subset \mathbb{R} \to \mathbb{R}$ *with* $\psi \in C^1(\varsigma_1, \varsigma_2)$ *and let* $w : [0,\infty) \to [0,\infty)$*,* $w \in C[0,\infty)$ *with the first and second order derivatives piecewise continuous on* $[0,\infty)$*. If* $\psi' \in L_1(\varsigma_1, \varsigma_2)$*, then equality holds:*

$$\begin{aligned}
&[\psi(\varsigma_1) + \psi(\varsigma_2)]\, w(0) - \left[\psi\left(\frac{\kappa\varsigma_1 + \varsigma_2}{1+\kappa}\right) + \psi\left(\frac{\varsigma_1 + \kappa\varsigma_2}{1+\kappa}\right)\right] w(1) \qquad (3)\\
&\qquad + \frac{1+\kappa}{\varsigma_2 - \varsigma_1}\left[J^w_{\left(\frac{\varsigma_1+\kappa\varsigma_2}{1+\kappa}\right)^+}\psi(\varsigma_2) + J^w_{\left(\frac{\kappa\varsigma_1+\varsigma_2}{1+\kappa}\right)^-}\psi(\varsigma_1)\right]\\
&= \frac{\varsigma_2 - \varsigma_1}{1+\kappa}\left[\int_0^1 w(\zeta)\psi'\left(\frac{\zeta}{1+\kappa}\varsigma_1 + \frac{1+\kappa-\zeta}{1+\kappa}\varsigma_2\right)d\zeta\right.\\
&\qquad \left. - \int_0^1 w(\zeta)\psi'\left(\frac{1+\kappa-\zeta}{1+\kappa}\varsigma_1 + \frac{\zeta}{1+\kappa}\varsigma_2\right)d\zeta\right].
\end{aligned}$$

Proof. For simplicity,

$$\begin{aligned}
&\int_0^1 w(\zeta)\psi'\left(\frac{\zeta}{1+\kappa}\varsigma_1 + \frac{1+\kappa-\zeta}{1+\kappa}\varsigma_2\right)d\zeta\\
&\qquad - \int_0^1 w(\zeta)\psi'\left(\frac{1+\kappa-\zeta}{1+\kappa}\varsigma_1 + \frac{\zeta}{1+\kappa}\varsigma_2\right)d\zeta := I_1 - I_2.
\end{aligned}$$

By integrating by parts in I_1, we have

$$\begin{aligned}
I_1 &= \frac{1+\kappa}{\varsigma_2 - \varsigma_1}\left[w(0)\psi(\varsigma_2) - w(1)\psi\left(\frac{\varsigma_1 + \kappa\varsigma_2}{1+\kappa}\right)\right]\\
&\qquad + \frac{(1+\kappa)}{(\varsigma_2 - \varsigma_1)}\int_0^1 w'(\zeta)\psi\left(\frac{\zeta}{1+\kappa}\varsigma_1 + \frac{1+\kappa-\zeta}{1+\kappa}\varsigma_2\right)d\zeta\\
&= \frac{1+\kappa}{\varsigma_2 - \varsigma_1}\left[w(0)\psi(\varsigma_2) - w(1)\psi\left(\frac{\varsigma_1 + \kappa\varsigma_2}{1+\kappa}\right)\right]\\
&\qquad + \frac{(1+\kappa)^2}{(\varsigma_2 - \varsigma_1)^2} J^w_{\left(\frac{\varsigma_1+\kappa\varsigma_2}{1+\kappa}\right)^+}\psi(\varsigma_2),
\end{aligned}$$

since

$$\int_0^1 w'(\zeta)\psi\left(\frac{\zeta}{1+\kappa}\varsigma_1+\frac{1+\kappa-\zeta}{1+\kappa}\varsigma_2\right)d\zeta=\frac{1+\kappa}{\varsigma_2-\varsigma_1}\int_{\frac{\varsigma_1+\kappa\varsigma_2}{1+\kappa}}^{\varsigma_2} w'\left[\frac{\varsigma_2-z}{\frac{\varsigma_2-\varsigma_1}{1+\kappa}}\right]\psi(z)dz.$$

Analogously for I_2, we have

$$I_2=\frac{1+\kappa}{\varsigma_2-\varsigma_1}\left[w(1)\psi(\frac{\kappa\varsigma_1+\varsigma_2}{1+\kappa})-w(0)\psi(\varsigma_1)\right]-\frac{(1+\kappa)^2}{(\varsigma_2-\varsigma_1)^2}J^w_{\left(\frac{\kappa\varsigma_1+\varsigma_2}{1+\kappa}\right)^-}\psi(\varsigma_1).$$

Substituting in I_1-I_2, grouping appropriately and multiplying by $\frac{\varsigma_2-\varsigma_1}{1+\kappa}$, we obtain the inequality (3). □

The following remarks have the mission of illustrating the scope and strength of the previous result. The choice of special weight functions in the Lemma 1 leads us to many interesting results published in the literature.

Remark 3. *Special cases of the Lemma 1:*

(i01) *If we take $\kappa=0$ in (3), then we get the following new general weighted identity.*

$$\begin{aligned}&[w(0)-w(1)]\left[\psi(\varsigma_2)+\psi(\varsigma_1)\right]+\frac{1}{\varsigma_2-\varsigma_1}\left(J^w_{\varsigma_1^+}\psi(\varsigma_2)+J^w_{\varsigma_2^-}\psi(\varsigma_1)\right)\\&=(\varsigma_2-\varsigma_1)\left[\int_0^1 w(\zeta)\psi'\left(\zeta\varsigma_1+(1-\zeta)\varsigma_2\right)d\zeta\right.\\&\qquad\left.-\int_0^1 w(\zeta)\psi'\left((1-\zeta)\varsigma_1+\zeta\varsigma_2\right)d\zeta\right].\end{aligned}\tag{4}$$

(i02) *If we take $\kappa=0, w(\zeta)=1-\zeta^\alpha$ in (3), substituting x for ς_1 in I_1 and x for ς_2 in I_2, then we will have [23, Lemma 2].*

(i03) *If we take $\kappa=0, w(\zeta)=(1-\zeta)^\alpha-\zeta^\alpha$ and work only with I_1, then we get [28, Lemma 2]; furthermore, if we substitute ς_2 for $r\in(\varsigma_1,\varsigma_2]$, then we will have [24, Lemma 3].*

Remark 4. *Other special cases of the Lemma 1:*

(i04) *If we take $\kappa=1, w(\zeta)=\zeta$ in (3), then we get*

$$\begin{aligned}&[w(0)-w(1)]\left[\psi(\varsigma_2)+\psi(\varsigma_1)\right]\\&\qquad+\frac{1}{\varsigma_2-\varsigma_1}\left(J^w_{\left(\frac{\varsigma_1+\varsigma_2}{2}\right)^+}\psi(\varsigma_2)+J^w_{\left(\frac{\varsigma_1+\varsigma_2}{2}\right)^-}\psi(\varsigma_1)\right)\\&=\frac{\varsigma_2-\varsigma_1}{4}\left[\int_0^1\zeta\psi'\left(\frac{\zeta}{2}\varsigma_1+\frac{2-\zeta}{2}\varsigma_2\right)d\zeta-\int_0^1\zeta\psi'\left(\frac{2-\zeta}{2}\varsigma_1+\frac{\zeta}{2}\varsigma_2\right)d\zeta\right].\end{aligned}\tag{5}$$

(i05) *If we take* $\kappa = 1, w(\zeta) = 1 - \zeta$, *by replacing in* (3) $1 - \zeta$ *by* ζ, *then we obtain*

$$\begin{aligned} &\frac{\psi(\varsigma_1) + \psi(\varsigma_2)}{2} - \frac{1}{\varsigma_2 - \varsigma_1} \int_{\varsigma_1}^{\varsigma_2} \psi(\zeta) d\zeta \\ &= \frac{\varsigma_2 - \varsigma_1}{4} \left[\int_0^1 \zeta \psi' \left(\frac{1-\zeta}{2} \varsigma_1 + \frac{1+\zeta}{2} \varsigma_2 \right) d\zeta \right. \\ &\qquad \left. - \int_0^1 \zeta \psi' \left(\frac{1+\zeta}{2} \varsigma_1 + \frac{1-\zeta}{2} \varsigma_2 \right) d\zeta \right], \end{aligned} \tag{6}$$

which is given as in [1, Lemma 2.1].

Our first and main result was obtained from Lemma 1 in the following theorem.

Theorem 2. *Let* $\psi : \left[\varsigma_1, \frac{\varsigma_2}{m}\right] \to \mathbb{R}$ *be a* L^1 *function. If* $|\psi'| \in N_{h,m}^{s,2} \left[\varsigma_1, \frac{\varsigma_2}{m}\right]$, *we have the following inequality:*

$$\begin{aligned} &\left| \mathbb{D} + \frac{1+\kappa}{\varsigma_2 - \varsigma_1} \left[J^w_{\left(\frac{\varsigma_1 + \kappa\varsigma_2}{1+\kappa}\right)^+} \psi(\varsigma_2) + J^w_{\left(\frac{\kappa\varsigma_1 + \varsigma_2}{1+\kappa}\right)^-} \psi(\varsigma_1) \right] \right| \\ &\leq \frac{\varsigma_2 - \varsigma_1}{1+\kappa} \left[\left|\psi'(\varsigma_1)\right| \mathbb{B} + m \left| \psi' \left(\frac{\varsigma_2}{m} \right) \right| \mathbb{C} \right], \end{aligned} \tag{7}$$

where

$$\mathbb{D} = [\psi(\varsigma_1) + \psi(\varsigma_2)]\, w(0) - \left[\psi \left(\frac{\kappa\varsigma_1 + \varsigma_2}{1+\kappa} \right) + \psi \left(\frac{\varsigma_1 + \kappa\varsigma_2}{1+\kappa} \right) \right] w(1),$$

$$\mathbb{B} = \int_0^1 |w(\zeta)| \left[h^s \left(\frac{\zeta}{1+\kappa} \right) + h^s \left(\frac{1+\kappa-\zeta}{1+\kappa} \right) \right] d\zeta,$$

$$\mathbb{C} = \int_0^1 |w(\zeta)| \left[\left(1 - h \left(\frac{1+\kappa-\zeta}{1+\kappa} \right) \right)^s + \left(1 - h \left(\frac{\zeta}{1+\kappa} \right) \right)^s \right] d\zeta.$$

Proof. From Lemma 1, we obtain

$$\begin{aligned} &\left| \int_0^1 w(\zeta) \psi' \left(\frac{\zeta}{1+\kappa} \varsigma_1 + \frac{1+\kappa-\zeta}{1+\kappa} \varsigma_2 \right) d\zeta \right. \\ &\qquad \left. - \int_0^1 w(\zeta) \psi' \left(\frac{1+\kappa-\zeta}{1+\kappa} \varsigma_1 + \frac{\zeta}{1+\kappa} \varsigma_2 \right) d\zeta \right| \\ &\leq \int_0^1 |w(\zeta)| \left| \psi' \left(\frac{\zeta}{1+\kappa} \varsigma_1 + \frac{1+\kappa-\zeta}{1+\kappa} \varsigma_2 \right) \right| d\zeta \\ &\qquad + \int_0^1 |w(\zeta)| \left| \psi' \left(\frac{1+\kappa-\zeta}{1+\kappa} \varsigma_1 + \frac{\zeta}{1+\kappa} \varsigma_2 \right) \right| d\zeta. \end{aligned} \tag{8}$$

Taking into account modified (h,m)–convexity of $|\psi'|$, we get

$$\left|\psi'\left(\frac{\zeta}{1+\kappa}\varsigma_1+\frac{1+\kappa-\zeta}{1+\kappa}\varsigma_2\right)\right|\le\left|\psi'(\varsigma_1)\right|h^s\left(\frac{\zeta}{1+\kappa}\right)+m\left|\psi'(\frac{\varsigma_2}{m})\right|\left(1-h\left(\frac{\zeta}{1+\kappa}\right)\right)^s \tag{9}$$

and

$$\left|\psi'\left(\frac{1+\kappa-\zeta}{1+\kappa}\varsigma_1+\frac{\zeta}{1+\kappa}\varsigma_2\right)\right|\le\left|\psi'(\varsigma_1)\right|h^s\left(\frac{1+\kappa-\zeta}{1+\kappa}\right)+m\left|\psi'(\frac{\varsigma_2}{m})\right|\left(1-h\left(\frac{1+\kappa-\zeta}{1+\kappa}\right)\right)^s. \tag{10}$$

By substituting (9) and (10) in (8), we have the required result. □

Remark 5. *If we take $\kappa=0$, we get some special cases of the Theorem 2:*

- (i01) *Theorem 2.1 from [3] (case $q=1$), using $w(\zeta)=1-2\zeta$ by working only with I_1 for $s=1, h(\zeta)=\zeta$ and $m-$convex functions.*
- (i02) *Theorem 2.2 from [12], by using I_2 with $w(\zeta)=1-\ 2\zeta$.*
- (i03) *Theorem 2.4 of [15] for $h(\zeta)=\zeta$ and $m=s=1$ with I_1 and $w(\zeta)=(1-\zeta)^\alpha-\zeta^\alpha$.*
- (i04) *Theorem 2.2 of [19], for an appropriate $w(\zeta)$ function.*
- (i05) *Theorem 2.3 of [20], with I_2 and $w(\zeta)=1-\ 2\zeta$,*
- (i06) *Theorem 7 from [23] (see Remark 3(ii)).*
- (i07) *Theorem 3.1 from [26], where I_2 was used and the interval $[0,1]$ was divided by using $w_1=\lambda-\zeta$ for $[0,\frac{1}{2}]$ and $w_2=\mu-\zeta$ for $[\frac{1}{2},1]$, where λ and μ real numbers such that $0\le\lambda\le\frac{1}{2}\le\mu\le1$.*
- (i08) *Theorem 3 of [28] by taking $w(\zeta)=(1-\zeta)^\alpha-\zeta^\alpha$ and I_2.*
- (i09) *Theorem 5 of [38] by working with $w(\zeta)=(1-\zeta)^\alpha-\zeta^\alpha$ and by using only I_2.*

Corollary 1. *If we choose $w(\zeta)=1-2\zeta, h(\zeta)=\zeta, m=s=1$ and $\kappa=1$ in Theorem 2, then we get the Bullen type inequality:*

$$\left|\frac{1}{2}\left[\frac{\psi(\varsigma_1)+\psi(\varsigma_2)}{2}+\psi\left(\frac{\varsigma_1+\varsigma_2}{2}\right)\right]-\frac{1}{\varsigma_2-\varsigma_1}\int_{\varsigma_1}^{\varsigma_2}\psi(\zeta)d\zeta\right| \le\frac{\varsigma_2-\varsigma_1}{16}\left(\left|\psi'(\varsigma_1)\right|+\left|\psi'(\varsigma_2)\right|\right). \tag{11}$$

Proof. From (7), we get

$$\begin{aligned}\mathbb{D}&=[\psi(\varsigma_1)+\psi(\varsigma_2)]\,w(0)-\left[\psi\left(\frac{\kappa\varsigma_1+\varsigma_2}{1+\kappa}\right)+\psi\left(\frac{\varsigma_1+\kappa\varsigma_2}{1+\kappa}\right)\right]w(1)\\&=[\psi(\varsigma_1)+\psi(\varsigma_2)]+\ 2\psi\left(\frac{\varsigma_1+\varsigma_2}{2}\right),\end{aligned}$$

since $w'(\zeta) = -2$,

$$\left| \mathbb{D} + \frac{2}{\varsigma_2 - \varsigma_1} \left[J^w_{\left(\frac{\varsigma_1+\varsigma_2}{2}\right)^+} \psi(\varsigma_2) + J^w_{\left(\frac{\varsigma_1+\varsigma_2}{2}\right)^-} \psi(\varsigma_1) \right] \right| \\ = \left| \psi(\varsigma_1) + \psi(\varsigma_2) + \ 2\psi\left(\frac{\varsigma_1+\varsigma_2}{2}\right) - \frac{4}{\varsigma_2 - \varsigma_1} \int_{\varsigma_1}^{\varsigma_2} \psi(\zeta) d\zeta \right|,$$

$$\mathbb{B} = \int_0^1 |1-2\zeta| \left[h\left(\frac{\zeta}{2}\right) + h\left(\frac{2-\zeta}{2}\right) \right] d\zeta = \int_0^1 |1-2\zeta| \left[\frac{\zeta}{2} + \frac{2-\zeta}{2} \right] d\zeta = \frac{1}{2},$$

$$\mathbb{C} = \int_0^1 |1-2\zeta| \left[\left(1 - h\left(\frac{2-\zeta}{2}\right)\right) + \left(1 - h\left(\frac{\zeta}{2}\right)\right) \right] d\zeta \\ = \int_0^1 |1-2\zeta| \left[1 - \frac{2-\zeta}{2} + 1 - \frac{\zeta}{2} \right] d\zeta = \int_0^1 |1- \ 2\zeta| \, d\zeta = \frac{1}{2},$$

and

$$\frac{\varsigma_2 - \varsigma_1}{1+\kappa} \left\{ |\psi'(\varsigma_1)| \, \mathbb{B} + m \left| \psi'\left(\frac{\varsigma_2}{m}\right) \right| \mathbb{C} \right\} = \frac{\varsigma_2 - \varsigma_1}{2} \left[\frac{|\psi'(\varsigma_1)|}{2} + \frac{|\psi'(\varsigma_2)|}{2} \right] \\ = \frac{\varsigma_2 - \varsigma_1}{4} \left(|\psi'(\varsigma_1)| + |\psi'(\varsigma_2)| \right).$$

Thus, we have

$$\left| \psi(\varsigma_1) + \psi(\varsigma_2) + \ 2\psi\left(\frac{\varsigma_1+\varsigma_2}{2}\right) - \frac{4}{\varsigma_2 - \varsigma_1} \int_{\varsigma_1}^{\varsigma_2} \psi(\zeta) d\zeta \right| \\ \leq \frac{\varsigma_2 - \varsigma_1}{4} \left(|\psi'(\varsigma_1)| + |\psi'(\varsigma_2)| \right).$$

By dividing both parts of the last inequality by 4, we obtain (11). The proof is completed. □

Corollary 2. *If we choose $w(\zeta) = 1 - \ 2\zeta, h(\zeta) = \zeta, m = s = 1$ and $\kappa = 0$ in Theorem 2, then we get Trapezoidal inequality (2).*

Proof. From (7), we get

$$\mathbb{D} = [\psi(\varsigma_1) + \psi(\varsigma_2)] \, w(0) - \left[\psi\left(\frac{\kappa\varsigma_1 + \varsigma_2}{1+\kappa}\right) + \psi\left(\frac{\varsigma_1 + \kappa\varsigma_2}{1+\kappa}\right) \right] w(1) \\ = [\psi(\varsigma_1) + \psi(\varsigma_2)] \, w(0) - [\psi(\varsigma_2) + \psi(\varsigma_1)] \\ = 2 \left[\psi(\varsigma_1) + \psi(\varsigma_2)\right] w(1),$$

since $w'(\zeta) = -2$,

$$\left| \mathbb{D} + \frac{1}{\varsigma_2 - \varsigma_1} \left(J^w_{\varsigma_1^+} \psi(\varsigma_2) + J^w_{\varsigma_2^-} \psi(\varsigma_1) \right) \right| \\ = \left| 2 \left[\psi(\varsigma_1) + \psi(\varsigma_2)\right] - \frac{4}{\varsigma_2 - \varsigma_1} \int_{\varsigma_1}^{\varsigma_2} \psi(\zeta) d\zeta \right|,$$

$$\mathbb{B}=\int_0^1|1-2\zeta|\left[h(\frac{\zeta}{2})+h(\frac{2-\zeta}{2})\right]d\zeta=\int_0^1|1-2\zeta|\left[\frac{\zeta}{2}+\frac{2-\zeta}{2}\right]d\zeta=\frac{1}{2},$$

$$\mathbb{C}=\int_0^1|1-2\zeta|\left[\left(1-h\left(\frac{2-\zeta}{2}\right)\right)+\left(1-h\left(\frac{\zeta}{2}\right)\right)\right]d\zeta$$
$$=\int_0^1|1-2\zeta|\left[1-\frac{2-\zeta}{2}+1-\frac{\zeta}{2}\right]d\zeta=\int_0^1|1-2\zeta|\,d\zeta=\frac{1}{2},$$

and

$$\frac{\varsigma_2-\varsigma_1}{1+\kappa}\left\{\left|\psi'(\varsigma_1)\right|\mathbb{B}+m\left|\psi'\left(\frac{\varsigma_2}{m}\right)\right|\mathbb{C}\right\}=(\varsigma_2-\varsigma_1)\left[\frac{|\psi'(\varsigma_1)|}{2}+\frac{|\psi'(\varsigma_2)|}{2}\right]$$
$$=\frac{\varsigma_2-\varsigma_1}{2}\left(\left|\psi'(\varsigma_1)\right|+\left|\psi'(\varsigma_2)\right|\right).$$

Thus, we have

$$\left|2\left[\psi(\varsigma_1)+\psi(\varsigma_2)\right]-\frac{4}{\varsigma_2-\varsigma_1}\int_{\varsigma_1}^{\varsigma_2}\psi(\zeta)d\zeta\right|\leq\frac{\varsigma_2-\varsigma_1}{2}\left(\left|\psi'(\varsigma_1)\right|+\left|\psi'(\varsigma_2)\right|\right).$$

By dividing both parts of the last inequality by 4, we obtain (2). The proof is completed. □

By imposing additional assumptions on $|\psi'|$, we can obtain new refinements of known results.

Theorem 3. *If $\psi:\left[\varsigma_1,\frac{\varsigma_2}{m}\right]\to\mathbb{R}$ is a L^1 function and if $|\psi'|^q\in N_{h,m}^{s,2}\left[\varsigma_1,\frac{\varsigma_2}{m}\right]$, then for $q\geq1$ the following inequality*

$$\left|\mathbb{D}+\frac{1+\kappa}{\varsigma_2-\varsigma_1}\left[J^w_{\left(\frac{\varsigma_1+\kappa\varsigma_2}{1+\kappa}\right)^+}\psi(\varsigma_2)+J^w_{\left(\frac{\kappa\varsigma_1+\varsigma_2}{1+\kappa}\right)^-}\psi(\varsigma_1)\right]\right| \tag{12}$$
$$\leq\frac{\varsigma_2-\varsigma_1}{1+\kappa}B_q\left\{\left(\left|\psi'(\varsigma_1)\right|^qC_{11}+m\left|\psi'\left(\frac{\varsigma_2}{m}\right)\right|^qC_{12}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left|\psi'(\varsigma_1)\right|^qC_{21}+m\left|\psi'\left(\frac{\varsigma_2}{m}\right)\right|^qC_{22}\right)^{\frac{1}{q}}\right\}$$

holds, with $\mathbb{D}$ as before,

$$B_q=\left(\int_0^1|w(\zeta)|^p\,d\zeta\right)^{\frac{1}{p}}=\left(\int_0^1|w(\zeta)|^{\frac{q}{q-1}}\,d\zeta\right)^{1-\frac{1}{q}},$$

$$C_{11}=\int_0^1h^s\left(\frac{\zeta}{1+\kappa}\right)d\zeta,\quad C_{12}=\int_0^1\left(1-h\left(\frac{\zeta}{1+\kappa}\right)\right)^sd\zeta,$$

$$C_{21}=\int_0^1h^s\left(\frac{1+\kappa-\zeta}{1+\kappa}\right)d\zeta\quad\text{and}\quad C_{22}=\int_0^1\left(1-h\left(\frac{1+\kappa-\zeta}{1+\kappa}\right)\right)^sd\zeta.$$

Proof. By proceeding as in the previous Theorem, from Lemma 1, we have

$$\begin{aligned}\bigg|\int_0^1 w(\zeta)\psi'\left(\frac{\zeta}{1+\kappa}\varsigma_1+\frac{1+\kappa-\zeta}{1+\kappa}\varsigma_2\right)d\zeta \\ -\int_0^1 w(\zeta)\psi'\left(\frac{1+\kappa-\zeta}{1+\kappa}\varsigma_1+\frac{\zeta}{1+\kappa}\varsigma_2\right)d\zeta\bigg| \\ \leq \int_0^1 |w(\zeta)|\left|\psi'\left(\frac{\zeta}{1+\kappa}\varsigma_1+\frac{1+\kappa-\zeta}{1+\kappa}\varsigma_2\right)\right|d\zeta \\ +\int_0^1 |w(\zeta)|\left|\psi'\left(\frac{1+\kappa-\zeta}{1+\kappa}\varsigma_1+\frac{\zeta}{1+\kappa}\varsigma_2\right)\right|d\zeta.\end{aligned} \tag{13}$$

By using the Hölder's inequality in each of the integrals of (13) leads us to

$$\begin{aligned}\int_0^1 |w(\zeta)|\left|\psi'\left(\frac{\zeta}{1+\kappa}\varsigma_1+\frac{1+\kappa-\zeta}{1+\kappa}\varsigma_2\right)\right|d\zeta \\ \leq \left(\int_0^1 |w(\zeta)|^p\,d\zeta\right)^{\frac{1}{p}}\left(\int_0^1\left|\psi'\left(\frac{\zeta}{1+\kappa}\varsigma_1+\frac{1+\kappa-\zeta}{1+\kappa}\varsigma_2\right)\right|^q d\zeta\right)^{\frac{1}{q}}\end{aligned} \tag{14}$$

and

$$\begin{aligned}\int_0^1 |w(\zeta)|\left|\psi'\left(\frac{1+\kappa-\zeta}{1+\kappa}\varsigma_1+\frac{\zeta}{1+\kappa}\varsigma_2\right)\right|d\zeta \\ \leq \left(\int_0^1 |w(\zeta)|^p\,d\zeta\right)^{\frac{1}{p}}\left(\int_0^1\left|\psi'\left(\frac{1+\kappa-\zeta}{1+\kappa}\varsigma_1+\frac{\zeta}{1+\kappa}\varsigma_2\right)\right|^q d\zeta\right)^{\frac{1}{q}}\end{aligned} \tag{15}$$

with $\frac{1}{p}+\frac{1}{q}=1$. By considering the (h,m)-convexity of the second type of $|\psi'|^q$, we obtain

$$\begin{aligned}\int_0^1\left|\psi'\left(\frac{\zeta}{1+\kappa}\varsigma_1+\frac{1+\kappa-\zeta}{1+\kappa}\varsigma_2\right)\right|^q d\zeta \leq \left|\psi'(\varsigma_1)\right|^q\int_0^1 h^s\left(\frac{\zeta}{1+\kappa}\right)d\zeta \\ +m\left|\psi'\left(\frac{\varsigma_2}{m}\right)\right|^q\int_0^1\left(1-h\left(\frac{\zeta}{1+\kappa}\right)\right)^s d\zeta\end{aligned} \tag{16}$$

and

$$\begin{aligned}\int_0^1\left|\psi'\left(\frac{1+\kappa-\zeta}{1+\kappa}\varsigma_1+\frac{\zeta}{1+\kappa}\varsigma_2\right)\right|^q d\zeta \leq \left|\psi'(\varsigma_1)\right|^q\int_0^1 h^s\left(\frac{1+\kappa-\zeta}{1+\kappa}\right)d\zeta \\ +m\left|\psi'\left(\frac{\varsigma_2}{m}\right)\right|^q\int_0^1\left(1-h\left(\frac{1+\kappa-\zeta}{1+\kappa}\right)\right)^s d\zeta.\end{aligned} \tag{17}$$

By denoting and required and substituting (16), (17) in (14) and (15), we obtain the desired result (12). □

Remark 6. *If we take $\kappa=0$, we get some special cases of the Theorem 3:*

(i.01) *Theorem 11 of [34] by using I_1, with $w(\zeta)=(1-\zeta)^\alpha-\zeta^\alpha$.*
(i.02) *Theorem 3.2 of [14] by using I_1, with $w(\zeta)=(1-\zeta)^\alpha-\zeta^\alpha$.*
(i.03) *Theorem 6 of [38] by using I_1 with $w(\zeta)=(1-\zeta)^\alpha-\zeta^\alpha$.*

(i.04) *Theorem 2.1 (second part) and Theorem 2.2 of [3] by using I_1 with $w(\zeta)=1-2\zeta$ and $m-$convex functions.*
(i.05) *Theorem 2.3 [12] by using I_1 with $w(\zeta)=1-2\zeta$.*
(i.06) *Theorem 8 of [23], for $s-$convex (see Remarks 3(ii)).*
(i.07) *Theorem 1 of [25] by using I_1 with $w(\zeta)=1-2\zeta$.*
(i.08) *Theorem 2.1 of [32] (see Remarks 5(i09)).*
(i.09) *Theorem 2 of [2] with $w(\zeta)=p(\zeta)$.*

Remark 7. *For $\kappa=1$ we have a new result for $(h,m)-$convex functions:*

Corollary 3. *If $\phi:\left[\varsigma_1,\frac{\varsigma_2}{m}\right]\to\mathbb{R}$ is a L^1 function and if $|\psi'|^q\in N^{s,2}_{h,m}\left[\varsigma_1,\frac{\varsigma_2}{m}\right]$, then for $q\ge 1$ the following inequality*

$$\begin{aligned}&\left|\mathbb{D}+\frac{2}{\varsigma_2-\varsigma_1}\left(J^w_{\varsigma_1+}\phi(\frac{\varsigma_1+\varsigma_2}{2})+J^w_{\varsigma_2-}\phi(\frac{\varsigma_1+\varsigma_2}{2})\right)\right| \\ &\le\frac{\varsigma_2-\varsigma_1}{2}\cdot B_q\cdot\left\{\left(\left|\phi'\left(\varsigma_1\right)\right|^qC_{11}+m\left|\phi'\left(\frac{\varsigma_2}{m}\right)\right|^qC_{12}\right)^{\frac{1}{q}}\right. \\ &\qquad\left.+\left(\left|\phi'\left(\varsigma_1\right)\right|^qC_{21}+m\left|\phi'\left(\frac{\varsigma_2}{m}\right)\right|^qC_{22}\right)^{\frac{1}{q}}\right\}\end{aligned}\tag{18}$$

holds, with $\mathbb{D}$ as before, $B_q=\left(\int_0^1|\ w(t)|^p\ dt\right)^{\frac{1}{p}}=\left(\int_0^1|w(t)|^{\frac{q}{q-1}}\ dt\right)^{1-\frac{1}{q}}$,

$$\begin{aligned}C_{11}&=\int_0^1h^s\left(\frac{\zeta}{2}\right)d\zeta,\qquad C_{12}=\int_0^1\left(1-h\left(\frac{\zeta}{2}\right)\right)^sd\zeta,\\ C_{21}&=\int_0^1h^s\left(\frac{2-\zeta}{2}\right)d\zeta,C_{22}=\int_0^1\left(1-h\left(\frac{2-\zeta}{2}\right)\right)^sd\zeta.\end{aligned}$$

Remark 8. *With $\kappa=1$ and the change of variables $t\to(t+1)$, we obtain Theorem 6 from [33] for convex functions (i.e. $m=s=1$ and $h(t)=t$), with $w_1(t)=\frac{tb+(1-t)a}{4}$ in I_1 and $w_2(t)=-\frac{ta+(1-t)b}{4}$ for I_2.*

Remark 9. *Under the above assumptions also we have the Theorem 11 of [34].*

Theorem 4. *If $\psi:\left[\varsigma_1,\frac{\varsigma_2}{m}\right]\to\mathbb{R}$ is a L^1 function and if $|\psi'|^q\in N^{s,2}_{h,m}\left[\varsigma_1,\frac{\varsigma_2}{m}\right]$, then for $q>1$ we have*

$$\begin{aligned}&\left|\mathbb{D}+\frac{1+\kappa}{\varsigma_2-\varsigma_1}\left(J^w_{\left(\frac{\varsigma_1+\kappa\varsigma_2}{1+\kappa}\right)^+}\psi(\varsigma_2)+J^w_{\left(\frac{\kappa\varsigma_1+\varsigma_2}{1+\kappa}\right)^-}\psi(\varsigma_1)\right)\right| \\ &\le\frac{\varsigma_2-\varsigma_1}{1+\kappa}D\left\{\left(\left|\psi'\left(\varsigma_1\right)\right|^qD_{11}+m\left|\psi'\left(\frac{\varsigma_2}{m}\right)\right|^qD_{12}\right)^{\frac{1}{q}}\right. \\ &\qquad\left.+\left(\left|\psi'\left(\varsigma_1\right)\right|^qD_{21}+m\left|\psi'\left(\frac{\varsigma_2}{m}\right)\right|^qD_{22}\right)^{\frac{1}{q}}\right\}\end{aligned}\tag{19}$$

with $\mathbb{D}$ *as before,* $D=\left(\int_0^1 |w(\zeta)|\,d\zeta\right)^{1-\frac{1}{q}}$,

$$D_{11}=\int_0^1 |w(\zeta)|\,h^s\left(\frac{\zeta}{1+\kappa}\right)d\zeta,\qquad D_{12}=\int_0^1 |w(\zeta)|\left(1-h\left(\frac{\zeta}{1+\kappa}\right)\right)^s d\zeta,$$

$$D_{21}=\int_0^1 |w(\zeta)|\,h^s\left(\frac{1+\kappa-\zeta}{1+\kappa}\right)d\zeta,\quad D_{22}=\int_0^1 |w(\zeta)|\left(1-h\left(\frac{1+\kappa-\zeta}{1+\kappa}\right)\right)^s d\zeta.$$

Proof. As before, taking into account Lemma 1, we have

$$\begin{aligned}&\left|\int_0^1 |w(\zeta)|\,\psi'\left(\frac{\zeta}{1+\kappa}\varsigma_1+\frac{1+\kappa-\zeta}{1+\kappa}\varsigma_2\right)d\zeta\right.\\&\qquad\left.-\int_0^1 |w(\zeta)|\,\psi'\left(\frac{1+\kappa-\zeta}{1+\kappa}\varsigma_1+\frac{\zeta}{1+\kappa}\varsigma_2\right)d\zeta\right|\\&\le\int_0^1 |w(\zeta)|\left|\psi'\left(\frac{\zeta}{1+\kappa}\varsigma_1+\frac{1+\kappa-\zeta}{1+\kappa}\varsigma_2\right)\right|d\zeta\\&\qquad+\int_0^1 |w(\zeta)|\left|\psi'\left(\frac{1+\kappa-\zeta}{1+\kappa}\varsigma_1+\frac{\zeta}{1+\kappa}\varsigma_2\right)\right|d\zeta\end{aligned}\tag{20}$$

and by using well-known power mean inequality, we have

$$\begin{aligned}&\int_0^1 |w(\zeta)|\left|\psi'\left(\frac{\zeta}{1+\kappa}\varsigma_1+\frac{1+\kappa-\zeta}{1+\kappa}\varsigma_2\right)\right|d\zeta\\&\le\left(\int_0^1 |w(\zeta)|\,d\zeta\right)^{1-\frac{1}{q}}\left(\int_0^1 |w(\zeta)|\left|\psi'\left(\frac{\zeta}{1+\kappa}\varsigma_1+\frac{1+\kappa-\zeta}{1+\kappa}\varsigma_2\right)\right|^q d\zeta\right)^{\frac{1}{q}}\end{aligned}\tag{21}$$

and

$$\begin{aligned}&\int_0^1 |w(\zeta)|\left|\psi'\left(\frac{1+\kappa-\zeta}{1+\kappa}\varsigma_1+\frac{\zeta}{1+\kappa}\varsigma_2\right)\right|d\zeta\\&\le\left(\int_0^1 |w(\zeta)|\,d\zeta\right)^{1-\frac{1}{q}}\left(\int_0^1 |w(\zeta)|\left|\psi'\left(\frac{1+\kappa-\zeta}{1+\kappa}\varsigma_1+\frac{\zeta}{1+\kappa}\varsigma_2\right)\right|^q d\zeta\right)^{\frac{1}{q}}.\end{aligned}\tag{22}$$

By using the modified (h,m)-convexity of $|\psi'|^q$, we get

$$\begin{aligned}&\int_0^1 |w(\zeta)|\left|\psi'\left(\frac{\zeta}{1+\kappa}\varsigma_1+\frac{1+\kappa-\zeta}{1+\kappa}\varsigma_2\right)\right|^q d\zeta\\&\le\int_0^1 |w(\zeta)|\left[h^s\left(\frac{\zeta}{1+\kappa}\right)|\psi'(\varsigma_1)|^q+m\left(1-h\left(\frac{\zeta}{1+\kappa}\right)\right)^s\left|\psi'(\frac{\varsigma_2}{m})\right|^q\right]d\zeta\\&=|\psi'(\varsigma_1)|^q\int_0^1 w(\zeta)h^s\left(\frac{\zeta}{1+\kappa}\right)d\zeta\\&\qquad+m\left|\psi'(\frac{\varsigma_2}{m})\right|^q\int_0^1 |w(\zeta)|\left(1-h\left(\frac{\zeta}{1+\kappa}\right)\right)^s d\zeta.\end{aligned}\tag{23}$$

Similarly,

$$\int_0^1 |w(\zeta)| \left|\psi'\left(\frac{1+\kappa-\zeta}{1+\kappa}\varsigma_1 + \frac{\zeta}{1+\kappa}\varsigma_2\right)\right|^q d\zeta \tag{24}$$
$$\leq \left|\psi'(\varsigma_1)\right|^q \int_0^1 |w(\zeta)|\, h^s\left(\frac{1+\kappa-\zeta}{1+\kappa}\right) d\zeta$$
$$+ m\left|\psi'(\frac{\varsigma_2}{m})\right|^q \int_0^1 |w(\zeta)| \left(1 - h\left(\frac{1+\kappa-\zeta}{1+\kappa}\right)\right)^s d\zeta.$$

If we put (23) and (24) in (21) and in (22), it allows us to obtain the inequality (19). The proof is completed. □

Remark 10. *Results known in the literature, which can be obtained from Theorem 4, by taking $\kappa = 0$, we can get the following special cases:*

- (i.01) *For the s-convex functions we can obtain the Theorem 12 of [34] by using I_1 with $w(\zeta) = (1-\zeta)^\alpha - \zeta^\alpha$.*
- (i.02) *Theorem 3.2 of [26] (see Remarks 5(i09)).*
- (i.03) *Theorem 5 of [2], by using I_1 for $w(t) = p(t)$.*
- (i.04) *Theorem 9 of [23] (see Remarks 3(ii)) and $|\psi|^q$ is $s-$convex.*
- (i.05) *Theorem 2.3 of [3] by using I_1 with $w(\zeta) = 1 - 2\zeta$.*
- (i.06) *Theorem 7 of [38] by using I_1 with $w(\zeta) = (1-\zeta)^\alpha - \zeta^\alpha$ and $|\psi|^q$ is $s-$convex.*
- (i.10) *Theorem 3.6 of [14] by using I_1 with $w(\zeta) = (1-\zeta)^{\frac{\alpha}{k}} - \zeta^{\frac{\alpha}{k}}$ and $|\psi|^q$ is $(h-m)-$convex.*

Remark 11. *With $\kappa = 1$ and using the new variables $\zeta \to t+1$, we obtain Theorem 7 from [33] for convex functions, $m = s = 1$ and $h(t) = t$, with $w(t) = \frac{(tb+(1-t)a}{4}$ for I_1 and $w(t) = -\frac{(ta+(1-t)b}{4}$ for I_2.*

Remark 12. *Under the assumptions of above remark, also we have the Theorem 12 of [34].*

Theorem 5. *Let $\psi : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ is a L^1 function. If $|\psi'|$ on the interval $[\varsigma_1, \varsigma_2]$ satisfies the condition of the Lagrange Theorem, then the following inequality holds:*

$$\left|\mathbb{D} + \frac{1+\kappa}{\varsigma_2 - \varsigma_1}\left(J^w_{\left(\frac{\varsigma_1+\kappa\varsigma_2}{1+\kappa}\right)^+}\psi(\varsigma_2) + J^w_{\left(\frac{\kappa\varsigma_1+\varsigma_2}{1+\kappa}\right)^-}\psi(\varsigma_1)\right)\right| \tag{25}$$
$$\leq \frac{(\varsigma_2-\varsigma_1)\,|\psi''(\zeta)|}{1+\kappa} \int_0^1 |w(\zeta)| \left|1 - \frac{2\zeta}{1+\kappa}\right| d\zeta,$$

where $\mathbb{D}$ was defined earlier,

$$\zeta \in \begin{cases} (c,d), & \text{for } \zeta \leq \frac{1+\kappa}{2}; \\ (d,c), & \text{for } \zeta > \frac{1+\kappa}{2}, \end{cases} c = \frac{\zeta}{1+\kappa}\varsigma_1 + \frac{1+\kappa-\zeta}{1+\kappa}\varsigma_2, \quad d = \frac{1+\kappa-\zeta}{1+\kappa}\varsigma_1 + \frac{\zeta}{1+\kappa}\varsigma_2.$$

Proof. From Lemma 1 and modulus properties, we can write

$$\left|\mathbb{D}+\frac{1+\kappa}{\varsigma_2-\varsigma_1}\left(J^w_{\left(\frac{\varsigma_1+\kappa\varsigma_2}{1+\kappa}\right)^+}\psi(\varsigma_2)+J^w_{\left(\frac{\kappa\varsigma_1+\varsigma_2}{1+\kappa}\right)^-}\psi(\varsigma_1)\right)\right| \tag{26}$$
$$\leq \int_0^1 |w(\zeta)|\left|\psi'\left(\frac{\zeta}{1+\kappa}\varsigma_1+\frac{1+\kappa-\zeta}{1+\kappa}\varsigma_2\right)-\psi'\left(\frac{1+\kappa-\zeta}{1+\kappa}\varsigma_1+\frac{\zeta}{1+\kappa}\varsigma_2\right)\right|d\zeta.$$

By using the Lagrange Theorem condition, for the right side of the inequality (26), we get

$$\left|\psi'\left(\frac{\zeta}{1+\kappa}\varsigma_1+\frac{1+\kappa-\zeta}{1+\kappa}\varsigma_2\right)-\psi'\left(\frac{1+\kappa-\zeta}{1+\kappa}\varsigma_1+\frac{\zeta}{1+\kappa}\varsigma_2\right)\right| \tag{27}$$
$$\leq \left|\psi''(\zeta)\right|\left|\frac{\zeta}{1+\kappa}\varsigma_1+\frac{1+\kappa-\zeta}{1+\kappa}\varsigma_2-\frac{1+\kappa-\zeta}{1+\kappa}\varsigma_1-\frac{\zeta}{1+\kappa}\varsigma_2\right|$$
$$=\left|\psi''(\zeta)\right|\left|\left(1-\frac{2\zeta}{1+\kappa}\right)(\varsigma_2-\varsigma_1)\right|.$$

By taking into account (26) and (27), we have (25). □

Corollary 4. *If we choose* $w(\zeta)=1-\ 2\zeta$ *and* $\kappa=0$ *in Theorem 5, then we get*

$$\left|\frac{\psi(\varsigma_2)+\psi(\varsigma_1)}{2}-\frac{1}{\varsigma_2-\varsigma_1}\int_{\varsigma_1}^{\varsigma_2}\psi(\zeta)d\zeta\right|\leq\frac{\varsigma_2-\varsigma_1}{12}\left|\left|\psi''\right|\right|_\infty, \tag{28}$$

where $\left|\left|\psi''\right|\right|_\infty=\max\limits_{\zeta\in[\varsigma_1,\varsigma_2]}\left(\left|\psi''(\zeta)\right|\right)$.

Proof. From (25), we get

$$\mathbb{D}=w(0)\left(\psi(\varsigma_1)+\psi(\varsigma_2)\right)-w(1)\left(\psi(\frac{\kappa\varsigma_1+\varsigma_2}{1+\kappa})+\psi(\frac{\varsigma_1+\kappa\varsigma_2}{1+\kappa})\right)$$
$$=\left(w(0)-w(1)\right)\left(\psi(\varsigma_2)+\psi(\varsigma_1)\right)=2\left(\psi(\varsigma_2)+\psi(\varsigma_1)\right),$$

since $w'(\zeta)=-2$,

$$\left|\mathbb{D}+\frac{2}{\varsigma_2-\varsigma_1}\left(J^w_{\varsigma_1^+}\psi(\varsigma_2)+J^w_{\varsigma_2^-}\psi(\varsigma_1)\right)\right|$$
$$=\left|2\left[\psi(\varsigma_2)+\psi(\varsigma_1)\right]-\frac{4}{\varsigma_2-\varsigma_1}\int_{\varsigma_1}^{\varsigma_2}\psi(\zeta)d\zeta\right|$$

and

$$\int_0^1|w(\zeta)|\left|1-\frac{2\zeta}{1+\kappa}\right|d\zeta=\int_0^1|1-2\zeta|\,|1-2\zeta|\,d\zeta=\int_0^1(1-2\zeta)^2\,d\zeta=\frac{1}{3}.$$

Thus, we have

$$\left|2\left[\psi(\varsigma_2)+\psi(\varsigma_1)\right]-\frac{4}{\varsigma_2-\varsigma_1}\int_{\varsigma_1}^{\varsigma_2}\psi(\zeta)d\zeta\right|\leq\frac{(\varsigma_2-\varsigma_1)\left|\left|\psi''\right|\right|_\infty}{3}.$$

By dividing both parts of the last inequality by 4, we obtain (28). □

Remark 13. *For $\varsigma_2 - \varsigma_1 > 1$, it is easy to verify that the upper estimate (28) was better than the estimate obtained in [27, Proposition 2] and confirm in a number of subsequent studies, for example, see [4, Corollary 3.1] and [5, Corollary 2], for more details.*

3 Some application to special means

Let us present some applications in which the obtained results can be used. Known mean values of the real numbers α and β; see Pearce and J. Pečarić [25]:

(i) Arithmetic mean: $A(\alpha,\beta) = \frac{\alpha+\beta}{2}$;
(ii) Quadratic mean: $Q(\alpha,\beta) = \sqrt{\alpha^2+\beta^2}$;
(iii) Geometric mean: $G(\alpha,\beta) = \sqrt{\alpha\beta}$, where $\alpha\beta \geq 0$;
(iv) Harmonic mean: $H(\alpha,\beta) = \frac{2\alpha\beta}{\alpha+\beta}$, where $\alpha+\beta \neq 0$;
(v) Logarithmic mean: $L(\alpha,\beta) = \frac{\beta-\alpha}{\ln\beta - \ln\alpha}$, where $\alpha,\beta > 0$ and $\alpha \neq \beta$.

Now, by using some results, we give special means of positive real numbers to some applications.

Example 1. *In Corollary 1, if we take $\psi(x) = \sqrt{1+x^2}$ defined on the interval $(0,\varsigma_2)$ as a function, then we get*

$$\left|A\left\{A\left[1,Q\left(\varsigma_2,1\right)\right],Q\left(\frac{\varsigma_2}{2},1\right)\right\} - A\left[Q\left(\varsigma_2,1\right),\ln\left(2A\left(\varsigma_2,Q\left(\varsigma_2,1\right)\right)\right)^{\frac{1}{\varsigma_2}}\right]\right| \leq \frac{\varsigma_2^2}{16}Q^{-1}\left(\varsigma_2,1\right)$$

and from (28), we have

$$\left|A(1,Q\left(\varsigma_2,1\right)) - A\left[Q\left(\varsigma_2,1\right),\ln\left(2A\left(\varsigma_2,Q\left(\varsigma_2,1\right)\right)\right)^{\frac{1}{\varsigma_2}}\right]\right| \leq \frac{\varsigma_2}{12}.$$

Example 2. *In Corollary 1, if we take $\psi(x) = \frac{1}{x^2}$ defined on the interval $(\varsigma_1,\varsigma_2)$, where $\varsigma_1,\varsigma_2 \in \mathbb{R}^+$ as a function, then we get*

$$\left|A\left[H^{-1}\left(\varsigma_1^2,\varsigma_2^2\right),A^{-2}\left(\varsigma_1,\varsigma_2\right)\right] - G^{-2}\left(\varsigma_1^2,\varsigma_2^2\right)\right| \leq \frac{\varsigma_2-\varsigma_1}{3}H^{-1}\left(\varsigma_1^{\frac{3}{2}},\varsigma_2^{\frac{3}{2}}\right)$$

and from (28), we have

$$\left|H^{-1}\left(\varsigma_1^2,\varsigma_2^2\right) - G^{-2}\left(\varsigma_1^2,\varsigma_2^2\right)\right| \leq \frac{\varsigma_2-\varsigma_1}{2\varsigma_1^4}.$$

Example 3. *In Corollary 1, if we take $\psi(x) = \frac{1}{x}$ defined on the interval $(\varsigma_1,\varsigma_2)$, where $\varsigma_1,\varsigma_2 \in \mathbb{R}^+$ as a function, then we get*

$$\left|A\left[H^{-1}\left(\varsigma_1,\varsigma_2\right),A^{-1}\left(\varsigma_1,\varsigma_2\right)\right] - L^{-1}\left(\varsigma_1,\varsigma_2\right)\right| \leq \frac{\varsigma_2-\varsigma_1}{16}H^{-1}\left(\varsigma_1^2,\varsigma_2^2\right)$$

and from (28), we have

$$\left|H^{-1}\left(\varsigma_1,\varsigma_2\right) - L^{-1}\left(\varsigma_1,\varsigma_2\right)\right| \leq \frac{\varsigma_2-\varsigma_1}{6\varsigma_1^3}.$$

4 Conclusions

In this article, we considered new integral inequalities of the Hadamard type for the weighted integrals. Throughout the article, we have shown that many known results are our particular cases. The generality of the results lies in the fact that they are valid for various classes of convex functions defined in some closed subset of a non-negative numerical semi–axis, for example, $s-$convex in the second sense, $h-$convex, $m-$convex, and for a number of other composite classes of convex functions. These results can easily be extended to the case of $(h,m)-$convex functions of the first type.

References

[1] M. Alomari, M. Darus, U.S. Kırmacı, *Refinements of Hadamard-type inequalities for quasi–convex functions with applications to trapezoidal formula and to special means*, Comput. Math. Appl., **59**:1 (2010), 225–232. Zbl 1189.26037

[2] M. Alomari, M. Darus, *Some Ostrowski type inequalities for convex functions with applications*, RGMIA Res. Rep. Coll., **13**:2 (2010), Article 3.

[3] M.K. Bakula, M.E. Ödemir, J. Pečarić, *Hadamard type inequalities for for m-convex and (α,m)-convex functions*, JIPAM, J. Inequal. Pure Appl. Math., **9**:4 (2008), Paper No. 96. Zbl 1163.26327

[4] B. Bayraktar, *Some integral inequalities of Hermite-Hadamard type for differentiable $(s,m)-$convex functions via fractional integrals*, TWMS J. App. Eng. Math., **10**:3 (2020), 625–637.

[5] B. Bayraktar, V. Kudaev, *Some new integral inequalities for (s,m)-convex and (α,m)-convex functions*, Bull. Karaganda Univ., Mathematics, **94**:2 (2019), 15–25.

[6] B. Bayraktar, J.E. Nápoles, *New generalized integral inequalities via (h,m)-convex modified functions*, Izv. Inst. Mat. Inform., Udmurt. Gos. Univ., **60** (2022), 3–15. Zbl 1524.26048

[7] B. Bayraktar, J.E. Nápoles, *Hermite–Hadamard weighted integral inequalities for (h,m)-convex modified functions*, Fract. Differ. Calc., **12**:2 (2022), 235–248. Zbl 1524.26049

[8] S. Bermudo, P. Kórus, J.E. Nápoles, *On $q-$Hermite-Hadamard inequalities for general convex functions*, Acta Math. Hungar., **162** (2020), 364–374.

[9] S.I. Butt, J. Pečarić, *Generalized Hermite-Hadamard's inequality*, Proc. A. Razmadze Math. Inst, **163** (2013), 9–27. Zbl 1298.26068

[10] S.I. Butt, S. Yousaf, A.O. Akdemir, M.A. Dokuyucu, *New Hadamard-type integral inequalities via a general form of fractional integral operators*, Chaos, Solitons & Fractals, **148** (2021), Article ID 111025. Zbl 1485.26023

[11] S.I. Butt, S. Yousaf, A. Asghar, K.A. Khan, H.R. Moradi, *New fractional Hermite-Hadamard-Mercer inequalities for harmonically convex function*, J. Funct. Spaces, **2021** (2021), Article ID 5868326. Zbl 1484.26055

[12] S.S. Dragomir, R.P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and trapezoidal formula*, Appl. Math. Lett., **11**:5 (1998), 91–95. Zbl 0938.26012

[13] S.S. Dragomir, C.E.M. Pearce, *Selected topics on Hermite-Hadamard inequalities and applications*, RGMIA Monographs, Victoria University, 2002.

[14] G. Farid, A.U. Rehman, Q.U. Ain, *k-fractional integral inequalities of Hadamard type for $(h-m)$-convex functions*, Comput. Methods Differ. Equ., **8**:1 (2020), 119–140. Zbl 1449.26028

[15] G. Farid, A. Rehman, M. Zahra, *On Hadamard inequalities for k-fractional integrals*, Nonlinear Funct. Anal. Appl., **21**:3 (2016), 463–478. Zbl 1358.26014

[16] P.M. Guzmán, J.E. Nápoles Valdés, Y. Gasimov, *Integral inequalities within the framework of generalized fractional integrals*, Fract. Differ. Calc., **11**:1 (2021), 69–84. Zbl 1499.26117

[17] H. Hudzik, L. Maligranda, *Some remarks on s-convex functions*, Aequationes Math., **48**:1 (1994), 100–111. Zbl 0823.26004

[18] E. Kikianty, *Hermite-Hadamard inequality in the geometry of Banach spaces.* Doctor Philosophy Thesis, School of Engineering and Science Faculty of Health, Engineering and Science Victoria University, 2010.

[19] U.S. Kırmacı, *Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula*, Appl. Math. Comput., **147**:1 (2004), 137–146. Zbl 1034.26019

[20] M. Muddassar, M.I. Bhatti, W. Irshad, *Generalisation of integral inequalities of Hermite-Hadamard type through convexity*, Bull. Aust. Math. Soc., **88**:2 (2013), 320–330. Zbl 1281.26016

[21] J.E. Nápoles, B. Bayraktar, *On the generalized inequalities of the Hermite-Hadamard type*, FILOMAT, **35**:14 (2021), 4917–4924.

[22] J.E. Nápoles Valdés, F. Rabossi, A.D. Samaniego, *Convex functions: Ariadne's thread or Charlotte's spiderweb?*, Adv. Math. Models Appl., **5**:2 (2020), 176–191.

[23] M.E. Özdemir, M. Avcı, H. Kavurmaci, *Hermite-Hadamard type inequalities for $s-$convex and $s-$ concave functions via fractional integrals*, Preprint, arXiv:1202.0380, 2012.

[24] M.E. Ödemir, S.S. Dragomir, Ç. Yıldız, *The Hadamard's inequality for convex function via fractional integrals*, Acta Math. Sci., Ser. B, Engl. Ed., **33**:5 (2013), 1293–1299. Zbl 1299.26013

[25] C.E.M. Pearce, J. Pečarić, *Inequalities for differentiable mappings with application to special means and quadrature formulae*, Appl. Math. Lett., **13**:2 (2000), 51–55. Zbl 0970.26016

[26] F. Qi, T.Y. Zhang, B.Y. Xi, *Hermite-Hadamard type integral inequalities for functions whose first derivatives are of convexity*, Ukr. Math. J., **67**:4 (2015), 555–567,

[27] M.Z. Sarıkaya, N. Aktan, *On the generalization of some integral inequalities and their applications.* Math. Comput. Modelling, **54**:9-10 (2011), 2175–2182. Zbl 1235.26015

[28] M.Z. Sarıkaya, E. Set, H. Yaldız, N. Başak, *Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities*, Math. Comput. Modelling, **57**:9-10 (2013), 2403–2407. Zbl 1286.26018

[29] E. Set, S.I. Butt, A.O. Akdemir, A. Karaoglan, T. Abdeljawad, *New integral inequalities for differentiable convex functions via Atangana-Baleanu fractional integral operators*, Chaos, Solitons & Fractals, **143** (2021), Article ID 110554.

[30] H.M. Srivastava, S. Mehrez, S.M. Sitnik, *Hermite-Hadamard-Type integral inequalities for convex functions and their applications*, Mathematics, **10**:17 (2022), Article ID 3127.

[31] G. Toader, *Some generalizations of the convexity*, in *Approximation and optimization*, Proc. Colloq., Cluj-Napoca/Rom. 1984, 1985, 329–338. Zbl 0582.26003

[32] M. Tunç, *On new inequalities for h-convex functions via Riemann-Liouville fractional integration*, Filomat, **27**:4 (2013), 559–565. Zbl 1324.26039

[33] M. Tunç, S. Balgeçti, *Some inequalities for differentiable convex functions with applications*, Preprint, arXiv:1406.7217, 2014.

[34] M. Tunç, S. Balgeçti, *Integral inequalities for mappings whose derivatives are $s-$convex in the second sense and applications to special means for positive real numbers*, Turkish J. Analysis Number Theory, **4**:2 (2016), 48–53.

[35] J.R. Wang, M. Fečkan, *Fractional Hermite-Hadamard inequalities*, Series: Fractional Calculus in Applied Sciences and Engineering, **5**, De Gruyter, Berlin, 2018. Zbl 1416.26002

[36] B.-Y. Xi, F. Qi, *Inequalities of Hermite-Hadamard type for extended s-convex functions and applications to means*, J. Nonlinear Convex Anal. **16**:5 (2015), 873–890. Zbl 1332.26020

[37] B.-Y. Xi, D.D. Gao, F. Qi, *Integral inequalities of Hermite-Hadamard type for (α, s)-convex and (α, s, m)-convex functions*, Italian J. Pure Appl. Math., **44** (2020), 499–510.

[38] C. Yıldız, M.E. Özdemir, H. Kavurmacı-Önalan, *Fractional integral inequalities via s-convex functions*, Turkish J. Analysis Number Theory, **5**:1 (2017), 18–22.

Juan E. Nápoles Valdés
UNNE FaCENA and UTN-FRRE,
Ave. Libertad 5450 and French 414 Resistencia,
Corrientes 3400 and Chaco 3500, Argentina
Email address: jnapoles@exa.unne.edu.ar
Email address: jnapoles@frre.utn.edu.ar

Bahtiyar Bayraktar
Bursa Uludag University Faculty of Education,
Gorukle Campus,
Bursa, Turkey
Email address: bbayraktar@uludag.edu.tr

Saad Ihsan Butt
COMSATS University,
Lahore Campus,
Islamabad, Pakistan
Email address: saadihsanbutt@gmail.com