S©MR

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ
Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru
ISSN 1813-3304

Vol. 21, No. 2, pp. 1097–1107 (2024)
https://doi.org/10.33048/semi.2024.21.071

УДК 517.977
MSC 49K15

INPUT RECONSTRUCTION PROBLEM FOR A NONLINEAR SYSTEM OF DIFFERENTIAL EQUATIONS: THE CASE OF INCOMPLETE MEASUREMENTS

V.L. ROZENBERG

Communicated by M.A. Shishlenin

Abstract: The problem of reconstructing an unknown input in a system of ordinary differential equations of a special kind is investigated by means of the approach of the theory of dynamic inversion. The input action should be reconstructed synchronously with the process of incomplete discrete measuring of a part of coordinates of the phase trajectory. A finite-step software-oriented solution algorithm based on the method of auxiliary closed-loop models is proposed, and its error is estimated. The novelty of the paper is that we study the inverse problem for a partially observed system with a nonlinear with respect to input equation describing the dynamics of the unmeasured coordinate.

Keywords: nonlinear system of ordinary differential equations, incomplete measurements, dynamic reconstruction, controlled model.

Rozenberg, V.L., Input reconstruction problem for a nonlinear system of differential equations: the case of incomplete measurements.

The work was performed as part of research conducted in the Ural Mathematical Center with the financial support of the Ministry of Science and Higher Education of the Russian Federation (Agreement number 075-02-2024-1377).

Received April, 12, 2022, published November, 25, 2024.

1 Introduction

The problem of reconstructing an unknown input (either a disturbance or a control) in a system of ordinary differential equations (ODEs) of a special kind on the basis of incomplete and inaccurate information on the phase state falls into the range of inverse problems of dynamics of controlled systems. This field has been intensively developing within the framework of identification theory due to its numerous applications. The first publications go back to the 60's of the previous century [1, 2], when some criteria of the unique solvability of inverse problems and of the continuous "input/output" dependence for systems described by ODEs were obtained. Inverse problems of dynamics, as a rule, are ill-posed and require the application of regularizing procedures. The huge amount of works is devoted to a posteriori approaches to constructing regularizing algorithms for solving different identification problems with the use of the whole history of output measurements. We list only some of related references. For an introduction to identification theory, see classical monograph [3]. In [4, 5], the emphasis is on the foundations of the theory of inverse and ill-posed problems.

Here, to solve the problem in question, we use the classical nowadays approach proposed and developed in the works by Kryazhimskii, Osipov, and their colleagues (see [6, 7, 8, 9] and bibliography in [8, 9]) and known now as the method of dynamic inversion. It is based on a combination of principles of the theory of positional control, first of all, of the Krasovskii principle of extremal aiming [10], and ideas of the theory of ill-posed problems [4]. The essence of the approach is that a reconstruction problem is reduced to a feedback control problem for an auxiliary dynamical system called a model. In the process, the adaptation of the model control to the results of current observations provides a required approximation of the unknown input. The method of dynamic inversion was applied many times to solving reconstruction, control, guidance problems in different statements for systems described by ODEs, functional differential equations, equations and variational inequalities with distributed parameters, equations with time delay, stochastic differential equations, fractional differential equations [8, 9, 11, 12, 13, 14, 15]. Stable algorithms operating for some classes of partially observed systems were designed [8, 11, 12, 13]; there the role of input signals can be played, for example, by measurements of a part of coordinates of the phase vector of a finite-dimensional system or by values of a solution on some subsets of the domain of definition in an infinite-dimensional problem. As an intrinsic feature of the considered systems, one should note their linearity with respect to input/control/disturbance.

The peculiarity of the dynamic reconstruction problem under incomplete information considered in the present paper is that an equation describing the dynamics of the unmeasured component of a system of ODEs is nonlinear with respect to input action. Note that a partial case of such problem was investigated in [16]. It seems that the results obtained will be useful to study the solvability of different nonlinear reconstruction problems.

2 Problem statement

Consider a nonlinear system of ODEs of the following form:

$$\begin{aligned} \dot y(t) &= f_1(t, y(t), z(t)) + g_1(t, y(t), z(t))u(t), \quad y(t_0) = y_0,\\ \dot z(t) &= f_2(t, y(t), z(t)) + g_2(t, y(t), z(t), u(t)), \quad z(t_0) = z_0. \end{aligned} \tag{1}$$

Here, $t \in T = [t_0, \vartheta]$, $(y(\cdot),\, z(\cdot))$ is the phase trajectory of the system, $y(t) \in \mathbb{R}^{n_1}$, $z(t) \in \mathbb{R}^{n_2}$; $u(\cdot)$ is an input action with values from a given compact convex set $P \subseteq \mathbb{R}^m$ and with bounded variation on T, i. e., $u(\cdot) \in U$, $U = \{u(\cdot) \in V(T; \mathbb{R}^m)\colon u(t) \in P \;\; \forall t \in T\}$; the vector functions f_1 and f_2 acting from $T \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ into $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively, and the matrix function g_1 acting from $T \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ into $\mathbb{R}^{n_1 \times m}$, and the vector function g_2 acting from $T \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m$ into $\mathbb{R}^{n_2}$ are Lipschitz with respect to their variables. A solution of the Cauchy problem is understood in the sense of Caratheodory.

The problem under discussion consists in the following. At discrete, frequent enough, times $\tau_i \in T$, $\tau_i = t_0 + i\delta$, $\delta = (\vartheta - t_0)/l$, $i \in [1 : (l-1)]$, the inaccurate information on the first component of the system is received. We assume that the initial state is known and $\tau_0 = t_0$. The measurement results, values $\xi_i \in \mathbb{R}^{n_1}$, satisfy the inequalities

$$\|\xi_i - y(\tau_i)\|_{n_1} \le h, \tag{2}$$

where $\|\cdot\|$ is the corresponding Euclidean norm, $h \in (0, 1)$ is the measurement error.

It is required to design an algorithm for the dynamic reconstruction of the unknown disturbance $u(t)$ from the information ξ_i, $i \in [1\colon (l-1)]$. The deviation of an approximation from the real input should be arbitrarily small in the metric of space $L_2(T; \mathbb{R}^m)$ for sufficiently small h and for time discretization step $\delta = \delta(h)$ concordant with h in a special way.

A finite-step software-oriented solution algorithm is based on the ideas of [6, 11]. In connection with the incomplete input information (only a part of the phase vector is measured at the times τ_i), first, we should construct a block of dynamic reconstructing of the unknown coordinate $z(t)$, which is treated as a provider of the information on the whole current phase state of the system. This information is operatively fed onto a block forming, by the feedback principle, a model control approximating the real input. The work of these blocks should be synchronized in time. As is said above, the novelty of the present paper consists namely in considering the inverse problem for dynamical system (1), when an input to be reconstructed and subject to known geometrical restrictions nonlinearly enters the equation for the unmeasured component $z(t)$.

3 Solution algorithm

The algorithm below is an application of the computational procedure from [11, 16] given the specific properties of system (1). At the initial moment $\tau_0 = t_0$, we fix a value h, determine parameters of the algorithm, including the value $l = l(h)$, and construct the uniform partition of the interval T with the step $\delta(h) = (\vartheta - t_0)/l(h)$: $\tau_i \in T$, $\tau_i = t_0 + i\delta(h)$, $i \in [0 : l(h)]$. We need a special condition restricting the dynamics of the system.

Condition 1. *The derivative $\dot{y}(\cdot)$ has bounded variation on T, $m \le n_1$, and the matrix g_1 is a matrix of full rank (i.e., of rank m) for all t, y, z.*

We introduce a controlled model system actually containing two blocks. The first block, identifier, using inaccurate measurements of the form (2), approximates the unmeasured component $z(t)$ in the continuous metric. The second block, controller, basing on the obtained information on the whole phase state of system (1), calculates a control approximating the desired input $u(t)$ in $L_2(T; \mathbb{R}^m)$-metric. Input model controls are produced by feedback laws based on the regularized Krasovskii principle of extremal aiming [10].

It is principal for us that, in virtue of the assumptions from Condition 1, we can formally solve the first equation of (1) with respect to $u(t)$:

$$u(t) = g_1^+(t, y(t), z(t))(\dot{y}(t) - f_1(t, y(t), z(t))), \tag{3}$$

where g_1^+ is the pseudoinverse matrix of dimension $m \times n_1$. The phase vector of the model is denoted by $w(t)$; it consists of two components with different destinations:

(i) an n_1-dimensional vector $w_y(t)$ and an n_2-dimensional vector $w_z(t)$ (identifier);

(ii) an n_1-dimensional vector $w_v(t)$ (controller).

The dynamics of the model and its initial state are defined by the relations

$$\begin{gathered}
\dot{w}_y(t) = \bar{u}_i, \\
\dot{w}_z(t) = f_2(\tau_i, \xi_i, w_z(\tau_i)) + \\
+ g_2(\tau_i, \xi_i, w_z(\tau_i), g_1^+(\tau_i, \xi_i, w_z(\tau_i))(\bar{u}_i - f_1(\tau_i, \xi_i, w_z(\tau_i)))), \\
\dot{w}_v(t) = f_1(\tau_i, \xi_i, w_z(\tau_i)) + g_1(\tau_i, \xi_i, w_z(\tau_i)) v_i, \\
w_y(t_0) = y_0,\ w_z(t_0) = z_0,\ w_v(t_0) = y_0.
\end{gathered} \tag{4}$$

Here, $t \in (\tau_i, \tau_{i+1}]$, $i \in [0\colon (l(h) - 1)]$; $\bar{u}_i$ and v_i are control actions of corresponding dimensions calculated (by the feedback principle) at the time τ_i by rules specified below.

Assuming the boundedness of the norms of the right-hand sides of system (1) by a constant $\bar{K}$ (its existence is evident), we find the value $\bar{u}_i$ from the relation

$$\bar{u}_i = \arg\min \left\{ 2\langle w_y(\tau_i) - \xi_i, u\rangle_{n_1} + \bar{\alpha}\|u\|_{n_1}^2\colon\ \|u\|_{n_1} \le \bar{K} \right\}. \tag{5}$$

The second model control v_i is defined as follows:

$$v_i = \arg\min \big\{ 2\langle w_v(\tau_i) - \xi_i, g_1(\tau_i, \xi_i, w_z(\tau_i))v\rangle_m + \alpha\|v\|_m^2 \colon v \in P \big\}. \qquad (6)$$

Here, $\bar\alpha = \bar\alpha(h)$ and $\alpha = \alpha(h)$ are regularization parameters, $\langle\cdot,\cdot\rangle$ is the corresponding scalar product. Obviously, situations are possible when the model controls can be found explicitly from formulas (5) and (6).

Dynamics (4) is chosen from the following argument. The motion of the component $w_y(t)$, for the choice of model control (5), provides the approximation of the derivative $\dot y(t)$ by this control in the $L_2(T;\mathbb{R}^{n_1})$-metric; thus overcoming the ill-posedness of the problem of numerical differentiation. This fact follows from the results of [6, 8, 17] concerning the principle of regularized extremal aiming applied to the first equation of (4). Then, using estimate (2) and formal expression (3) of the disturbance $u(t)$ from the first equation of the given system with the change of $\dot y(t)$ for $\bar u_i$, we expect the closeness of $w_z(t)$ to $z(t)$. In turn, this allows us to track the coordinate $y(t)$ by the component $w_v(t)$, and to approximate the desired input by model control (6), which form is explained (again as (5)) by the application of the regularized extremal aiming to the third equation of (4).

Let us choose regularization functions $\bar\alpha(h)$, $\alpha(h) : (0,1) \to \mathbb{R}^+$ and a family of partitions of the interval T with step $\delta(h)$, $h \in (0,1)$, with the properties

$$\delta(h) \to 0, \quad \bar\alpha(h) \to 0, \quad \alpha(h) \to 0,$$

$$\bar\rho(h) = \Big(\frac{(h+\delta(h))^2}{\bar\alpha^2(h)} + \bar\alpha(h)\Big)^{1/2}, \quad \bar\rho(h) \to 0,$$

$$\rho(h) = \Big(\frac{(h+\delta(h)+\bar\rho(h))^2}{\alpha^2(h)} + \alpha(h)\Big)^{1/2}, \quad \rho(h) \to 0 \quad \text{as} \quad h \to 0. \qquad (7)$$

Note that the relation between $\bar\rho(h)$ and $\rho(h)$ in (7) substantiates the fact that we need two regularization parameters, $\bar\alpha(h)$ and $\alpha(h)$. The control process for the model is organized as follows. At the initial time t_0, we fix h, $\delta(h)$, $\bar\alpha(h)$, and $\alpha(h)$. The work of the algorithm is decomposed into $l(h)$ identical steps. At the ith step performed on the interval $(\tau_i, \tau_{i+1}]$, the input data for calculations are the measurement ξ_i and the model state $w(\tau_i)$ obtained by this moment. The following operations are fulfilled. First, the block-identifier calculates model controls (5), then the block-controller, controls (6), after that the model state $w(\tau_{i+1})$ is recomputed. Actually, during the interval $(\tau_i, \tau_{i+1}]$, the constant controls

$$\bar u(t) = \bar u_i, \quad v(t) = v_i, \qquad (8)$$

are fed onto the input of system (4), thus forming the piece-wise constant functions $\bar u(t)$, $v(t)$, $t \in T$. At the next, $(i+1)$th, step, analogous actions are repeated. The work of the algorithm stops at the terminal time $t = \vartheta$. Let us formulate the main result of the paper.

Theorem 1. *Let conditions (7) of concordance of the parameters be fulfilled. Then, for model control (6), (8), we have the following estimate for the approximation error:*

$$\|u(\cdot) - v(\cdot)\|_{L_2(T;\mathbb{R}^m)} \leq C\rho(h), \tag{9}$$

where C is a constant independent of the values under estimation.

Proof. We use results obtained earlier for the reconstruction problem for a system of ODEs in the case of measuring the whole phase vector [8, 17]. Under the assumption that the variation of $\dot{y}(\cdot)$ is bounded, according to the general scheme from [17], we derive the estimate

$$\|\dot{y}(\cdot) - \bar{u}(\cdot)\|_{L_2(T;\mathbb{R}^{n_1})} \leq C_1\bar{\rho}(h). \tag{10}$$

Here and below, we denote by C_i auxiliary constants, which are independent of estimated values and can be written explicitly.

Using estimate (10), first, we show that the model variable $w_z(\cdot)$ approximates the unmeasured component $z(\cdot)$:

$$\|z(\tau_i) - w_z(\tau_i)\|_{n_2} \leq C_2(h + \delta(h) + \bar{\rho}(h)) \quad \forall i \in [0 : l(h)]. \tag{11}$$

Then, we complete the proof by means of the reapplication of an estimate like (10) to the third equation of model (4).

So, consider $t \in (\tau_i,\, \tau_{i+1}]$. Using relation (3), for almost all t, we write the equality:

$$\dot{z}(t) = f_2(t, y(t), z(t)) + g_2(t, y(t), z(t), g_1^+(t, y(t), z(t))(\dot{y}(t) - f_1(t, y(t), z(t)))).$$

Subtracting the similar equation for the model component $w_z(t)$, we obtain

$$\begin{gathered}\dot{z}(t) - \dot{w}_z(t) = f_2(t, y(t), z(t)) - f_2(\tau_i, \xi_i, w_z(\tau_i)) \\ + g_2(t, y(t), z(t), g_1^+(t, y(t), z(t))(\dot{y}(t) - f_1(t, y(t), z(t)))) \\ - g_2(\tau_i, \xi_i, w_z(\tau_i), g_1^+(\tau_i, \xi_i, w_z(\tau_i))(\bar{u}_i - f_1(\tau_i, \xi_i, w_z(\tau_i)))).\end{gathered}$$

Note that this relation, due to the arbitrariness of index i, is fulfilled for almost all $t \in T$. Taking into account that $w_z(t_0) = z_0$ and integrating, we have for all $t \in T$

$$\begin{gathered}z(t) - w_z(t) = \sum_{k=0}^{i-1} \int_{\tau_k}^{\tau_{k+1}} \Big(f_2(\tau, y(\tau), z(\tau)) - f_2(\tau_k, \xi_k, w_z(\tau_k)) \\ + g_2(\tau, y(\tau), z(\tau), g_1^+(\tau, y(\tau), z(\tau))(\dot{y}(\tau) - f_1(\tau, y(\tau), z(\tau)))) \\ - g_2(\tau_k, \xi_k, w_z(\tau_k), g_1^+(\tau_k, \xi_k, w_z(\tau_k))(\bar{u}_k - f_1(\tau_k, \xi_k, w_z(\tau_k)))) \Big)\, d\tau \\ + \int_{\tau_i}^{t} \Big(f_2(\tau, y(\tau), z(\tau)) - f_2(\tau_i, \xi_i, w_z(\tau_i)) \\ + g_2(\tau, y(\tau), z(\tau), g_1^+(\tau, y(\tau), z(\tau))(\dot{y}(\tau) - f_1(\tau, y(\tau), z(\tau)))) \\ - g_2(\tau_i, \xi_i, w_z(\tau_i), g_1^+(\tau_i, \xi_i, w_z(\tau_i))(\bar{u}_i - f_1(\tau_i, \xi_i, w_z(\tau_i)))) \Big)\, d\tau.\end{gathered}$$

The fact that the functions f_i, g_i, $i = 1, 2$, are Lipschitz and bounded and the matrix g_1 is always of full rank implies the boundedness and Lipschitz property of all their possible products and combinations as well as of the matrix g_1^+ (the desired properties of the latter are proved, for example, in [11]). Using these results, the inequality $\|w_z(t) - w_z(\tau_k)\|_{n_2} \le C_3\delta(h)$ for $t \in (\tau_k,\, \tau_{k+1}]$, and the boundedness of the model controls, we get an estimate similar to (11). Namely,

$$\|z(t) - w_z(t)\|_{n_2} \le C_4 \sum_{k=0}^{i-1} \int\limits_{\tau_k}^{\tau_{k+1}} \Big(\delta(h) + h + \|z(\tau) - w_z(\tau_k)\|_{n_2}$$

$$+\|g_2(\tau, y(\tau), z(\tau), g_1^+(\tau, y(\tau), z(\tau))(\dot y(\tau) - f_1(\tau, y(\tau), z(\tau))))$$

$$-g_2(\tau_k, \xi_k, w_z(\tau_k), g_1^+(\tau_k, \xi_k, w_z(\tau_k))(\bar u_k - f_1(\tau_k, \xi_k, w_z(\tau_k))))\|_{n_2}\Big)\, d\tau$$

$$+C_5 \int\limits_{\tau_i}^{t} \Big(\delta(h) + h + \|z(\tau) - w_z(\tau_i)\|_{n_2}$$

$$+\|g_2(\tau, y(\tau), z(\tau), g_1^+(\tau, y(\tau), z(\tau))(\dot y(\tau) - f_1(\tau, y(\tau), z(\tau))))$$

$$-g_2(\tau_i, \xi_i, w_z(\tau_i), g_1^+(\tau_i, \xi_i, w_z(\tau_i))(\bar u_i - f_1(\tau_i, \xi_i, w_z(\tau_i))))\|_{n_2}\Big)\, d\tau.$$

Let us estimate separately the following integrals under the summation sign in the right-hand side of the inequality above:

$$\int\limits_{\tau_k}^{\tau_{k+1}} \|g_2(\tau, y(\tau), z(\tau), g_1^+(\tau, y(\tau), z(\tau))(\dot y(\tau) - f_1(\tau, y(\tau), z(\tau))))$$

$$-g_2(\tau_k, \xi_k, w_z(\tau_k), g_1^+(\tau_k, \xi_k, w_z(\tau_k))(\bar u_k - f_1(\tau_k, \xi_k, w_z(\tau_k))))\|_{n_2}\, d\tau$$

$$\le C_6 \int\limits_{\tau_k}^{\tau_{k+1}} \Big(\delta(h) + h + \|z(\tau) - w_z(\tau_k)\|_{n_2} + \|g_1^+(\tau, y(\tau), z(\tau))(\dot y(\tau) -$$

$$-f_1(\tau, y(\tau), z(\tau))) - g_1^+(\tau_k, \xi_k, w_z(\tau_k))(\bar u_k - f_1(\tau_k, \xi_k, w_z(\tau_k)))\|_m\Big)\, d\tau$$

$$= C_6 \int\limits_{\tau_k}^{\tau_{k+1}} \Big(\delta(h) + h + \|z(\tau) - w_z(\tau_k)\|_{n_2}$$

$$+\|g_1^+(\tau, y(\tau), z(\tau))(\dot y(\tau) - \bar u_k) + (g_1^+(\tau, y(\tau), z(\tau)) - g_1^+(\tau_k, \xi_k, w_z(\tau_k)))\bar u_k$$

$$+g_1^+(\tau_k, \xi_k, w_z(\tau_k)) f_1(\tau_k, \xi_k, w_z(\tau_k)) - g_1^+(\tau, y(\tau), z(\tau)) f_1(\tau, y(\tau), z(\tau))\|_m\Big)\, d\tau$$

$$\le C_7 \int\limits_{\tau_k}^{\tau_{k+1}} \Big(\delta(h) + h + \|z(\tau) - w_z(\tau_k)\|_{n_2} + \|\dot y(\tau) - \bar u_k\|_{n_1}\Big)\, d\tau$$

$$\leq C_8 \int_{\tau_k}^{\tau_{k+1}} \Big(\delta(h) + h + \|z(\tau) - w_z(\tau)\|_{n_2} + \|\dot{y}(\tau) - \bar{u}(\tau)\|_{n_1}\Big)\, d\tau.$$

After the similar estimation of the same integral over the segment $[\tau_i,\, t]$, summarizing all such estimates, we obtain

$$\|z(t)-w_z(t)\|_{n_2} \leq C_9 \sum_{k=0}^{i-1} \int_{\tau_k}^{\tau_{k+1}} \Big(\delta(h)+h+\|z(\tau)-w_z(\tau)\|_{n_2}+\|\dot{y}(\tau)-\bar{u}(\tau)\|_{n_1}\Big)\, d\tau$$

$$+C_{10} \int_{\tau_i}^{t} \Big(\delta(h) + h + \|z(\tau) - w_z(\tau)\|_{n_2} + \|\dot{y}(\tau) - \bar{u}(\tau)\|_{n_1}\Big)\, d\tau$$

$$\leq C_{11} \int_{t_0}^{t} \Big(\delta(h) + h + \|z(\tau) - w_z(\tau)\|_{n_2} + \|\dot{y}(\tau) - \bar{u}(\tau)\|_{n_1}\Big)\, d\tau.$$

Using (10), we derive

$$\|z(t) - w_z(t)\|_{n_2} \leq C_{12} \int_{t_0}^{t} \|z(\tau) - w_z(\tau)\|_{n_2}\, d\tau + C_{10}(\delta(h) + h + \bar{\rho}(h)).$$

The application of the Gronwall lemma results in desired relation (11).

Now, we can declare that the second component of the model solves the problem of reconstructing the input action $u(t)$ on the base of measurements of the phase state of accuracy (11). We use again results of [17], namely, an estimate similar to (10), where the role of measurement accuracy is played by the right-hand part of (11). Thus, the final estimate of approximation quality (9) from the assertion of the theorem is proved:

$$\|u(\cdot) - v(\cdot)\|_{L_2(T;\mathbb{R}^m)} \leq C\Big(\frac{(h + \delta(h) + \bar{\rho}(h))^2}{\alpha^2(h)} + \alpha(h)\Big)^{1/2}.$$

□

Setting $\delta(h) = h$, $\bar{\alpha}(h) = h^{2/3}$, and $\alpha(h) = h^{2/9}$, we can easily see that the estimate is of order $O(h^{1/9})$. Note that its optimality is not investigated; here, it is important that the sequence of model controls converges to the real input as $h \to 0$.

4 Numerical example

As a model example, we consider the problem for the following system of ODEs:

$$\begin{aligned}
&\dot{x}_1(t) = -x_1(t) - x_2(t) - x_3(t) + (x_3^2(t) + 1)u_1(t),\\
&\dot{x}_2(t) = -x_1^2(t) + 2t^2 + (x_1^2(t) + x_3^2(t))u_2(t),\\
&\dot{x}_3(t) = (2x_1(t) - 1)x_3(t) - (1/t)x_2(t) - 2te^{-t} + 1 + (u_1^2(t) + t)u_2(t),\\
&t \in T = [0,1],\ x_1(0) = 1,\ x_2(0) = 0,\ x_3(0) = 0,\ u_1, u_2 \in [0, 1.5].
\end{aligned}$$

At discrete times, the vector $y(t) = (x_1(t), x_2(t))$ is inaccurately measured, i.e., we know $\xi(t) = (\xi_1(t), \xi_2(t))$. The coordinate $z(t) = x_3(t)$ and disturbance $u(t) = (u_1(t), u_2(t))$ are to be reconstructed.

In the terms of system (1), we have

$$f_1(t) = \begin{pmatrix} -x_1(t) - x_2(t) - x_3(t) \\ -x_1^2(t) + 2t^2 \end{pmatrix},\ g_1(t) = \begin{pmatrix} x_3^2(t) + 1 & 0 \\ 0 & x_1^2(t) + x_3^2(t) \end{pmatrix},$$

$$f_2(t) = (2x_1(t) - 1)x_3(t) - (1/t)x_2(t) - 2te^{-t} + 1, \quad g_2(t) = (u_1^2(t) + t)u_2(t),$$

$$g_1^+(t) = \begin{pmatrix} 1/(x_3^2(t) + 1) & 0 \\ 0 & 1/(x_1^2(t) + x_3^2(t)) \end{pmatrix}.$$

Note that $n_1 = 2$, $n_2 = 1$, $m = 2$, and Condition 1 is evidently fulfilled.

Let us briefly describe the analog of model (4), where the vectors $w_y(t)$ and $w_v(t)$ are two-dimensional, $w_z(t)$ is one-dimensional:

$$\dot{w}_{y1}(t) = \bar{u}_{1i},\ \dot{w}_{y2}(t) = \bar{u}_{2i},$$

$$\dot{w}_z(t) = (2\xi_{1i} - 1)w_{zi} - (1/\tau_i)\xi_{2i} - 2\tau_i e^{-\tau_i} + 1$$

$$+ \left(\left(\frac{\bar{u}_{1i} + \xi_{1i} + \xi_{2i} + w_{zi}}{w_{zi}^2 + 1} \right)^2 + \tau_i \right) \left(\frac{\bar{u}_{2i} + \xi_{1i}^2 - 2\tau_i^2}{\xi_{1i}^2 + w_{zi}^2} \right),$$

$$\dot{w}_{v1}(t) = -\xi_{1i} - \xi_{2i} - w_{zi} + (w_{zi}^2 + 1)v_{1i},$$

$$\dot{w}_{v2}(t) = -\xi_{1i}^2 + 2\tau_i^2 + (\xi_{1i}^2 + w_{zi}^2)v_{2i},$$

$$t \in (\tau_i, \tau_{i+1}],\ w_{y1}(0) = 1,\ w_{y2}(0) = 0,\ w_z(0) = 0,\ w_{v1}(0) = 1,\ w_{v2}(0) = 0.$$

At each time τ_i, the control values $\bar{u}_{1i}$, $\bar{u}_{2i}$, v_{1i}, v_{2i} are calculated explicitly from (5), (6) with the use of the values of measurements and model variables obtained till this moment.

In the computational experiment, we choose the unknown functions $u_1(t) = t$ and $u_2(t) = 1$; they generate the solution of the Cauchy problem $x_1(t) = e^{-t}$, $x_2(t) = t^3$, $x_3(t) = t$.

The results of reconstructing $u(t)$ for different sets of parameters of the algorithm are presented in Figs. 1, 2, where the real function $u(t) = (u_1(t), u_2(t))$ is shown by the dashed line, and the result of its reconstruction, the function $v(t) = (v_1(t), v_2(t))$, by the solid one. This is in agreement with the main assertion of the paper; convergence (9) takes place provided relations (7) between the parameters are fulfilled.

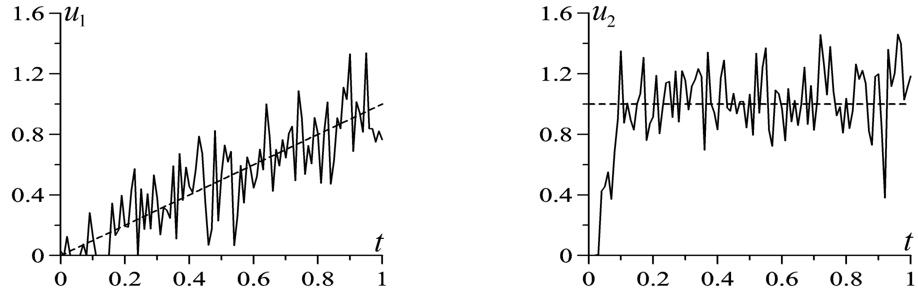

Fig. 1. Parameters: $h = 0.01$, $\delta = 0.01$, $\bar{\alpha} = 0.045$, $\alpha = 0.35$; approximation error: $\|u(\cdot) - v(\cdot)\|_{L_2(T;\mathbb{R}^2)} = 0.443$.

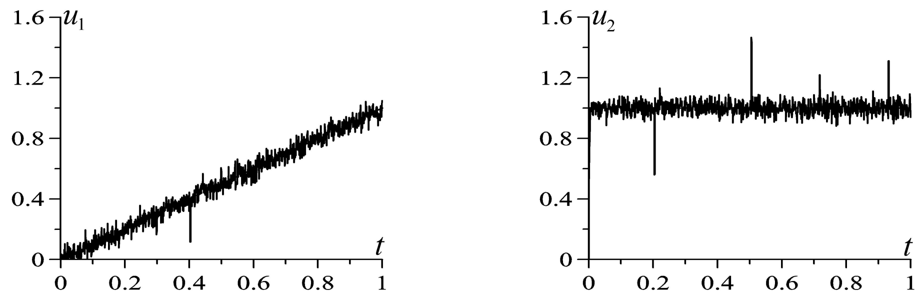

Fig. 2. Parameters: $h = 0.0001$, $\delta = 0.0001$, $\bar{\alpha} = 0.002$, $\alpha = 0.12$; approximation error: $\|u(\cdot) - v(\cdot)\|_{L_2(T;\mathbb{R}^2)} = 0.029$.

5 Conclusions

In the paper, we consider the problem of dynamic reconstruction of an unknown input action in a partially observed system of ODEs with a nonlinear with respect to input equation describing the dynamics of the unmeasured coordinate. A finite-step software-oriented solution algorithm based on the method of auxiliary closed-loop models is proposed; its error is estimated.

As a perspective direction of further development of the topic, we plan to apply the results obtained to solving the problem of reconstructing disturbances in a quasilinear stochastic differential equation by means of the method of moments under assumption that the information on a number of realizations of a part of coordinates of the stochastic process is available.

References

[1] R.W. Brockett, M.D. Mesarović, *The reproducivility of multivariable systems*, J. Math. Anal. Appl., **11**:1-3 (1965), 548–563. Zbl 0135.31301

[2] M.K. Sain, J.L. Massey, *Invertibility of linear time-invariant dynamical systems*, IEEE Trans. Automat. Contr., **14**:2 (1969), 141–149. MR0246664

[3] J.P. Norton, *An Introduction to identification*, Acad. Press, London etc., 1986. Zbl 0617.93064

[4] A.N. Tikhonov, V.Ya. Arsenin, *Solutions of ill-posed problems*, Wiley, New York etc., 1977. Zbl 0354.65028

[5] S.I. Kabanikhin, *Inverse and Ill-posed problems*, De Gruyter, Berlin, 2012. Zbl 1247.65077

[6] A.V. Kryazhimskii, Yu.S. Osipov, *Modelling of a control in a dynamic system*, Eng. Cybern., **21**:2 (1984), 38–47. Zbl 0699.49035

[7] Yu.S. Osipov, A.V. Kryazhimskii, *Inverse problems for ordinary differential equations: dynamical solutions*, Gordon and Breach, Amsterdam, 1995. Zbl 0884.34015

[8] Yu.S. Osipov, A.V. Kryazhimskii, V.I. Maksimov, *Some algorithms for the dynamic reconstruction of inputs*, Proc. Steklov Inst. Math., **275**, Suppl. 1, 2011, S86–S120. Zbl 1305.93093

[9] V.I. Maksimov, *The methods of dynamical reconstruction of an input in a system of ordinary differential equations*, J. Inverse Ill-Posed Problems, **29**:1 (2021), 125–156. Zbl 1465.34026

[10] N.N. Krasovskii, A.I. Subbotin, *Game-theoretical control problems*, Springer, New York etc., 1988. Zbl 0649.90101

[11] A.V. Kryazhimskii, Yu.S. Osipov, *On stable positional renewal of control according to measurements of a part of coordinates*, In: A.V. Kim and V.I. Maksimov eds., *Some control and stability problems*, UrO Akad. Nauk SSSR Press, Sverdlovsk, 1989, 33–47. Zbl 0788.93055

[12] V.I. Maksimov, *Problems of dynamic reconstruction of the inputs of infinite-dimensional systems*, UrO RAN Press, Ekaterinburg, 2000.

[13] V.L. Rozenberg, *On a problem of dynamical input reconstruction for a system of special type under conditions of uncertainty*, AIMS Math., **5**:5 (2020), 4108–4120. Zbl 1487.49044

[14] M.S. Blizorukova, V.I. Maksimov, *Dynamic discrepancy method in the problem of reconstructing the input of a system with time delay control*, Comput. Math. Math. Phys., **61**:3 (2021), 359–367. Zbl 1467.93156

[15] P.G. Surkov, *Approximate calculation of the Caputo-type fractional derivative from inaccurate data. Dynamical approach*, Fract. Calc. Appl. Anal., **24**:3 (2021), 895–922. Zbl 1498.26016

[16] V.L. Rozenberg, *On a problem of dynamic disturbance reconstruction in a nonlinear system of differential equations*, Tr. Inst. Mat. Mekh. UrO RAN, **27**:2 (2021), 197–207. MR4276733

[17] A.Yu. Vdovin, *On the problem of reconstruction of perturbation in a dynamical system*, Ph.D. Thesis, UrO Akad. Nauk SSSR Press, Sverdlovsk, 1989.

Valeriy Lvovich Rozenberg
Krasovskii Institute of Mathematics and Mechanics UB RAS,
S. Kovalevskoi str. 16,
620108, Yekaterinburg, Russia;
Ural State University of Railway Transport,
Kolmogorova str. 66,
620034, Yekaterinburg, Russia
Email address: rozen@imm.uran.ru