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ON RIGID INCLUSIONS AND CAVITIES IN ELASTIC BODY WITH A CRACK: NON-COERCIVE CASE

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Abstract: In the paper, we consider an equilibrium problem for an elastic body with a crack in a case of Neumann boundary conditions at the external boundary. The Neumann boundary conditions imply a non-coercivity of the problem. Inequality constraints are imposed on the solution providing a mutual non-penetration between the crack faces. Various passages to limit with respect to the parameter characterizing a rigidity of the body are analyzed, and limit models are investigated. In particular, an existence of solutions is proved for all cases considered; necessary and sufficient conditions imposed on the external forces are found. The limit models describe the elastic body with a volume rigid inclusion and the body with a cavity. These results are obtained both for the case when the crack is located inside the elastic body and for the case when it extends to the outer boundary.

Keywords: elastic body, crack, non-coercive boundary value problem, volume rigid inclusion, cavity.

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1 Introduction

Last years, a lot of papers related to crack problems in elastic bodies are published. Describing cracked bodies with non-penetration boundary conditions imposed on the crack faces, these papers cover a wide range of tasks: existence of solutions, a solution dependence on physical parameters, a dependence of solutions on geometry of problems (in particular, a dependence on the shape and crack length, etc.) as well as other qualitative properties of solutions [1-12], see also the book [13] and the literature therein. As usually, the results mentioned are obtained for situations where the bodies are fixed on the external boundaries what implies a coercivity of the problems. The present paper deals with Neumann type conditions at the external boundary, and thus, with non-coercive boundary value problems. Together with inequality type conditions on the crack faces, this implies specific difficulties in the analysis. We prove an existence of solutions for suitable external forces providing necessary and sufficient conditions for the solvability. In so doing, Sobolev spaces are presented as direct sums of orthogonal subspaces, and solutions are found being elements of subspaces. Assuming that the elasticity tensor is changed in a subdomain, we justify passages to limits as a rigidity parameter of the elastic body tends to infinity and to zero. The limit models are characterized by rigid inclusions and cavities in elastic bodies. One can refer the reader to papers [14, 15, 16, 17, 18, 19] related to elastic bodies with rigid inclusions and cavities for Dirichlet boundary conditions on external boundaries. As for general approaches to analysis of non-coercive boundary value problems we can mention the works [20, 21, 22]. Various non-coercive boundary problems for elastic bodies with cracks and thin inclusions can be found in [23, 24, 25, 26, 27]. Other models for describing composite bodies are presented in [28, 29, 30].

The paper is structured as follows. In Section 2, we provide variational and differential formulations of the problem considered and find suitable conditions imposed on external forces providing a solvability of the problem. Passages to limits, as a rigidity parameter of the elastic moduli tends to infinity and to zero, are investigated in Sections 3 and 4. Limit models corresponding to rigid inclusions and cavities inside the elastic body are analyzed. A case of the crack crossing the external boundary of the elastic body is investigated in Section 5.

2 Setting the problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary Γ, and $\gamma \subset \Omega$ a smooth curve without self-intersections, $\bar{\gamma} \cap \Gamma = \emptyset$. Denote by $n = (n_1, n_2)$, $\nu = (\nu_1, \nu_2)$ unit normal vectors to Γ and γ, respectively; $\Omega_\gamma = \Omega \backslash \bar{\gamma}$, see Fig.1.

Consider an elasticity tensor $A = \{a_{ijkl}\}$ with the usual symmetry and positive definiteness properties

$$a_{ijkl}\xi_{ij}\xi_{kl} \geq c_0|\xi|^2 \quad \forall \xi_{ij},\ c_0 = \text{const} > 0;\ a_{ijkl} = a_{jikl} = a_{klij},$$
$$a_{ijkl} \in L^\infty(\Omega),\ i,j,k,l = 1,2.$$

Summation convention over repeated indices is used; all functions with two lower indices are assumed to be symmetric in those indices.

Problem formulation to be analyzed is as follows. We have to find a displacement field $u = (u_1, u_2)$ and a stress tensor $\sigma = \{\sigma_{ij}\}, i,j = 1,2$, defined in Ω_γ, such that

$$-\text{div}\sigma = f,\ \sigma = A\varepsilon(u) \text{ in } \Omega_\gamma, \tag{1}$$

$$\sigma n = 0 \quad \text{on } \Gamma, \tag{2}$$

$$[u_\nu] \geq 0,\ [\sigma_\nu] = 0,\ \sigma_\nu[u_\nu] = 0 \text{ on } \ \gamma, \tag{3}$$

$$\sigma_\nu \leq 0,\ \sigma_\tau = 0 \text{ on } \gamma^\pm, \tag{4}$$

$$\int\limits_{\Omega_\gamma} u = 0, \int\limits_{\Omega_\gamma} (u_{1,2} - u_{2,1}) = 0, \tag{5}$$

where $u_\nu = u\nu$; $\sigma_\nu = \sigma_{ij}\nu_j\nu_i$ is a normal stress, $\sigma_\tau = \sigma_{ij}\nu_j\tau_i,\ \tau = (\tau_1, \tau_2) = (\nu_2, -\nu_1)$; $[u] = u^+ - u^-$ is the jump of the function u at γ, where the signs $\pm$ correspond to positive and negative directions of ν, and $u^\pm$ are traces of u at $\gamma^\pm$; $\varepsilon(u) = \{\varepsilon_{ij}(u)\}$ is the stain tensor, $\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}),\ i,j = 1,2$.

Relations (1) are the equilibrium equation and the constitutive law for the elastic body. Boundary conditions (3)-(4) are well known for the crack model with inequality type constraints at the crack faces, see [13]. The inequality in (3) provides a mutual non-penetration between the crack faces $\gamma^\pm$. The model (1)-(5) corresponds to a free boundary approach. This means that the contact set between the crack faces is unknown a priori. The Neumann type condition (2) at the external boundary Γ implies a non-coercivity of the problem (1)-(4). In view of this boundary condition, we have to find a solution of the problem in a suitable subspace. Relations (5) provide a uniqueness of the solution.

The problem (1)-(5) admits a variational formulation. To this end, we introduce the space

$$W = \{u \in H^1(\Omega_\gamma)^2 \mid \int\limits_{\Omega_\gamma} u = 0, \int\limits_{\Omega_\gamma} (u_{1,2} - u_{2,1}) = 0,\ u = (u_1, u_2)\},$$

and for any set $S \subset \mathbb{R}^2$ the following space of infinitesimal rigid displacements

$$R(S) = \{\rho = (\rho_1, \rho_2) \mid \rho(x) = (c^1, c^2) + b(x_2, -x_1),\ x = (x_1, x_2) \in S;\ b, c^1, c^2 \in \mathbb{R}\}$$

with arbitrary $b, c^1, c^2 \in \mathbb{R}$.

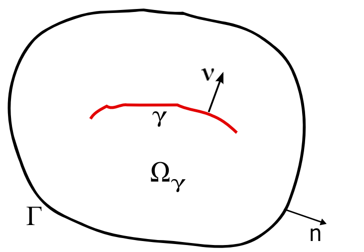

FIG. 1. Elastic body with crack γ

In the space W, we consider the inner product

$$\langle u, \bar{u} \rangle = \int\limits_{\Omega_\gamma} u_i \int\limits_{\Omega_\gamma} \bar{u}_i + \int\limits_{\Omega_\gamma} u_{i,j} \bar{u}_{i,j}, \;\; u = (u_1, u_2), \, \bar{u} = (\bar{u}_1, \bar{u}_2). \tag{6}$$

The inner product (6) induces the norm in W being equivalent to the standard one. Suitable arguments can be found in [13], Section 1.1.5. A presence of the cut γ in the domain Ω_γ does not provide any difficulties since the domain Ω_γ can be divided into two subdomains with Lipschitz boundaries.

The following statement takes place.

Proposition 1. The space $H^1(\Omega_\gamma)^2$ can be presented as a direct sum of two orthogonal subspaces with respect to the inner product (6),

$$H^1(\Omega_\gamma)^2 = W \oplus R(\Omega_\gamma).$$

Proof. Take any elements $\rho \in R(\Omega_\gamma)$ and $u \in H^1(\Omega_\gamma)^2$. Let $\rho(x) = (c^1, c^2) + b(x_2, -x_1), \; c^1, c^2, b \in \mathbb{R}$. Then we have

$$\langle u, \rho \rangle = c^i |\Omega| \int\limits_{\Omega_\gamma} u_i + b \int\limits_{\Omega_\gamma} (u_{1,2} - u_{2,1}) + b \Big(\int\limits_{\Omega_\gamma} u_1 \int\limits_{\Omega_\gamma} x_2 - \int\limits_{\Omega_\gamma} u_2 \int\limits_{\Omega_\gamma} x_1 \Big); \; u = (u_1, u_2). \tag{7}$$

From (7) it follows that sufficient and necessary conditions for the identity

$$\langle u, \rho \rangle = 0 \;\; \forall \rho \in R(\Omega_\gamma)$$

are of the form

$$\int_{\Omega_\gamma} u = 0, \int_{\Omega_\gamma} (u_{1,2} - u_{2,1}) = 0,\ u = (u_1, u_2),$$

i.e. $u \in W$. Proposition 1 is proved.

Consider the set of permissible displacements

$$K = \{v \in H^1(\Omega_\gamma)^2 \ |\ [v_\nu] \geq 0 \text{ on } \gamma\},$$

and the energy functional $E: \ H^1(\Omega_\gamma)^2 \to \mathbb{R}$,

$$E(v) = \frac{1}{2} \int_{\Omega_\gamma} \sigma(v)\varepsilon(v) - \int_{\Omega_\gamma} fv,$$

where $f = (f_1, f_2) \in L^2(\Omega_\gamma)^2$ is a given external force acting on the elastic body, $\sigma(u) = A\varepsilon(u)$. To simplify notations we write $\sigma(u)\varepsilon(u)$ instead of $\sigma_{ij}(u)\varepsilon_{ij}(u)$. In this case the minimization problem

$$\inf_{v \in K \cap W} E(v). \tag{8}$$

has a solution u satisfying the variational inequality

$$u \in K \cap W, \tag{9}$$

$$\int_{\Omega_\gamma} \sigma(u)\varepsilon(\bar{u} - u) - \int_{\Omega_\gamma} f(\bar{u} - u) \geq 0 \quad \forall \bar{u} \in K \cap W. \tag{10}$$

Indeed, the coercivity of the functional E on the space W follows from the inequality

$$\int_{\Omega_\gamma} \varepsilon(v)\varepsilon(v) \geq c_1 \{ \int_{\Omega_\gamma} v_{i,j} v_{i,j} + \int_{\Omega_\gamma} v_i \int_{\Omega_\gamma} v_i \} = c_1 \|v\|^2_{H^1(\Omega_\gamma)^2} \ \ \forall v \in W \tag{11}$$

with a positive constant c_1 what is a consequence of Proposition 1.4 from [13]. The set $W \cap K$ is weakly closed. Hence, we can use the general statement concerning a solution existence for minimization problems, see for example Theorem 1.13 in [13]. Thus the problem (8) has a solution.

We need to prove one more statement.

Proposition 2. For any $\tilde{u} \in K$ there exist $\bar{u} \in K \cap W$ and $\rho \in R(\Omega_\gamma)$ such that

$$\tilde{u} = \bar{u} + \rho.$$

Proof. We take any $\tilde{u} \in K$. By Proposition 1, we have

$$\tilde{u} = \bar{u} + \rho,\ \bar{u} \in W,\ \rho \in R(\Omega_\gamma).$$

Moreover

$$[\tilde{u}_\nu] = [(\rho + \bar{u})_\nu] = [\bar{u}_\nu] \geq 0 \text{ on } \gamma.$$

Thus, we have $\bar{u} \in K$. On the other hand, $\bar{u} \in W$, and consequently, $\bar{u} \in K \cap W$. Proposition 2 is proved.

Assume that the external forces f satisfy the condition

$$\int_{\Omega_\gamma} f\rho = 0 \ \ \forall \rho \in R(\Omega_\gamma). \tag{12}$$

In this case, the relation (10) can be rewritten as

$$\int_{\Omega_\gamma} \sigma(u)\varepsilon(\bar{u} + \rho - u) - \int_{\Omega_\gamma} f(\bar{u} + \rho - u) \geq 0 \quad \forall \bar{u} \in K \cap W, \ \forall \rho \in R(\Omega_\gamma),$$

and consequently, by Proposition 2, the problem (9), (10) reduces to the variational inequality

$$u \in K \cap W, \tag{13}$$

$$\int_{\Omega_\gamma} \sigma(u)\varepsilon(\hat{u} - u) - \int_{\Omega_\gamma} f(\hat{u} - u) \geq 0 \quad \forall \hat{u} \in K. \tag{14}$$

Thus, the following statement has been proved.

Theorem 1. There exists a (unique) solution of the problem (13)-(14) provided that the condition (12) holds.

To prove a uniqueness of the solution to (13)-(14), we assume that there are two solutions $u^1, u^2 \in K \cap W$. Then (13)-(14) imply the following inequality for $u = u^1 - u^2$,

$$\int_{\Omega_\gamma} \varepsilon(u)\varepsilon(u) \leq 0.$$

Hence, by inequality (11), it follows $u \equiv 0$ in Ω_γ that proves the statement.

We can also check that problem formulations (1)-(5) and (13)-(14) are equivalent for smooth solutions. The proof of this statement is omitted here since it reminds that for the case of Dirichlet conditions at the external boundary Γ (i.e. $u = 0$ instead of $\sigma n = 0$), see details in [13].

To conclude this section, we notice that the condition (12) is not only sufficient for the solvability of the problem (13)-(14) but it is necessary. Indeed, assume that the problem (13)-(14) has a solution. We can substitute in (14) test functions of the form $\bar{u} = u \pm \rho, \ \rho \in R(\Omega_\gamma)$. This implies (12).

3 Formation of rigid inclusions in elastic body

Assume that a subdomain $\omega \subset \Omega$ has a smooth boundary Γ_0 such that $\gamma \subset \Gamma_0$, meas $(\Gamma_0 \setminus \gamma) > 0$, $\Gamma \cap \Gamma_0 = \emptyset$, see Fig. 2. This subdomain can be seen as an elastic inclusion in the elastic body Ω. A unit normal vector to Γ_0 is denoted by ν. The condition (12) is assumed to be fulfilled in this section.

We introduce a positive parameter λ into the model (1)-(5). This parameter would be responsible for the rigidity of the inclusion ω. To be more precise,

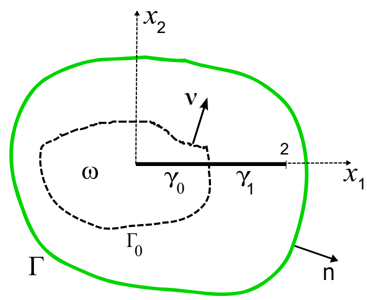

FIG. 2. Elastic body with inclusion ω and crack γ

introduce the elasticity tensor

$$a_{ijkl}^{\lambda} = \begin{cases} a_{ijkl} & \text{in} \quad \Omega \setminus \bar{\omega} \\ \lambda^{-1} a_{ijkl} & \text{in} \quad \omega \end{cases} \tag{15}$$

with a positive parameter λ. Denote $A^{\lambda} = \{a_{ijkl}^{\lambda}\}$ and consider the problem similar to (1)-(5) with the elasticity tensor A^{λ}. Namely, we have to find a displacement field $u^{\lambda} = (u_1^{\lambda}, u_2^{\lambda})$ and a stress tensor $\sigma^{\lambda} = \{\sigma_{ij}^{\lambda}\}, i, j = 1, 2,$ defined in Ω_{γ}, such that

$$-\text{div}\sigma^{\lambda} = f,\ \sigma^{\lambda} = A^{\lambda}\varepsilon(u^{\lambda}) \text{ in } \Omega_{\gamma}, \tag{16}$$

$$\sigma^{\lambda} n = 0 \ \text{ on } \Gamma, \tag{17}$$

$$[u_{\nu}^{\lambda}] \geq 0,\ [\sigma_{\nu}^{\lambda}] = 0,\ \sigma_{\nu}^{\lambda}[u_{\nu}^{\lambda}] = 0 \text{ on } \ \gamma, \tag{18}$$

$$\sigma_{\nu}^{\lambda} \leq 0,\ \sigma_{\tau}^{\lambda} = 0 \text{ on } \gamma^{\pm}, \tag{19}$$

$$\int\limits_{\Omega_{\gamma}} u^{\lambda} = 0, \int\limits_{\Omega_{\gamma}} (u_{1,2}^{\lambda} - u_{2,1}^{\lambda}) = 0. \tag{20}$$

We know that the problem (16)-(20) can be written in the variational form

$$u^\lambda \in K \cap W, \tag{21}$$

$$\int\limits_{\Omega_\gamma} \sigma^\lambda(u^\lambda)\varepsilon(\hat{u} - u^\lambda) - \int\limits_{\Omega_\gamma} f(\hat{u} - u^\lambda) \geq 0 \quad \forall \hat{u} \in K. \tag{22}$$

In what follows, we plan to justify a passage to limit as $\lambda \to 0$ in (21)-(22). The limit model would correspond to a rigid inclusion ω inside Ω with a crack γ on the boundary Γ_0.

From (21)-(22) we obtain

$$\int\limits_{\Omega\setminus\bar{\omega}} \sigma(u^\lambda)\varepsilon(u^\lambda) + \frac{1}{\lambda}\int\limits_{\omega} \sigma(u^\lambda)\varepsilon(u^\lambda) - \int\limits_{\Omega_\gamma} f u^\lambda = 0, \tag{23}$$

where $\sigma(u^\lambda) = A\varepsilon(u^\lambda)$. The relation (23) implies for small λ

$$\int\limits_{\Omega_\gamma} \sigma(u^\lambda)\varepsilon(u^\lambda) \leq \int\limits_{\Omega_\gamma} f u^\lambda.$$

Thus, taking into account the positive definiteness of the tensor A and the inequality (11), we have uniformly for small λ,

$$\|u^\lambda\|^2_{H^1(\Omega_\gamma)^2} \leq c. \tag{24}$$

Indeed, for small λ the inequality $\frac{1}{\lambda} > 1$ holds, hence, the arguments used for the proof of the energy functional coercivity can be applied. Choosing a subsequence, if necessary, by (24), we assume that as $\lambda \to 0$,

$$u^\lambda \to u \text{ weakly in } H^1(\Omega_\gamma)^2. \tag{25}$$

In addition to this, the relation (23) implies

$$\int\limits_{\omega} \sigma(u^\lambda)\varepsilon(u^\lambda) \leq c\lambda,$$

and consequently, for the limit function u we obtain

$$\varepsilon_{ij}(u) = 0 \text{ in } \omega,\ i,j = 1,2.$$

As a result, it provides

$$u|_\omega = \rho^0 \in R(\omega). \tag{26}$$

Introduce two more functional spaces

$$H^{1,\omega}(\Omega_\gamma)^2 = \{v \in H^1(\Omega_\gamma)^2 \mid v|_\omega \in R(\omega)\},$$

$$V = \{v \in H^{1,\omega}(\Omega_\gamma)^2 \mid \int\limits_{\Omega_\gamma} v = 0, \int\limits_{\Omega_\gamma}(v_{1,2} - v_{2,1}) = 0,\ v = (v_1, v_2)\}.$$

Let K_ω be a set of permissible displacements,

$$K_\omega = \{v \in H^{1,\omega}(\Omega_\gamma)^2 \mid [v_\nu] \geq 0 \text{ on } \gamma\}.$$

Take any element $\hat{u} \in K_\omega$; then $\hat{u} \in K$. This element can be substituted as a test one in (22). By (25), (26), the passage to the limit, as $\lambda \to 0$, is possible what implies

$$u \in K_\omega \cap V, \tag{27}$$

$$\int\limits_{\Omega\setminus\bar{\omega}} \sigma(u)\varepsilon(\hat{u}-u) - \int\limits_{\Omega_\gamma} f(\hat{u}-u) \geq 0 \quad \forall \hat{u} \in K_\omega. \tag{28}$$

Thus, the following assertion is proved.

Theorem 2. Solutions of the problems (21)-(22) converge in the sense (25), (26) as $\lambda \to 0$ to the solution of the problem (27)-(28) provided that the condition (12) holds.

The problem (27)-(28) describes an equilibrium of the elastic body $\Omega \setminus \bar{\omega}$ with the rigid inclusion ω in the presence of the crack γ located on the boundary Γ_0. This problem admits a differential formulation. We have to find a displacement field $u = (u_1, u_2)$ and a stress tensor $\sigma = \{\sigma_{ij}\}, i, j = 1, 2$, defined in Ω_γ, $\Omega \setminus \bar{\omega}$, respectively, as well as $\rho^0 \in R(\omega)$, such that

$$-\mathrm{div}\sigma = f,\ \sigma = A\varepsilon(u) \text{ in } \Omega \setminus \bar{\omega}, \tag{29}$$

$$\sigma n = 0 \ \text{ on } \Gamma, \tag{30}$$

$$u = \rho^0 \text{ on } \omega;\ [u] = 0 \text{ on } \Gamma_0 \setminus \gamma, \tag{31}$$

$$[u_\nu] \geq 0,\ \sigma_\nu^+ \leq 0,\ \sigma_\tau^+ = 0,\ \sigma_\nu^+[u_\nu] = 0 \text{ on } \ \gamma, \tag{32}$$

$$\int\limits_{\Gamma_0} \sigma^+\nu \cdot \rho + \int\limits_{\omega} f\rho = 0 \ \forall \rho \in R(\omega), \tag{33}$$

$$\int\limits_{\Omega_\gamma} u = 0, \int\limits_{\Omega_\gamma} (u_{1,2} - u_{2,1}) = 0. \tag{34}$$

The following statement takes place.

Theorem 3. Problem formulations (27)-(28) and (29)-(34) are equivalent for smooth solutions.

The proof of this theorem reminds that for the case of elastic body with the rigid inclusion ω and Dirichlet conditions on the external boundary Γ, see [13], and we omit it.

Let us demonstrate that an existence of the solution to (27)-(28) can be proved directly by minimizing the energy functional over a suitable set. The inner product in the space $H^{1,\omega}(\Omega_\gamma)^2$ is introduced by the formula (6). Similar to Proposition 1, the following statement takes place.

Proposition 3. The space $H^{1,\omega}(\Omega_\gamma)^2$ can be presented as a direct sum of two orthogonal subspaces with respect to the inner product (6),

$$H^{1,\omega}(\Omega_\gamma)^2 = V \oplus R(\Omega_\gamma).$$

In addition to this, similar to Proposition 2, we have

Proposition 4. For any $\tilde{u} \in K_\omega$ there exist $\bar{u} \in K_\omega \cap V$ and $\rho \in R(\Omega_\gamma)$ such that

$$\tilde{u} = \bar{u} + \rho.$$

Now, consider the energy functional $E_\omega : H^{1,\omega}(\Omega_\gamma)^2 \to \mathbb{R}$,

$$E_\omega(v) = \frac{1}{2} \int\limits_{\Omega_\gamma} \sigma(v)\varepsilon(v) - \int\limits_{\Omega_\gamma} fv.$$

Then the minimization problem

$$\inf_{v \in K_\omega \cap V} E_\omega(v)$$

has a solution u satisfying the variational inequality

$$u \in K_\omega \cap V, \tag{35}$$

$$\int\limits_{\Omega_\gamma} \sigma(u)\varepsilon(\bar{u} - u) - \int\limits_{\Omega_\gamma} f(\bar{u} - u) \geq 0 \quad \forall \bar{u} \in K_\omega \cap V. \tag{36}$$

The coercivity of the functional E_ω follows from the reasoning of Section 2 related to the coercivity of the functional E. Taking into account the condition (12) and Proposition 4, we rewrite the problem (35)-(36) in the form

$$u \in K_\omega \cap V, \tag{37}$$

$$\int\limits_{\Omega_\gamma} \sigma(u)\varepsilon(\bar{u} - u) - \int\limits_{\Omega_\gamma} f(\bar{u} - u) \geq 0 \quad \forall \bar{u} \in K_\omega, \tag{38}$$

what coincides with (27)-(28).

To conclude the section, note that the condition (12) is also necessary for the solvability of the problem (37)-(38). Indeed, let the problem (37)-(38) have the solution. Substituting in (38) test functions of the form $\bar{u} = u \pm \rho$, $\rho \in R(\Omega_\gamma)$, we obtain (12).

4 Formation of cavities in elastic body

Assume that the subdomain $\omega \subset \Omega$ is chosen like in the previous section, see Fig 2. Introduce the elasticity tensor

$$A_\lambda = \begin{cases} A & \text{in} \quad \Omega \setminus \bar{\omega} \\ \lambda A & \text{in} \quad \omega \end{cases} \tag{39}$$

with a positive parameter λ. We consider the problem similar to (1)-(5) with the elasticity tensor A_λ. Namely, we have to find a displacement field $u^\lambda = (u_1^\lambda, u_2^\lambda)$ and a stress tensor $\sigma^\lambda = \{\sigma_{ij}^\lambda\}, i, j = 1, 2$, defined in Ω_γ, such that

$$-\mathrm{div}\sigma^\lambda = f,\ \sigma^\lambda = A_\lambda \varepsilon(u^\lambda) \text{ in } \Omega_\gamma, \tag{40}$$

$$\sigma^\lambda n = 0 \ \text{ on } \Gamma, \tag{41}$$

$$[u^\lambda_\nu] \ge 0,\ [\sigma^\lambda_\nu] = 0,\ \sigma^\lambda_\nu [u^\lambda_\nu] = 0 \text{ on } \ \gamma, \tag{42}$$

$$\sigma^\lambda_\nu \le 0,\ \sigma^\lambda_\tau = 0 \text{ on } \gamma^\pm, \tag{43}$$

$$\int\limits_{\Omega_\gamma} u^\lambda = 0, \int\limits_{\Omega_\gamma} (u^\lambda_{1,2} - u^\lambda_{2,1}) = 0. \tag{44}$$

As we know, the problem (40)-(44) admits the variational formulation

$$u^\lambda \in K \cap W, \tag{45}$$

$$\int\limits_{\Omega_\gamma} \sigma^\lambda(u^\lambda)\varepsilon(\hat{u} - u^\lambda) - \int\limits_{\Omega_\gamma} f(\hat{u} - u^\lambda) \ge 0 \quad \forall \hat{u} \in K. \tag{46}$$

In this section we assume that the function f satisfies the condition (12), and moreover, $f \equiv 0$ in ω. In what follows, a passage to the limit as $\lambda \to 0$ is justified in (45)-(46). The limit model would correspond to the elastic body with the cavity ω.

The relations (45)-(46) imply

$$\int\limits_{\Omega \setminus \bar{\omega}} \sigma(u^\lambda)\varepsilon(u^\lambda) + \lambda \int\limits_{\omega} \sigma(u^\lambda)\varepsilon(u^\lambda) - \int\limits_{\Omega \setminus \bar{\omega}} f u^\lambda = 0, \tag{47}$$

where $\sigma(u^\lambda) = A\varepsilon(u^\lambda)$. This equality gives

$$\int\limits_{\Omega \setminus \bar{\omega}} \sigma(u^\lambda)\varepsilon(u^\lambda) \le \int\limits_{\Omega \setminus \bar{\omega}} f u^\lambda. \tag{48}$$

Consider the space $H^1(\Omega \setminus \bar{\omega})^2$ with the inner product

$$\{u, \bar{u}\} = \int\limits_{\Omega \setminus \bar{\omega}} u_i \int\limits_{\Omega \setminus \bar{\omega}} \bar{u}_i + \int\limits_{\Omega \setminus \bar{\omega}} u_{i,j}\bar{u}_{i,j},\ \ u = (u_1, u_2),\ \bar{u} = (\bar{u}_1, \bar{u}_2). \tag{49}$$

As we know (Section 2), the space $H^1(\Omega \setminus \bar{\omega})^2$ can be presented as a direct sum of two orthogonal subspaces with respect to the inner product (49). Namely,

$$H^1(\Omega \setminus \bar{\omega})^2 = \tilde{W} \oplus R(\Omega \setminus \bar{\omega}),$$

where

$$\tilde{W} = \{\tilde{u} \in H^1(\Omega \setminus \bar{\omega})^2 \mid \int\limits_{\Omega \setminus \bar{\omega}} \tilde{u} = 0, \int\limits_{\Omega \setminus \bar{\omega}} (\tilde{u}_{1,2} - \tilde{u}_{2,1}) = 0,\ \tilde{u} = (\tilde{u}_1, \tilde{u}_2)\}.$$

Then for the solution u^λ of (45)-(46) we obtain the following representation in the domain $\Omega \setminus \bar{\omega}$,

$$u^\lambda = \hat{u}^\lambda + \rho^\lambda,\ \hat{u}^\lambda \in \tilde{W},\ \rho^\lambda \in R(\Omega \setminus \bar{\omega}).$$

In this case, the inequality (48) gives

$$\int\limits_{\Omega\setminus\bar{\omega}} \sigma(\hat{u}^\lambda)\varepsilon(\hat{u}^\lambda) \le \int\limits_{\Omega\setminus\bar{\omega}} f\hat{u}^\lambda. \tag{50}$$

Since $\hat{u}^\lambda \in \tilde{W}$, the inequality (50) implies the estimate being uniform in λ,

$$\|\hat{u}^\lambda\|^2_{H^1(\Omega\setminus\bar{\omega})^2} \le c. \tag{51}$$

For small λ, from (47) it follows

$$\lambda \int\limits_{\Omega\setminus\bar{\omega}} \sigma(u^\lambda)\varepsilon(u^\lambda) + \lambda \int\limits_{\omega} \sigma(u^\lambda)\varepsilon(u^\lambda) \le \int\limits_{\Omega\setminus\bar{\omega}} f\hat{u}^\lambda.$$

Taking into account a boundedness of $\hat{u}^\lambda$ in the space $H^1(\Omega \setminus \bar{\omega})^2$ and the property $u^\lambda \in W$ the previous relation gives the estimate

$$\lambda\|u^\lambda\|^2_{H^1(\Omega_\gamma)^2} \le c. \tag{52}$$

Choosing a subsequence, if necessary, by (51), (52), we assume that as $\lambda \to 0$,

$$\hat{u}^\lambda \to u \text{ weakly in } H^1(\Omega \setminus \bar{\omega})^2, \tag{53}$$

$$\sqrt{\lambda}u^\lambda \to \tilde{u} \text{ weakly in } H^1(\Omega_\gamma)^2. \tag{54}$$

Let us rewrite the variational inequality (45)-(46) in the form

$$u^\lambda \in K \cap W, \tag{55}$$

$$\int\limits_{\Omega\setminus\bar{\omega}} \sigma(\hat{u}^\lambda)\varepsilon(\hat{u} - \hat{u}^\lambda) + \lambda \int\limits_{\omega} \sigma(\hat{u}^\lambda)\varepsilon(\hat{u} - \hat{u}^\lambda) - \int\limits_{\Omega\setminus\bar{\omega}} f(\hat{u} - \hat{u}^\lambda) \ge 0 \quad \forall \hat{u} \in K. \tag{56}$$

By the convergences (53)-(54), we can pass to the limit in (55)-(56) as $\lambda \to 0$ what implies

$$u \in \tilde{W}, \tag{57}$$

$$\int\limits_{\Omega\setminus\bar{\omega}} \sigma(u)\varepsilon(\hat{u} - u) - \int\limits_{\Omega\setminus\bar{\omega}} f(\hat{u} - u) \ge 0 \quad \forall \hat{u} \in K. \tag{58}$$

By arbitrariness of the test functions in the domain $\Omega \setminus \bar{\omega}$, we can rewrite the inequality (57)-(58) in the form

$$u \in \tilde{W}, \quad \int\limits_{\Omega\setminus\bar{\omega}} \sigma(u)\varepsilon(\hat{u}) - \int\limits_{\Omega\setminus\bar{\omega}} f\hat{u} = 0 \quad \forall \hat{u} \in H^1(\Omega \setminus \bar{\omega})^2. \tag{59}$$

Hence, the following statement has been proved.

Theorem 4. Solutions of the problems (45)-(46) converge in the sense (53), (54) as $\lambda \to 0$ to the solution of the problem (59) provided that the condition (12) holds and $f \equiv 0$ in ω.

To conclude the section, we provide an equivalent differential formulation of the problem (59). It is necessary to find functions $u = (u_1, u_2)$ and $\sigma = \{\sigma_{ij}\}, i, j = 1, 2$, defined in $\Omega \setminus \bar{\omega}$, such that

$$-\mathrm{div}\sigma = f, \ \sigma = A\varepsilon(u) \text{ in } \Omega \setminus \bar{\omega}, \tag{60}$$

$$\sigma n = 0 \ \text{ on } \Gamma; \ \sigma\nu = 0 \ \text{ on } \Gamma_0, \tag{61}$$

$$\int\limits_{\Omega\setminus\bar{\omega}} u = 0, \int\limits_{\Omega\setminus\bar{\omega}} (u_{1,2} - u_{2,1}) = 0. \tag{62}$$

The problem (60)-(62) describes an equilibrium of the elastic body $\Omega \setminus \bar{\omega}$ containing the cavity ω. Boundary conditions (61) of the Neumann type are imposed at the external boundaries Γ and Γ_0.

Notice that the solution of the problem (59) is unique.

5 Crack crossing the external boundary

Similar results can be obtained in the case of a crack crossing the external boundary Γ at the point $x^0 \in \Gamma$, see Fig. 3. We do not go in details in such the case just providing problem formulations for the original equilibrium problem as well as for two limit cases. In so doing, it is assumed that the angle between γ and Γ is nonzero at the point x^0.

As before, we assume that γ is a smooth curve without selfintersections, $\Omega_\gamma = \Omega \setminus \bar{\gamma}, \ \bar{\gamma} \cap \Gamma = x^0$, and f satisfies the condition (12). A nonzero angle between γ and Γ allows us to divide the domain Ω_γ into two subdomains with Lipschitz boundaries and use the arguments of the previous sections. First, consider the equilibrium problem for the case corresponding to Fig. 3. We have to find functions $u = (u_1, u_2)$, $\sigma = \{\sigma_{ij}\}, i, j = 1, 2$, defined in Ω_γ,

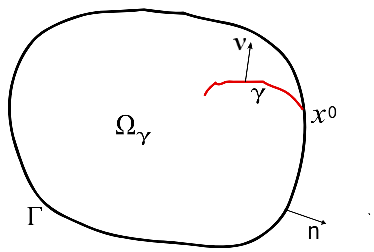

FIG. 3. Elastic body with crack γ

such that

$$-\mathrm{div}\sigma = f,\ \sigma = A\varepsilon(u) \text{ in } \Omega_\gamma, \tag{63}$$

$$\sigma n = 0 \ \text{ on } \Gamma, \tag{64}$$

$$[u_\nu] \geq 0,\ [\sigma_\nu] = 0,\ \sigma_\nu[u_\nu] = 0 \text{ on } \ \gamma, \tag{65}$$

$$\sigma_\nu \leq 0,\ \sigma_\tau = 0 \text{ on } \gamma^\pm, \tag{66}$$

$$\int\limits_{\Omega_\gamma} u = 0, \int\limits_{\Omega_\gamma} (u_{1,2} - u_{2,1}) = 0. \tag{67}$$

The problem (63)-(67) admits the variational formulation. Introducing the subspace $W \subset H^1(\Omega_\gamma)^2$ and the set K similar to Section 2, we can prove a unique solution of the problem

$$u \in K \cap W, \tag{68}$$

$$\int\limits_{\Omega_\gamma} \sigma(u)\varepsilon(\hat{u} - u) - \int\limits_{\Omega_\gamma} f(\hat{u} - u) \geq 0 \quad \forall \hat{u} \in K. \tag{69}$$

Problem formulations (63)-(67) and (68)-(69) are equivalent for smooth solutions.

The next step is related to the analysis of the limit procedure as the elasticity tensor is changed in ω according to the formula (15), and the domain ω is chosen as depicted in Fig. 4. In such a case, the boundary of the domain ω assumed to be smooth consists of three parts: $\gamma, \gamma_1, \Gamma_1$. In particular, the angle between γ_1 and Γ is nonzero at the point $x^1 \in \Gamma$. The equilibrium problem corresponding to the elasticity tensor A^λ coincides with (21)-(22). The passage to the limit as $\lambda \to 0$ can be justified in (21)-(22), and the limit problem reads as follows: find functions $u = (u_1, u_2)$, $\sigma = \{\sigma_{ij}\}, i, j = 1, 2$, defined in Ω_γ, $\Omega \setminus \bar{\omega}$, respectively, as well as $\rho^0 \in R(\omega)$, such that

$$-\mathrm{div}\sigma = f,\ \sigma = A\varepsilon(u) \text{ in } \Omega \setminus \bar{\omega}, \tag{70}$$

$$\sigma n = 0 \ \text{ on } \Gamma \setminus \Gamma_1, \tag{71}$$

$$u = \rho^0 \text{ on } \omega;\ [u] = 0 \text{ on } \gamma_1, \tag{72}$$

$$[u_\nu] \geq 0,\ \sigma_\nu^+ \leq 0,\ \sigma_\tau^+ = 0,\ \sigma_\nu^+ [u_\nu] = 0 \text{ on } \ \gamma, \tag{73}$$

$$\int\limits_{\gamma \cup \gamma_1} \sigma^+ \nu \cdot \rho + \int\limits_\omega f \rho = 0 \ \ \forall \rho \in R(\omega), \tag{74}$$

$$\int\limits_{\Omega_\gamma} u = 0, \int\limits_{\Omega_\gamma} (u_{1,2} - u_{2,1}) = 0. \tag{75}$$

The variational formulation of the problem (70)-(75) coincides with (37)-(38) with suitably introduced notations for K_ω, V in the case of geometry corresponding to Fig. 4. The differential and variational formulations of the problem are equivalent for smooth solutions. Notice that the limiting case corresponds to the elastic body $\Omega \setminus \bar{\omega}$ conjugating with the rigid inclusion ω.

To conclude the section, consider the elasticity tensor A_λ according to the formula (39). In this case, the differential problem formulation coincides with (40)-(44), and we aim to pass to the limit in (45)-(46) as $\lambda \to 0$ assuming that f satisfies (12) and $f \equiv 0$ in ω. Omitting the arguments, we provide the differential statement of the limit problem: it is necessary to find functions $u = (u_1, u_2)$ and $\sigma = \{\sigma_{ij}\}, i, j = 1, 2$, defined in $\Omega \setminus \bar{\omega}$, such that

$$-\mathrm{div}\sigma = f,\ \sigma = A\varepsilon(u) \text{ in } \Omega \setminus \bar{\omega}, \tag{76}$$

$$\sigma n = 0 \ \text{ on } \Gamma \setminus \Gamma_1;\ \sigma \nu = 0 \ \text{ on } \gamma \cup \gamma_1, \tag{77}$$

$$\int\limits_{\Omega \setminus \bar{\omega}} u = 0, \int\limits_{\Omega \setminus \bar{\omega}} (u_{1,2} - u_{2,1}) = 0. \tag{78}$$

The equivalent variational formulation of the problem (76)-(78) coincides with (59) for the suitably introduced space $\tilde{W}$ corresponding to Fig. 4.

The problem (76)-(78) describes an equilibrium state of the elastic body occupying the domain $\Omega \setminus \bar{\omega}$ with the Neumann boundary conditions at the external boundary.

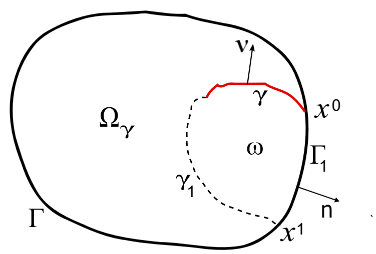

Fig. 4. Elastic body with inclusion ω and crack γ

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