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СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ
Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru
ISSN 1813-3304

Vol. 21, No. 2, pp. 810–822 (2024) УДК 512.643
https://doi.org/10.33048/semi.2024.21.054 MSC 15B48

CHAINABLE PROPERTIES OF SEMIGROUPS OF NONNEGATIVE MATRICES

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Communicated by S.V. Sudoplatov

Abstract: The theorem by Protasov and Voynov on the combinatorial structure of semigroups of nonnegative matrices extends a well-known result of Frobenius on the canonical form of an irreducible nonnegative matrix. We generalize the Protasov — Voynov theorem to not necessarily irreducible semigroups of matrices. For this purpose, an extensions of the concepts of imprimitivity index and canonical partition are introduced which are based on the chain properties of nonnegative matrices.

Keywords: nonnegative matrices, chainable matrices, chainable index

1 Introduction

By the famous Frobenius theorem any nonnegative irreducible matrix is either primitive or after a suitable permutation similarity transformation can be reduced to a so-called Frobenius form (see, for example [6]).

In the work [8] Protasov and Voynov proved the generalization of the Frobenius theorem to irreducible semigroups of nonnegative matrices. For this they have proposed a semigroup analogues to the notions of irreducible and primitive matrix and formulated an important theorem on the composition of irreducible semigroups

Alpin Yu.A., Guterman A.E., Shafeev E.R., Chainable properties of semigroups of nonnegative matrices.

Received Febraury, 13, 2024, Published November, 20, 2024.

of nonnegative matrices [8, Theorem 1], see also the Theorem 4 in Section 6 which describes a combinatorial structure of an irreducible semigroup of nonnegative matrices without zero rows and columns. The key notion of this theorem is the imprimitivity index of a semigroup.

The main result of this work is a generalization of the Protasov — Voynov theorem. We use the notions of potentially chainable matrix and insolidarity index, which correspondingly are the natural extentions of notions of primitive matrix and imprimitivity index, and show, see Theorem 3, that any semigroup of matrices without zero rows and columns, and without the additional assumption of its irreducibility, also has a special combinatorial structure which depends on its insolidarity index.

In the Theorem 7 it is proven that for an irreducible semigroup a partition into insolidarity classes coincides with partition into compatibility classes. This shows that the Theorem 3 is a natural generalization of the Protasov—Voynov Theorem. Informally speaking, this Theorem shows what can be saved from the Protasov—Voynov Theorem if we avoid the assumption of irreducibility, leaving only the assumption of absence of zero rows and columns.

Chainable matrices were first considered in [7], where their structure and algebraic properties were studied. Later, chainable and potentially chainable matrices, along with the chainable rank were considered in the works [4] and [5].

In paragraph 2 we introduce main definitions and formulate the basic statements. In paragraph 3 we formulate the theorem on the upper bound for the chainable rank of a product of two matrices and study the case of equality in given inequality. Paragraph 4 describes matrices that act on solidarity classes as permutations in terms of the chainable rank. In paragraph 5 new definitions are introduced for semigroups of matrices and the main theorem of this paper is proven. In paragraph 6 we compare obtained result with known facts about irreducible matrices and with the Protasov — Voynov Theorem.

2 On the chainable rank

The set of nonnegative matrices without zero rows and columns is denoted by $\mathbb{P}$, while by $\mathbb{P}_n$ we denote the subset of $\mathbb{P}$ consisting of matrices of order n. Let $A \in \mathbb{P}$ be an $n \times m$ matrix. We say that i-th and j-th rows of A *intersect* if they have positive elements in some common column. The set of indices $\{1, 2, \dots, n\}$ is denoted by $\mathbf{n}$.

Every $n \times m$ matrix $A \in \mathbb{P}$ has the following binary relations on the set $\mathbf{n}$ associated with it.

Definition 1. Indices i and j are compatible by a matrix A (A-compatible), if i-th and j-th rows of A intersect.

Definition 2. Indices i and j are in solidarity relation in the matrix A (are in A-solidarity relation), if there exists a sequence of indices $i = i_1, i_2, \dots, i_s = j$, such that rows i_k, i_{k+1} are compatible by the matrix A for $k = 1, \dots, s-1$.

The relation of A-solidarity is obviously an equivalence relation on $\mathbf{n}$.

In terms of matrix A the definition 2 means the following:

Proposition 1. *Indices $i, j \in \boldsymbol{n}$ are in solidarity relation by a matrix A if and only if for some integer s the following inequality holds:*

$$(AA^T)^s_{ij} > 0.$$

Proof. Indices i_1 and i_2 are compatible by A if and only if $(AA^T)_{i_1 i_2} > 0$. Correspondingly, the sequence from Definition 2 exists if and only if there are indices $i_2, \dots i_{s-1}$, such that

$$(AA^T)_{i_1,i_2}(AA^T)_{i_2,i_3} \dots (AA^T)_{i_{s-1},i_s} > 0, \tag{2.1}$$

Existence of such a sequence by the definition of matrix product is equivalent to the condition $(AA^T)^s_{i_1,i_s} > 0$. □

Definition 3. A matrix $A \in \mathbb{P}$ of size $n \times m$ is called chainable, if any indices $i, j \in \mathbf{n}$ are in solidarity relation by A, i.e. $(AA^T)^s_{ij} > 0$ for some s.

Definition 4. Chainable rank of an $n \times m$ matrix $A \in \mathbb{P}$ is a number of classes of A-solidarity and is denoted by $\mathrm{crk}(A)$.

From the Definition 4 one can see, that the chainable rank of $n \times m$ matrix $A \in \mathbb{P}$ is bounded by the following inequalities: $1 \le \mathrm{crk}(A) \le n$. Also $\mathrm{crk}(A) = 1$ only if A is a chainable matrix (Theorem 2.1 from [4]). Maximal value of $\mathrm{crk}(A) = n$ is achieved if A has no intersecting rows, the latter holds true precisely when each column of A has exactly one positive entry. In this case A has in total m positive entries. Let r_i denote the number of positive entries in i-th row. It is clear that $r_1 + \dots + r_n = m$. This equality is possible only when $n \le m$, and if we have $m = n$, then it means $r_1 = \dots = r_n = 1$. Let us formulate this useful statement separately (see different proof in [5, Lemma 3.7]):

Proposition 2. *If $A \in \mathbb{P}_n$, then $\mathrm{crk}(A) = n$ if and only if a matrix A contains exactly one positive entry in each row and in each column, i.e., it is a monomial matrix.*

By Theorem 3.1 in [4] chainable rank of a product of two nonnegative matrices satisfies the following inequality

$$\mathrm{crk}(AB) \le \min(\mathrm{crk}(A), \mathrm{crk}(B)). \tag{2.2}$$

Recall (see, for example, [4]) that any nonnegative matrix has an *associated* operator A, mapping the set $2^{\mathbf{n}}$ into the set $2^{\mathbf{m}}$ defined by the formula

$$\alpha \mapsto \alpha A = \{j | (A)_{ij} > 0, i \in \alpha\}. \tag{2.3}$$

As can be seen from (2.3), the set αA consists of indices of columns of a matrix A that have positive entries in rows with indices from α. To complete this definition we set $\varnothing A = \varnothing$.

Lemma 1. *Operator $A : 2^{\boldsymbol{n}} \to 2^{\boldsymbol{m}}$, associated with an $n \times m$ matrix $A \in \mathbb{P}$, have the following properties:*

1) $\alpha \ne \varnothing \Rightarrow \alpha A \ne \varnothing$, i.e., non-empty sets map to non-empty sets;

2) for any $j \in \boldsymbol{m}$ there exists a subset α such that $j \in \alpha A$.

Proof. The first statement is true because A has no zero rows, the second one — because A has no zero columns. □

Using the Definition of the solidarity relation and Lemma 1 we can describe non-chainable matrix in the following way.

Proposition 3. *Let $A \in \mathbb{P}$, and the solidarity classes corresponding to A are $\alpha_1, \ldots, \alpha_r$ $(r \geq 2)$. Then for $s = 1, \ldots, r$ we have:*

1) intersection of the rows with the indices from the set α_s and the columns with the indices from the set $\alpha_s A$ gives a chainable submatrix, let us denote it by A_s^{ch}.

2) all entries of the rows with the indices from α_s and the columns with the indices from $\alpha_s A$ which does not belong to A_s^{ch}, are equal to zero.

3 The equality $\operatorname{crk}(AB) = \operatorname{crk}(A) = \operatorname{crk}(B)$

Lemma 2. *Suppose we have a chainable matrix with n rows. Given any partition of the set $\mathbf{n} = \{1, \ldots, n\}$ for any set π_1 from this partition there exists a set $\pi_2, \pi_1 \neq \pi_2$, such that there exists a row with the index from the set π_1 intersecting a row with the index from the set π_2.*

Proof. Assume the opposite. Then there exists a partition of the set $\mathbf{n}$ such that rows of some set do not intersect with rows not belonging to that set. It is clear that such a matrix cannot be chainable. □

Theorem 1. *Let A be an $n \times m$ matrix and B be an $m \times p$ matrix from $\mathbb{P}$. Suppose that chainable ranks of matrices A, B are the same and equal to $r \geq 2$. We have the partition of the set $\boldsymbol{n}$ into A-solidarity classes:*

$$\alpha = (\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_r), \tag{3.1}$$

and the partition of the set $\boldsymbol{m}$ into B-solidarity classes:

$$\beta = (\beta_1,\ \beta_2,\ \ldots,\ \beta_r). \tag{3.2}$$

Equality $\operatorname{crk}(AB) = r$ holds if and only if for each class β_j there exists exactly one class α_i such that $\alpha_i A = \beta_j$. In other words, equality holds when partition (3.2) coincides with partition

$$\alpha_1 A,\ \alpha_2 A,\ \ldots,\ \alpha_r A. \tag{3.3}$$

up to the order of elements.

Proof. Without loss of generality we can assume that matrix A is partitioned into horizontal stripes in accordance with classes (3.1):

$$A = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_r \end{pmatrix}. \tag{3.4}$$

Indeed, this can be achieved by left multiplying A with an appropriate permutation matrix. Chainable rank of A is invariant under this multiplication. By Proposition 3 matrix A_s contains a chainable submatrix A_s^{ch} located in columns with indices from the set $\alpha_s A$ while all the other elements of A_s are equal to zero.

The sets

$$\beta_1 B,\ \beta_2 B,\ \ldots,\ \beta_r B \tag{3.5}$$

define a partition of the matrix B into vertical submatrices. Matrix constructed from the columns with indices from $\beta_t B$ is denoted by B_t. Without loss of generality we can assume that these vertical submatrices form full stripes in the order (3.5):

$$B = \left(B_1|B_2| \ \dots \ |B_r\right). \tag{3.6}$$

This can be achieved by right multiplying B by an appropriate permutation matrix. Chainable rank of B does not change after that. By Proposition 3 matrix B_t contains a chainable submatrix B_t^{ch} located in rows with the indices from class β_t while all the other elements of B_t are equal to zero.

We can multiply A and B blockwise:

$$AB = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_r \end{pmatrix} \left(B_1|B_2| \ \dots \ |B_r\right) = \begin{pmatrix} A_1B_1 & A_1B_2 & \dots & A_1B_r \\ A_2B_1 & A_2B_2 & \dots & A_2B_r \\ \cdot & \cdot & \cdot & \cdot \\ A_rB_1 & A_rB_2 & \dots & A_rB_r \end{pmatrix}. \tag{3.7}$$

Suppose that condition of the theorem is satisfied, i.e., for any class β_t there exists exactly one class α_s, such that $\alpha_s A = \beta_t$. Then

$$A_sB_t = \begin{cases} A_s^{ch}B_t^{ch}, \text{ if } \alpha_s A = \beta_t; \\ 0 \text{ , otherwise.} \end{cases}$$

Therefore, the matrix (3.7) is a block-monomial matrix: each block row and each block column has exactly one non-zero and, moreover, chainable block. Chainable rank of such a matrix is equal to r.

Suppose that the condition of the theorem is not satisfied. This condition (regarding two partitions of the same finite set) can be reformulated as follows: each class β_j is contained in one of the classes $\alpha_i A$. Violation of this condition means that the elements of some class β_k from (3.2) belong to some classes of partition (3.3). Let us denote these sets by

$$\alpha_{j_1}A \ \dots, \ \alpha_{j_h}A.$$

It means, that the class β_k is partitioned into subsets

$$\beta_k \cap (\alpha_{j_1}A), \ \dots, \ \beta_k \cap (\alpha_{j_h}A). \tag{3.8}$$

Correspondingly, the matrix B_k can be partitioned into horizontal submatrices corresponding to the subsets (3.8). The same can be said about the chainable submatrix B_k^{ch} of the matrix B_k located in rows with indices from β_k. By Lemma 2 for a subset $\beta_k \cap \alpha_{j_1}A$ there exists a subset $\beta_k \cap \alpha_{j_g}A$ such that some pair of rows with indices from these two sets intersect. Let

$$u \in \beta_k \cap \alpha_{j_1}A, \ v \in \beta_k \cap \alpha_{j_g}A$$

be the indices of these intersecting rows. Let us denote by $p \in \alpha_{j_1}$ and $q \in \alpha_{j_g}$ the rows of the matrix A that contain elements $(A)_{pu} > 0, (A)_{qv} > 0$.

By multiplying $p-$th row of the matrix A with the matrix B we obtain the $p-$th row of the matrix AB. This row has the u-th row of B as a term with a positive coefficient $(A)_{pu}$. Correspondingly, the q-th row of the matrix AB has the v-th row of B as a term with a positive coefficient $(A)_{qv}$. Therefore p-th and q-th rows of AB intersect by the same column by which u-th and v-th rows of B intersect. Thus for AB we have:

a) indices from α_{j_1} are in solidarity relation, since these indices are in solidarity relation in A, and for the same reason indices from α_{j_g} are in solidarity relation (see [4, Lemma 3.1]);

b) the row $p \in \alpha_{j_1}$ intersects with the row $q \in \alpha_{j_g}$.

From the properties a) and b) we can conclude that sets α_{j_1} and α_{j_g} are subsets of the same AB-solidarity class, therefore, the number of classes of AB-solidarity is smaller than the number of classes of A-solidarity, i.e., $\operatorname{crk}(AB) < \operatorname{crk}(A)$.

□

4 The equiality $\operatorname{crk}(A^2) = \operatorname{crk}(A)$

Definition 5. Suppose we have a matrix $A \in \mathbb{P}_n$ and some partition $\alpha_1,\ \alpha_2,\ \dots,\ \alpha_r$ of the set $\mathbf{n}$. Following [8], we say that *matrix A acts as a permutation on the sets of partition*, if for each set α_j there exists exactly one set α_i, such that $\alpha_i A = \alpha_j$.

Example 1. Consider partition $\alpha_1 = \{1, 2\}, \alpha_2 = \{3\}$ and matrix

$$M = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

It is clear that $\alpha_1 M = \alpha_2, \alpha_2 M = \alpha_1$. Therefore, M acts as a permutation on sets of this partition.

Index j is said to be an A-*successor* of index i, if $(A)_{ij} > 0$.

Lemma 3. *Matrix $A \in \mathbb{P}_n$ acts as a permutation on sets of the partition $(\alpha_1, \alpha_2, \dots, \alpha_r)$ of the set $\boldsymbol{n}$ if for any $i, j \in \boldsymbol{n}$ belonging to different sets of the partition their A-successors also belong to different sets of this partition.*

Proof. Let us choose one representative in each set of the partition:

$$i_1 \in \alpha_1, \dots, i_r \in \alpha_r.$$

Let j_s be an A-successor of index i_s, $s = 1, \dots, r$. By condition of the Lemma, A-successors belong to pairwise distinct sets of the partition. It means that there exists a permutation σ of the set $\{1, \dots, r\}$ such that

$$j_1 \in \alpha_{\sigma(1)}, \dots, j_r \in \alpha_{\sigma(r)}. \tag{4.1}$$

Let us prove that if $j_s \in \alpha_{\sigma(s)}$, then *all* A-successors of the index i_s belong to $\alpha_{\sigma(s)}$, i.e., $\alpha_s A \subseteq \alpha_{\sigma(s)}$. Assume the opposite: some A-successor of i_s, for example, index k belongs to $\alpha_{\sigma(t)}, t \neq s$. But we also have $j_t \in \alpha_{\sigma(t)}$. Thus, A-successors of indices i_s and i_t belong to the same set of the partition while i_s and i_t do not belong to the same set of the partition. This contradicts the condition of the Lemma. We have proven that $\alpha_s A \subseteq \alpha_{\sigma(s)}$, $s = 1, \dots, r$. Since A have no zero columns, all inclusions are in fact equalities:

$$\alpha_s A = \alpha_{\sigma(s)},\ \ s = 1, \dots, r.$$

□

Lemma 4. *Suppose a matrix $A \in \mathbb{P}_n$ acts as a permutation on sets of some partition $\alpha_1, \alpha_2, \dots, \alpha_r$ of the set $\boldsymbol{n}$. Then matrix A is permutationally similar to a block-monomial matrix with r non-zero blocks.*

Proof. Let matrix P permute the rows of A so that first positions are occupied by rows with indices from α_1, then followed by rows with indices from α_2 and so on. Then PAP^T is block-monomial with r non-zero blocks. □

Corollary 1. *For a matrix $A \in \mathbb{P}_n$ equality*

$$\operatorname{crk}(A^2) = \operatorname{crk}(A) \tag{4.2}$$

holds if and only if the following two conditions are satisfied:

1) matrix A acts on the solidarity classes as a permutation,

2) matrix A is permutationally similar to a block-monomial matrix with $r = \operatorname{crk}(A)$ non-zero blocks which are chainable matrices.

Proof. Suppose equality (4.2) holds, then Property 1) is obtained by applying the Theorem 1 to the case $A = B$. Property 2) is obtained by applying the Lemma 4, and the statement regarding non-zero blocks. The chainable rank of a block-monomial matrix is obviously equal to the sum of chainable ranks of its non-zero blocks. Since the number of such blocks is equal to $r = \operatorname{crk}(A)$, chainable ranks of all the blocks are equal to one, i.e., these blocks are chainable matrices.

Let conditions 1) and 2) be satisfied. The square of a block-monomial matrix with chainable blocks is a block-monomial matrix, each non-zero block of which is a product of two chainable blocks of the original matrix, therefore each block is itself chainable. Obviously, $\operatorname{crk}(A^2) = \operatorname{crk}(A)$. □

5 Chainable properties of semigroups

First, we provide definitions that extend Definitions 1 and 2 to semigroups.

Definition 6. ([3]) Indices $i, j \in \mathbf{n}$ are *compatible* by a semigroup $\mathcal{P} \subseteq \mathbb{P}_n$, if they are compatible by some matrix $A \in \mathcal{P}$.

Definition 7. Indices i and j are *in solidarity relation* in a semigroup $\mathcal{P} \subseteq \mathbb{P}_n$, if there exists a sequence of indices

$$i = i_1, i_2, \ldots, i_s = j, \tag{5.1}$$

in which adjacent indices i_k, i_{k+1} are compatible by a semigroup $\mathcal{P}$, i.e., compatible by some matrix $A_k \in \mathcal{P}$, $(k = 1, \ldots, s-1)$.

The relation of solidarity for indices in $\mathcal{P}$ is obviously an equivalence relation on $\mathbf{n}$. We assert the corresponding equivalence classes as *solidarity classes.*

Lemma 5. *Suppose for matrices $A, B \in \mathbb{P}_n$ we have*

$$\operatorname{crk}(A) = \operatorname{crk}(A^2) = \operatorname{crk}(B) = \operatorname{crk}(B^2) = \operatorname{crk}(AB) = r \geq 2. \tag{5.2}$$

Then

1) the partitions into solidarity classes for matrices A and B coincide, and matrices A and B act on these classes as permutations;

2) matrices A and B after a common permutation similarity transformation can be reduced to a block-monomial form with r non-zero blocks, where all non-zero blocks are chainable matrices.

Proof. Consider the following partitions of the set $\mathbf{n}$

$$\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_r, \tag{5.3}$$

$$\alpha_1 A,\ \alpha_2 A,\ \ldots,\ \alpha_r A, \tag{5.4}$$

$$\beta_1,\ \beta_2,\ \ldots,\ \beta_r, \tag{5.5}$$

$$\beta_1 B,\ \beta_2 B,\ \ldots,\ \beta_r B. \tag{5.6}$$

Here (5.3) and (5.5) are the solidarity classes for matrices A and B. If (5.2) holds, four partitions above coincide (i.e., are the same up to the order of elements). Indeed, partitions (5.3) and (5.4), and also (5.5) and (5.6) coincide by Corollary 1 (pg. 816); (5.4) and (5.5) coincide by Theorem 1 (pg.813). Since matrices A and B define the same partition of the set $\mathbf{n}$ into solidarity classes, it is possible to choose one permutation (as it is shown in the proof of the Lemma 4) that transforms these matrices into block-monomial form. By Lemma 4 non-zero blocks in this form are chainable matrices. □

Let $I(\mathcal{P})$ be the set of matrices from a semigroup $\mathcal{P} \subseteq \mathbb{P}_n$ that have the minimal chainable rank:

$$I(\mathcal{P}) = \{A \in \mathcal{P} \mid \operatorname{crk}(A) = \min_{M \in \mathcal{P}} \operatorname{crk}(M)\}. \tag{5.7}$$

It follows from (2.2) that $I(\mathcal{P})$ is a (two-sided) ideal in $\mathcal{P}$.

Corollary 2. *For all matrices in the ideal $I(\mathcal{P})$ the solidarity classes are the same.*

Proof. It is clear that for any $A, B \in I(\mathcal{P})$ the condition of the Lemma 5 is satisfied. Therefore, if for some matrix $A \in I(\mathcal{P})$ its solidarity classes are known, these classes are the solidarity classes for every matrix in the ideal $I(\mathcal{P})$. □

Lemma 6. [3, Lemma 2] *Suppose for matrices $A, B \in \mathbb{P}$ there exists a product AB. If indices i and j are A-compatible, then they are AB-compatible.*

Proof. Let $(A)_{ik} > 0, (A)_{jk} > 0, (B)_{kl} > 0$. Then

$$(AB)_{il} \geq (A)_{ik}(B)_{kl} > 0,\ \ (AB)_{jl} \geq (A)_{jk}(B)_{kl} > 0.$$

□

Lemma 7. *Indices $i, j \in \boldsymbol{n}$ are compatible by a semigroup $\mathcal{P}$ if and only if they are compatible by (any) matrix from the ideal $I(\mathcal{P})$.*

Proof. By Corollary 2 it is sufficient for us to prove that indices i, j are compatible by some matrix from $I(\mathcal{P})$. Let indices $i, j \in \mathbf{n}$ be compatible by a matrix $B \in \mathcal{P}$. By Lemma 6, if $C \in I(\mathcal{P})$ these indices are compatible by $BC \in I(\mathcal{P})$.

Converse statement is obvious. □

Theorem 2. *Indices $i, j \in \boldsymbol{n}$ are in solidarity relation in $\mathcal{P}$ if and only if they are in solidarity relation in any matrix from the ideal $I(\mathcal{P})$.*

Proof. Let indices i, j be in solidarity relation in $\mathcal{P}$. Then in the sequence (5.1) adjacent indices are compatible by $\mathcal{P}$ and by Lemma 7 they are in A-solidarity relation for any matrix $A \in I(\mathcal{P})$. Since solidarity relation on the set of indices is transitive, the indices i, j are also in A-solidarity relation.

Converse statement is obvious. □

Definition 8. Index of insolidarity of a semigroup $\mathcal{P} \subseteq \mathbb{P}_n$ is the maximum number $r = r(\mathcal{P})$ of indices, any two of which are not in solidarity in $\mathcal{P}$. In other words, index $r(\mathcal{P})$ is equal to the number of classes of $\mathcal{P}$-solidarity.

By Theorem 2, solidarity classes of semigroup $\mathcal{P}$ coincide with solidarity classes of any matrix from the ideal $I(\mathcal{P})$. From this fact and the description (5.7) of the ideal $I(\mathcal{P})$ it follows, that index of insolidarity of a semigroup $\mathcal{P} \subseteq \mathbb{P}_n$ is equal to the smallest value of chainable rank of matrices from this semigroup:

$$r(\mathcal{P}) = \min_{A \in \mathcal{P}} \operatorname{crk}(A). \tag{5.8}$$

Theorem 3. *Let $\mathcal{P} \subseteq \mathbb{P}_n$ be a matrix semigroup with insolidarity index equal to r.*

1. If $r = 1$, then semigroup $\mathcal{P}$ is chainable (i.e. contains a chainable matrix).

2. If $r \geq 2$, then all matrices from $\mathcal{P}$ act on solidarity classes as permutations. All matrices from semigroup $\mathcal{P}$ after a common permutation similarity transformation become block-monomial with r blocks. After this transformation, semigroup contains matrices in which all non-zero blocks are chainable matrices.

Proof. 1. Directly follows from (5.8).

2. By Lemma 5 all matrices from the ideal $I(\mathcal{P})$ act as permutations on the equivalence classes of the solidarity relation. Also by Lemma 5 there exists a permutation similarity transformation after which all matrices from the ideal $I(\mathcal{P})$ become block-monomial with r chainable non-zero blocks.

Now we prove that not only matrices from $I(\mathcal{P})$, but any matrix $A \in \mathcal{P}$ acts as a permutation on the set of solidarity classes. By Lemma 3, it is sufficient to show that for indices that are not in the solidarity relation, their A-successors are also not in the solidarity relation. Assume the opposite: for some indices i and j that are not in the solidarity relation we have $(A)_{ik} > 0, (A)_{jl} > 0$ but the indices k and l are in the solidarity relation. Hence if $B \in I(\mathcal{P})$ and $(B)_{ks} > 0, (B)_{lt} > 0$, then s and t are in the solidarity relation. For the matrix $AB \in I(\mathcal{P})$ and the indices i and j we have:

$$(AB)_{is} \geq (A)_{ik}(B)_{ks} > 0, (AB)_{jt} \geq (A)_{jl}(B)_{lt} > 0.$$

Therefore, indices i and j should be in the solidarity relation since they have successors that are in the solidarity relation. A contradiction. □

6 The connection between Protasov—Voynov theorem and the Theorem 3

Recall that a nonnegative matrix A of order n is called irreducible if for any indices i, j from the set $\mathbf{n} = \{1, \ldots, n\}$ there exists an integer l such that (i, j)-entry in the matrix A^l is positive.

From the Perron—Frobenius theory of nonnegative matrices it is known that the spectral radius $\rho(A)$ of a nonnegative matrix A is its simple eigenvalue. The number $r = r(A)$ of eigenvalues equal by their absolute value to $\rho(A)$ is called the imprimitivity index of an irreducible matrix. If $r = 1$, then matrix is primitive, while if $r > 1$ then it is imprimitive.

The combinatorial (i.e. defined by position of positive entries) characteristic of primitive matrix is that some power of this matrix is a positive matrix. By Frobenius theorem, an irreducible matrix is either primitive or after a suitable permutation similarity transfromation it can be reduced to the so-called Frobenius form:

$$A = \begin{pmatrix} 0 & A_{12} & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & A_{r-1,r} \\ A_{r1} & 0 & \dots & 0 \end{pmatrix}, \tag{6.1}$$

while in the matrix $A^r = \mathrm{diag}(A_{11}^{(r)}, A_{22}^{(r)}, \dots, A_{rr}^{(r)})$ the diagonal blocks are primitive (see, fro example [6, chapter 8, paragraph 2]).

In the work [8] Protasov and Voynov proved a generalization of the Frobenius theorem onto semigroups of nonnegative matrices. The following semigroup analogues of notions of irreducibility and imprimitivity were used to formulate their results. A semigroup $\mathcal{P}$ of nonnegative matrices of order n is called irreducible if for any $i, j \in \mathbf{n}$ there exists a matrix from $\mathcal{P}$ with a positive (i, j)-entry. It is clear that irreducibility of a nonnegative matrix A is equivalent to irreducibility of the semigroup $\langle A \rangle$. A semigroup is called primitive if it contains at least one positive matrix. Irreducible semigroups that are not primitive are called imprimitive.

The Protasov—Voynov theorem ([8]) describes a combinatorial structure of irreducible semigroup of nonnegative matrices without zero rows and columns. The key notion of the theorem is the imprimitivity index of a semigroup. We shall provide a corresponding definition and formulate the theorem in the way that somewhat differs from the original but is more fitting to the context of this work.

Definition 9. Index of imprimitivity of a semigroup $\mathcal{P} \subseteq \mathbb{P}_n$ is the maximum integer r, such that there exist r indices, any two of which are not compatible by $\mathcal{P}$.

Theorem 4. *([8, Theorem 1]) Let $\mathcal{P} \subseteq \mathbb{P}_n$ be an irreducible semigroup with imprimitivity index r.*

If $r = 1$, then the semigroup $\mathcal{P}$ contains a positive matrix.

If $r \geq 2$, then there exists a unique partition of the set $\boldsymbol{n}$ into r sets, on which all matrices of the semigroup act as permutations. This partition is called canonical.

All matrices from the semigroup $\mathcal{P}$ after a common permutation similarity transformation can be reduced to a block-monomial form with r non-zero blocks. After this transformation the semigroup contains matrices in which all non-zero blocks are positive.

The fact that canonical partition can be described in terms of compatibility relation is significant due to the following results.

Theorem 5. *([2]) For an irreducible semigroup $\mathcal{P} \subseteq \mathbb{P}_n$ a partition of the set $\boldsymbol{n}$ into the compatibility classes is the canonical partition.*

Theorem 6. *A partition of the set $\boldsymbol{n}$ into the solidarity classes is a subpartition of any partition on which matrices from a semigroup $\mathcal{P} \subseteq \mathbb{P}_n$ act as permutations.*

Proof. Suppose that matrices from $\mathcal{P}$ act on the partition π as permutations and indices i and j are in solidarity relation in $\mathcal{P}$. Then, by Definition 7, there exists a sequence of indices

$$i = i_1, i_2, \ldots, i_s = j, \tag{6.2}$$

such that indices i_k, i_{k+1} are compatible by some matrix $A_k \in \mathcal{P},\ k = 1, \ldots, s-1$.

Indices that belong to different π-classes are obviously not compatible by any matrix from $\mathcal{P}$. In other words, indices that are compatible by a matrix from $\mathcal{P}$ necessarily belong to the same π-class. Therefore, any two adjacent indices in the sequence (6.2) belong to the same π-class, hence indices i and j that are in solidarity relation belong to this class. It follows that partition into the solidarity classes is a subpartition of π.

□

Theorem 7. *For an irreducible semigroup $\mathcal{P} \subseteq \mathbb{P}_n$ the partition into the solidarity classes coincides with the partition into the compatibility classes.*

Proof. 1) By Theorem 6 partition into $\mathcal{P}$-solidarity classes is a subpartition of partition into $\mathcal{P}$-compatibility classes, i.e. indices that are in $\mathcal{P}$-solidarity are $\mathcal{P}$-compatible. 2) From the Definition 7 it is clear that $\mathcal{P}$-compatible indices are in $\mathcal{P}$-solidarity relation. The statement of our theorem follows now from 1) and 2). □

By Theorem 7 index of insolidarity of an irreducible semigroup $\mathcal{P} \subseteq \mathbb{P}_n$ equals its index of imprimitivity. In particular, index of insolidarity of an irreducible matrix $A \in \mathbb{P}_n$ is equal to its index of imprimitivity. Let us apply obtained results to the semigroup $\langle A \rangle$ which is generated by single matrix $A \in \mathbb{P}_n$. From the inequality (2.2) we obtain a chain of inequalities

$$\operatorname{crk}(A) \geq \operatorname{crk}(A^2) \geq \cdots \geq \operatorname{crk}(A^k) \geq \operatorname{crk}(A^{k+1}) \geq \cdots \tag{6.3}$$

moreover, the following theorem holds

Theorem 8. *([5], Theorem 3.5) If* $\operatorname{crk}(A^{k+1}) = \operatorname{crk}(A^k)$, *then* $\operatorname{crk}(A^{k+t}) = \operatorname{crk}(A^k)$ *for each* $t \in \mathbb{N}$.

Let $\varkappa$ denote the minimum power for which in the chain (6.3) equality $\operatorname{crk}(A^{\varkappa+1}) = \operatorname{crk}(A^{\varkappa})$ takes place.

Proposition 4. $\varkappa \leq n - 1$.

Proof. First, we note the obvious inequality: $\varkappa \leq n$. Suppose that $\varkappa = n$. This equality is possible only if $\operatorname{crk}(A) = n$, hence, by Proposition 2, A is a monomial matrix. The set of monomial matrices is closed under multiplication, so $\operatorname{crk}(A^k) = n$ for $k = 1, 2, \ldots$ But in this case $\varkappa = 1$. Because of this contradiction we conclude that $\varkappa \leq n - 1$. □

From the Proposition 4 it follows that matrix A^{n-1} belongs to the ideal $I(\langle A \rangle)$. Considering that, from the Theorem 2 we derive:

Corollary 3. *Let $A \in \mathbb{P}_n$. Indices $i, j \in \mathbf{n}$ are in solidarity relation in a semigroup $\langle A \rangle$ if and only if they are in solidarity relation in the matrix A^{n-1}.*

Let us denote the insolidarity index of the semigroup $\langle A \rangle$ by $r(A)$. Then the statement of Corollary 3 can be written as equality:

$$r(A) = \operatorname{crk}(A^{n-1}). \tag{6.4}$$

Theorem 9. *Let $A \in \mathbb{P}_n$.*

1. If $r(A) = 1$, then A is a potentially chainable matrix, namely matrices A^k, $k \geq n - 1$ are chainable.

2. If $r = r(A) \geq 2$, then the matrix A after a suitable permutation similarity transformation could be reduced to a block-monomial form with r blocks. In matrices A^k, $k \geq n-1$, non-zero blocks are chainable matrices.

Suppose that A is irreducible. By Theorem 7 solidarity classes of the semigroup $\langle A \rangle$ coincide with compatibility classes. If $r(A) = 1$, i.e., any two indices are compatible, then A is a primitive matrix, see [4, Proposition 6.1]. Now let $r = r(A) \geq 2$. Then classes of compatibility form so called cyclic partition of the set $\mathbf{n}$. Therefore, there is a numeration of compatibility classes: $\alpha_1, \alpha_2, \ldots, \alpha_r$, such that $\alpha_1 A = \alpha_2, \alpha_2 A = \alpha_3, \ldots, \alpha_r A = \alpha_1$. This numeration corresponds to the form (6.1). Thus, in case of irreducibility; block-monomial form from Theorem 9 turns into the Frobenius form. Detailed proof of this fact with the use of the compatibility relation can be found in [1].

Acknowledgments

The authors are grateful to Vladimir Protasov for interesting discussions.

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