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ON SOME TYPES OF ALGEBRAS OF A JONSSON SPECTRUM

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Abstract: We work in the framework of study of Jonsson spectra for concerning classes of structures. In this paper, the issues of cosemantic Jonsson theories and Jonsson spectrum are discussed. There are shown some results on the opportunity of introducing some types of algebras on the Jonsson spectrum. Also, it is proved that the finite axiomatizability of the Kaiser hull of the Jonsson theory implies the finiteness of cosemantic Jonsson theories.

Keywords: Existentially closed models, amalgamation property, joint embedding property, Jonsson theory, Jonsson spectrum, cosemantic Jonsson theories, cosemanticness classes, cosemantic structures, the lattice of Jonsson theories, Jonsson equivalence, finitely axiomatizable Jonsson theory.

1 Introduction

As is well known, in the classical first-order Model Theory, there are two directions, conventionally called "Eastern"and "Western"in honor of the places of residence of their founders Abraham Robinson and Alfred Tarski. The tasks of the "Western"Model Theory, as a rule, are formulated within the

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study of complete theories, which gives this area an advantage in the range of tools and techniques used to solve these problems. As for the "Eastern"Model Theory, the situation is more complicated, since research in this field focuses on the study of, generally speaking, incomplete theories and, as a result, often more general tasks.

This work refers specifically to the "Eastern"Model Theory. At the same time, due to the complexity of the issues facing us, we need to follow detailed restrictions in order to obtain more fruitful results. Actually, we emphasize the basic syntactic and semantic properties of the classical algebras and limit ourselves to the study of inductive, or, equivalently, $\forall\exists$-axiomatized theories. In the class of inductive theories, we distinguish its special subclass, more exactly the subclass of Jonsson theories, which occupy a special place in the research of Model Theory in the spirit of Abraham Robinson. Classical examples of Jonsson theories are group theory, the theory of abelian groups, the theory of fields of the fixed characteristic, module theory, lattice theory, and others. The existence of such algebraic examples leads us to the firm conviction that the study of model-theoretic issues of the Jonsson theories is an urgent task.

In recent decades, a special apparatus has been created to study the Jonsson theories. More essential papers by this topic, which one should pay attention, are linked from the following list of references: [1, 2, 3, 4]. More relevant and complete information on Jonsson theories apparatus, and techniques, and other interesting things, a reader can obtain from [5, 6, 7, 8, 9].

When studying Jonsson theories, we use so-called "semantic method" named as a semantic model of the theory under consideration. The essence of the semantic method is to transfer the properties of the center of the Jonsson theory to the Jonsson theory itself. The center is understood as the complete theory of the ω^+-homogeneous ω^+-universal model of this Jonsson theory. Such a method allows us to introduce and successfully apply the techniques of "Western"Model Theory in our research.

Another technique used by us was proposed by the first author of this paper. In classical Model Theory, as a rule, when studying the syntactic and semantic properties of various algebraic structures, the class of models of a given theory is studied, that is, the transition from theory to its models is considered. In our research, we often use the opposite principle: when studying some algebraic structure, we consider the spectrum of its Jonsson theories. This technique allows us to identify many interesting properties of theories that are relevant not only in Model Theory, but also, for example, Universal Algebra. And since the methods of Universal Algebra have been developing more and more recently and are actively used in related fields, the described method allows not only to obtain new results, but also to establish new interdisciplinary connections.

The article consists of 4 sections: an introduction and 3 main sections. Section 1 provides basic information on Jonsson theories, describes the apparatus for studying Jonsson theories, and also shows some important results concerning the cosemantic Jonsson theories. Section 2 is devoted to the study of the structural properties of the Jonsson spectrum of a class of first-order language structures; we consider the union of the Jonsson L-theories regarding the preservation of Jonsonness and then apply the obtained results to the structure of Jonsson spectrum. Finally, in Section 3, generalizations of some well-known results on the Jonsson spectrum are given.

Let us introduce the technical agreements. The numbering of Definitions, Theorems, Lemmas, Propositions, and Corollaries is independent and end-to-end within each section. $\square$ denotes the end of the proof.

Now let us introduce the notation and determine the frame of our study.

We work in a first-order countable language L. By theory, we mean a consistent set of sentences in the given language. By $Cn(T)$, we denote a deductive closure of T, which is a set of all L-sentences φ such as of $T \vdash \varphi$. All theories and structures in this paper are considered within the framework of L.

If T is an L-theory, then E_T denotes a class of existentially closed models of the theory T.

Let K be a class of L-structures. By "$K \models T$"we mean that $A \models T$ for any $A \in K$, or $K \subseteq Mod(T)$.

Let T and T' be L-theories such as T is logically equivalent to T', i.e., for any φ such that $T \vdash \varphi$, there exists ψ such that $\psi \leftrightarrow \varphi$ and $T' \vdash \psi$, and vice versa. It also means that $Mod(T) = Mod(T')$. In this paper, we do not differ logically equivalent theories, that is we consider $T = T'$.

Let A be an L-structure. By $T^0(A)$, we mean the theory $Th_{\forall\exists}(A)$ that is a set of all $\forall\exists$-sentences of L true for the structure A. The theory $T^0(A)$ is called a Kaiser hull of A.

2 Cosemantic Jonsson theories

We start with some basic information on Jonsson theories and related concepts. When considering the class of models of inductive theories, we always deal with the class of existentially closed models of the given theory.

Definition 1. [10] *A structure A is called an existentially closed model of T, if $A \models T$ and, for any model B of T,*

$$A \subseteq B \Rightarrow A \prec_1 B.$$

It is well-known that, if T is an inductive theory each model of T can be extended to some existentially closed model of T.

The following facts are well-known.

Theorem 1. [10] *If $A \prec_1 B$, where $A \models T$ and B is an existentially closed model of T, then A is also an existentially closed model of T.*

Theorem 2. [10] *Let T be any L-theory. For any model A of T the following are equivalent:*
(i) A is existentially closed over T;
(ii) A is existentially closed over $Cn(T \cap \forall_1)$.

Before presenting the concept of Jonsson theories let us remind the definitions of two properties that are said to be algebraic originally, but take a great place in studying Model Theory.

Definition 2. [11, p.80] *A theory T has the joint embedding property, if, for any models A and B of T, there exists a model M of T and isomorphic embeddings $f : A \to M$, $g : B \to M$.*

Definition 3. [11, p.80] *A theory T has the amalgamation property, if for any models A, B_1, B_2 of T and isomorphic embeddings $f_1 : A \to B_1$, $f_2 : A \to B_2$ there are $M \models T$ and isomorphic embeddings $g_1 : B_1 \to M$, $g_2 : B_2 \to M$, such that $g_1 \circ f_1 = g_2 \circ f_2$.*

Further, we will write "JEP" and "AP" as shorter forms for the joint embedding and amalgamation properties, correspondingly.

Now let us describe Jonsson theories. Note that we work with the following definition that was introduced in the Russian edition of [11].

Definition 4. [11, p.80] *A theory T is called a Jonsson theory, if the following conditions hold:*
1) T has at least one infinite model;
2) T is an inductive theory (or $\forall\exists$-axiomatizable, which is equivalent);
3) T admits JEP;
4) T admits AP.

There are a lot of algebraic structures whose theories are Jonsson. Classical examples of Jonsson theories include
1) group theory;
2) the theory of abelian groups;
3) the theory of Boolean algebras;
4) the theory of linear orders;
5) field theory of characteristic p, where p is zero or a prime number;
6) the theory of ordered fields;
7) the theory of modules.

One can note that some of the given examples represent universally axiomatized theories. Indeed, such theories occupy a significant place, especially in Universal Algebra. Some classes of structures as varieties, as it is known, are axiomatized by universal sentences, and some of them perform the classes of models of Jonsson theories (for example, groups). Following the paper of A. Pillay [12], we call such theories Robinsonian theories.

Definition 5. *A theory T is said to be a Robinsonian theory if it is Jonsson and $\forall$-axiomatizable.*

Discussing existentially closed models of Jonsson theories, it is necessary to mention the following fact.

Theorem 3. [13] *Suppose T be an L-theory, and let T admit JEP. Let A and B be existentially closed model of T. Then each $\forall\exists$-sentence that is true in A is true in B as well.*

In other words, for any theory T admitting JEP, any two of its existentially closed models are elementary equivalent by $\forall\exists$-sentences. It follows from this fact that this is true for any Jonsson theory, since Jonsson theories, by definition, admit the joint embedding property.

The following definitions and theorems were formulated by T.G. Mustafin and form an essential apparatus for studying Jonsson theories.

Definition 6. [4] *A model C_T of a Jonsson theory T is called a semantic model of this theory, if $|C_T| = 2^\omega$ and C_T is ω^+-universal ω^+-homogeneous model of T.*

Actually, the semantic model of the theory T is a natural semantic invariant of T, which is demonstrated by Theorem 4.

Theorem 4. [4] *An inductive theory T is Jonsson iff it has a ω^+-universal ω^+-homogeneous model (which is its semantic model).*

Definition 7. [5] *Let T be a Jonsson theory. Then theory $T^* = Th(C)$ is called a center of T.*

The following fact follows from Definition 7.

Corollary 1. [5, 155] *Let T be a Jonsson theory, T^* be its center. Then T and T^* are mutually model consistent.*

One of the most important tools for comparing models of the same theory among themselves in classical Model Theory is the elementary equivalence relation. If two arbitrary models do not differ from each other by the sentences of a given language regarding theory, then such a theory is obviously complete. When studying models of an arbitrary Jonsson theory, as noted earlier, this tool may not work due to the, generally speaking, incompleteness of the Jonsson theories. On the other hand, morphisms are no less important tools for comparing two models of the same Jonsson theory, which in this case, as a rule, are either isomorphisms or homomorphisms. In some cases, they coincide with elementary monomorphisms. Perfect Jonsson theories are a special case of such situations. Let us remind the definition:

Definition 8. [4] *A Jonsson theory T is called perfect, if C_T is ω^+-saturated.*

When considering a positive Jonsson theory [2, 3] with homomorphisms, immersion is a specific type of morphism, and then the saturation of the semantic model also plays an important role, as in the case of perfect Jonsson theory. And the models of the center of such Jonsson theories are limited to classes of existentially closed models of the considered Jonsson theory.

A natural generalization of the concept of elementary equivalence when working with Jonsson theories is the concept of the cosemanticness of models. To describe this notion, we use the concept of a Jonsson spectrum.

Definition 9. [14] *Let K be a class of L-structures. A Jonsson spectrum $JSp(K)$ of K is the followng set of theories*

$$JSp(K) = \{T \mid T \text{ is a Jonsson theory and } \forall A \in K\ A \models T\}.$$

For the first time, Jonsson spectrum was introduced in [14], where the authors presented the Jonsson analogue of Schroder-Bernstein property over abelian groups by applying the concept of Jonsson spectrum. It was also used in [15] for describing the properties of R-modules and generalization of elementary equivalence between structures. Some of important properties of the Jonsson spectrum are also demonstrated in [16].

Similarly, it was introduced the notion of Robinson spectrum.

Definition 10. [15] *Let K be a class of L-structures. A Robinson spectrum $RSp(K)$ of K is the following set of theories*

$$RSp(K) = \{T \mid T \text{ is a Robinsonian theory and } \forall A \in K\ A \models T\}.$$

Now let us present the definition of the cosemanticness of two L-structures.

Definition 11. *L-structures A and B are called cosemantic (denoted by $A \bowtie B$), if $JSp(A) = JSp(B)$.*

When considering this binary relation between two models of an arbitrary Jonsson theory, we obtain a partition into disjoint classes, that is, it is an equivalence relation. On the other hand, due to the definition of the cosemanticness of two structures, we see the involvement of the concepts of the Jonsson spectra of these models in this. Thus, the concept of the cosemanticness of models is closely related to the previously introduced by T.G. Mustafin concept of the cosemanticness of theories. In fact, on the set of the Jonsson spectrum, one can also consider some equivalence relation, which will be the cosematicness relation of two theories of this spectrum.

Definition 12. [5] *Let T_1 and T_2 be Jonsson L-theories, C_1 and C_2 be their semantic models, correspondingly. T_1 and T_2 are said to be cosemantic (denoted by "$T_1 \bowtie T_2$"), if $C_1 = C_2$.*

The practical significance of the cosematicness of two models can be illustrated by two following classical examples: abelian groups and coherent rings. In [14, 15], the criteria of cosemanticness for these classes of algebras are given. As can be easily seen from the results of these works, the concept of cosematicness not only generalizes, but also refines the concept of elementary equivalence.

A lot of properties of cosemantic Jonsson theories was described by Ye.T. Mustafin in [4]. Some of them will be used in this paper, and here the first one:

Proposition 1. [4] *If T is a Jonsson theory and T' is an inductive theory such that $T_\forall = T'_\forall$ then T' is a Jonsson theory that is cosemantic to T.*

When introducing the cosemanticness relation on the Jonsson spectrum $JSp(K)$ of some fixed class K of L-structures, we obtain a factor-set, which we denote by $JSp(K)_{/\bowtie}$. Then $[T]$ denotes the cosemanticness class of a theory $T \in JSp(K)$ in this factor-set. Indeed, we can consider the Jonsson theories as one-element cosemanticness classes of a certain Jonsson spectrum. Such Jonsson spectra do exists. For example, the factor-set of the Robinson spectrum consists of one-element cosemanticness classes, which is shown below in this section.

Now let us demonstrate the connection between cosemantic Jonsson theories and their classes of models.

Proposition 2. *Let T_1 and T_2 be Jonsson theories. Then $T_1 \bowtie T_2$ iff the class E_{T_1} of existentially closed structures of T_1 coincides with the class E_{T_2} of existentially closed structures of T_2.*

Proof. $\to$ Firstly, we need to prove that $E_{T_1} = E_{T_2}$, if $T_1 \bowtie T_2$, i.e. that $E_{T_1} \subseteq E_{T_2}$ and $E_{T_2} \subseteq E_{T_1}$. Note that since $T_1 \bowtie T_2$, $C_1 = C_2 = C$, C is an existentially closed model both for T_1 and T_2. Besides C is ω^+-universal for T_1 and T_2. Let A be an existentially closed model of T_1. Then, by Theorem 3, $A \equiv_{\forall\exists} C$. Consequently A is a model of T_2. Moreover, according to Definition 1, $A \prec_1 C$. It follows that, by Theorem 1, A is an existentially closed model of T_2. In force of arbitrariness of A we may state that $E_{T_1} \subseteq E_{T_2}$. The converse can be proved by analogy, i.e. that $E_{T_2} \subseteq E_{T_1}$.

$\leftarrow$ It is easy to see that, for any two inductive theories T and T', T and T' are mutually model consistent iff $E_T = E_{T'}$. Indeed, the fact that T and T' are mutually model consistent is equivalent to the fact $T_\forall = T'_\forall$. So, if $T_\forall = T'_\forall$ then, according to Theorem 2, $E_T = E_{T_\forall}$ and $E_{T'} = E_{T'_\forall}$, therefore $E_T = E_{T'}$. Converse, if $E_T = E_{T'}$, then for any model $A \in Mod(T)$, there is an existentially closed model $M \in E_T$ such that $A \subseteq M$. But $M \in E_{T'}$ as well. The same is for an arbitrary model $B \in ModT'$: there is $M' \in E_{T'}$ such that $B \subseteq M'$, and $M' \in E_T$ as well. Thus, T and T' are mutually model consistent. From this, we can conclude that if $E_{T_1} = E_{T_2}$ then T_1 and T_2 are mutually model consistent. Then, by Proposition 1, $T_1 \bowtie T_2$. □

When considering the cosemanticness of Robinsonian theories, we can get the following result.

Theorem 5. *Let T be a Robinsonian theory, and let T' be a Jonsson theory that is cosemantic with T. Then $T \subseteq T'$.*

Proof. Since T and T' are cosemanitic, $T^* = T'^*$. As we know from Theorem 1, T and T' are mutually model-consistent with their common center, and therefore with each other. This means that $T_\forall = T'_\forall$. But T is a Robinsonian theory, which means that all sentences deduced in T are deduced in T'. Hence, $T \subseteq T'$. □

Basing on Theorem 5, we get the following corollaries, which gives us some information on the structure of Jonsson spectrum and Robinson spectrum of an arbitrary class of L-structures.

Corollary 2. *Let K be an arbitrary class of L-structures (possibly, it consists of one structure), $JSp(K)_{/\bowtie}$ be a factor set of the Jonsson spectrum of K with respect to cosemanticness, $[\Delta]$ be an arbitrary cosemanticness class from $JSp(K)_{/\bowtie}$, $T \in [\Delta]$ be a Robinsonian theory. Then T is the only Robinsonian theory in $[\Delta]$.*

Proof. Suppose the opposite. Let T' be a Robinsonian theory such that $T' \neq T$ and let $T' \in [\Delta]$. Since $T \bowtie T'$, by Theorem 5 it is true that $T \subseteq T'$, but it is also true that $T' \subseteq T$. Therefore, $T = T'$. □

Corollary 3. *Let K be an arbitrary class of L-structures (possibly, it consists of one structure), $RSp(K)_{/\bowtie}$ be a factor set of the Robinson spectrum of K with respect to cosemanticness. Then every cosemanticness class $[\Delta]$ contains exactly one theory. In other words, for any two Robinsonian L-theories T and T', the relation of cosemanticness is equivalent to the equality (logical equivalence) of theories, i.e. $T \bowtie T' \Leftrightarrow T = T'$.*

Proof. The proof follows directly from Corollary 2. □

Now let us present a theorem, which not only demonstrates some specific features of inductive and Jonsson theories in terms of cosemanticness, but also is a necessary tool for obtaining the further results.

Theorem 6. *Let T be a Jonsson L-theory, C_T be its semantic model, and let T' be a theory of the same language such as T' is inductive, $T \subseteq T'$ and $C_T \models T'$. Then T' is a Jonsson theory that is cosemantic to T.*

There are several ways to prove this theorem. We show a more detailed proof to make a reader familiar with the technique of working with Jonsson theories.

Proof. Firstly, we show that T' is a Jonsson theory. $C_T \models T'$, so T' has it least one infinite model. It is inductive by the condition. Note that all models of T' are models of T, therefore any two models of T' can be embedded to C_T, which means that T' admits JEP. As for AP, let A be a model of T' and $A \to B$, $A \to C$, where B and C are models of T'. If we consider A, B, C as models of T, there is a model $D \models T$ such that $B \to D$, $C \to D$ and this diagram of embeddings is commutative, as soon as T admits AP. T is an inductive theory, so there is an existentially closed model $M \models T$ such that D can be embedded to M. But, according to Theorem 3, $M \equiv_{\forall\exists} C_T$, therefore $M \models T'$. It follows that T' admits AP. Now let us show that $T \bowtie T'$, i.e. C_T is a cosemantic model for T'. It is easy to see that C_T is ω^+-universal for T', since $Mod(T') \subseteq Mod(T)$. Similarly, C_T is ω^+-homogeneous for T', as all necessary isomorphic maps are provided by their presence in the class of models of T. □

The following result seems important to us, since it touches on the issues of finite axiomatizability, but on the basis of which in the future we can consider quite a lot of interesting questions related to the structure of the Jonsson spectrum in connection with the classical problems of finite axiomatizability in various classes of algebras.

Theorem 7. *Let T be a Jonsson theory and $T^0 = Th_{\forall\exists}(C_{[T]})$ be a finitely axiomatizable theory. Then there are finitely many theories that are cosemantic to T.*

Proof. Let us prove the theorem by converse. By condition, T^0 is finitely axiomatizable. As we mentioned before, we do not differ logically equivalent theories, so we consider a finite list of L-sentences (i.e. the list of axioms) as the theory T^0. Suppose there is an infinite number of theories cosemantic to T. In force of Theorem 6, $T \bowtie T^0$. Let $\Delta_1, \Delta_2, ..., \Delta_i, ...$ be a list of all Jonsson theories that are cosemantic to T and T^0. As soon as T^0 is finitely axiomatizable there is an L-sentence φ that is equivalent to the conjunction of the axioms of T^0. It is clear that, for any Δ_i, $\Delta_i \subseteq T^0$. It means that each Δ_i is also finitely axiomatizable and there are L-sentences $\psi_1, \psi_2, ..., \psi_i, ...$ that are equivalent to the conjunctions of axioms of $\Delta_1, \Delta_2, ..., \Delta_i, ...$, correspondingly. Consider a theory $\mathbb{T} = \bigcup_i \Delta_i$. Note that $\mathbb{T}$ is inductive, $C \in Mod(\mathbb{T})$, so, by Theorem 6, it is a Jonsson theory cosemantic to T and T^0. For each Δ_i, $\Delta_i \subseteq T^0$, then $\mathbb{T} \subseteq T^0$. But T^0 is finitely axiomatizable, then the set $\mathbb{T}$ of sentences has to be finite. Therefore, the number of $\Delta_i, i \geq 1$, is finite. □

3 The algebra of the Jonsson spectrum

In this section, we will describe the structural properties of the Jonsson spectrum of an arbitrary class K of L-structures. We consider Jonsson theories as elements of some algebraic structures and present the specific form of these structures.

As soon as theories are syntactic concepts, we use a syntactic approach. When dealing with Jonsson theories, we should pay attention to the syntactic characterization of the main features of Jonssonness: inductiveness, AP and JEP of these theories. It is well known that the inductiveness of a theory is equivalent to its $\forall\exists$-aziomatizability. As for AP and JEP, our main syntactic tools are the following two theorems. Theorem 8, which was shown by A. Robinson, is a syntactic criterion of joint embedding property. Theorem 9 describes the syntactic nature of AP and was proved by D. Bryars.

Theorem 8. [17] *For the first order theory T of the language L (of arbitrary cardinality) the following conditions are equivalent:*

(1) *T has JEP;*
(2) *For all universal sentences α, β of L, if $T \vdash \alpha \vee \beta$ then $T \vdash \alpha$ or $T \vdash \beta$.*
(3) *If φ and ψ are existential L-sentences such that $T \cup \{\varphi\}$ and $T \cup \{\psi\}$ are consistent then $T \cup \{\varphi, \psi\}$ is consistent.*

Theorem 9. [18] *The following are equivalent:*

(1) *T has the Amalgamation property;*
(2) *for all universal L-formulas $\alpha_1(\overline{x}), \alpha_2(\overline{x})$ with $T \vdash \forall x(\alpha_1(x) \vee \alpha_2(x))$ there are existential L-sentences $\beta_1(\overline{x}), \beta_2(\overline{x})$ such that*

$$T \vdash \forall x(\beta_i(x) \to \alpha_i(x)),\ i = 1, 2,$$

and

$$T \vdash \forall x(\beta_1(x) \vee \beta_2(x)).$$

Now let us show some results on the closedness of the union of Jonsson theories regarding JEP and AP. Let T_1, T_2 be some L-theories such that $T_1 \cup T_2$ is consistent and there is $M \in Mod(T_1 \cup T_2)$ such that M is an infinite model. Then Theorems 10-12 are true.

Theorem 10. *$T_1 \cup T_2$ admits the joint embedding property.*

Proof. To prove this theorem we use the syntactic criterion of JEP, particularly, p.3 of Theorem 8. Let φ and ψ be existential sentences such that $T_1 \cup T_2 \cup \{\varphi\}$ and $T_1 \cup T_2 \cup \{\psi\}$ are consistent. Then it follows from $T_1 \cup T_2 \cup \{\varphi\}$ that $T_1 \cup \{\varphi\}$ and $T_2 \cup \{\varphi\}$ are consistent. Similarly, from $T_1 \cup T_2 \cup \{\psi\}$ it follows that $T_1 \cup \{\psi\}$ and $T_2 \cup \{\psi\}$ are consistent. Therefore, due to the fact that T_1 and T_2 are Jonsson theories and admit JEP, $T_1 \cup \{\varphi, \psi\}$ and $T_2 \cup \{\varphi, \psi\}$ are consistent. From this, it follows that $Cn(T_1) \cup \{\varphi, \psi\}$ and $Cn(T_2) \cup \{\varphi, \psi\}$ are consistent sets of sentences. By Compactness Theorem, $Cn(T_1) \cup Cn(T_2) \cup \{\varphi, \psi\}$ is a consistent set of sentences. Since $T_1 \cup T_2 \subseteq Cn(T_1) \cup Cn(T_2)$ then $T_1 \cup T_2 \cup \{\varphi, \psi\}$ is also consistent. Thus $T_1 \cup T_2$ admits JEP. □

Theorem 11. *$T_1 \cup T_2$ admits the amalgamation property.*

Proof. To prove this, we will use the syntactic criterion of AP, that is Theorem 9. Let $T_1 \cup T_2 \vdash \forall x(\alpha_1(x) \vee \alpha_2(x))$, where $\alpha_1(x)$ and $\alpha_2(x)$ are universal L-formulas. Suppose $T_1 \nvdash \forall x(\alpha_1(x) \vee \alpha_2(x))$ and $T_2 \nvdash \forall x(\alpha_1(x) \vee \alpha_2(x))$. It means that $T_1 \cup \{\neg\forall x(\alpha_1(x) \vee \alpha_2(x))\}$ and $T_2 \cup \{\neg\forall x(\alpha_1(x) \vee \alpha_2(x))\}$ are consistent and. consequently, $Cn(T_1) \cup \{\neg\forall x(\alpha_1(x) \vee \alpha_2(x))\}$ and $Cn(T_2) \cup \{\neg\forall x(\alpha_1(x) \vee \alpha_2(x))\}$ are consistent. Therefore the set $Cn(T_1) \cup Cn(T_2) \cup \{\neg\forall x(\alpha_1(x) \vee \alpha_2(x))\}$ is consistent. But $T_1 \cup T_2$ is a subset of $Cn(T_1) \cup Cn(T_2)$, so $T_1 \cup T_2 \cup \{\neg\forall x(\alpha_1(x) \vee \alpha_2(x))\}$ has to be consistent, which is impossible, because $T_1 \cup T_2 \vdash \forall x(\alpha_1(x) \vee \alpha_2(x))$. Thus $T_1 \vdash \forall x(\alpha_1(x) \vee \alpha_2(x))$ or $T_2 \vdash \forall x(\alpha_1(x) \vee \alpha_2(x))$. At the first case, there are such existential L-sentences $\beta_1(x)$ and $\beta_2(x)$ such that $T_1 \vdash \forall x(\beta_1(x) \vee \beta(x))$, and $T_1 \vdash \forall x(\beta_1(x) \to \alpha_1(x))$, and $T_1 \vdash \forall x(\beta_2(x) \to \alpha_2(x))$, since T_1 is Jonsson and admits JEP. Hence $T_1 \cup T_2 \vdash \forall x(\beta_1(x) \vee \beta(x))$, and $T_1 \cup T_2 \vdash \forall x(\beta_1(x) \to \alpha_1(x))$, and $T_1 \cup T_2 \vdash \forall x(\beta_2(x) \to \alpha_2(x))$ for the same existential formulas $\beta_1(x)$ and $\beta_2(x)$. Thus $T_1 \cup T_2$ admits AP. Similarly, if $T_2 \vdash \forall x(\alpha_1(x) \vee \alpha_2(x))$ then $T_1 \cup T_2$ also admits AP. □

Note that the theory $T_1 \cup T_2$ is inductive. Then, by virtue of these facts, and Theorems 10 and 11, and Definition 4, we obtain the following result.

Theorem 12. *$T_1 \cup T_2$ is a Jonsson theory.*

Now we move to considering the algebraic structure of a Jonsson spectrum. Let us consider some fixed class K of L-structures, such that there is an infinite structure $M \in K$, and the Jonsson spectrum $JSp(K)$. By $Th_{\forall\exists}(K)$, we denote a theory consisting of all propositions true for each model $A \in K$.

Lemma 1. *$T \in JSp(K)$ if and only if T is a Jonsson theory and $T \subseteq Th_{\forall\exists}(K)$.*

Proof. $\rightarrow$ Firstly, note that all Jonsson theories are $\forall\exists$-axiomatizable. Let $T \in JSp(K)$, then T, by Definition 9, must be Jonsson. Now suppose that $T \nsubseteq Th_{\forall\exists}(K)$. Then T may contain some sentences from $Th_{\forall\exists}(K)$ (at least tautologies), as well as it contains some other sentences that are not in $Th_{\forall\exists}(K)$. That is, T can be represented by the form of

$$T = \Gamma_1 \cup \Gamma_2,$$

where $\Gamma_1 \subset Th_{\forall\exists}(K)$, $\Gamma_2 \nsubseteq Th_{\forall\exists}(K)$, Γ_2 is a set of $\forall\exists$-sentences. But, since it is not true that, for any $A \in K$, $A \models \Gamma_2$, there is such a model $A' \in K$, that $A' \nvDash T$. Hence, $T \notin JSp(K)$. Thus, we have come to a contradiction. So, $T \subseteq Th_{\forall\exists}(K)$.

$\leftarrow$ Let T be a Jonsson theory, and let $T \subseteq Th_{\forall\exists}(K)$. From the latter it follows that $K \subseteq Mod(T)$, which means that for any model $A \in K$ it is true that $A \in Mod(T)$. Adding to this the fact that T is Jonsson, we get by Definition 9 that $T \in JSp(K)$. □

It is known [4] that the empty theory is a Jonsson theory. Then the following is obvious by Lemma 1:

Corollary 4. *The empty theory is included in $JSp(K)$.*

Let $\Delta_1, \Delta_2 \in JSp(K)$. Then, by Lemma 1, $\Delta_1 \subseteq Th_{\forall\exists}(K)$ and $\Delta_2 \subseteq Th_{\forall\exists}(K)$. Obviously, $\Delta_1 \cup \Delta_2 \subseteq Th_{\forall\exists}(K)$. From this we can conclude that the theory of $\Delta_1 \cup \Delta_2$ is consistent, since its superset $Th_{\forall\exists}(K)$ is a consistent set of sentences. Then for this case we can apply Theorems 10 and 11 and get the following fact:

Corollary 5. *Let $\Delta_1, \Delta_2 \in JSp(K)$. Then $\Delta_1 \cup \Delta_2$ has AP and JEP.*

In addition, it is easy to see that the theory $\Delta_1 \cup \Delta_2$ is inductive and has infinite models. From the above we get

Corollary 6. *Let $\Delta_1, \Delta_2 \in JSp(K)$. Then $\Delta_1 \cup \Delta_2 \in JSp(K)$.*

The following theorem provides information about the structure of the Jonsson spectrum of the class K as an algebra.

Theorem 13. *Let L be a first-order language, and let K be a fixed class of L-structures, $JSp(K)$ be its Jonsson spectrum. Then $(JSp(K), \cup)$ is a commutative monoid.*

Proof. According to Theorem 6, for any $\Delta_1, \Delta_2 \in JSp(K)$ $\Delta_1 \cup \Delta_2 \in JSp(K)$, i.e. the operation of union of theories (as for sets) of the Jonsson spectrum is closed. The associativity and commutativity of this operation is obvious. The single element is an empty theory according to Corollary 4. □

Now let us consider the structure properties of an arbitrary cosemanticness class in the Jonsson spectrum.

Let us introduce the operations "$\vee$"and "$\wedge$"for theories as follows. Let T and T' be L-theories. Let $T \wedge T' = \{\varphi \wedge \varphi' \mid \varphi \in T, \varphi' \in T'\}$, if this theory is consistent. It is easy to see that this theory is logically equivalent to $T \cup T'$, and the class of models of this theory consists from L-structures that are models of T and T' simultaneously. Similarly, let $T \vee T' = \{\varphi \vee \varphi' \mid \varphi \in T, \varphi' \in T'\}$. The class of models of $T \vee T'$ is represented by L-structures that are models of T or T'. Note that this theory is not logically equivalent to $T \cap T'$ in general case. Let us demonstrate it by the following example.

Let L be the language of fields, T be the theory of fields of characteristic 2 and T' be the theory of fields of characteristic 3. Then $T \cap T'$ is the theory of fields. The class of models of $T \cap T'$ contains fields of any characteristic p, where p is zero or a prime number, whereas the models of $T \vee T'$ are fields of characteristic 2 or 3.

The following theorem is well-known and was proved by Ye.T. Mustafin.

Theorem 14. [4] *Let T and T' be cosemantic Jonsson theories. Then $T \vee T'$ is also a Jonsson theory that is cosemantic to T and T'.*

The author notes in [4] that, when T and T' are Jonsson theories such that T is not cosemantic with T', $T \vee T'$ is not a Jonsson theory in general case. It follows from this fact that $JSp(K)$ is generally not closed with respect to the operation "$\vee$".

Let K be a class of L-structures, $JSp(K)_{/\bowtie}$ be a factor-set of its Jonsson spectrum with respect to cosemanticness, $[T] \in JSp(K)_{/\bowtie}$. Then we have the following result.

Theorem 15. *Each cosemanticness class $[T] \in JSp(K)_{/\bowtie}$ is a lattice with respect to operations "$\vee$"and "$\wedge$".*

Proof. We need to show the closedness of $[T]$ with respect to "$\vee$"and "$\wedge$". Let $T_1, T_2 \in [T]$. By Theorem 14, $T_1 \vee T_2$ is a Jonsson theory that is cosemantic to T_1 and T_2. In addition, it is obvious that $T_1 \vee T_2 \subset Th_{\forall\exists}(K)$, so $T_1 \vee T_2 \in [T]$. As it is noted above, $T_1 \wedge T_2 = T_1 \cup T_2$. According to Theorem 13, $T_1 \wedge T_2 \in JSp(K)$. Since $T_1 \in T_1 \wedge T_2$ and $T_2 \in T_1 \cup T_2$, we can apply Theorem 6 and obtain that $T_1 \wedge T_2 \bowtie T_1$ and $T_1 \wedge T_2 \bowtie T_2$. Therefore $T_1 \wedge T_2 \in [T]$. □

4 Further results on Jonsson spectrum

In this section, we generalize some well-known theorems on Jonsson theories considering the cosemanticness classes of these theories regarding some fixed Jonsson spectrum. Actually, the results of this section represents some additional information on the structure of the Jonsson spectrum and assist to describe the connection between different cosemanticness classes.

Previously, in [5], it was defined the concept of a binary J-equivalence relation for L-structures. Later, the first author of this paper proposed the notion of cosemanticness of L-structures, mentioned before, which became a more preferred tool for studying models and their theories by some reasons. One of these reasons is that the technique of describing the models through their Jonsson spectra seems to us more prospective in the sense of developing of apparatus for studying Jonsson spectrum. Another reason is the formulation essence of the notion of cosemanticness of models. However, there is a direct connection between the concepts of Jonsson equivalence and cosemanticness, which we demonstrate in this Section.

Let us describe the notion of Jonsson equivalence.

Definition 13. *Let A and B be L-structures. A and B are said to be Jonsson equivalent, if, for any Jonsson L-theory T,*

$$A \models T \Leftrightarrow B \models T.$$

In this paper, we define the Jonsson equivalence for two classes of L-structures. Let us do it as follows.

Definition 14. *The classes K_1 and K_2 of L-structures are called Jonsson equivalent (denoted by "$K_1 \equiv_J K_2$"), if the following holds for any Jonsson L-theory T:*

$$K_1 \models T \Leftrightarrow K_2 \models T.$$

The following proposition connects the Jonsson equivalence of two classes of L-structures with their Jonsson spectra.

Proposition 3. *Let K_1 and K_2 be two classes of L-structures. Then*

$$K_1 \equiv_J K_2 \Leftrightarrow JSp(K_1) = JSp(K_2).$$

Proof. Let $K_1 \equiv_J K_2$ and let T_1 be an arbitrary Jonsson theory such that $K_1 \models T_1$. Then $T \in JSp(K_1)$. But $K_2 \models T$ as well, then $T \in JSp(K_2)$. It follows that $JSp(K_1) \subseteq JSp(K_1)$ and, by analogy, vice versa. Now let $JSp(K_1) = JSp(K_2)$. All Jonsson theories T_i such that $K_1 \models T_i$ and $K_2 \models T_i$ are in $JSp(K_1)$ and $JSp(K_2)$, so $K_1 \equiv_J K_2$. □

In [5], one can find some interesting facts on the connection between two models with respect to the their equivalence relation. Here we obtain some generalization of that theorems concerning the classes of models instead of single models and the cosemanticness classes instead of single theories.

Again, let K be an arbitrary class of L-structures, then $JSp(K)_{/\bowtie}$ is a factor-set of the Jonsson spectrum of K. Let $[T] \in JSp(K)_{/\bowtie}$. In Theorem 2, it was shown that, for any two Jonsson theories T_1 and T_2, $T_1 \bowtie T_2$ iff $E_{T_1} = E_{T_2}$. It means that, for any theories T and T' from $[T] \in JSp(K)_{/\bowtie}$, $E_T = E_{T'}$, i.e., for any cosemanticness class there is the class of existentially closed models. Let us denote it as $E_{[T]}$.

The following fact is well-known.

Proposition 4. [5, p.161] *If T is an $\forall$-complete Jonsson theory then, for any infinite model A of T, $T^0(A)$ is a Jonsson theory.*

The following theorem shows the connection between the classes of existentially closed models of two cosemanticness classes and their Kaiser hulls.

Theorem 16. *Let $[T_1], [T_2] \in JSp(K)_{/\bowtie}$, $K_1 \subseteq E_{[T_1]}$ and $K_2 \subseteq E_{[T_2]}$. Then the following conditions are equivalent:*
1) $K_1 \equiv_J K_2$;
2) $T^0(K_1) = T^0(K_2)$.

Proof. The implication (2) $\to$ (1) is obvious due the inductiveness of any theory T such that $K_1 \models T$ or $K_2 \models T$. Now we show (1) $\to$ (2). According to Proposition 3, (1) is equivalent to the fact that $JSp(K_1) = JSp(K_2)$. In force of Theorem 6, $T^0(C_{[T_1]})$ and $T^0(C_{[T_2]})$ are Jonsson theories. Besides, for any $A \in K_1$, $A \models T^0(C_{[T_1]})$ and, for any $B \in K_2$, $B \models T^0(C_{[T_2]})$ by Theorem 3. Therefore $T^0(C_{[T_1]}) \in JSp(K_1)$ and $T^0(C_{[T_2]}) \in JSp(K_1)$, which means, by Lemma 1, $T^0(C_{[T_1]}) \subseteq T^0(K_2)$ and $T^0(C_{[T_2]}) \subseteq T^0(K_1)$. According to Theorem 3, $T^0(C_{[T_1]}) = T^0(K_1)$ and $T^0(C_{[T_2]}) = T^0(K_2)$, so $T^0(C_{[T_1]}) \subseteq T^0(C_{[T_2]})$ and $T^0(C_{[T_2]}) \subseteq T^0(C_{[T_1]})$. Thus $T^0(C_{[T_1]}) = T^0(C_{[T_2]})$ and $T^0(K_1) = T^0(K_2)$. □

When studying the structure of the Jonsson spectrum, we always need some tools that allow to identify the specific difference between cosemanticness classes. The following proposition and corollary show when the cosemanticness classes coincide.

Proposition 5. *Let $[T_1], [T_2] \in JSp(K)_{/\bowtie}$, $C_{[T_1]}$ and $C_{[T_2]}$ be semantic models of the classes $[T_1]$ and $[T_2]$, correspondingly. Let $C_{[T_1]} \models T_2$ for some $T_2 \in [T_2]$, $C_{[T_2]} \models T_1$ for some $T_1 \in [T_1]$. Then the classes $[T_1]$ and $[T_2]$ coincide.*

Proof. Let us consider the theories $T_1 \in [T_1]$ and $T_2 \in [T_2]$. Then $T' = T_1 \cup T_2$ is a Jonsson theory by Theorem 12. As $T_1 \subseteq T'$ and $C_{[T_1]} \models T'$, $C_{[T_1]}$ is a semantic model of T' in force of Theorem 6. Similarly, $C_{[T_2]}$ is also a semantic models of T'. It means that $C_{[T_1]}$ and $C_{[T_2]}$ are both semantic models of T', and $T_1 \bowtie T', T_2 \bowtie T'$. Consequently, $T_1 \bowtie T_2$ and $[T_1] = [T_2]$. □

Corollary 7. *Let $[T_1], [T_2] \in JSp(K)_{/\bowtie}$, $M \in E_{[T_1]}$ and $M \in E_{[T_2]}$. Then the classes $[T_1]$ and $[T_2]$ coincide.*

Proof. It is true that $C_{[T_1]} \equiv_{\forall\exists} M$ and $C_{[T_2]} \equiv_{\forall\exists} M$ by Theorem 3. It means that $C_{[T_1]} \models T$ for any $T \in [T_2]$ and $C_{[T_2]} \models T'$ for any $T' \in [T_1]$. So, by Proposition 5, $[T_1] = [T_2]$. □

And finally, we have the following theorem, which shows the connection of two cosemanticness classes by the relations of cosemanticness, Jonsson equivalence and equality between their semantic models.

Theorem 17. *Let* $[T_1], [T_2] \in JSp(K)_{/\bowtie}$, $C_{[T_1]}$ *and* $C_{[T_2]}$ *be the semantic models of the classes* $[T_1]$ *and* $[T_2]$*, correspondingly. Then the following conditions are equivalent:*
1) $C_{[T_1]} \bowtie C_{[T_2]}$*;*
2) $C_{[T_1]} \equiv_J C_{[T_2]}$*;*
3) $C_{[T_1]} = C_{[T_2]}$.

Proof. The equivalence (1) → (2) is obvious by Proposition 3. The implication (3) → (1) is also obvious. We have to prove (2) → (3), or $[T_1] = [T_2]$, which is equivalent. Let $C_{[T_1]} \bowtie C_{[T_2]}$, which means that $JSp(C_{[T_1]}) = JSp(C_{[T_2]})$ by Definition 9. It is easy to see that all theories from $[T_1]$ are in $JSp(C_{[T_1]})$ and all theories from $[T_2]$ are in $JSp(C_{[T_2]})$. Since $JSp(C_{[T_1]}) = JSp(C_{[T_2]})$, $[T_2] \subseteq JSp(C_{[T_2]})$ and $[T_2] \subseteq JSp(C_{[T_1]})$. Consequently, $C_{[T_1]} \models T_2$ for any $T_2 \in [T_2]$ and $C_{[T_2]} \models T_1$ for any $T_1 \in [T_1]$. By Proposition 5, $[T_1] = [T_2]$. □

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