S©MR

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ
Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru
ISSN 1813-3304

Vol. 21, No. 2, pp. 882–896 (2024)
https://doi.org/10.33048/semi.2024.21.058

УДК 517.987
MSC 28A80, 40A05

SEQUENTIAL LABYRINTH FRACTALS

HARSHA GOPALAKRISHNAN AND SRIJANANI ANURAG PRASAD

Communicated by E.S. Doubtsov

Abstract: This paper introduces the concept of a sequential labyrinth fractal constructed on a unit square using two sequences. It also explains how the obtained fractal differs from the classical labyrinth fractal, mixed labyrinth fractal and supermixed labyrinth fractal. The Hausdorff and the box-counting dimension of sequential labyrinth fractals, which are constructed using convergent sequences, are also examined. Besides that, it gives the dimension of fractals on the unit square, which are generated from converging sequences with or without having the labyrinth conditions.

Keywords: Fractals; Labyrinth fractals; Hausdorff dimension; Box-counting dimension; Sequences.

1 Introduction

In the literature, there are various methods for the construction of fractals. Some such techniques include fractals generated by iterated function systems, fractals generated by finite subdivisions and so on. This paper deals with fractals constructed on a unit square by the finite subdivision method, where the division is characterised by a sequence. Besides that, we are imposing extra conditions on this construction to produce a particular type of fractal called sequential labyrinth fractal.

Gopalakrishnan, Harsha, Prasad, Srijanani Anurag, Sequential Labyrinth Fractals.

© 2023 Harsha Gopalakrishnan, Srijanani Anurag Prasad.

The work of Harsha Gopalakrishnan is supported by PMRF (grant number: 3201486).

Received February, 20, 2023 , Published November, 1, 2024.

A fractal using finite subdivision on the unit square was introduced and studied by Bedford [1] and McMullen [2] simultaneously in 1984, and such fractals are known as Bedford-McMullen carpet. The ratio of division is preserved at each stage of construction. It can be seen in the literature that there exists a collection of fractals using different kinds of subdivisions in the unit square, such as Lalley-Gatzouras carpets [3], Baranski carpets [4] and so on. Those can be considered as generalizations of the Bedford-McMullen carpet.

Cristea and Stiensky introduced the concept of labyrinth fractal on the unit square by the finite subdivision of unit square into m^2 squares and by choosing a set of squares with tree property, corner property and exit property [5, 6]. This process is continued on each chosen square to obtain the classical labyrinth fractal. This kind of fractal has further extensions, such as mixed [7] and supermixed [8] labyrinth fractals. In mixed labyrinth fractals, different division ratios and patterns are used at different stages of iteration and in supermixed cases, different patterns are used within a particular stage. But in each of these cases, the ratio of division remains the same at a particular stage or for a specific pattern. In [9], Cristea and Leobacher studied some properties of mixed labyrinth fractals. Labyrinth fractals are further constructed in triangles [10] and quadrilaterals [11]. In this paper, a labyrinth fractal is constructed on the unit square using two sequences of tuples.

Labyrinth fractals on a square have broad applications in Physics. They are used in the study of planar nanostructures [12], the fractal reconstruction of complicated images, signals and radar backgrounds [13] and the construction of prototypes of ultra-wide band radar antennas [14] . Besides, labyrinth fractals have applications in telecommunication [15].

The organization of the paper is as follows: Section 2 gives the construction of a sequential labyrinth fractal using two sequences of k-tuple and l-tuple. The construction uses the finite subdivision on the unit square, resulting in a set of rectangles. From the smaller rectangles obtained at each stage, a set of rectangles is chosen such that they satisfy tree property, corner property and exit property. This collection is said to be a sequential labyrinth set of respective order, and its limit is defined as the sequential labyrinth fractal. Sections 3 and 4 mainly deal with the converging sequence. In section 3, we begin with two sequences of k-tuple, which converge to the k-tuple of $1/k$ and examine whether the corresponding fractal coincides with the classical $k \times k$ labyrinth fractal given in [5] and [6]. Section 4 is devoted to finding both the box-counting dimension and the Hausdorff dimension of the sequential labyrinth fractal, which are generated from converging sequences.

2 Labyrinth fractal on a unit square generated by two sequences

This section gives the construction of sequential labyrinth fractals on a unit square. It can be seen that the classical labyrinth fractal is a particular case of the sequential labyrinth fractal.

We let I denote the unit square $[0,1] \times [0,1]$ throughout this paper. Now, define two sequences of k-tuple and l-tuple with some specific properties as given in Definition 1. These sequences will play an important role in constructing sequential labyrinth fractals since they act as the contraction factor in the construction.

Definition 1. *Let $\{\lambda_n\}$ be a sequence of k-tuples $(k \geq 3)$, say $\lambda_n = (\lambda_n^1, \lambda_n^2, \ldots, \lambda_n^k)$ with $0 < \lambda_n^i < 1$ for $i = 1, 2, \ldots, k$ and $\{\gamma_n\}$ be a sequence of l-tuples $(l \geq 3)$, say $\gamma_n = (\gamma_n^1, \gamma_n^2, \ldots, \gamma_n^l)$ with $0 < \gamma_n^j < 1$ with $j = 1, 2, \ldots, l$ such that $\sum_{i=1}^{k} \lambda_n^i = 1$ and $\sum_{j=1}^{l} \gamma_n^j = 1$ for all $n \geq 1$.*

Now, we define a class of functions using the sequences defined in Definition 1.

Definition 2. *Let $\{\lambda_n\}$ and $\{\gamma_n\}$ be the sequences of k-tuple and l-tuple as given in Definition 1. For $1 \leq i \leq k$,$1 \leq j \leq l$ and $n \geq 1$, define the functions $P_n^{i,j} : I \to I$ as,*

$$P_n^{i,j}(x,y) = (\lambda_n^i x, \gamma_n^j y) + (\lambda_n^1 + \lambda_n^2 + \ldots + \lambda_n^{i-1}, \gamma_n^1 + \gamma_n^2 + \ldots + \gamma_n^{j-1}).$$

The next theorem shows that the functions defined in Definition 2 are contractions.

Theorem 1. *For $1 \leq i \leq k, 1 \leq j \leq l$ and $n \geq 1$, $P_n^{i,j}$ is a contraction in the Euclidean metric.*

Proof. Let d represent the Euclidean metric in $\mathbb{R}^2$. Then, we have

$$\begin{aligned} d(P_n^{i,j}(x,y), P_n^{i,j}(x',y')) &= d((\lambda_n^i x, \gamma_n^j y), (\lambda_n^i x', \gamma_n^j y')) \\ &= ||(\lambda_n^i (x - x'), \gamma_n^j (y - y'))|| \\ &\leq \max\{\lambda_n^i, \gamma_n^j\} ||(x - x', y - y')|| \\ &= \max\{\lambda_n^i, \gamma_n^j\} d((x,y),(x',y')). \end{aligned}$$

Since $0 < \lambda_n^i, \gamma_n^j < 1$, $0 < \max\{\lambda_n^i, \gamma_n^j\} < 1$ and hence $P_n^{i,j}$ is a contraction for $1 \leq i \leq k, 1 \leq j \leq l$ and $n \geq 1$. □

Now, we construct two sequences $\{\mu_n\}$ and $\{\delta_n\}$ from the sequences $\{\lambda_n\}$ and $\{\gamma_n\}$ respectively. $\{\mu_n\}$ (respectively $\{\delta_n\}$) is a sequence of sequences and for each μ_n (respectively δ_n), after the $(k^n)^{th}$ (respectively $(l^n)^{th}$) element, all entries will be zero. So $\{\mu_n\}$ and $\{\delta_n\}$ are sequences of elements from c_{00}, where $c_{00} = \{x \in \ell^p :$ all but finitely many $x(j)'$s are equal to zero$\}$. The sequences $\{\mu_n\}$ and $\{\delta_n\}$ are explicitly defined as follows: The i^{th} term of μ_n is

$$\mu_n^i = \begin{cases} \lambda_1^{i_1}\lambda_2^{i_2}\dots\lambda_{n-1}^{i_{n-1}}\lambda_n^{i-sk} & \text{if } 1 \leq i \leq k^n \\ 0 & \text{otherwise} \end{cases}, \tag{1}$$

where $i_t = \text{Rem}\Big(\frac{\left[\frac{i-1}{k^{n-t}}\right]}{k}\Big) + 1$ for $t = 1, 2, \dots, n-1$ and $s = \left[\frac{i-1}{k}\right]$. The j^{th} term of δ_n is

$$\delta_n^j = \begin{cases} \gamma_1^{j_1}\gamma_2^{j_2}\dots\gamma_{n-1}^{j_{n-1}}\gamma_n^{j-vk} & \text{if } 1 \leq j \leq l^n \\ 0 & \text{otherwise} \end{cases}, \tag{2}$$

where $j_t = \text{Rem}\Big(\frac{\left[\frac{j-1}{l^{n-t}}\right]}{l}\Big) + 1$ for $t = 1, 2, \dots, n-1$ and $v = \left[\frac{j-1}{l}\right]$. Here, Rem denotes the remainder function and is defined as $\text{Rem}\big(\frac{a}{b}\big) :=$ The remainder obtained when b divides a, for any two integers a and b. Also, for any real number x, $[x]$ denotes the greatest integer less than or equal to x.

The following proposition states that the sum of entries of each term in sequences $\{\mu_n\}$ and $\{\delta_n\}$ equals 1.

Proposition 1. *For each* $n \geq 1$, $\sum\limits_{i=1}^{\infty} \mu_n^i = 1$ *and* $\sum\limits_{j=1}^{\infty} \delta_n^j = 1$.

Proof. Consider the case $n = 1$. Clearly, $\Sigma_{i=1}^{\infty}\mu_1^i = \Sigma_{i=1}^{k}\mu_1^i = \Sigma_{i=1}^{k}\lambda_1^i = 1$. For any n, consider,

$$\begin{aligned}
\sum_{i=1}^{\infty}\mu_n^i &= \sum_{i=1}^{k^n}\mu_n^i \\
&= \sum_{i=1}^{k}\mu_n^i + \sum_{i=k+1}^{2k}\mu_n^i + \ldots + \sum_{i=(k-1)k+1}^{k^2}\mu_n^i + \sum_{i=k^2+1}^{k^2+k}\mu_n^i + \ldots + \\
&\quad \sum_{i=2k^2-k+1}^{2k^2}\mu_n^i + \ldots + \sum_{i=k^n-k^{n-1}+1}^{k^n-k^{n-1}+k}\mu_n^i + \ldots + \sum_{i=k^n-k+1}^{k^n}\mu_n^i \\
&= \sum_{i=1}^{k}\lambda_1^1\lambda_2^1\ldots\lambda_{n-1}^1\lambda_n^i + \sum_{i=k+1}^{2k}\lambda_1^1\lambda_2^1\ldots\lambda_{n-2}^1\lambda_{n-1}^2\lambda_n^{i-k} + \ldots + \\
&\quad \sum_{i=(k-1)k+1}^{k^2}\lambda_1^1\lambda_2^1\ldots\lambda_{n-2}^1\lambda_{n-1}^k\lambda_n^{i-(k-1)k} + \\
&\quad \sum_{i=k^2+1}^{k^2+k}\lambda_1^1\lambda_2^1\ldots\lambda_{n-2}^2\lambda_{n-1}^1\lambda_n^{i-k^2} + \ldots + \\
&\quad \sum_{i=2k^2-k+1}^{2k^2}\lambda_1^1\lambda_2^1\ldots\lambda_{n-2}^2\lambda_{n-1}^k\lambda_n^{i-(2k^2-k)} + \ldots + \\
&\quad \sum_{i=k^n-k^2+1}^{k^n-k^2+k}\lambda_1^k\lambda_2^k\ldots\lambda_{n-1}^1\lambda_n^{i-(k^n-k^2)} + \ldots + \\
&\quad \sum_{i=k^n-k+1}^{k^n}\lambda_1^k\ldots\lambda_{n-1}^k\lambda_n^{i-(k^n-k)} \\
&= \sum_{i=1}^{k}\lambda_1^1\lambda_2^1\ldots\lambda_{n-2}^1\lambda_{n-1}^i + \sum_{i=1}^{k}\lambda_1^1\lambda_2^1\ldots\lambda_{n-3}^1\lambda_{n-2}^2\lambda_{n-1}^i + \ldots \\
&\quad + \sum_{i=1}^{k}\lambda_1^k\lambda_2^k\ldots\lambda_{n-2}^k\lambda_{n-1}^i \\
&\quad \vdots \\
&= \lambda_1^1 + \lambda_1^2 + \ldots\lambda_1^k \\
&= 1.
\end{aligned}$$

□

Next, we will define a class of functions using the sequences $\{\mu_n\}$ and $\{\delta_n\}$ as follows:

Definition 3. *Let* $\{\mu_n\}$ *and* $\{\delta_n\}$ *be the sequences as given in Equations* (1) *and* (2)*, which are defined using* $\{\lambda_n\}$ *and* $\{\gamma_n\}$*. For each* $n \geq 1$ *and* $1 \leq i \leq k^n, 1 \leq j \leq l^n$*, the functions* $Q_n^{i,j} : I \to I$ *are defined as*
$Q_n^{i,j}(x,y) = (\mu_n^i x, \delta_n^j y) + (\mu_n^1 + \mu_n^2 + \ldots + \mu_n^{i-1}, \delta_n^1 + \delta_n^2 + \ldots + \delta_n^{j-1})$.

Each of the functions defined above is found to be a contraction and it is stated formally as a theorem as follows:

Theorem 2. $Q_n^{i,j}$ *is a contraction in Euclidean metric for* $n \geq 1$ *and* $1 \leq i \leq k^n, 1 \leq j \leq l^n$.

Proof. If d denotes the Euclidean metric, we obtain

$$\begin{aligned} d(Q_n^{i,j}(x,y), Q_n^{i,j}(x',y')) &= d((\mu_n^i x, \delta_n^j y), (\mu_n^i x', \delta_n^j y')) \\ &= ||(\mu_n^i (x-x'), \delta_n^j (y-y'))|| \\ &\leq \max\{\mu_n^i, \delta_n^j\} ||(x-x', y-y')|| \\ &= \max\{\mu_n^i, \delta_n^j\} d((x,y),(x',y')). \end{aligned}$$

Since $\mu_n^i = \lambda_1^{i_1}\lambda_2^{i_2}\ldots\lambda_n^{i_n}$ and $\delta_n^j = \gamma_1^{j_1}\gamma_2^{j_2}\ldots\gamma_n^{j_n}$ for some $i_1, i_2, \ldots, i_n \in \{1,2,\ldots,k\}$ and $j_1, j_2, \ldots, j_n \in \{1,2,\ldots,l\}$ and $0 < \lambda_i^j, \gamma_i^j < 1$, $0 < \mu_n^i, \delta_n^j < 1$ for all i, j and n. Hence, $0 < \max\{\mu_n^i, \delta_n^j\} < 1$. Thus, $Q_n^{i,j}$ is a contraction for all i, j and n. □

Remark 1. *Denote the set* $\{Q_m^{i,j}(I) : 1 \leq i \leq k^m, 1 \leq j \leq l^m\}$ *by* $\mathcal{S}_m$ *for any* $m \geq 1$*. It is easy to see that*
$\mathcal{S}_1 = \{Q_1^{i,j}(I) : 1 \leq i \leq k, 1 \leq j \leq l\} = \{P_1^{i,j}(I) : 1 \leq i \leq k, 1 \leq j \leq l\}$.

Definition 4. *For an* $m \geq 1$*, let* $\mathcal{W} \subseteq \mathcal{S}_m$*. The graph corresponding to* $\mathcal{W}$ *is denoted by* $\mathcal{G}(\mathcal{W})$ *and is defined as follows: The vertices of* $\mathcal{G}(\mathcal{W})$ *are elements of* $\mathcal{W}$ *and there exists an edge between two vertices of* $\mathcal{G}(\mathcal{W})$ *if and only if the corresponding elements in* $\mathcal{W}$ *have a common side.*

Definition 5. *Let* $A_m = \{(i,j) : 1 \leq i \leq k^m, 1 \leq j \leq l^m\}$*. Then, for a fixed* $\mathcal{W} \subseteq \mathcal{S}_m$*, let* $B_m = \{(i,j) \in A_m : Q_m^{i,j}(I) \in \mathcal{W}\}$.

Now, we shall define sequential labyrinth sets.

Definition 6. *A subset* $\mathcal{W} \subseteq \mathcal{S}_m$ *is said to be a sequential labyrinth set of order* m *corresponding to the sequences* $\{\mu_n\}$ *and* $\{\delta_n\}$ *if it satisfies the following three properties.*

(1) *Tree property :* $\mathcal{G}(\mathcal{W})$ *is a tree.*
(2) *Exit property : There exists exactly one* i *such that* $Q_m^{i,1}(I), Q_m^{i,l^m}(I) \in \mathcal{W}$ *and exactly one* j *such that* $Q_m^{1,j}(I), Q_m^{k^m,j}(I) \in \mathcal{W}$. $Q_m^{i,1}(I)$ *is called the bottom exit,* $Q_m^{i,l^m}(I)$ *is called the top exit,* $Q_m^{1,j}(I)$ *is called the left exit and* $Q_m^{k^m,j}(I)$ *is called the right exit.*
(3) *Corner property: There exists at most one element in* $\mathcal{W}_m$ *from each of the sets* $\{Q_m^{1,1}(I), Q_m^{k^m,l^m}(I)\}$ *and* $\{Q_m^{k^m,1}(I), Q_m^{1,l^m}(I)\}$. $Q_m^{1,1}(I)$ *is*

called the bottom-left corner, $Q_m^{k^m,l^m}(I)$ *is called the top-right corner,* $Q_m^{k^m,1}(I)$ *is called the bottom-right corner and* $Q_m^{1,l^m}(I)$ *is called the top-left corner.*

In this case, an element $W \in \mathcal{W} \subseteq \mathcal{S}_m$ *is said to be a white rectangle of order* m*.*

The next definition provides a particular way to choose a subset of $\mathcal{S}_m$ from a given subset of $\mathcal{S}_1$.

Definition 7. *Let* $\mathcal{W}_1 \subseteq \mathcal{S}_1$ *and let* B_1 *be taken as in Definition 5. Define* $\mathcal{W}_m = \{P_1^{i_1,j_1} \circ P_2^{i_2,j_2} \circ \ldots \circ P_m^{i_m,j_m}(I) : (i_1,j_1),(i_2,j_2),\ldots,(i_m,j_m) \in B_1\}$.

If a sequential labyrinth set of order 1 is chosen, Theorem 3 suggests to construct a sequential labyrinth set of order m for any $m \geq 2$.

Theorem 3. *If* $\mathcal{W}_1$ *is a sequential labyrinth set of order* 1 *corresponding to the sequence* $\{\mu_n\}$ *and* $\{\delta_n\}$*, then* $\mathcal{W}_m$ *will be a sequential labyrinth set of order* m *corresponding to* $\{\mu_n\}$ *and* $\{\delta_n\}$*.*

Proof. Here, we need to prove the 4 conditions below.

(1) $\mathcal{W}_m \subseteq \mathcal{S}_m$

To prove this, choose an element $P_1^{k_1,l_1} \circ P_2^{k_2,l_2} \circ \ldots \circ P_m^{k_m,l_m}(I) \in \mathcal{W}_m$. We should be able to write this as a $Q_m^{i,j}(I)$ for some $1 \leq i \leq k^m$, $1 \leq j \leq l^m$. For $(x,y) \in I$,

$$\begin{aligned}
&P_1^{k_1,l_1} \circ P_2^{k_2,l_2} \circ \ldots \circ P_m^{k_m,l_m}(x,y)\\
&= P_1^{k_1,l_1} \circ P_2^{k_2,l_2} \circ \ldots \circ P_{m-1}^{k_{m-1},l_{m-1}}\big((\lambda_m^{k_m}x, \gamma_m^{l_m}y)+\\
&\qquad (\lambda_m^1 + \ldots + \lambda_m^{k_m-1}, \gamma_m^1 + \ldots + \gamma_m^{l_m-1})\big)\\
&= P_1^{k_1,l_1} \circ P_2^{k_2,l_2} \circ \ldots \circ P_{m-2}^{k_{m-2},l_{m-2}}\big((\lambda_{m-1}^{k_{m-1}}\lambda_m^{k_m}x, \gamma_{m-1}^{l_{m-1}}\gamma_m^{l_m}y)\\
&\quad + (\lambda_{m-1}^{k_{m-1}}(\lambda_m^1 + \ldots + \lambda_m^{k_m-1}), \gamma_{m-1}^{k_{m-1}}(\gamma_m^1 + \ldots + \gamma_m^{l_m-1}))\\
&\quad + (\lambda_{m-1}^1 + \ldots + \lambda_{m-1}^{k_{m-1}-1}, \gamma_{m-1}^1 + \ldots + \gamma_{m-1}^{l_{m-1}-1})\big)\\
&\vdots\\
&= (\lambda_1^{k_1}\lambda_2^{k_2}\ldots\lambda_m^{k_m}x, \gamma_1^{l_1}\gamma_2^{l_2}\ldots\gamma_m^{l_m}y)+\\
&\quad (\lambda_1^{k_1}\ldots\lambda_{m-1}^{k_{m-1}}(\lambda_m^1 + \ldots + \lambda_m^{k_m-1}), \gamma_1^{l_1}\ldots\gamma_{m-1}^{l_{m-1}}(\gamma_m^1 + \ldots + \gamma_m^{l_m-1}))+\\
&\quad (\lambda_1^{k_1}\ldots\lambda_{m-2}^{k_{m-2}}(\lambda_{m-1}^1 + \ldots + \lambda_{m-1}^{k_{m-1}-1}), \gamma_1^{l_1}...\gamma_{m-2}^{l_{m-2}}(\gamma_{m-1}^1 + \ldots + \gamma_{m-1}^{l_{m-1}-1}))\\
&\quad + \ldots + (\lambda_1^{k_1}\lambda_2^{k_2}(\lambda_3^1 + \ldots + \lambda_3^{k_3-1}), \gamma_1^{l_1}\gamma_2^{l_2}(\gamma_3^1 + \ldots + \gamma_3^{l_3-1}))\\
&\quad + (\lambda_1^{k_1}(\lambda_2^1 + \ldots + \lambda_2^{k_2-1}), \gamma_1^{l_1}(\gamma_2^1 + \ldots + \gamma_2^{l_2-1}))\\
&\quad + (\lambda_1^1 + \ldots + \lambda_1^{k_1-1}, \gamma_1^1 + \ldots + \gamma_1^{l_1-1})\\
&= (\mu_m^p x, \delta_m^q y) + (\mu_m^1 + \mu_m^2 + \ldots + \mu_m^{p-1}, \delta_m^1 + \delta_m^2 + \ldots + \delta_m^{q-1})\\
&= (\mu_m^p x, \delta_m^q y) + (\mu_m^1 + \mu_m^2 + \ldots + \mu_m^{p-1}, \delta_m^1 + \delta_m^2 + \ldots + \delta_m^{q-1})\\
&= Q_m^{p,q}(x,y),
\end{aligned}$$

where
$p = (k_1 - 1)k^{m-1} + (k_2 - 1)k^{m-2} + \ldots + (k_{m-1} - 1)k + k_m$ and
$q = (l_1 - 1)l^{m-1} + (l_2 - 1)l^{m-2} + \ldots + (l_{m-1} - 1)l + l_m$.

(2) Exit property : If $P_1^{i,1}(I)$ and $P_1^{i,l}(I) \in \mathcal{W}_1$ forms unique bottom and top exits respectively in the first stage, $P_1^{i,1} \circ P_2^{i,1} \circ \ldots \circ P_m^{i,1}(I)$ and $P_1^{i,l} \circ P_2^{i,l} \circ \ldots \circ P_m^{i,l}(I)$ forms the bottom and top exits respectively in the m^{th} stage. Similar proof holds for unique pair of left and right exits.

(3) $\mathcal{G}(\mathcal{W}_m)$ is a tree: The connectedness is ensured by exit property. Also, there does not exist a cycle in $\mathcal{W}_1$ by definition. Assume that there does not exist a cycle till $\mathcal{W}_{m-1}$, for $m \geq 1$. Suppose there exists a cycle in the m^{th} stage. Let $C = \{a_0, a_1, \ldots, a_s\}$ be a cycle in $\mathcal{G}(\mathcal{W}_m)$. For each $a \in \mathcal{W}_m$, let $t(a)$ be an element in $\mathcal{W}_{m-1}$, which contains a. Let $j_0 = 0$,$b_0 = t(a_0)$ and $j_k = \min\{i : t(a_i) \neq b_{k-1}, j_{k-1} < i \leq s\}$, $b_k = t(a_{j_k})$ for $k \geq 1$. Choose r minimal such that the set $\{i : t(a_i) \neq b_r, j_r < i \leq s\}$ is empty. For $i = 1, 2, \ldots, r$ the vertex b_{i-1} is a neighbour of b_i in $\mathcal{G}(\mathcal{W}_{m-1})$. The set $\{b_0, b_1, \ldots, b_r\}$ can not contain a cycle by the induction hypothesis. Also, if $r = 0$, then all a_i's are contained in b_0 and it will contradict the fact that $\mathcal{G}(\mathcal{W}_1)$ is a tree. Hence $r \geq 1$. Thus, the graph induced by $\mathcal{G}(\mathcal{W}_{m-1})$ on the set $\{b_0, b_1, \ldots, b_r\}$ is a tree with more than one vertex. This implies that the cycle C returns to a_0 through the same side where it leaves a_0. But the exit property gives that $a_i = a_j$ for some $j \neq i$, $a_i, a_j \in \{a_0, a_1, \ldots, a_s\}$, which is a contradiction. Hence, $\mathcal{G}(\mathcal{W}_m)$ is a tree.

(4) Corner property: Since there exists at most one element from opposite corners in $\mathcal{W}_1$, there will exist at most one element from opposite corners in $\mathcal{W}_m$ also. That is, if $P_1^{1,1}(I) \in \mathcal{W}_1$ from $\{P_1^{1,1}(I), P_1^{k,l}(I)\}$, then $P_1^{1,1} \circ P_2^{1,1} \circ \ldots \circ P_m^{1,1}(I)$ will be the corner in $\mathcal{W}_m$ and the opposite corner will not be in $\mathcal{W}_m$.

□

Now, if we consider the subsequences $\{\mu_{nm}\} = \{\mu_m, \mu_{2m}, \mu_{3m} \ldots\}$ of $\{\mu_n\}$ and $\{\delta_{nm}\} = \{\delta_m, \delta_{2m}, \delta_{3m} \ldots\}$ of $\{\delta_n\}$, it can be seen that $\mathcal{W}_m$ as defined in Definition 7 will be a labyrinth set of order 1 for the subsequences.

Theorem 4. *If $\mathcal{W}_1$ is a sequential labyrinth set of order 1 corresponding to the sequences $\{\mu_n\}$ and $\{\delta_n\}$, then $\mathcal{W}_m$ is a sequential labyrinth set of order 1 corresponding to the subsequences $\{\mu_{nm}\}$ and $\{\delta_{nm}\}$.*

Proof. It is clear that $\mathcal{W}_m$ is a sequential labyrinth set from the proof of Theorem 3. That means it satisfies tree property, exit property and corner property. The white rectangle $\mathcal{W}_m$ corresponds to the terms μ_m and δ_m of the sequences $\{\mu_n\}$ and $\{\delta_n\}$ respectively. Also, μ_m and δ_m are the first term of the subsequences $\{\mu_{nm}\}$ and $\{\delta_{nm}\}$ respectively. Hence, clearly $\mathcal{W}_m$ is a sequential labyrinth set of order 1 corresponding to the subsequences $\{\mu_{nm}\}$ and $\{\delta_{nm}\}$. □

The main concept of this paper is sequential labyrinth fractal and it is defined as follows:

Definition 8. *If $\mathcal{W}_1$ is a sequential labyrinth set of order 1, the m^{th} stage pre-fractal associated with $\mathcal{W}_1$ is denoted by $\mathcal{L}_m$ and is defined as* $L_m = \bigcup\limits_{W_m \in \mathcal{W}_m} W_m$ *for all $m \geq 1$. The sequential labyrinth fractal is denoted by L_∞ and is defined as* $L_\infty = \bigcap\limits_{m=1}^{\infty} L_m$.

Now, we will provide an example of the construction. Note that it is different from the classical labyrinth fractal.

Example 1. *Consider the sequences $\{\lambda_n\}$ of 4-tuple and $\{\gamma_n\}$ of 3-tuple as follows:*
$\{\lambda_n\} = \{(\frac{1}{4}, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}), (\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{3}), (\frac{1}{4}, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}), (\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{3}), \ldots\}$
$\{\gamma_n\} = \{(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (\frac{3}{5}, \frac{1}{5}, \frac{1}{5}), (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}), \ldots\}$.
Then, the corresponding sequences $\{\mu_n\}$ and $\{\delta_n\}$ are $\{\mu_n\} = \big\{(\frac{1}{4}, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, 0, 0, ...), (\frac{1}{24}, \frac{1}{24}, \frac{1}{12}, \frac{1}{12}, \frac{1}{48}, \frac{1}{48}, \frac{1}{24}, \frac{1}{24}, \frac{1}{24}, \frac{1}{24}, \frac{1}{12}, \frac{1}{12}, \frac{3}{48}, \frac{3}{48}, \frac{3}{24}, \frac{3}{24}, 0, 0, \ldots), \ldots\big\}$ *and* $\{\delta_n\} = \big\{(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0, 0, \ldots), (\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, 0, 0, \ldots), \ldots\big\}$. *Choose* $\mathcal{W}_1 = \{P_1^{1,1}(I), P_1^{1,2}(I), P_1^{2,2}(I), P_1^{3,1}(I), P_1^{3,2}(I), P_1^{3,3}(I), P_1^{4,2}(I)\}$,*a sequential labyrinth set of order* 1 *and take $\mathcal{W}_m$ as in Definition 7. Then,* $L_\infty = \bigcap\limits_{m=1}^{\infty} \bigcup\limits_{W_m \in \mathcal{W}_m} W_m$ *is a sequential labyrinth fractal. The first three stages of this sequential labyrinth fractal are given in Figure 1.*

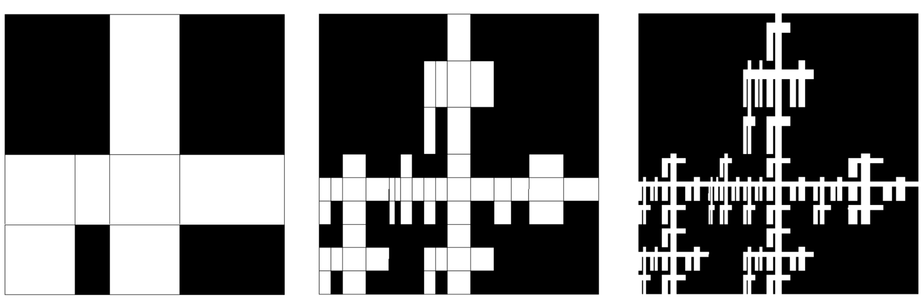

FIG. 1. An example for the first three stages of a sequential labyrinth fractal

Remark 2. *If $\{\lambda_n\}$ and $\{\gamma_n\}$ are constant sequences of k-tuples with each coordinate $\frac{1}{k}$ and let $\{\mu_n\}$ and $\{\delta_n\}$ be sequences constructed from $\{\lambda_n\}$ and $\{\gamma_n\}$ as in Equations (1) and (2). Then, the corresponding sequential labyrinth fractal is the same as the classical $k \times k$-labyrinth fractal on the unit square.*

Remark 3. *We claim that the sequential labyrinth fractal defined above does not fall in any of the existing case labyrinth fractals in the literature. The classical labyrinth fractal given in [5] and [6] uses a same division factor, say* $\frac{1}{m}$, $m \geq 4$, $m \in \mathbb{N}$ *at each stage, and the same pattern is used throughout the construction. The mixed labyrinth fractal defined in [7] uses a division factor* $\frac{1}{m_k}$ *for the* k^{th} *stage, where* $m_k \in \mathbb{N}$, $m_k \geq 4$ *and different patterns at different stages. The supermixed labyrinth fractal [8] uses different patterns within a particular stage, but again the division factor is of the form* $\frac{1}{m}$, $m \in \mathbb{N}$, $m \geq 4$ *at every stage. Note that the sequential labyrinth fractal does not assume any such restriction. The division depends on the sequence of tuples, where the entries in a tuple can be any number between* 0 *and* 1 *such that the sum of entries of each tuple is* 1. *So, this is a more general class of labyrinth fractal, where the division along the horizontal and the vertical axis are completely different, and at each stage, the division factor, which depends on the sequence, keeps on changing. But the pattern remains the same throughout. So, unlike the earlier cases, we obtained a set of different-sized rectangles at every stage instead of squares.*

3 Sequential labyrinth fractals generated from converging sequences

Let $\{\lambda_n\}$ and $\{\gamma_n\}$ be two sequences of k-tuples such that both $\{\lambda_n\}$ and $\{\gamma_n\}$ converges to $(\frac{1}{k}, \frac{1}{k}, \ldots, \frac{1}{k})$. In this section, we shall prove that the sequential labyrinth fractal generated from these sequences and the classical $k \times k$-labyrinth fractal (*i.e.*, the fractal generated from constant sequences) need not be the same.

Example 2. *Consider the sequences* $\{\lambda_n\} = \{(\frac{1}{4} - \frac{1}{4(n+1)}, \frac{1}{4} + \frac{1}{4(n+1)}, \frac{1}{4}, \frac{1}{4})\}$ *and* $\{\gamma_n\} = \{(\frac{1}{4} - \frac{1}{2^{n+2}}, \frac{1}{4}, \frac{1}{4} + \frac{1}{2^{n+2}}, \frac{1}{4})\}$. *Then, both the sequences* $\{\lambda_n\}$ *and* $\{\gamma_n\}$ *converges to* $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ *as* $n \to \infty$. *Let* $\{\mu_n\}$ *and* $\{\delta_n\}$ *be the sequences constructed as in Equations (1) and (2) . We will consider the first stage of a sequential labyrinth fractal generated from the sequences* $\{\mu_n\}$ *and* $\{\delta_n\}$. *Note that* $\mu_1 = (\frac{1}{8}, \frac{3}{8}, \frac{1}{4}, \frac{1}{4}, 0, 0, \ldots)$ *and* $\delta_1 = (\frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{4}, 0, 0, \ldots)$. *Choose* $\mathcal{W}_1 = \{P_1^{1,1}(I), P_1^{1,2}(I), P_1^{2,2}(I), P_1^{3,2}(I), P_1^{4,2}(I), P_1^{2,3}(I), P_1^{2,4}(I), P_1^{1,4}(I)\}$ *as the first stage of sequential labyrinth set and let* L_∞ *be the labyrinth fractal corresponding to* $\{\mu_n\}$ *and* $\{\delta_n\}$.

Let L^*_∞ *be the sequential labyrinth fractal generated from the constant sequence (*i.e., *the classical labyrinth fractal) with the position of whites in the first stage being the same as that of the position of whites in the first stage of sequential labyrinth set. It can be explicitely given as follows: Take* $\{\lambda^*_n\} = \{\gamma^*_n\} = \{(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})\}$ *and* $\{\mu^*_n\}$ *and* $\{\delta^*_n\}$ *are defined as in Equations (1) and (2) . Take the first stage as*
$\mathcal{W}^*_1 = \{P*_1^{1,1}(I), P*_1^{1,2}(I), P*_1^{2,2}(I), P*_1^{3,2}(I), P*_1^{4,2}(I), P*_1^{2,3}(I),$
$P*_1^{2,4}(I), P*_1^{1,4}(I)\}$ *and all these functions are defined as in Definition 2 and corresponding to* $\{\lambda^*_n\}$ *and* $\{\gamma^*_n\}$.

Even though the sequences $\{\lambda_n\}$ and $\{\gamma_n\}$ converge to $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, the fractals L_∞ and L^*_∞ are different. For this, consider the first stage of both L_∞ and L^*_∞, which are given in the Figure 2. Since $(\frac{1}{2}, \frac{1}{8})$ is the bottom-left

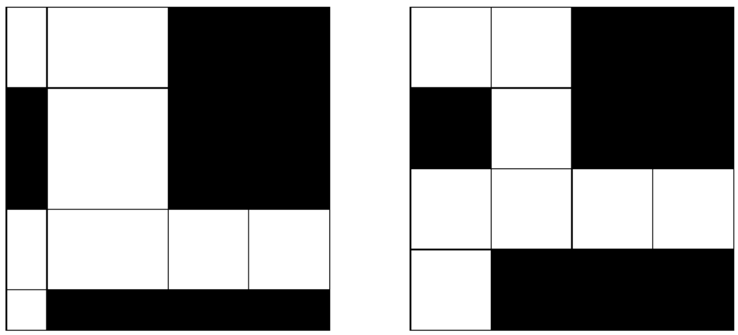

FIG. 2. The first stage of L_∞ and L^*_∞ respectively

corner of $P_1^{3,2}(I)$, and bottom left corner of first stage is a white rectangle, $(\frac{1}{2}, \frac{1}{8}) \in L_m$ for each m and hence $(\frac{1}{2}, \frac{1}{8}) \in L_\infty$. But $(\frac{1}{2}, \frac{1}{8})$ does not belongs to first stage of L^*_∞ and hence cannot belong to L^*_∞. Hence, L_∞ and L^*_∞ are different.

Even though these fractals are different, in the next section, we shall see that the box-counting dimensions of both fractals coincide.

4 Fractal dimension of sequential labyrinth fractal generated from converging sequences

This section examines the box-counting dimension and the Hausdorff dimension of the sequential labyrinth fractal corresponding to the converging sequences.

Theorem 5 provides the box-counting dimension of a sequential labyrinth fractal.

Theorem 5. *Let L_∞ be a sequential labyrinth fractal generated from the nonconstant sequences $\{\lambda_n\} = \{(\lambda_n^1, \lambda_n^2, ..., \lambda_n^k)\}$ and $\{\gamma_n\} = \{(\gamma_n^1, \gamma_n^2, ..., \gamma_n^k)\}$, both converges to the k-tuple $(\frac{1}{k}, \frac{1}{k}, \dots, \frac{1}{k})$, $k \geq 3$. Let $\{\mu_n\}$ and $\{\gamma_n\}$ be defined as in Equations 1 and 2. Let T be the number of white rectangles in the first stage of L_∞. Then, the box-counting dimension of L_∞ is $\frac{\log T}{\log k}$.*

Proof. Define,

$$\alpha_n = \max\left\{\sup_{1\le i\le k^n} \mu_n^i, \sup_{1\le i\le k^n} \delta_n^i\right\}$$

and

$$\beta_n = \min\left\{\inf_{1\le i\le k^n} \mu_n^i, \inf_{1\le i\le k^n} \delta_n^i\right\}$$

Since both sequences $\{\lambda_n^i\}$ and $\{\gamma_n^i\}$ converge to $\frac{1}{k}$ as n goes to infinity for all $1 \le i \le k$, for any $\epsilon > 0$, there exist an N_ϵ such that,

$$\left|\lambda_n^i - \frac{1}{k}\right| < \epsilon \text{ and } \left|\gamma_n^i - \frac{1}{k}\right| < \epsilon \text{ for all } n \ge N_\epsilon.$$

So,

$$\frac{1}{k} - \epsilon \le \lambda_n^i \le \frac{1}{k} + \epsilon \text{ and } \frac{1}{k} - \epsilon \le \gamma_n^i \le \frac{1}{k} + \epsilon \text{ for all } n \ge N_\epsilon.$$

Consider a sequence $\{\lambda_1^{i_1}, \lambda_2^{i_2}, \lambda_3^{i_3}, \ldots\}$, where $i_1, i_2, i_3, \ldots \in \{1, 2, 3, \ldots, k\}$. It is clear that this sequence converges to $\frac{1}{k}$. In particular, if $\epsilon = \frac{1}{k}$, there exist an $N_{\frac{1}{k}} = N$ (say) such that $\lambda_n^{i_n} \le \frac{2}{k}$ for all $n \ge N$. Since $k > 2$, $\prod_{n=N}^{\infty} \lambda_n^{i_n} \le \prod_{n=N}^{\infty} \frac{2}{k} = 0$. Now consider,

$$\prod_{n=1}^{\infty} \lambda_n^{i_n} = \lambda_1^{i_1} \lambda_2^{i_2} \ldots \lambda_{N-1}^{i_{N-1}} \prod_{n=N}^{\infty} \lambda_n^{i_n} \le \Lambda^{N-1} \prod_{n=N}^{\infty} \lambda_n^{i_n} = 0,$$

where $\Lambda = \max\{\lambda_1^{i_1}, \lambda_2^{i_2}, \ldots, \lambda_{N-1}^{i_{N-1}}\}$.

Note that, each μ_n^i can be written as $\mu_n^i = \lambda_1^{i_1} \lambda_2^{i_2} \ldots \lambda_n^{i_n}$ for some $i_1, i_2, \ldots, i_n \in \{1, 2, \ldots, k\}$. Thus, $\lim_{n\to\infty} \mu_n^i = 0$ for any $1 \le i \le k^n$. Similarly $\lim_{n\to\infty} \delta_n^i = 0$ for any $1 \le i \le k^n$. Hence $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} \beta_n = 0$.

The minimal number of squares of side length α_n, which covers L_∞, cannot exceed T^n, where T is the number of white rectangles in the first stage. Hence, an upper bound for the upper box-counting dimension of sequential labyrinth fractal is given by,

$$\overline{\dim}_B(L_\infty) \le \lim_{n\to\infty} \frac{\log T^n}{-\log \alpha_n}.$$

Also, for any fixed $\varepsilon > 0$, and for all $n \ge N_\epsilon$, we obtain

$$\alpha_n \le S^{N_\epsilon}\left(\frac{1}{k} + \epsilon\right)^{n-N_\epsilon},$$

where $S = \max\{\lambda_m^i, \gamma_m^i : m = 1, 2, \ldots N_\epsilon, 1 \le i \le k\}$. Hence,

$$\begin{aligned}\overline{\dim}_B(L_\infty) &\le \lim_{n\to\infty} \frac{\log T^n}{-\log \alpha_n} \\ &\le \lim_{n\to\infty} \frac{\log T^n}{-[\log(S^{N_\epsilon}(\frac{1}{k}+\epsilon)^{n-N_\epsilon})]} \\ &= \lim_{n\to\infty} \frac{\log T}{-[\frac{N_\epsilon}{n}\log S + (1-\frac{N_\epsilon}{n})\log(\frac{1}{k}+\epsilon)]} \\ &= \frac{\log T}{-\log(\frac{1}{k}+\epsilon)}.\end{aligned}$$

This is true for every $\epsilon > 0$ and hence,

$$\overline{\dim}_B(L_\infty) \le \frac{\log T}{\log k}.$$

Similarly, the number of squares of side length β_n, required to cover L_∞ will be greater than or equal to T^n. Hence, the lower box-counting dimension of sequential labyrinth fractal, $\underline{\dim}_B(L_\infty)$, is bounded below by $\lim\limits_{n\to\infty} \frac{\log T^n}{-\log \beta_n}$. Also,

$$\beta_n \ge s^{N_\epsilon}\big(\frac{1}{k}-\epsilon\big)^{n-N_\epsilon},$$

where $s = \min\{\lambda_m^i, \gamma_m^i : m = 1, 2, \ldots N_\epsilon, 1 \le i \le k\}$. Then,

$$\begin{aligned}\underline{\dim}_B(L_\infty) &\ge \lim_{n\to\infty} \frac{\log T^n}{-\log \beta_n} \\ &\ge \lim_{n\to\infty} \frac{\log T^n}{-[\log(s^{N_\epsilon}(\frac{1}{k}-\epsilon)^{n-N_\epsilon})]} \\ &= \lim_{n\to\infty} \frac{\log T}{-[\frac{N_\epsilon}{n}\log s + (1-\frac{N_\epsilon}{n})\log(\frac{1}{k}-\epsilon)]} \\ &= \frac{\log T}{-\log(\frac{1}{k}-\epsilon)}.\end{aligned}$$

As earlier,

$$\underline{\dim}_B(L_\infty) \ge \frac{\log T}{\log k}.$$

Thus, $\frac{\log T}{\log k} \le \underline{\dim}_B(L_\infty) \le \overline{\dim}_B(L_\infty) \le \frac{\log T}{\log k}$. Hence, the box-counting dimension of the sequential labyrinth fractal exists and it is given by $\dim_B(L_\infty) = \frac{\log T}{\log k}$. □

The following theorem gives the bounds for the Hausdorff dimension of a sequential labyrinth fractal.

Theorem 6. *Let L_∞ be a sequential labyrinth fractal generated from the nonconstant sequences $\{\lambda_n\} = \{(\lambda_n^1, \lambda_n^2, ..., \lambda_n^k)\}$ and $\{\gamma_n\} = \{(\gamma_n^1, \gamma_n^2, ..., \gamma_n^k)\}$, both converging to the k-tuple $(\frac{1}{k}, \frac{1}{k}, \ldots, \frac{1}{k})$, $k \ge 3$. Let T be the number of* *white rectangles in the first stage of* L_∞*. Then,* $1 \leq \dim_H(L_\infty) \leq \frac{\log T}{\log k}$*, where* $\dim_H(L_\infty)$ *denotes the Hausdorff dimension of* L_∞*.*

Proof. Consider the projection $P : L_\infty \to [0,1]$. Due to the connectedness and exit property of L_∞, it is easy to see that P is surjective. Also, P is a Lipschitz map. Thus

$$1 = \dim_H([0,1]) = \dim_H(P(L_\infty)) \leq \dim_H(L_\infty).$$

Since the box-counting dimension is an upper bound of the Hausdorff dimension, we obtain $\dim_H(L_\infty) \leq \dim_B(L_\infty) = \frac{\log T}{\log k}$. □

Remark 4. *The selection of white rectangles does not play a role in finding the box-counting dimension. Even if we are choosing the white rectangles, which do not satisfy the tree property, the corner property and the exit property, the box-counting dimension of the limiting set will be* $\frac{\log T}{\log k}$*, where* T *is the number of whites in the first stage. But in any case, the convergence of sequences is used to find the dimension. However, while finding the lower bound of the Hausdorff dimension, we have used the conditions of the labyrinth fractal.*

5 Conclusion

This paper mainly deals with constructing the sequential labyrinth fractal and finding the dimension of the sequential labyrinth fractal generated from converging sequences. Besides that, it is possible to construct fractals on the unit square from sequences without imposing the labyrinth condition.

Acknowledgement. The authors are grateful to Mr. Hareesh Gopikrishnan for the images produced in this paper. We also thank the anonymous referee for the useful comments and suggestions, which improved the manuscript.

References

[1] Tim Bedford, *Crinkly curves, Markov partitions and dimension*, PhD thesis, University of Warwick, 1984.

[2] C. McMullen, *The Hausdorff dimension of general Sierpiński carpets*, Nagoya Math. J., **96** (1984), 1–9. Zbl 0539.28003

[3] S.P. Lalley, D. Gatzouras, *Hausdorff and box dimensions of certain self-affine fractals*, Indiana Univ. Math. J., **41**:2 (1992), 533–568. Zbl 0757.28011

[4] Krzysztof Barański, *Hausdorff dimension of the limit sets of some planar geometric constructions*, Adv. Math., **210**:1 (2007), 215–245. Zbl 1116.28008

[5] L.L. Cristea, B. Steinsky, *Curves of infinite length in 4×4-labyrinth fractals*, Geom. Dedicata, **141** (2009), 1–17. Zbl 1226.28008

[6] L.L. Cristea, B. Steinsky, *Curves of infinite length in labyrinth fractals*, Proc. Edinb. Math. Soc., II. Ser., **54**:2 (2011), 329–344. Zbl 1228.28008

[7] L.L. Cristea, B. Steinsky, *Mixed labyrinth fractals*, Topology Appl., **229** (2017), 112–125. Zbl 1381.28008

[8] L.L. Cristea, G. Leobacher, *Supermixed labyrinth fractals*, J. Fractal Geom., **7**:2 (2020), 183–218. Zbl 1445.28007

[9] L.L. Cristea, G. Leobacher, *On the length of arcs in labyrinth fractals*, Monatsh. Math., **185**:4 (2018), 575–590. Zbl 1391.28008
[10] L.L. Cristea, P. Surer, *Triangular labyrinth fractals*, Fractals, **27**:8 (2019), Article ID 1950131. Zbl 1434.28015
[11] Harsha Gopalakrishnan and Srijanani Anurag Prasad, *Quadrilateral labyrinth fractals*, arXiv preprint, arXiv:2108.11686, 2021.
[12] A.A. Potapov, V.A. German, V.I. Grachev, *Fractal labyrinths as a basis for reconstruction planar nanostructures*, International Conference on Electromagnetics in Advanced Applications (ICEAA), IEEE, 2013, 949–952.
[13] A.A. Potapov, V.A. German, V.I. Grachev, *"Nano-" and radar signal processing: Fractal reconstruction complicated images, signals and radar backgrounds based on fractal labyrinths*, 14th International Radar Symposium (IRS), IEEE, 2013, 941–946.
[14] A.A. Potapov, W. Zhang., *Simulation of new ultra-wide band fractal antennas based on fractal labyrinths*, CIE International Conference on Radar (RADAR), IEEE, 2016, 1–5.
[15] Alexander Potapov and Victor Potapov, *Fractal radioelement's, devices and systems for radar and future telecommunications: Antennas, capacitor, memristor, smart 2d frequency-selective surfaces, labyrinths and other fractal metamaterials*, J. International Sci. Publications: Materials, Methods & Technologies, **11** (2017), 492–512.

Harsha Gopalakrishnan
Department of Mathematics and Statistics,
Indian Institute of Technology, Tirupati
517619, Andhra Pradesh, India
Email address: harshagopal3@gmail.com

Srijanani Anurag Prasad
Department of Mathematics and Statistics,
Indian Institute of Technology, Tirupati
517619, Andhra Pradesh, India
Email address: srijanani@iittp.ac.in