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СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ
Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru
ISSN 1813-3304

Vol. 21, No. 2, pp. 1011–1023 (2024)
https://doi.org/10.33048/semi.2024.21.067

УДК 517.9
MSC 44A12, 46F12

THE RAY TRANSFORM OF SYMMETRIC TENSOR FIELDS WITH INCOMPLETE PROJECTION DATA ON A CONVEX NON-SMOOTH DOMAIN

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Communicated by E.M. Rudoy

Abstract: We consider the ray transform I_Γ that integrates symmetric rank m tensor fields on $\mathbb{R}^n$ supported in a bounded convex domain $D \subset \mathbb{R}^n$ over lines. The integrals are known for the family Γ of lines l such that endpoints of the segment $l \cap D$ belong to a given part $\gamma = \partial D \cap \mathbb{R}^n_+$ of the boundary, for some half-space $R^n_+ \subset \mathbb{R}^n$. In this work, we assume that the domain D is convex with a non-smooth boundary. In this case, we prove that the kernel of the operator I_Γ coincides with the space of γ-potential tensor fields, which is a generalization of the results obtained in [2].

Keywords: tomography with incomplete data, ray transform, tensor analysis.

1 Introduction

The ray transform I integrates rank m symmetric tensor fields on $\mathbb{R}^n$ over lines. In the cases of $m = 0$ and $m = 1$, the ray transform is the main mathematical tool of Computer tomography and Doppler tomography respectively. In the case of $m \geq 2$, the ray transform is used in different problems of tomography of anisotropic media.

Vaytsel N.A., The ray transform of symmetric tensor fields with incomplete projection data on convex nonsmooth domain.

Received March, 7, 2023, published November, 8, 2024.

For $m > 0$, the operator I has a non-zero kernel consisting of so called potential tensor fields. A symmetric rank m tensor field f is potential if it can be represented in the form $f = dv$ for some rank $m-1$ symmetric tensor field v, where $d = \sigma\nabla$ is the symmetrized covariant derivative. The definition also involves boundary condition(or decay condition at infinity): the potential v must vanish either on the whole boundary of the domain under consideration or on a part of the boundary (or must certainly decay at infinity).

In problems with incomplete projection data the ray transform If is known not for all lines of $\mathbb{R}^n$ but for some family of lines. Such a situation appears if the domain involves a non-transparent inclusion such that the sounding radiation does not transmit the inclusion.

In the present work, we generalize results of the recent paper [2] by V.A. Sharafutdinov in the following two directions. First of all, we do not assume the domain under consideration to be smooth. Smoothness arguments of [2] are replaced by the usage of the Saint-Venant operator introduced in [1]. Second, the strict convexity of the domain is replaced with the standard convexity. The latter replacement is possible due to some generalization of the support theorem for the Radon transform. The possibility is mentioned in [2, Remark 3.5].

The author is grateful to V.A. Sharafutdinov for wise guidance and helpful suggestions and to K.V. Storozhuk for useful remarks.

2 A generalization of the support theorem for the Radon transform

Let us recall the definition of the Radon transform and the support theorem for the Radon transform.

Let $C_0(\mathbb{R}^n)$ be the space of continuous complex-valued functions with compact supports on $\mathbb{R}^n$ with the norm $\|f\| = \sup\limits_{x\in\mathbb{R}^n} |f(x)|$. We define similarly $C_0(\mathbb{P}^n)$, where $\mathbb{P}^n$ is the manifold of hyperplanes in $\mathbb{R}^n$. The *Radon transform* is the linear operator

$$\mathcal{R} : C_0(\mathbb{R}^n) \to C_0(\mathbb{P}^n)$$

defined by

$$\mathcal{R}f(\xi) = \int\limits_{\xi} f(x)\, dm(x) \quad (\xi \in \mathbb{P}^n)$$

for $f \in C_0(\mathbb{R}^n)$ where dm is the Lebesgue $(n-1)$-dimensional measure on the hyperplane ξ. The *formal adjoint of the Radon transform* is the linear operator

$$\mathcal{R}^* : C(\mathbb{P}^n) \to C(\mathbb{R}^n)$$

defined by

$$\mathcal{R}^*\varphi(x) = \int\limits_{x\in\xi} \varphi(\xi)\, d\mu(\xi),$$

where $d\mu$ is the measure on the compact set $\{\xi \in \mathbb{P}^n \,|\, x \in \xi\}$ which is invariant under the group of rotations around x and such that the measure of the whole set is 1.

We now formulate the standard support theorem for the Radon transform of compactly supported continuous functions. Let $\|\cdot\|$ be the standard norm on $\mathbb{R}^n$. For $\xi \in \mathbb{P}^n$, the distance from the origin to the hyperplane ξ is denoted by $d(0,\xi)$.

Theorem 1. *Let $f \in C_0(\mathbb{R}^n)$ satisfy the following condition: there exists a constant $A > 0$ such that $\mathcal{R}f(\xi) = 0$ if $d(0,\xi) > A$. Then $f(x) = 0$ for $\|x\| > A$.*

Define $\mathcal{D}(\mathbb{R}^n)$ as the space of test functions, i.e. infinitely differentiable complex-valued functions with compact supports in $\mathbb{R}^n$. Let $\mathcal{D}'(\mathbb{R}^n)$ be the space of all distributions on $\mathbb{R}^n$ and $\mathcal{E}'(\mathbb{R}^n)$ be the space of compactly supported distributions on $\mathbb{R}^n$. We denote the value of a distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ on a test function $\varphi \in \mathcal{D}(\mathbb{R}^n)$ by $\langle f|\varphi\rangle$.

For a distribution $f \in \mathcal{E}'(\mathbb{R}^n)$, the *Radon transform* is defined by

$$\langle \mathcal{R}f|\varphi\rangle = \langle f|\mathcal{R}^*\varphi\rangle \quad (\varphi \in \mathcal{D}(\mathbb{P}^n))\,.$$

We consider distributions $f \in \mathcal{D}'(\mathbb{R}^n)$ that admit a decomposition of the form

$$f = f_1 + f_2 \quad \big(f_1 \in \mathcal{E}'(\mathbb{R}^n),\ f_2 \in C(\mathbb{R}^n)\big)\,, \tag{1}$$

where f_2 is a fast decaying function, i.e., for every $k \in \mathbb{N}$, there exist a constant $C_k = C_k(f_2)$ such that

$$(1+|x|)^k |f_2(x)| \le C_k \quad \text{for all } x \in \mathbb{R}^n.$$

For such a distribution f, the *Radon transform* is defined by

$$\langle \mathcal{R}f|\varphi\rangle = \langle \mathcal{R}f_1|\varphi\rangle + \langle \mathcal{R}f_2|\varphi\rangle \quad (\varphi \in \mathcal{D}(\mathbb{P}^n))\,, \tag{2}$$

where

$$\langle \mathcal{R}f_2|\varphi\rangle = \int\limits_{\mathbb{P}^n} (\mathcal{R}f_2(\xi))\, \varphi(\xi)\, d\xi \quad (\varphi \in \mathcal{D}(\mathbb{P}^n))\,.$$

Here $d\xi$ is a suitable measure on $\mathbb{P}^n$ invariant with respect to the group of isometries of Euclidean space $\mathbb{R}^n$.

The decomposition (1) is not unique. Let us demonstrate that the definition (2) is independent of the choice of the decomposition (1). Given $f \in \mathcal{D}'(\mathbb{R}^n)$, let $f = \widetilde{f_1} + \widetilde{f_2}$ ($\widetilde{f_1} \in \mathcal{E}'(\mathbb{R}^n)$, $\widetilde{f_2} \in C(\mathbb{R}^n)$) be another decomposition of the form (1). Then, for an arbitrary test function $\varphi \in \mathcal{D}(\mathbb{P}^n)$,

$$\langle \mathcal{R}f_1|\varphi\rangle + \langle \mathcal{R}f_2|\varphi\rangle = \langle f_1|\mathcal{R}^*\varphi\rangle + \langle f_2|\mathcal{R}^*\varphi\rangle = \langle f|\mathcal{R}^*\varphi\rangle.$$

The equality $\langle \mathcal{R}\widetilde{f}_1|\varphi\rangle + \langle \mathcal{R}\widetilde{f}_2|\varphi\rangle = \langle f|\mathcal{R}^*\varphi\rangle$ is proved in the same way. From two last equalities,

$$\langle \mathcal{R}f_1|\varphi\rangle + \langle \mathcal{R}f_2|\varphi\rangle = \langle \mathcal{R}\widetilde{f}_1|\varphi\rangle + \langle \mathcal{R}\widetilde{f}_2|\varphi\rangle.$$

Thus, the correctness of the definition (2) has been verified. Introduce the notation $\beta_A(x) = \{\xi \in \mathbb{P}^n \,|\, d(x,\xi) \le A\}$, where d is the Euclidean distance in $\mathbb{R}^n$.

Theorem 2. *Let $f \in \mathcal{D}'(\mathbb{R}^n)$ be a distribution admitting a decomposition of the form (1). If $\operatorname{supp}(\mathcal{R}f) \subset \beta_A(x)$ for some $x \in \mathbb{R}^n$, then $\operatorname{supp}(f) \subset \overline{B_A(x)}$, where by $B_A(x)$ is the ball of radius A centered at x.*

We omit the proof of this theorem which follows the same way as the proof of the support theorem for compactly supported distributions [1, Chapter I, Theorem 5.4] since the function f_2 in decomposition (1) satisfies the hypotheses of the support theorem for the Radon transform [1, Chapter I, Theorem 2.6 and Corollary 2.8].

Now we give a corollary of Theorem 2 which will be used later. Here $n = 2$ in notation $\beta_A(x)$

Corollary 1. *Let D be a bounded closed convex domain in $\mathbb{R}^2$ and $f \in C(D)$. Extend f by zero outside D. If $\operatorname{supp}(\mathcal{R}f) \subset \beta_A(x)$ for some $x \in \mathbb{R}^2$, then $\operatorname{supp}(f) \subset \overline{B_A(x)}$, where $B_A(x)$ is the disk of radius A centered at x.*

3 The two-dimensional ray transform with incomplete data

Given an integer $m \ge 0$, by $T^m = T^m\mathbb{R}^n$ we denote the complex vector space of all functions $\underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{m \text{ factors}} \to \mathbb{C}$ that are $\mathbb{R}$-linear in each of arguments. If $e_1, ..., e_n$ is a basis for $\mathbb{R}^n$, then the numbers $u_{i_1\dots i_m} = u(e_{i_1}, ..., e_{i_m})$ are called the *components* of the tensor $u \in T^m$ with respect to the basis. Assuming the choice of a basis to be clear from the context, we shall denote this fact by the record $u = (u_{i_1\dots i_m})$.

Let $S^m\mathbb{R}^n$ be the subspace of T^m consisting of rank m symmetric tensors on $\mathbb{R}^n$. Its dimension is $\binom{n+m-1}{m}$. In particular, $S^0\mathbb{R}^n = \mathbb{C}$ and $S^1\mathbb{R}^n = \mathbb{C}^n$. It is also convenient to assume that $S^{-1}\mathbb{R}^n = 0$. Let $\sigma : T^m \to S^m\mathbb{R}^n$ be the symmetrization defined by the equality:

$$\sigma u_{i_1\dots i_m} = \frac{1}{m!}\sum_{\pi\in\Pi_m} u_{i_{\pi(1)}\dots i_{\pi(m)}},$$

where the summation is performed over the group Π_m of all permutations of the set $\{1, \dots, m\}$. We also define the symmetrization with respect to a part of indices by the formula:

$$\sigma(i_1 \dots i_p)u_{i_1\dots i_m} = \frac{1}{p!}\sum_{\pi\in\Pi_p} u_{i_{\pi(1)}\dots i_{\pi(p)} i_{p+1}\dots i_m}.$$

We shall need the alternation with respect to two indices

$$\alpha(i_1 i_2) u_{i_1 i_2 j_1 \dots j_p} = \frac{1}{2}\left(u_{i_1 i_2 j_1 \dots j_p} - u_{i_2 i_1 j_1 \dots j_p}\right).$$

A record of the type

$$\text{sym } u : (i_1 \dots i_{k-1}) i_k j_1 (j_2 \dots j_l),$$

is convenient for notation the partial symmetry of the tensor u. It means that the tensor u is symmetric with respect to each group of indices in parentheses.

For an integer $k \geq 0$, the space of k times continuously differentiable $S^m\mathbb{R}^n$-valued functions on $\mathbb{R}^n$ is denoted by $C^k(\mathbb{R}^n; S^m\mathbb{R}^n)$. Its elements are called k times continuously differentiable symmetric tensor field of rank m on $\mathbb{R}^n$. The space $C^k(\mathbb{R}^n; T^m)$ is the defined in the same way.

Let $D \subset \mathbb{R}^n$ be a closed convex domain. We say that a function $f \in C(D)$ belongs to $C^k(D)$ if, for every point of D, there exist an open neighborhood $U \subset \mathbb{R}^n$ and function $g \in C^k(U)$ such that $g|_{D\cap U} = f|_{D\cap U}$. Now the space $C^k(D; S^m\mathbb{R}^n)$ of symmetric tensor fields of the class C^k is defined as the set of continuous maps $D \to S^m\mathbb{R}^n$ whose all coordinates belong to $C^k(D)$.

We define operator of *differentiation*

$$\nabla : C^k(D; T^m) \to C^{k-1}(D; T^{m+1})$$

by the equalities

$$\nabla u = (u_{i_1 \dots i_m\,;\,j}); \quad u_{i_1 \dots i_m\,;\,j} = \frac{\partial u_{i_1 \dots i_m}}{\partial x^j},$$

where $(x^1, \dots, x^n)$ are Cartesian coordinates in $\mathbb{R}^n$. The iterated derivative is denoted as $\nabla^k u = (u_{i_1 \dots i_m\,;\,j_1 \dots j_k})$. The first order differential operator

$$d = \sigma\nabla : C^k(D; S^m\mathbb{R}^n) \to C^{k-1}(D; S^{m+1}\mathbb{R}^n)$$

is called the *inner derivative*.

The operator d obeys the following hypoellipticity. If the right-hand side of the equation $dv = f$ is of the class C^k in an open domain $U \subset \mathbb{R}^n$, then any solution to the equation is of the class C^{k+1} in U. This follows from [3, Theorem 2.2.2]

We identify the family of all oriented lines in $\mathbb{R}^n$ with the tangent bundle of the unit sphere $\mathbb{S}^{n-1}$

$$T\mathbb{S}^{n-1} = \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \mid \|\xi\| = 1, \langle x, \xi\rangle = 0\},$$

by identifying $(x, \xi) \in T\mathbb{S}^{n-1}$ with the line $l_{x,\xi} = \{x + t\xi \mid t \in \mathbb{R}\}$ through the point x in the direction ξ. Hereinafter $\langle\cdot,\cdot\rangle$ is the standard dot product on $\mathbb{R}^n$ and $\|\cdot\|$ is the corresponding norm. Since $T\mathbb{S}^{n-1}$ is a smooth manifold, the spaces of (complex-valued) functions $C^k(T\mathbb{S}^{n-1})$ and $C^k_0(T\mathbb{S}^{n-1})$ are well defined.

Here and further till the end of the section $n = 2$, let $D \subset \mathbb{R}^2$ be a closed bounded convex domain. The *ray transform* is initially defined as the linear operator

$$I : C^k(D; S^m\mathbb{R}^2) \to L^\infty(T\mathbb{S}^1) \tag{3}$$

by

$$If(x,\xi) = \int\limits_{-\infty}^{+\infty} f_{i_1\dots i_m}(x+t\xi)\xi^{i_1}\dots\xi^{i_m}\,dt =$$

$$= \int\limits_{-\infty}^{+\infty} \langle f(x+t\xi), \xi^m\rangle\,dt \quad \big((x,\xi)\in T\mathbb{S}^1\big)\,, \quad (4)$$

where f is extended by zero outside D. Here the space $L^\infty(T\mathbb{S}^1)$ is introduced with the help of the measure $dxd\xi$ on $T\mathbb{S}^1$ which is defined by

$$\int\limits_{T\mathbb{S}^1} \varphi(x,\xi)dxd\xi = \int\limits_{\mathbb{S}^1}\int\limits_{\xi^\perp} \varphi(x,\xi)dxd\xi \quad \big(\varphi\in C_0(T\mathbb{S}^1)\big)\,.$$

On the right-hand side of the latter formula, dx is the length element of the line $\xi^\perp = \{x\in\mathbb{R}^2 \,|\, \langle x,\xi\rangle = 0\}$ and $d\xi$ is the length element of the circle $\mathbb{S}^1$.

Now, let $D\subset\mathbb{R}^2$ be a closed convex bounded domain containing an inner point. Choose a line $l_0\subset\mathbb{R}^2$ through an inner point of D. By $\mathbb{R}^2_+$ we denote one of two closed half planes bounded by l_0 and by $\mathbb{R}^2_-$, the second one. Set $D_\pm = D\cap\mathbb{R}^2_\pm$ and $\gamma = \partial D\cap\mathbb{R}^2_+$, see Fig. 1. Let Γ be the closed set in $T\mathbb{S}^1$ consisting of $(x,\xi)\in T\mathbb{S}^1$ such that the intersection of the line $l_{x,\xi} = \{x+t\xi \,|\, t\in\mathbb{R}\}$ with D is a non-empty segment with both endpoints belonging to the curve γ. We introduce the linear continuous operator

$$I_\Gamma : C^k(D;S^m\mathbb{R}^2)\to L^\infty(\Gamma) \quad (5)$$

by $I_\Gamma f = (If)|_\Gamma$ where If is the value of the operator (3) on the tensor field $f\in C^k(D;S^m\mathbb{R}^2)$. The operator (5) is called the *ray transform with incomplete projection data determined on the domain* $\Gamma\subset T\mathbb{S}^1$.

We say that $f\in C^k(D;S^m\mathbb{R}^2)$ is a γ-*potential* tensor field if there exists a field $v\in C^{k+1}(D_+;S^{m-1}\mathbb{R}^2)$, satisfying the boundary condition

$$v|_\gamma = 0$$

and the equation

$$dv = f \quad \text{in} \quad D_+.$$

Theorem 3. *Let $k\ge m\ge 0$, $D\subset\mathbb{R}^2$ be a closed bounded convex domain, domains $D_\pm$ and the curve $\gamma\subset\partial D$ be chosen as above. The kernel of the operator I_Γ coincides with the space of γ-potential tensor fields.*

Starting the proof, first of all we choose Cartesian coordinates (x_1,x_2) on $\mathbb{R}^2$ so that the line l_0 participating in above-presented definitions coincides with the x_1-axis $\{x_2=0\}$. Then $D_\pm = \{(x_1,x_2)\in D \,|\, \pm x_2\ge 0\}$ and $\gamma = \{(x_1,x_2)\in\partial D \,|\, x_2\ge 0\}$. The proof is going by induction in m.

In case of $n=2$ and $m=0$ the ray transform I coincides up to notations with the Radon transform $\mathcal{R}$ and Theorem 5 reduces to Corollary 1 due to the fact that a compact convex set D_- coincides with the intersection of closed balls containing D_-.

Proposition 1. *Theorem 3 is true for* $m = 1$.

Proof. Let a vector field $f = (f_1, f_2) \in C^k(D, \mathbb{C}^2)\,(k \geq 1)$ belong to the kernel of the operator I_Γ. We are going to prove that the 1-form

$$\omega = f_1 dx_1 + f_2 dx_2 \tag{6}$$

is exact on the domain D_+.

We parametrize the curve γ by the arc length

$$\gamma(t) = (\gamma_1(t), \gamma_2(t)) \quad (0 \leq t \leq T),$$

Recall that the boundary curve of a convex domain $D \subset \mathbb{R}^2$ is differentiable at almost all points; therefore γ_1 and γ_2 are differentiable almost everywhere and $\dot\gamma_1^2 + \dot\gamma_2^2 = 1$ at any point of differentiability. For $t_1, t_2 \in [0, T]$ let $[\gamma(t_1), \gamma(t_2)]$ be the straight line segment joining the points $\gamma(t_1)$ and $\gamma(t_2)$. By $S \subset [0, T]$ we denote the set of t such that $\gamma(t)$ is the point of strict convexity, i.e.

$$S = \{t \in [0, T] \,|\, \nexists\, t_1,\, t_2 : \quad t_1 < t < t_2 \quad \text{and} \quad [\gamma(t_1), \gamma(t_2)] \subset \gamma\}.$$

Let us make sense to the integral

$$\int\limits_{\gamma|_{[t_1,t_2]}} \omega = \int\limits_{t_1}^{t_2} (f_1(\gamma(t))\dot\gamma_1(t) + f_2(\gamma(t))\dot\gamma_2(t))\, dt \tag{7}$$

Formally speaking, the integrand on the right-hand side of (7) is not defined since the tangent vector $\dot\gamma(t) = (\dot\gamma_1(t), \dot\gamma_2(t))$ does not exist for all $t \in [0, T]$. Nevertheless, the *left tangent vector*

$$\dot\gamma^-(t) = \lim_{\Delta t \to +0} \frac{\gamma(t - \Delta t) - \gamma(t)}{\Delta t}$$

and the *right tangent vector*

$$\dot\gamma^+(t) = \lim_{\Delta t \to +0} \frac{\gamma(t + \Delta t) - \gamma(t)}{\Delta t}$$

exist for every $t \in [0, T]$ since γ is a convex curve. Observe that $\|\dot\gamma^-(t)\| = \|\dot\gamma^+(t)\| = 1$ and the functions $\dot\gamma^-$ and $\dot\gamma^+$ coincide almost everywhere on $[0, T]$ since a convex curve is differentiable almost everywhere. Thus, $\dot\gamma^-$ and $\dot\gamma^+$ are bounded measurable functions with countable sets of discontinuities. Therefore the integrals

$$\int\limits_{t_1}^{t_2} \big(f_1(\gamma(t))\dot\gamma_1^-(t) + f_2(\gamma(t))\dot\gamma_2^-(t)\big)\, dt, \int\limits_{t_1}^{t_2} \big(f_1(\gamma(t))\dot\gamma_1{}^+(t) + f_2(\gamma(t))\dot\gamma_2{}^+(t)\big)\, dt$$

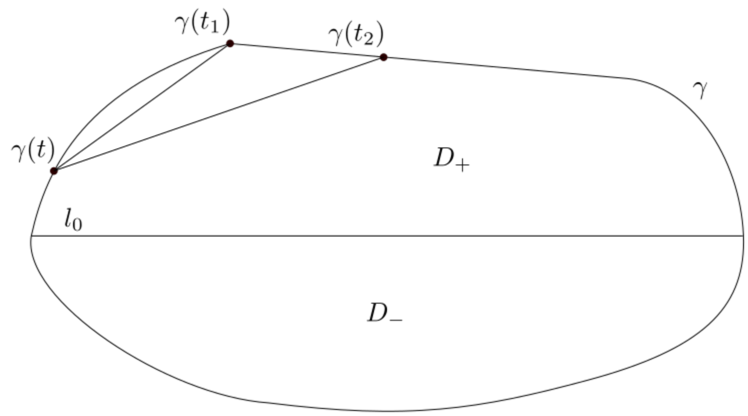

Fig. 1

are well define and coincide. We define the integral (7) as any of two latter integrals. The hypothesis $I_\Gamma f = 0$ can be written in the form: for $t_1, t_2 \in [0, T]$

$$\left.\begin{aligned} &\int\limits_{[\gamma(t_1),\gamma(t_2)]} \omega = 0 \quad \text{if } [\gamma(t_1), \gamma(t_2)] \not\subset \gamma \\ &\int\limits_{[\gamma(t_1),\gamma(t_2)]} \omega = 0 \quad \text{if } t_1,\ t_2 \in S \text{ and } [\gamma(t_1), \gamma(t_2)] \subset \gamma \end{aligned}\right\} \tag{8}$$

Let us show that

$$\int\limits_{\gamma|_{[t_1,t_2]}} \omega = 0 \quad (t_1, t_2 \in [0, T]). \tag{9}$$

In the case $[t_1, t_2] \subset S$ the statement (9) is proved similarly to [2, Section 3, Proposition 2]. In view of (8), we infer that (9) is true for arbitrary $t_1, t_2 \in S$.

Let us prove (9) for $t_1 \in S$ and $t_2 \notin S$ such that $0 < t_1 < t_2$ and $[\gamma(t_1), \gamma(t_2)] \subset \gamma$. To this end we choose $t \in (0, t_1)$. Let D_{t,t_1,t_2} be the triangle with vertices at points $\gamma(t), \gamma(t_1), \gamma(t_2)$, see Fig.1. By the Green formula

$$\iint\limits_{D_{t,t_1,t_2}} d\omega = \int\limits_{[\gamma(t),\gamma(t_1)]} \omega + \int\limits_{[\gamma(t_1),\gamma(t_2)]} \omega - \int\limits_{[\gamma(t),\gamma(t_2)]} \omega. \tag{10}$$

Observe that the second integral on the right-hand side is independent of t. The integral on the left-hand side and and the first integral on the right-hand side tend to 0 as $t \to t_1$. The last integral on the right-hand side is equal to 0 for $t \in [0, t_1)$ because of (8). Thus, passing to limit in (10) as $t \to t_1$, we obtain

$$\int\limits_{[\gamma(t_1),\gamma(t_2)]} \omega = 0 \quad (t_1 \le t_2,\ [\gamma(t_1), \gamma(t_2)] \subset \gamma,\ t_1 \in S,\ t_2 \notin S).$$

Together with additivity of integration, this gives (9).

For $t_1, t_2 \in [0,T]$ $(t_1 \le t_2)$, let D_{t_1,t_2} be the domain bounded by the part $\gamma|_{t_1,t_2}$ of γ and the straight line segment $[\gamma(t_1), \gamma(t_2)]$. By the Green formula,

$$\iint\limits_{D_{t_1,t_2}} d\omega = \int\limits_{\gamma|_{[t_1,t_2]}} \omega - \int\limits_{[\gamma(t_1),\gamma(t_2)]} \omega.$$

Both integrals on the right-hand side are equal to zero by (8) and (9). Thus,

$$\iint\limits_{D_{t_1,t_2}} d\omega = 0.$$

Since $t_1, t_2 \in [0,T]$ are arbitrary, this implies that

$$\left.\begin{aligned} \int\limits_{[\gamma(t_1),\gamma(t_2)]} d\omega &= 0 \quad \text{if } [\gamma(t_1),\gamma(t_2)] \not\subset \gamma \\ \int\limits_{[\gamma(t_1),\gamma(t_2)]} d\omega &= 0 \quad \text{if } t_1,\, t_2 \in S \text{ and } [\gamma(t_1),\gamma(t_2)] \subset \gamma \end{aligned}\right\}$$

Since $d\omega = \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) dx_1 \wedge dx_2$, the latter equality is equivalent to

$$I_\Gamma\Big(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\Big) = 0.$$

Since Theorem 3 has been already proven for $m = 0$, the latter equality implies that

$$d\omega = 0 \quad \text{in the domain } D_+, \tag{11}$$

i.e., the form ω is closed in the domain D_+. Since the domain is simply connected, the form ω is exact in D_+, i.e., there exists a function $v \in C^1(D_+)$ satisfying

$$dv = \omega \quad \text{in the domain } D_+. \tag{12}$$

The function v is determined by the equation (12) uniquely up to an additive constant. Choose the constant such that $v(\gamma(0)) = 0$. Again using (9), we obtain

$$v(\gamma(t)) = \int\limits_{\gamma|_{[0,t]}} \omega = 0 \quad (t \in [0,T]),$$

i.e.,

$$v|_\gamma = 0. \tag{13}$$

In equations (11) and (12), d is the exterior derivative on differential forms. But for the form (6), the equation (12) is equivalent to

$$dv = f \quad \text{in the domain } D_+, \tag{14}$$

where d is the inner derivative.

It remains to use the hypoellipticity of the operator d: a solution v to the equation (14) belongs to $C^{k+1}(D_+)$ if $k \geq m$ and $f \in C^k(D_+; \mathbb{C}^2)$. Equations (13) and (14) mean that f is a γ-potential field. □

The proof of Theorem 3 in the case of $m \geq 2$ coincides with [2, Section 3, Theorem 1], because the strict convexity hypothesis and smoothness of the domain under consideration was not used in the mentioned proof.

4 The Saint-Venant operator and the equation $dv = f$

Let $D \subset \mathbb{R}^n$ $(n \geq 2)$ be a closed convex domain bounded by a closed hypersurface ∂D.

Recall [3] that the *Saint-Venant operator*

$$W : C^k(D; S^m\mathbb{R}^n) \to C^k(D; S^m\mathbb{R}^n \otimes S^m\mathbb{R}^n) \quad (k \geq m > 0),$$

defined by the equality

$$(Wu)_{i_1\dots i_m j_1\dots j_m} = \sigma(i_1 \dots i_m)\sigma(j_1 \dots j_m) \sum_{l=0}^{m} (-1)^l \binom{m}{l} \times$$
$$\times u_{i_1\dots i_{m-l} j_1\dots j_l\, ;\, j_{l+1}\dots j_m i_{m-l+1}\dots i_m}.$$

The Saint-Venant operator has the following property. Let $f \in C^k(D; S^m\mathbb{R}^n)$ for a closed domain $D \subset \mathbb{R}^n$. If the restriction of the field Wf to every two-dimensional plane is equal to zero, then $Wf = 0$ in D. This follows from the fact that the field Wf has the following symmetry

$$\text{sym } (Wf)_{i_1\dots i_m j_1\dots j_m} : (i_1 \dots i_m)(j_1 \dots j_m).$$

We now formulate the following version of [3, Theorem 2.2.2].

Theorem 4. *Let Ω be a closed convex set in $\mathbb{R}^n$, $x_0 \in \partial\Omega$, $k \geq m \geq 1$ be integers, $f \in C^k(\Omega; S^m\mathbb{R}^n)$, $v^l \in T^{m+l-1}$ $\quad (0 \leq l \leq m-1)$. The equation*

$$dv = f \tag{15}$$

has at most one solution $v \in C^{k+1}(D_+; S^{m-1}\mathbb{R}^n)$ satisfying the initial conditions

$$\nabla^l v(x_0) = v^l \quad (l = 0, 1, \dots, m-1) \tag{16}$$

For existence of a solution $v \in C^{k+1}(D_+; S^{m-1}\mathbb{R}^n)$ to the problem (15)–(16) it is necessary that the right-hand side of the equation (15) satisfies the condition

$$Wf = 0 \text{ in } \Omega, \tag{17}$$

and the tensors v^l have the symmetries

$$\text{sym } v^l : (i_1 \dots i_{m-1})(i_m \dots i_{m+l-1}) \quad (l = 0, 1, \dots, m-1) \tag{18}$$

and satisfy the relations

$$\sigma(i_1 \dots i_m) v^l_{i_1\dots i_{m+l-1}} = f_{i_1\dots i_m\, ;\, i_{m+1}\dots i_{m+l-1}}(x_0) \quad (l = 1, \dots, m-1). \tag{19}$$

If U is a simply-connected domain, then conditions (17)–(19) are suffcient for existence of a solution $v \in C^{k+1}(D_+; S^{m-1}\mathbb{R}^n)$ to the problem (15)–(16).

Unlike [3, Theorem 2.2.2], in Theorem 4 the initial conditions are taken at a point on the boundary of the Ω, but Theorem 4 can be proved in the same way as the original theorem, because the Ω is closed convex set.

5 The multi-dimensional ray transform with incomplete projection data

Let $D \subset \mathbb{R}^n$ $(n \geq 2)$ be a closed convex domain bounded by a closed hypersurface ∂D. For such a domain, the ray transform

$$I : C^k(D; S^m\mathbb{R}^n) \to L^\infty(T\mathbb{S}^{n-1}) \tag{20}$$

is defined by the same formula (4), where f is extended by zero outside D. Choose an affine hyperplane $P_0 \subset \mathbb{R}^n$ through an inner point of the domain D. It splits $\mathbb{R}^n$ into two closed half spaces bounded by P_0. We denote them by $\mathbb{R}^n_+$ and $\mathbb{R}^n_-$. Set $D_\pm = D \cap \mathbb{R}^n_\pm$ and $\gamma = \partial D \cap \mathbb{R}^n_+$. Let Γ be a closed set in $T\mathbb{S}^{n-1}$ consisting of $(x, \xi) \in T\mathbb{S}^{n-1}$ such that the intersection of the line $l_{x,\xi} = \{x + t\xi \,|\, t \in \mathbb{R}\}$ with D is a non-empty segment with both endpoints belonging to the curve γ. We introduce the linear operator

$$I_\Gamma : C^k(D; S^m\mathbb{R}^n) \to L^\infty(\Gamma) \tag{21}$$

by $I_\Gamma f = (If)|_\Gamma$ where If is the value of the operator (20) on the field $f \in C^k(D; S^m\mathbb{R}^n)$. The operator (21) is called the *ray transform with incomplete projection data determined on the domain* $\Gamma \subset T\mathbb{S}^{n-1}$.

We say that $f \in C^k(D; S^m\mathbb{R}^n)$ is a γ-*potential* tensor field if there exists a field $v \in C^{k+1}(D; S^{m-1}\mathbb{R}^n)$ satisfying the boundary condition

$$v|_\gamma = 0$$

and the equation

$$dv = f \quad \text{in } D_+.$$

Theorem 5. *Let $k \geq m \geq 0$, $D \subset \mathbb{R}^n$ be a closed bounded convex domain, domains $D_\pm$ and hypersurface $\gamma \subset \partial D$ be chosen as above. The kernel of the operator I_Γ coincides with the space of γ-potential tensor fields.*

Proof. We present the proof for $m \geq 1$. For $m = 0$ the proof is the same with many simplifications. For $n = 2$, Theorem 5 coincides with Theorem 3. Now assume $n \geq 3$.

For an affine 2D-plane $P \subset \mathbb{R}^n$, we set $D^P = D \cap P$, $D^P_\pm = D_\pm \cap P$, $\gamma^P = \gamma \cap P$. Then D^P is a closed bounded convex domain in P. We do not consider the cases, when the domain D^P is empty or consists of one point. We also denote by Γ^P the set of $(x, \xi) \in T\mathbb{S}^{n-1}$, such that the intersection of the line $l_{x,\xi} = \{x + t\xi \,|\, t \in \mathbb{R}\}$ with D^P is a non-empty segment with endpoints belonging γ^P.

For an affine 2D-plane $P \subset \mathbb{R}^n$ by $\mathbb{R}^2_P \subset \mathbb{R}^n$ we denote the vector 2D subspace parallel to the plane P. Let us consider a tensor field $f \in C^k(D; S^m\mathbb{R}^n)$. The restriction $f^P \in C^k(D^P; S^m\mathbb{R}^2_P)$ of f to a plane P is defined as follows. The field f can be identified with the function

$$f(x,\xi) = f_{i_1,\dots i_m}(x)\xi^{i_1}\dots\xi^{i_m} \quad (x \in D,\, \xi \in \mathbb{R}^n),$$

which is a homogeneous polynomial of degree m in ξ with coefficients $f_{i_1,\dots i_m} \in C^k(D)$. The tensor field f^P is identified with the restriction of the polynomial to $D^P \times \mathbb{R}^2_P$.

Let a tensor field $f \in C^k(D; S^m\mathbb{R}^n)$ belong to the kernel of the operator (21). Then, for every affine 2D-plane $P \subset \mathbb{R}^n$ the tensor field $f^P \in C^k(D^P; S^m\mathbb{R}^2_P)$ belongs to the kernel of the *ray transform on the 2D-plane* P

$$I_{\Gamma^P} : C^k(D^P; S^m\mathbb{R}^2_P) \to L^\infty(\Gamma^P). \tag{22}$$

If $D^P_- = D_- \cap P = \emptyset$, then (22) is the ray transform with complete projection data. If $D^P_- \neq \emptyset$, (22) is the ray transform with incomplete data.

Applying Theorem 3, we obtain, for every affine 2D-plane $P \subset \mathbb{R}^n$, a tensor field $v^P \in C^{k+1}(D^P_+; S^{m-1}\mathbb{R}^2_P)$ satisfying the boundary condition

$$v^P|_{\gamma^P} = 0 \tag{23}$$

and the equation

$$d^P v^P = f^P \quad \text{in the domain } D^P_+, \tag{24}$$

where d^P is the inner derivative on the plane P. Applying Theorem 4 with $\Omega = D^P_+$ we obtain

$$Wf^P = 0 \quad \text{in } D^P_+$$

and, for arbitrary fixed $x_0 \in \gamma$, tensors

$$v^0 = 0, \quad v^l_{i_1\dots i_{m+l-1}} = f_{i_1\dots i_m\,;\,i_{m+1}\dots i_{m+l-1}}(x_0) \quad (l = 1,\dots,m-1)$$

have the symmetries

$$\text{sym } (v^l)^P : (i_1\dots i_{m-1})(i_m\dots i_{m+l-1}) \quad (l = 0,1,\dots,m-1)$$

and satisfy the relations

$$\sigma(i_1\dots i_m)(v^l)^P_{i_1\dots i_{m+l-1}} = f^P_{i_1\dots i_m\,;\,i_{m+1}\dots i_{m+l-1}}(x_0) \quad (l = 1,\dots,m-1).$$

From the above-mentioned property of the Saint-Venant operator Wf is uniquely determined by its restrictions to 2D-planes. We see that

$$Wf = 0 \text{ in } D_+. \tag{25}$$

Using the symmetries of $(v^l)^P$, we obtain that

$$\alpha(i_m i_k) f_{i_1\dots i_m\,;\,i_{m+1}\dots i_{m+l-1}}(x_0) = 0 \quad (k = m+1,\dots,m+l-1).$$

Now we set $\Omega = D_+$ in Theorem 4. Inserting tensors v^l into the initial conditions (16), we see that v^l satisfy the conditions (18) and (19). By (25), f satisfies the condition (17). Applying Theorem 4, we obtain that the equation $dv = f$ has a unique solution $v \in C^{k+1}(D_+; S^{m-1}\mathbb{R}^n)$ satisfying the initial conditions (16). Since v is also the solution of the equation (24) for an arbitrary hyperplane $P \subset \mathbb{R}^n$, it satisfies the boundary condition (23). Thus, $v|_\gamma = 0$ and v is a γ-*potential tensor field*. □

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