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SPHERICALLY ORDERED GROUPS

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Abstract: We introduce and study the class of spherically ordered groups. Axioms of spherical orders used for these groups are examined and their (in)dependence is shown. The notions of spherically orderable groups and their spectra of spherical orderability are defined. Values of these spectra are found for a series of certain groups.

Keywords: spherical order, group, spectrum of spherical orderability.

1 Introduction

Well known linearly ordered groups and circularly ordered groups are both deeply investigated and described [1, 2] and admit various generalizations and modifications for partial, left and right orderings [1, 3], betweenness and separation groups [4, 5, 6], semigroups [7], and T-generically ordered groups [8].

We continue to study n-sphericall orders and spread the notions of linearly ordered groups and circularly ordered groups to n-spherically ordered ones, where linearly and circularly ordered groups have the values $n = 2$ and $n = 3$, respectively. We examine axioms of spherical orders, used for these groups, and show their (in)dependence. We introduce the notion of spectrum

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of spherical orderability for an arbitrary group. Values of these spectra are found for a series of known groups.

2 Spherical orders

Let $\overline{x}$ be a n-tuple $(x_1, x_2, \ldots, x_n)$, σ be a permutation of degree n. Then the tuple $(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ is denoted by $\overline{x}_\sigma$.

We consider the following generalization of linear and circular orders based both on axioms for these orders [1, 2, 9, 10], and on the orientation of simplicial complexes [14, 15], in particular, on directed triangles and tetrahedrons [16].

Definition (cf. [11, 12, 13]). An n-ary relation K_n on a set A is called a n-*spherical* order relation, for $n \geq 2$, if it satisfies the following conditions:

(nso1) If $\overline{x} \in A^n$ and σ is a transposition on $\{1, 2, \ldots, n\}$, then $\overline{x} \in K_n$ or $\overline{x}_\sigma \in K_n$;

(nso2) If $\overline{x} \in A^n$ and σ is a transposition on $\{1, 2, \ldots, n\}$, then $\overline{x} \in K_n$ and $\overline{x}_\sigma \in K_n$ iff there are distinct indices i and j such that $x_i = x_j$;

(nso3) For any $\overline{x} \in K_n$ and any element $t \in A$, there is an index i such that $(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n) \in K_n$.

A structure $\mathcal{M}$ provided with a n-spherical order is called n-*spherically ordered*.

Remark 1. In view of the axiom (nso2) any spherical order K_n on a set A contains all n-tuples in A^n with some repeated coordinates.

Now we argue to prove the following proposition asserting that the additional forth axiom used before for the definition of n-spherical order is deduced from (nso1) and (nso2).

Proposition 1. *For any n-spherical order K_n the following condition holds:*

(nso4) *For any even permutation σ on $\{1, 2, \ldots, n\}$, if $\overline{x} \in K_n$ then $\overline{x}_\sigma \in K_n$.*

Proof. Since each even permutation is represented as a composition of even many transpositions it suffices to show that if $\overline{x} \in K_n$ and $\sigma = (i\,j)(k\,l)$ then $\overline{x}_\sigma \in K_n$. In view of Remark 1 we can assume that $\overline{x}$ consists of pairwise distinct coordinates. Now as $\overline{x} \in K_n$ then $\overline{x}_{(i\,j)} \notin K_n$ by the axiom (nso2). Applying the axiom (nso1) with the transposition $(k\,l)$ we obtain $\overline{x}_\sigma = (\overline{x}_{(i\,j)})_{(k\,l)} \in K_n$.

Remark 2. By the axioms (nso1) and (nso2) odd permutations for tuples in K_n with pairwise distinct coordinated are forbidden. So we admit cyclic permutations in the axiom (nso4) iff n is odd, In view of this circumstance the axiom (nso4) is modified from cyclic permutations to even ones, where cyclic permutations were used in the previous version of the definition of n-spherical order in [11, 12, 13].

Thus for any n-tuple $\overline{x}$ with pairwise distinct coordinates in a given set either this tuple belongs to K_n, together with all its even permutations, or any its odd permutation belongs to K_n, otherwise. These even permutations are covered by cyclic ones iff $n = 3$, since there are n cyclic permutations and $\frac{n!}{2}$ even ones, including identical one, and $n = \frac{n!}{2}$ iff $n = 3$, i.e. only circular orders are exhausted by cyclic permutations of given tuples.

Remark 3. Since each permutation is composed by transpositions we can replace transpositions in axioms (nso1) and (nso2) by odd permutations obtaining the same classes of n-spherical orders, $n \geq 2$.

Remark 4. The axioms above produce all possible linear orders K_2 and circular orders K_3. Here (2so2) gives the reflexivity: $\forall x K_2(x, x)$, and the antisymmetry:

$$\forall x_1, x_2(K_2(x_1, x_2) \wedge K_2(x_2, x_1) \rightarrow x_1 = x_2),$$

(2so2) and (2so3) give the transitivity:

$$\forall x_1, x_2, x_3(K_2(x_1, x_2) \wedge K_2(x_2, x_3) \rightarrow K_2(x_1, x_3)),$$

and the axiom (2so1) gives the linearity: $\forall x_1, x_2(K_2(x_1, x_2) \vee K_2(x_2, x_1))$. For the transitivity it suffices to take pairwise distinct elements a, b, c with $(a, b) \in K_2$ and $(b, c) \in K_2$. By (2so3) we have $(a, c) \in K_2$ or $(c, b) \in K_2$, and $(b, a) \in K_2$ or $(a, c) \in K_2$. But the cases $(c, b) \in K_2$ and $(b, a) \in K_2$ are impossible in view of (nso2), implying the required $(a, c) \in K_2$.

The axioms for circular orders [1, 10] are immediately implied by the axioms (3so1), (3so2), (3so3), and Proposition 1.

Remark 5. Like the case of linear orders any n-spherical order K_n on a set A has the dual one consisting of all n-tuples in $A^n \setminus K_n$ and all n-tuples in A^n with some repeated coordinates. We denote this dual order by $\overline{K_n}$.

The following theorem describes possibilities of independence for the axioms (nso1), (nso2) and (nso3).

Theorem 1. 1. *For any* $n \geq 3$ *the axioms* (nso1), (nso2), (nso3) *are independent: for each axiom* (nsoi), $i = 1, 2, 3$, *there is a relation* K_n *that violates this axiom and satisfies the others.*

2. *The axioms* (2so2) *and* (2so3) *are independent and imply the axiom* (2so1).

Proof. Let $n \geq 2$. If $K_n = A^n$ with $|A| \geq n$ then it satisfies (nso1) and (nso3) whereas (nso2) does not hold since in such a case K_n contains all n-tuples $\overline{x}$ and $\overline{x}_\sigma$ with pairwise distinct coordinates. Thus (nso2) is independent from the others.

Now we argue to show that the axiom (nso3) can be violated for a relation K_n satisfying (nso1) and (nso2). For this purpose we take the set $A = \{1, 2, \ldots, n, n+1\}$ and generate the relation $K_n \subset A^n$ as follows. We include to K_n all n-tuples with some repeated coordinates. Besides we include the tuple $(1,2,3,\dots,n)$ and, for $t=n+1$, the tuples $(2,t,3,\dots,n)$, $(1,3,t,\dots,n)$, …, $(t,1,2,\dots,n-1)$ with all their even permutations and arbitrarily extend the obtained relation by n-tuples with pairwise distinct coordinates such that these tuples are included together with their even permutations only. The obtained relation K_n satisfies (nso1) and (nso2), and does not satisfy (nso3) which is witnessed by the tuple $(1,2,3,\dots,n)$ and the value $t=n+1$.

Thus the axioms (nso2) and (nso3) are independent for each $n\geq 2$.

Now we consider the (in)dependence of the axiom (nso1) from the others.

1. If $n\geq 3$, $A=\{1,2,\dots,n\}$ and K_n consists both of all n-tuples in A^n with some repeated coordinates and of the tuple $(1\,2\,\dots\,n)$ then K_n satisfies the axioms (nso2) and (nso3), since it does not contain transpositions for $(1\,2\,\dots\,n)$, and each tuple in K_n either contains all elements of A or has repeated coordinates such that preserving two repeated coordinates other coordinates can be replaced by an arbitrary element t and the obtained tuple again belongs to K_n. At the same time neither $(2,1,3,4,\dots,n)$ nor $(2,3,1,4,\dots,n)$ belongs to K_n implying that (nso1) does not hold. Thus the axiom (nso1) does not depend from the others.

2. Let $n=2$ and K_2 satisfy the axioms (2so2) and (2so3). By the axiom (2so2) any element x_1 of the given set A belongs to the pair $(x_1,x_1)\in K_2$. Taking an arbitrary element $t\in A$ we have $(t,x_1)\in K_2$ or $(x_1,t)\in K_2$ by the axiom (2so3) that confirms the axiom (2so1) and its dependence from (2so2) and (2so3).

3 Spherically orderable groups and their spherical spectra

Definition. A group G is called (agreed) *n-spherically ordered*, or *n-s-ordered*, if G is provided with a n-spherical order K_n such that for any $(x_1,\dots,x_n)\in K_n$ and any $y\in G$ the tuples $(x_1y,\dots,x_ny)$ and $(yx_1,\dots,yx_n)$ belong to K_n.

Further we assume the coherency of the given n-spherical order with the group operation and omit that it is "agreed". In general case it is more narrow than simply n-spherical orders but it is unified with the notions of linearly ordered and circularly ordered groups.

A group G is called *n-spherically orderable*, or *n-s-orderable*, if G has a n-spherically ordered expansion. A group G is called *spherically orderable* if it is n-spherically orderable for some $n>1$.

For a group G we define its *spectrum* $\mathrm{Sp}_{\mathrm{so}}$ *of spherical orderability*, or *spherical spectrum*, as follows:

$$\mathrm{Sp}_{\mathrm{so}}(G)=\{n\in\omega\setminus\{0,1\}\mid G\text{ is }n\text{-spherically orderable}\}.$$

A group G is called *totally spherically orderable*, or *totally s-orderable*, if G has maximal spectrum of spherical orderability, i.e. $\mathrm{Sp}_{\mathrm{so}}(G)=\omega\setminus\{0,1\}$.

A group G is called *almost totally spherically orderable*, or *almost totally s-orderable*, if $\mathrm{Sp}_{\mathrm{so}}(G)$ is a cofinite subset of ω.

A group G is (*almost*) *not s-orderable in any way* if $\mathrm{Sp}_{\mathrm{so}}(G)$ is empty (respectively, finite).

The notions above for the spherical orderability and its spectra can be naturally spread for an arbitrary structure. Besides, the spherical orderability admits similar variations of orderability as for linear (bi-)orderability such as left-orderability and right-orderability [3, 17].

A natural **problem** arises on description of spherical spectra for groups and related structures.

By the definition a group G is linearly ordered iff G is 2-spherically ordered, and G is cyclically ordered iff G is 3-spherically ordered. Here $2 \in \mathrm{Sp}_{\mathrm{so}}(G)$ and $3 \in \mathrm{Sp}_{\mathrm{so}}(G)$, respectively.

Again by the definition a group G is spherically orderable iff $\mathrm{Sp}_{\mathrm{so}}(G) \neq \emptyset$.

Remark 6. If a group G is n-spherically orderable then each subgroup of G is n-spherically orderable, too, since any restriction $\mathcal{M}$ of a n-spherically ordered structure $\mathcal{N}$, with a n-spherical order K_n, is again n-spherically ordered, with the n-spherical order $K_n \cap M^n$.

In view of Remark 6 we have the following *Monotonicity property* for the spectrum of spherical orderability:

Proposition 2. *For any groups G_1, G_2 if $G_1 \leqslant G_2$ then $\mathrm{Sp}_{\mathrm{so}}(G_1) \supseteq \mathrm{Sp}_{\mathrm{so}}(G_2)$.*

Proposition 2 immediately implies:

Corollary 1. *If G is an* (*almost*) *totally s-orderable group then any its subgroup is also* (*almost*) *totally s-orderable.*

Corollary 2. *If G is* (*almost*) *not s-orderable in any way then any its supergroup is also* (*almost*) *not s-orderable in any way.*

Proposition 2 and Corollary 1 can be also deduced from the following criterion of n-spherical orderability:

Proposition 3. *A group G is n-spherically ordered by a n-spherical order K_n iff for any n-tuple $(a_1, \ldots, a_n) \in K_n$ with pairwise distinct coordinates and for any $b \in G$ the tuples $(a_1 b, \ldots, a_n b)$ and $(ba_1, \ldots, ba_n)$ are even permutations, i.e. are not odd permutations of some tuples in K_n.*

Proof. By the definition of n-spherical order K_n it consists of all n-tuples with some repeated coordinates and of all even permutations of given n-tuples in K_n such that for any n-tuple, in the universe, with pairwise distinct coordinate either this tuple or all its odd permutations belong to K_n. Now if G is n-spherically ordered by K_n it is forbidden to include into K_n its odd permutations of forms $(a_1 b, \ldots, a_n b)$ and $(ba_1, \ldots, ba_n)$, and conversely permitted to include its even permutations of forms $(a_1 b, \ldots, a_n b)$ and $(ba_1, \ldots, ba_n)$.

Remark 7. Proposition 3 shows that the only obstacle for a n-spherical order K_n on a group G to make this group to be n-spherical ordered is the possibility for the multiplications $(a_1b,\dots,a_nb)$ and $(ba_1,\dots,ba_n)$ to produce an odd permutation for a tuple $(a'_1,\dots,a'_n)$ in K_n with pairwise distinct coordinates, which may differ from the tuple $(a_1,\dots,a_n)$. In particular, elements of stabilizers for n-element sets $\{a_1,\dots,\, a_n\}\subseteq G$ can not produce odd permutations of tuples $(a_1,\dots,a_n)\in K_n$. Here $b\neq e$ and therefore the maps $a_i\mapsto a_ib$ and $a_i\mapsto ba_i$ do not have fixed elements.

In general, K_n is uniquely defined by its subrelation K^0_n consisting of all n-tuples in K_n with pairwise distinct coordinates, since $K_n = K^0_n\,\dot\cup\, K^1_n$, where K^1_n consists of all tuples in G^n with some repeated coordinates. The subrelation K^0_n produces an algebra $\mathcal{K}^0_n$ with $\dfrac{n!}{2}$ unary operations of even permutations forming the alternating group A_n, and unary operations r_b and l_b, for $b\in G$, carrying out the maps $a_i\mapsto a_ib$ and $a_i\mapsto ba_i$, respectively. Since all these operations are invertible, with $(r_b)^{-1}=r_{b^{-1}}$ and $(l_b)^{-1}=l_{b^{-1}}$, and form a *derivative* group $S=S(G,n)$ with the identical even permutation which is equal both to r_e and l_e, we obtain an S-act $\mathcal{K}^0_n=\langle K^0_n,s\rangle_{s\in S}$, for $|G|\geq n$. Here actions r_b and l_b can be even permutations if, for instance, $|G|=n$, which implies $S=A_n$, They may not belong to A_n, if $b\neq e$ and G is torsion-free. And they coincide if b belongs to the center of G. The group S is generated by its two subgroups A_n and $S'=\{r_b,l_b\mid b\in G\}$, and further, elements in $f\in A_n$ and in $s\in S'$ commute: $fs=sf$. Besides, $r_b=l_b$ and $\alpha_{bb'}=\alpha_b\alpha_{b'}$ for any $\alpha\in\{r,l\}$ and $b,b'\in G$, if G is abelian. Since A_n is abelian iff $n\leq 3$, the algebra $\mathcal{K}^0_n$ has the commutative derivative group S iff $n\leq 3$ and G is abelian.

Clearly, if G is a group, n-spherically ordered by an order K_n then G is n-spherically ordered by the dual order $\overline{K_n}=(G^n\setminus K_n)\cup K^1_n$, too.

In view of Proposition 3 and Remark 7 we have the following:

Corollary 3. *Let K_2 be a binary relation on a group G. Then the pair (G,K_2) is a linearly ordered group, i.e. it is 2-spherically ordered, iff for any pair $(a_1,a_2)\in K_2$ with $a_1\neq a_2$ and for any $b\in G$ the pairs (a_1b,a_2b) and (ba_1,ba_2) are not transpositions of some pairs in K_2.*

Since all odd permutations for the group S_3 are transpositions we additionally have:

Corollary 4. *Let K_3 be a ternary relation on a group G. Then the pair (G,K_3) is a circularly ordered group, i.e. it is 3-spherically ordered, iff for any triple $(a_1,a_2,a_3)\in K^0_3$ and for any $b\in G$ the triples (a_1b,a_2b,a_3b) and (ba_1,ba_2,ba_3) are not transpositions of some triples in K^0_3.*

Remark 8. Each orbit O of the group S on K^0_n connects all even permutations of tuples in O and possibly tuples based on distinct n-element sets if some r_b or l_b are not permutations of these sets.

The number r of these orbits O is called the *rank of generation* of K_n^0 with respect to S and denoted by $\mathrm{rk}_S(K_n^0)$. This rank is said to be the rank of generation for K_n, denoted by $\mathrm{rk}_S(K_n)$.

If the group G is abelian then $\mathrm{rk}_S(K_n^0)$ is finite iff G is finite, since for tuples $(x_1,\ldots,x_n)$ and $(y_1,\ldots,y_n)$ based on distinct n-element sets the relation $r_b = l_b$ is defined by $b = y_1 - x_1$. For an infinite non-abelian group one may put defining relations connecting n-tuples into finitely many orbits via chains of tuples. Indeed, the connection of tuples $(x_1,\ldots,x_n)$ and $(y_1,\ldots,y_n)$ can be organized via intermediate tuples $(z_1,\ldots,z_n)$ formed by new defining elements and with appropriate r_b and l_b, where the elements b are composed with some new defining elements. Thus there are infinite non-abelian groups with finite $\mathrm{rk}_S(K_n^0)$. Moreover, finite values $\mathrm{rk}_S(K_n^0)$ can be realized arbitrarily in $\omega\setminus\{0\}$ using defining relations connecting step-by-step orbits by r_b and l_b with new defining elements b.

In view of Remark 8 for an abelian group G the operators $r_b = l_b$ on G^3 either fix all coordinates of a triple or move all its coordinates, i.e. can not generate odd permutations (transpositions). Thus Corollary 4 immediately implies:

Corollary 5. *For any abelian group G, $3\in \mathrm{Sp}_{\mathrm{so}}(G)$.*

The following construction based on Proposition 3 shows how the spectrum $\mathrm{Sp}_{\mathrm{so}}(G)$ can be reduced to an empty one, i.e. with G which is not s-orderable in any way.

For a required group G we take a generating set $\{a_n \mid n\in\omega\}\cup\{b,c\}$. Now we consider the following defining relations: $ba_0c = a_1$, $ba_1c = a_0$, $ba_nc = a_n$ for $n\geq 2$. The group G has an infinite graph, which is generated both by the minimal set $\{a_n \mid n\in\omega\setminus\{0\}\}\cup\{b\}$, with $c = a_n^{-1}b^{-1}a_n$ and $a_0 = ba_1c = a_n^{-1}b^{-1}a_n$, $n>1$, by the minimal set $\{a_n \mid n\in\omega\setminus\{0\}\}\cup\{c\}$, with $b = a_nc^{-1}a_n^{-1}$ and $a_0 = ba_1c = a_nc^{-1}a_n^{-1}a_1c$, as well as by the minimal sets $\{a_n \mid n\in\omega\setminus\{1\}\}\cup\{b\}$ and $\{a_n \mid n\in\omega\setminus\{1\}\}\cup\{c\}$. Thus G is infinite, too, with torsion-free subgroups $\langle a_n\rangle$, $n\in\omega$, $\langle b\rangle$, $\langle c\rangle$. The defining relations show that any m-tuple $(ba_{i_1}c, ba_{i_2}c,\ldots, ba_{i_m}c)$ consisting of pairwise distinct elements and with some $i_j = 0$ and $i_k = 1$ is a transposition of $(a_{i_1}, a_{i_2},\ldots,a_{i_m})$. Therefore m-spherical orders on G coordinated with left and right group actions can not be formed for any m. Hence we have the following:

Theorem 2. *There exists a group G such that $\mathrm{Sp}_{\mathrm{so}}(G)=\emptyset$.*

Definition. Let $<$ be a strict linear order on a group G, C be the *cyclification* of $<$ consisting of all tuples (x,y,z) with $x<y<z\vee y<z<x\vee z<x<y$. A n-ary relation K_n^0 on G, for $n\geq 3$, is called *$<$-coordinated* if K_n^0 consists of all n-tuples $(a_0,\ldots,a_{n-1})$ with $a_0<\ldots<a_{n-1}$, and for any $b\in G$ the tuples $(a_0b,\ldots,a_{n-1}b)$ and $(ba_0,\ldots,ba_{n-1})$ satisfy $C\left(a_ib, a_{(i+1)\,(\mathrm{mod}\,n)}b, a_{(i+2)\,(\mathrm{mod}\,n)}b\right)$ and $C\left(ba_i, ba_{(i+1)(\mathrm{mod}\,n)}, ba_{(i+2)(\mathrm{mod}\,n)}\right)$, $i=0,\ldots,n-1$.

Remark 9. By Remark 7 any $<$-coordinated relation K_n^0 is uniquely extensible to its closure K_n under even permutations of tuples and addition of all tuples in G^n with some repeated coordinates. In view of Remark 7 this closure satisfies the axioms of n-spherical order iff even permutations of tuples $(a_1,\dots,a_n)$ in K_n do not meet their odd permutations under group actions $(a'_1b,\dots,a'_nb)$ and $(ba''_1,\dots,ba''_n)$ in K_n, where $(a'_1,\dots,a'_n),(a''_1,\dots,a''_n)\in K_n$, $b\in G$.

Remark 10. If G is a finite group with $|G|=m$ and $m<n$ then G is n-spherically orderable. Indeed, in such a case the relation K_n consisting of all tuples with some repeated coordinates satisfies the axioms of n-spherical order implying that $K_n=K_n^1$ and $\langle G,K_n\rangle$ is a n-spherically ordered group.

In view of Remark 10 we have the following:

Proposition 4. *If G is a group with $|G|=m\in\omega$ then $\mathrm{Sp}_{\mathrm{so}}(G)\supseteq\{n\in\omega\mid n>m\}$, in particular, G is almost totally s-orderable.*

The following assertion is a reformulation of well-known folklore fact that nonunit linearly ordered groups are infinite and, moreover, torsion-free. Thus we have the following:

Proposition 5. *A finite group G is 2-spherically orderable iff $|G|=1$.*

Proposition 6. *Any 2-spherically orderable group G is torsion-free, that is, if G contains an element of finite positive order then $2\notin\mathrm{Sp}_{\mathrm{so}}(G)$.*

Remark 11. Since each group $\mathbb{Z}_m$ is circularly orderable [1], i.e. $3\in\mathrm{Sp}_{\mathrm{so}}(\mathbb{Z}_m)$, then Propositions 4 and 6 imply that $\mathrm{Sp}_{\mathrm{so}}(\mathbb{Z}_2)=\mathrm{Sp}_{\mathrm{so}}(\mathbb{Z}_3)=\omega\setminus\{0,1,2\}$.

The groups $\mathbb{Z}_m$, for even m, are not m-spherically orderable since the definition of m-spherically orderable group implies that K_m is closed both under even and odd permutations of m pairwise distinct elements: odd permutations are obtained by actions of elements in $\mathbb{Z}_m$ on m-tuples of pairwise distinct elements of $\mathbb{Z}_m$. It contradicts the axiom (nso2). Thus $m\notin\mathrm{Sp}_{\mathrm{so}}(\mathbb{Z}_m)$ for each even m. In particular, by Propositions 4 and 6, $\mathrm{Sp}_{\mathrm{so}}(\mathbb{Z}_4)=\omega\setminus\{0,1,2,4\}$.

The arguments above show that any group G containing a subgroup $\mathbb{Z}_m$, for even m, can not be m-spherically ordered by a relation K_m since, in view of (nso4), it should contain a tuple $(i_1,\dots,i_m)$, where $\{i_1,\dots,i_m\}=\{0,\dots,m-1\}$. In particular, if G contains subgroups $\mathbb{Z}_m$, for each even m, then $\mathrm{Sp}_{\mathrm{so}}(G)$ is contained in the set of odd numbers.

At the same time the groups $\mathbb{Z}_m$, for odd $m\geq 5$, are m-spherically orderable since the definition of m-spherically orderable group implies even permutations of elements in K_m, generated by the tuple $(0,1,\dots,m-1)$ only. Thus $m\in\mathrm{Sp}_{\mathrm{so}}(\mathbb{Z}_m)$ for each odd $m\geq 5$.

In view of Proposition 2 and Remark 11 each element a of even order m in a given group G implies $m\notin\mathrm{Sp}_{\mathrm{so}}(G)$. Thus we have the following:

Proposition 7. *For any group G, $\mathrm{Sp}_{\mathrm{so}}(G)$ does not contain even numbers which are equal to orders of elements in G.*

In particular, we have the following:

Corollary 6. *If a group G contains elements of each even order then*

$$\mathrm{Sp}_{\mathrm{so}}(G) \cap 2\mathbb{Z} = \emptyset.$$

Proposition 8. *For any natural m, n with $2 < n < m$, and $n \nmid m$ if n is even, then the group $\mathbb{Z}_m$ is n-spherically orderable.*

Proof. We form the n-spherical order K_n on the universe $\mathbb{Z}_m$ adding to the set of n-tuples with some repeated coordinates all even permutations of tuples $(k_1(\mathrm{mod}\, m), \ldots, k_n(\mathrm{mod}\, m))$ with $k_1 < \ldots < k_n$ and even permutations of tuples $(k_1 + q(\mathrm{mod}\, m), \ldots, k_n + q(\mathrm{mod}\, m))$ with $q \in \mathbb{Z}_m$, i.e. K_n is generated by its $<$-coordinated subrelation K_n^0, where $<$ is the natural strict order on $\mathbb{Z}_m$.

Since $n \nmid m$ for even n, $\mathbb{Z}_m$ does not contain a subgroup $\mathbb{Z}_n$ violating the n-spherical orderability as in Remark 11. Thus even permutations of tuples $(k_1 + q(\mathrm{mod}\, m), \ldots, k_n + q(\mathrm{mod}\, m))$ are coordinated with even permutations of tuples $(k_1, \ldots, k_n)$, i.e. they do not produce odd permutations. Since these permutations cover all possibilities for tuples with even permutations only, Proposition 3 guarantees that $\mathbb{Z}_m$ is n-spherically orderable by K_n.

Propositions 4, 5, 8 and Remark 11 immediately imply the following description of spherical spectra for the groups $\mathbb{Z}_m$:

Theorem 3. *Let $m \in \omega \setminus \{0, 1\}$. Then*

$$\mathrm{Sp}_{\mathrm{so}}(\mathbb{Z}_m) = \omega \setminus (\{0, 1, 2\} \cup \{n \mid n|m \textit{ and } n \textit{ is even}\}).$$

Proposition 9. *The group $\mathbb{Z}$ is totally s-orderable.*

Proof. Let $n \in \omega \setminus \{0, 1\}$. We form the n-spherical order K_n on the universe $\mathbb{Z}$ adding to the set of n-tuples with some repeated coordinates all even permutations of tuples $(k_1, \ldots, k_n)$ with $k_1 < \ldots < k_n$, i.e. generate K_n by the $<$-coordinated relation K_n^0 with the natural order $<$. Clearly, for any $m \in \mathbb{Z}$, $(k_1 + m, \ldots, k_n + m)$ preserves the set of these tuples and satisfies the axioms of n-spherical orders in view of Proposition 3, as required. Thus, $\mathrm{Sp}_{\mathrm{so}}(\mathbb{Z}) = \omega \setminus \{0, 1\}$, i.e. $\mathbb{Z}$ is totally s-orderable.

It is known that any torsion-free abelian group can be lexicographically ordered with respect to its generators. Therefore the arguments for Proposition 9 imply the following:

Theorem 4. *Any torsion-free abelian group is totally s-orderable.*

Remark 12. The group confirming Theorem 2 is torsion-free and it is not s-orderable at all. It illustrates that the commutativity of a group is essential in Theorem 4.

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