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TRANSPOSED POISSON STRUCTURES ON THE EXTENDED SCHRÖDINGER-VIRASORO AND THE ORIGINAL DEFORMATIVE SCHRÖDINGER-VIRASORO ALGEBRAS

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Abstract: We compute $\frac{1}{2}$-derivations on the extended Schrödinger-Virasoro [1] algebras and the original deformative Schrödinger-Virasoro algebras. The extended Schrödinger-Virasoro algebras have neither nontrivial $\frac{1}{2}$-derivations nor nontrivial transposed Poisson algebra structures. We demonstrate that the original deformative Schrödinger-Virasoro algebras have nontrivial $\frac{1}{2}$-derivations, indicating that they possess nontrivial transposed Poisson structures.

Keywords: Lie algebra, extended Schrödinger-Virasoro algebra, original deformative Schrödinger-Virasoro algebra, transposed Poisson algebra, $\frac{1}{2}$-derivation.

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[1]We refer the notion of "Witt" algebra to the simple Witt algebra and the notion of "Virasoro" algebra to the central extension of the simple Witt algebra.

1 Introduction

Poisson algebras appear in a variety of geometric and algebraic contexts, including Poisson manifolds, algebraic geometry, noncommutative geometry, operads, quantization theory, quantum groups, etc. The study of Poisson algebras also led to other algebraic structures, such as generic Poisson algebras, algebras of Jordan brackets and generalized Poisson algebras, Gerstenhaber algebras, Novikov-Poisson algebras, Quiver Poisson algebras, etc. In the recent paper [2], the authors initiated a study of a notion of a transposed Poisson algebra by reversing the roles of the two operations in the Leibniz rule that defines a Poisson algebra. A transposed Poisson algebra defined in this manner not only retains some characteristics of a Poisson algebra, such as closure under tensor products and Koszul self-duality as an operad, but also encompasses a diverse range of identities [24, 5, 21, 22]. It is noteworthy that the authors provided several constructions of transposed Poisson algebras from Novikov-Poisson algebras, commutative associative algebras and pre-Lie algebras [2]. Moreover, transposed Poisson algebras are related to weak Leibniz algebras [4].

Let R_z (resp., L_z) denote the operator of the right (resp., left) multiplication by an element $z \in L$. We see from definition of transposed Poisson algebra that both R_z and L_z are $\frac{1}{2}$-derivations on Lie algebra. Actually, $R_z = L_z$ for all $z \in L$. This motivated to define all transposed Poisson structures on Witt and Virasoro algebras in [6]; on twisted Heisenberg-Virasoro, Schrödinger-Witt and extended Schrödinger-Witt algebras in [29]; on Schrödinger algebra in $(n+1)$-dimensional space-time in [28]; on the n-th Schrödinger algebra in [26]; on solvable Lie algebra with filiform nilradical in [1]; on oscillator Lie algebras in [17]; on Witt type Lie algebras in [15]; on generalized Witt algebras in [16], [18]; on Virasoro-type algebras in [19]; on loop Heisenberg-Virasoro algebras [27]; Block Lie algebras in [14, 16] and on Lie incidence algebras (for all references, see the survey [13]).

The characterization of transposed Poisson structures derived on the Witt algebra [6] raises the question of identifying algebras related to the Witt algebra that possess nontrivial transposed Poisson structures. Consequently, several algebras associated with the Witt algebra are examined in prior works [15, 14, 16]. This paper extends that line of research. Specifically, we detail transposed Poisson structures on central extensions of the extended Schrödinger-Witt algebras and the original deformative Schrödinger-Witt algebras.

The Schrödinger-Witt algebra $\mathfrak{so}$, originally introduced by Henkel [9] during his study on the invariance of the free Schrödinger equation, is a vector space over the complex field $\mathbb{C}$ with a basis $\{L_n, M_n, Y_{n+\frac{1}{2}} \mid n \in \mathbb{Z}\}$ satisfying the following non-vanishing relations

$$[L_m, L_n] = (n-m)L_{m+n}, \quad [Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}] = (m-n)M_{m+n+1}$$
$$[L_m, M_n] = nM_{m+n}, \qquad [L_m, Y_{n+\frac{1}{2}}] = \left(n + \tfrac{1-m}{2}\right) Y_{m+n+\frac{1}{2}}.$$

It is easy to see that $\mathfrak{so}$ is a semi-direct product of the Witt algebra and the two-step nilpotent infinite-dimensional Lie algebra. The structure and representation theory of $\mathfrak{so}$ have been extensively studied by Roger and Unterberger [23]. In order to investigate vertex representations of $\mathfrak{so}$, Unterberger [25] introduced a class of new infinite-dimensional Lie algebras $\widetilde{\mathfrak{so}}$ called the extended Schrödinger-Witt algebra, which can be viewed as an extension of $\mathfrak{so}$ by conformal current with conformal weight 1. In [8], authors studied the derivations, the central extensions and the automorphism group of the extended Schrödinger-Witt algebra. In [30], Lie bialgebra structure on the extended Schrödinger-Witt algebra was obtained. The notion of n-derivation of the extended Schrödinger-Witt algebra was investigated in [31], and the main result when $n = 2$ was applied to characterize the linear commuting maps and the commutative post-Lie algebra structures on $\widetilde{\mathfrak{so}}$.

Both original and twisted Schrödinger-Witt algebras, and also their deformations were introduced by Henkel [9], Unterberger [10] and Roger [23], in the context of non-equilibrium statistical physics, closely related to both Schrödinger Lie algebras and the Virasoro algebras, which are known to be important in many areas of mathematics and physics. Unterberger [25] constructed the explicit non-trivial vertex algebra representations of the original sector. In [11] the derivation algebra and the automorphism group of the original deformative Schrödinger-Witt algebras $L_{\lambda,\mu}$ were described. Moreover, the second cohomology group of $L_{\lambda,\mu}$ were determined in [20].

In this paper, we first point out if a Lie algebra L is Z-graded, then the space consisting of all $\frac{1}{2}$-derivations of L is naturally Z-graded. With this simple but key observation, we calculate $\frac{1}{2}$-derivations on central extensions of the extended Schrödinger-Witt algebras $\widetilde{\mathfrak{so}}$ and the original deformative Schrödinger-Witt algebras $L_{\lambda,\mu}$ by directly calculating any homogeneous subspace of $\frac{1}{2}$-derivations on them. In addition, we prove that central extension of $L_{\lambda,\mu}$ admits nontrivial transposed Poisson structures only for $\lambda = 1$; the extended Schrödinger-Virasoro algebras do not admit nontrivial $\frac{1}{2}$-derivations, also have no nontrivial transposed Poisson structures.

2 Preliminaries

In this section, we recall some definitions and known results for studying transposed Poisson structures. Although all algebras and vector spaces are considered over the complex field, many results can be proven over other fields without modifications of proofs.

Definition 1. *Let $\mathfrak{L}$ be a vector space equipped with two nonzero bilinear operations $\cdot$ and $[\cdot,\cdot]$. The triple $(\mathfrak{L},\cdot,[\cdot,\cdot])$ is called a transposed Poisson algebra if $(\mathfrak{L},\cdot)$ is a commutative associative algebra and $(\mathfrak{L},[\cdot,\cdot])$ is a Lie algebra that satisfies the following compatibility condition*

$$2z\cdot[x,y] = [z\cdot x,y] + [x,z\cdot y].$$

Definition 2. *Let* $(\mathfrak{L},[\cdot,\cdot])$ *be a Lie algebra. A transposed Poisson structure on* $(\mathfrak{L},[\cdot,\cdot])$ *is a commutative associative multiplication* $\cdot$ *in* $\mathfrak{L}$ *which makes* $(\mathfrak{L},\cdot,[\cdot,\cdot])$ *a transposed Poisson algebra.*

Definition 3. *Let* $(\mathfrak{L},[\cdot,\cdot])$ *be a Lie algebra,* $\varphi:\mathfrak{L}\to\mathfrak{L}$ *be a linear map. Then* φ *is a* $\frac{1}{2}$*-derivation if it satisfies*

$$\varphi([x,y])=\tfrac{1}{2}\big([\varphi(x),y]+[x,\varphi(y)]\big).$$

Observe that $\frac{1}{2}$-derivations are a particular case of δ-derivations introduced by Filippov in 1998 [7] and recently the notion of $\frac{1}{2}$-derivations of algebras was generalized to $\frac{1}{2}$-derivations from an algebra to a module [32]. The main example of $\frac{1}{2}$-derivations is the multiplication by an element from the ground field. Let us call such $\frac{1}{2}$-derivations as trivial $\frac{1}{2}$-derivations. It is easy to see that $[\mathfrak{L},\mathfrak{L}]$ and $\mathrm{Ann}(\mathfrak{L})$ are invariant under any $\frac{1}{2}$-derivation of $\mathfrak{L}$.

Let G be an abelian group, $\mathfrak{L}=\bigoplus\limits_{g\in G}\mathfrak{L}_g$ be a G-graded Lie algebra. We say that a $\frac{1}{2}$-derivation φ has degree g $(deg(\varphi)=g)$ if $\varphi(\mathfrak{L}_h)\subseteq\mathfrak{L}_{g+h}$. Let $\triangle(\mathfrak{L})$ denote the space of $\frac{1}{2}$-derivations and write $\triangle_g(\mathfrak{L})=\{\varphi\in\triangle(\mathfrak{L})\mid \deg(\varphi)=g\}$. The following trivial lemmas are useful in our work.

Lemma 1. *Let* $\mathfrak{L}=\bigoplus\limits_{g\in G}\mathfrak{L}_g$ *be a* G*-graded Lie algebra. Then*

$$\triangle(\mathfrak{L})=\bigoplus_{g\in G}\triangle_g(\mathfrak{L}).$$

Lemma 2. *(see [6]) Let* $(\mathfrak{L},\cdot,[\cdot,\cdot])$ *be a transposed Poisson algebra and* z *an arbitrary element from* $\mathfrak{L}$*. Then the left multiplication* L_z *in the commutative associative algebra* $(\mathfrak{L},\cdot)$ *gives a* $\frac{1}{2}$*-derivation of the Lie algebra* $(\mathfrak{L},[\cdot,\cdot])$*.*

Lemma 3. *(see [6]) Let* $\mathfrak{L}$ *be a Lie algebra without non-trivial* $\frac{1}{2}$*-derivations. Then every transposed Poisson structure defined on* $\mathfrak{L}$ *is trivial.*

3 Transposed Poisson structures on the extended Schrödinger-Virasoro algebras

Unterberger [25] introduced a class of new infinite-dimensional Lie algebras $\widetilde{\mathfrak{sv}}$ called the extended Schrödinger-Witt algebra.

Definition 4. *The extended Schrödinger-Witt Lie algebra* $\widetilde{\mathfrak{sv}}$ *is a vector space spanned by a basis* $\{L_n,M_n,N_n,Y_{n+\frac{1}{2}}\mid n\in\mathbb{Z}\}$ *with the following brackets:*

$$\begin{aligned}
&[L_m,L_n]=(n-m)L_{m+n}, && [L_m,M_n]=nM_{m+n},\\
&[L_m,N_n]=nN_{m+n}, && [N_m,M_n]=2M_{m+n},\\
&[L_m,Y_{n+\frac{1}{2}}]=(n+\tfrac{1-m}{2})Y_{m+n+\frac{1}{2}}, && [N_m,Y_{n+\frac{1}{2}}]=Y_{m+n+\frac{1}{2}},\\
&[Y_{m+\frac{1}{2}},Y_{n+\frac{1}{2}}]=(m-n)M_{m+n+1},
\end{aligned}$$

for all $m, n \in \mathbb{Z}$.

Here, half-integer indices were chosen for the basis elements Y_s. This is because Y has a conformal weight of $\frac{3}{2}$ under the action of the Virasoro field L (see, e.g., [3] or [12]). Note, in particular, that although its weight is a half-integer, Y is a bosonic field. This would contradict the spin-statistics theorem if it were not for the fact that Y is not intended to represent a relativistic field. These half-integer indices are very useful for describing derivations or $\frac{1}{2}$-derivation spaces using the grading of the algebras [29].

Theorem 1. *(see [29]) Every $\frac{1}{2}$-derivation on $\widetilde{\mathfrak{so}}$ is trivial.*

In this section we consider a central extension of the extended Schrödinger-Witt algebra $\widetilde{\mathfrak{so}}$. In [8] it is referred that $\widetilde{\mathfrak{so}}$ has only three independent classes of central extensions. Let $\hat{\mathfrak{so}} = \widetilde{\mathfrak{so}} \oplus \mathbb{C}C_L \oplus \mathbb{C}C_{LN} \oplus \mathbb{C}C_N$ be the vector space over the complex field $\mathbb{C}$ with a basis $\{L_n, M_n, N_n, Y_{n+\frac{1}{2}}, C_L, C_{LN}, C_N | n \in \mathbb{Z}\}$ satisfying the following relations

$$
\begin{aligned}
&[L_m, L_n] = (n-m)L_{m+n} + \delta_{m+n,0}\tfrac{m^3-m}{12}C_L, && [L_m, M_n] = nM_{m+n},\\
&[L_m, N_n] = nN_{m+n} + \delta_{m+n,0}(m^2-m)C_{LN}, && [N_m, M_n] = 2M_{m+n},\\
&[L_m, Y_{n+\frac{1}{2}}] = (n + \tfrac{1-m}{2})Y_{m+n+\frac{1}{2}}, && [N_m, Y_{n+\frac{1}{2}}] = Y_{m+n+\frac{1}{2}},\\
&[Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}] = (m-n)M_{m+n+1}, && [N_m, N_n] = n\delta_{m+n,0}C_N,
\end{aligned}
$$

for all $m, n \in \mathbb{Z}$. The infinite-dimensional Lie algebra $\hat{\mathfrak{so}}$ considered in this paper called *the extended Schrödinger-Virasoro algebra*. Denote

$$
H = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}N_n \oplus \mathbb{C}C_N, \quad \mathbf{Vir} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}L_n \oplus \mathbb{C}C_L, \quad \mathcal{HV} = H \oplus \mathbf{Vir} \oplus \mathbb{C}C_{LN},
$$

$$
\mathcal{S} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}M_n \bigoplus_{n\in\mathbb{Z}} \mathbb{C}Y_{n+\frac{1}{2}}, \quad \mathcal{H}_{\mathcal{S}} = H \oplus \mathcal{S}.
$$

They are subalgebras of $\hat{\mathfrak{so}}$, where H is an infinite-dimensional Heisenberg algebra, $\mathbf{Vir}$ is the classical Virasoro algebra, $\mathcal{HV}$ is the twisted Heisenberg-Virasoro algebra, $\mathcal{S}$ is a two-step nilpotent Lie algebra and $\mathcal{H}_{\mathcal{S}}$ is the semi-direct product of H and $\mathcal{S}$. Then $\hat{\mathfrak{so}}$ is the semi-direct product of the twisted Heisenberg-Virasoro algebra $\mathcal{HV}$ and $\mathcal{S}$, and $\mathcal{S}$ is an ideal of $\hat{\mathfrak{so}}$.

The description of $\frac{1}{2}$-derivations on Lie algebra $\mathcal{HV}$ is given in the following theorem.

Theorem 2. *(see [29]) Every $\frac{1}{2}$-derivation on $\mathcal{HV}$ is trivial.*

There is a $\frac{1}{2}\mathbb{Z}$-grading on $\hat{\mathfrak{so}}$ by

$$
\hat{\mathfrak{so}}_0 = \langle L_0, M_0, N_0, C_L, C_{LN}, C_N \rangle
$$

$$
\hat{\mathfrak{so}}_n = \langle L_n, M_n, N_n \rangle, \; n \neq 0, \quad \hat{\mathfrak{so}}_{n+\frac{1}{2}} = \langle Y_{n+\frac{1}{2}} \rangle,
$$

then

$$\hat{\mathfrak{so}} = \left(\bigoplus_{n\in\mathbb{Z}} \hat{\mathfrak{so}}_n\right) \bigoplus \left(\bigoplus_{n\in\mathbb{Z}} \hat{\mathfrak{so}}_{n+\frac{1}{2}}\right).$$

Hence $\Delta(\hat{\mathfrak{so}})$ has a natural $\frac{1}{2}\mathbb{Z}$-grading, i.e.,

$$\Delta(\hat{\mathfrak{so}}) = \left(\bigoplus_{n\in\mathbb{Z}} \Delta_n(\hat{\mathfrak{so}})\right) \bigoplus \left(\bigoplus_{n\in\mathbb{Z}} \Delta_{n+\frac{1}{2}}(\hat{\mathfrak{so}})\right).$$

Now we will compute $\frac{1}{2}$-derivations of the algebra $\hat{\mathfrak{so}}$.

Lemma 4. $\Delta_0(\hat{\mathfrak{so}}) = \langle \mathrm{Id}\rangle$ *and* $\Delta_{j\in\mathbb{Z}\setminus\{0\}}(\hat{\mathfrak{so}}) = 0$.

Proof. Suppose that $\varphi_j \in \Delta_j(\hat{\mathfrak{so}})$ be a homogeneous $\frac{1}{2}$-derivation. In this case, we have

$$\varphi_j(\hat{\mathfrak{so}}_n) \subseteq \hat{\mathfrak{so}}_{n+j}, \quad \varphi_j(\hat{\mathfrak{so}}_{n+\frac{1}{2}}) \subseteq \hat{\mathfrak{so}}_{n+j+\frac{1}{2}}. \tag{1}$$

By Theorem 2 and the relation (1) we can get

$$\begin{aligned}
&\varphi_j(L_n) = \delta_{j,0}\lambda L_n + a_{j,n}M_{n+j}, && \varphi_j(N_n) = \delta_{j,0}\lambda N_n + b_{j,n}M_{n+j},\\
&\varphi_j(M_n) = c^1_{j,n}L_{n+j} + c^2_{j,n}M_{n+j} + c^3_{j,n}N_{n+j} + && \delta_{n+j,0}(c^1_nC_L + c^2_nC_{LN} + c^3_nC_N),\\
&\varphi_j(C_L) = \delta_{j,0}\lambda C_L, && \varphi_j(C_{LN}) = \delta_{j,0}\lambda C_{LN},\\
&\varphi_j(Y_{n+\frac{1}{2}}) = d_{j,n}Y_{n+j+\frac{1}{2}}, && \varphi_j(C_N) = \delta_{j,0}\lambda C_N,
\end{aligned}$$

for all $n \in \mathbb{Z}$, and for some $\lambda \in \mathbb{C}$.

(1) If $j \neq 0$, then applying φ_j to both side of

$$[L_m, L_n] = (n-m)L_{m+n} + \delta_{m+n,0}\frac{m^3-m}{12}C_L,$$

we obtain

$$2(n-m)a_{j,m+n} = (n+j)a_{j,n} - (m+j)a_{j,m}. \tag{2}$$

Setting $n = 0$ in (2), we get $(j-m)a_{j,m} = ja_{j,0}$. Then taking $m = j$ in this equation, we derive $a_{j,0} = 0$. Consequently, we have $a_{j,m} = 0$ for $m \neq j$. Letting $m = j$ and $n = -j$ in (2), we obtain $a_{j,j} = 0$, this shows $a_{j,m} = 0$ for all $m \in \mathbb{Z}$. Now applying φ_j to both side of $[L_m, N_n] = nN_{m+n} + \delta_{m+n,0}(m^2 - m)C_{LN}$, we obtain

$$2nb_{j,m+n} = (n+j)b_{j,n}. \tag{3}$$

Setting $n = 0$ in (3), we get $b_{j,0} = 0$. Then taking $m = -n$ in this equation, we derive $b_{j,n} = 0$ for $n \neq -j$. Letting $m = -2j$ and $n = j$ in (3), this implies $b_{j,-j} = 0$, this shows $b_{j,n} = 0$ for all $n \in \mathbb{Z}$. Next applying φ_j to both side of $[N_m, M_n] = 2M_{m+n}$, it gives

$$c^1_{j,m+n} = 0,\ 2c^2_{j,m+n} = c^2_{j,n},\ 4c^3_{j,m+n} = -mc^1_{j,n}, \tag{4}$$

$$\begin{aligned}\delta_{m+n+j,0}(4c^1_{m+n}C_L + (4c^2_{m+n} + ((n+j)^2 - (n+j))c^1_{j,n})C_{LN} +\\ (4c^3_{m+n} - (n+j)c^3_{j,n})C_N) = 0\end{aligned} \tag{5}$$

Taking $m = 0$ in (4), we obtain $c^1_{j,n} = c^2_{j,n} = c^3_{j,n} = 0$ for all $n \in \mathbb{Z}$. Setting $m = 0$, $n = -j$ in (5) it gives $c^1_{-j} = c^2_{-j} = c^3_{-j} = 0$. Then applying φ_j to both side of the multiplication $[Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}] = (m-n)M_{m+n+1}$, we have

$$(m-n+j)d_{j,m} + (m-n-j)d_{j,n} = 0.$$

Putting $n = m-j$ in this equation, we deduce $d_{j,m} = 0$ for all $m \in \mathbb{Z}$. It proves $\varphi_j = 0$ for $j \in \mathbb{Z} \setminus \{0\}$.

(2) If $j = 0$, then applying φ_0 to both side of $[L_m, L_n] = (n-m)L_{m+n} + \delta_{m+n,0}\frac{m^3-m}{12}C_L$, we obtain (2) for $j = 0$. Taking $m = 0$ in (2), it gives $a_{0,n} = 0$ for $n \neq 0$. Setting $m = -n \neq 0$ in (2), we have $a_{0,0} = 0$. Now applying φ_0 to both side of $[L_m, N_n] = nN_{m+n} + \delta_{m+n,0}(m^2-m)C_{LN}$, we obtain (3) for $j = 0$. Setting $m = 0$ in (3), we get $b_{0,n} = 0$ for $n \neq 0$. Then taking $m = -n \neq 0$ in this equation, we derive $b_{0,0} = 0$. Next applying φ_0 to both side of $[N_m, M_n] = 2M_{m+n}$, it gives

$$c^1_{0,m+n} = 0,\ 2c^2_{0,m+n} = c^2_{0,n} + \lambda,\ 4c^3_{0,m+n} = -mc^1_{0,n}, \tag{6}$$

and the relation (5) for $j = 0$. Taking $m = 0$ in (6), this implies $c^1_{0,n} = c^3_{0,n} = 0$ and $c^2_{0,n} = \lambda$ for all $n \in \mathbb{Z}$. Setting $m = -n$ in (5) it gives $c^1_0 = c^2_0 = c^3_0 = 0$. Similarly, applying φ_0 to both side of the multiplication $[Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}] = (m-n)M_{m+n+1}$, we have

$$(m-n)(d_{0,m} + d_{0,n}) = 2(m-n)\lambda.$$

Putting $m = 0$ in this equation, we deduce $d_{0,n} = 2\lambda - d_{0,0}$ for all $n \in \mathbb{Z}$. So we can get $d_{0,n} = d_{0,0} = \lambda$ for $n \in \mathbb{Z}$. Hence $\varphi_0 = \lambda\ \mathrm{Id}$.

□

Lemma 5. $\Delta_{j+\frac{1}{2}}(\hat{\mathfrak{so}}) = 0$.

Proof. Let $\varphi_{j+\frac{1}{2}} \in \Delta_{j+\frac{1}{2}}(\hat{\mathfrak{so}})$ be a homogeneous $\frac{1}{2}$-derivation. Then we have

$$\varphi_{j+\frac{1}{2}}(\hat{\mathfrak{so}}_n) \subseteq \hat{\mathfrak{so}}_{n+j+\frac{1}{2}}, \quad \varphi_{j+\frac{1}{2}}(\hat{\mathfrak{so}}_{n+\frac{1}{2}}) \subseteq \hat{\mathfrak{so}}_{n+j+1}. \tag{7}$$

By (7) we can assume that

$$\varphi_{j+\frac{1}{2}}(L_n) = \alpha_{j,n}Y_{n+j+\frac{1}{2}}, \quad \varphi_{j+\frac{1}{2}}(N_n) = \beta_{j,n}Y_{n+j+\frac{1}{2}},$$
$$\varphi_{j+\frac{1}{2}}(M_n) = \gamma_{j,n}Y_{n+j+\frac{1}{2}},$$
$$\varphi_{j+\frac{1}{2}}(Y_{n+\frac{1}{2}}) = \sigma_{j,n}L_{n+j+1} + \mu_{j,n}N_{n+j+1} + \tau_{j,n}M_{n+j+1}.$$

where $\alpha_{j,n},\ \beta_{j,n},\ \gamma_{j,n},\ \sigma_{j,n},\ \mu_{j,n},\ \tau_{j,n} \in \mathbb{C}$.

Let us say $\varphi = \varphi_{j+\frac{1}{2}}$. The algebra $\hat{\mathfrak{so}}$ admits a $\mathbb{Z}_2$-grading. Namely,

$$\hat{\mathfrak{so}}_{\bar{0}} = \langle \hat{\mathfrak{so}}_j \rangle_{j\in\mathbb{Z}} \quad \text{and} \quad \hat{\mathfrak{so}}_{\bar{1}} = \langle \hat{\mathfrak{so}}_{j+\frac{1}{2}} \rangle_{j\in\mathbb{Z}}.$$

The mapping φ changes the grading components. It is known, that the commutator of one derivation and one $\frac{1}{2}$-derivation gives a new $\frac{1}{2}$-derivation. Hence, $[\varphi, \mathrm{ad}_{Y_{n+\frac{1}{2}}}]$ is a $\frac{1}{2}$-derivation which preserve the grading components. Namely, it is a $\frac{1}{2}$-derivation, described in Lemma 4, i.e., $[\varphi, \mathrm{ad}_{Y_{n+\frac{1}{2}}}] = \alpha_n\mathrm{Id}$.

It is easy to see, that

$$\alpha_n M_m = [\varphi, \mathrm{ad}_{Y_{n+\frac{1}{2}}}](M_m) = \varphi([M_m, Y_{n+\frac{1}{2}}]) - [\varphi(M_m), Y_{n+\frac{1}{2}}]$$

$$= -\gamma_{j,m}[Y_{m+j+\frac{1}{2}}, Y_{n+\frac{1}{2}}] = -(m+j-n)\gamma_{j,m}M_{m+n+j+1}.$$

For fixed elements m and j, we can choose an element n, such that $n \neq m+j$ and $n \neq -1-j$. The next observations give $\gamma_{j,m} = 0$ for each $(j,m) \in \mathbb{Z} \times \mathbb{Z}$. Hence, $\alpha_n = 0$ for all $n \in \mathbb{Z}$. Then we consider

$$0 = [\varphi, \mathrm{ad}_{Y_{n+\frac{1}{2}}}](N_m) = \varphi([N_m, Y_{n+\frac{1}{2}}]) - [\varphi(N_m), Y_{n+\frac{1}{2}}]$$

$$= \varphi(Y_{m+n+\frac{1}{2}}) - \beta_{j,m}[Y_{m+j+\frac{1}{2}}, Y_{n+\frac{1}{2}}] = \sigma_{j,m+n}L_{m+n+j+1} + \mu_{j,m+n}N_{m+n+j+1}$$

$$+\tau_{j,m+n}M_{m+n+j+1} - \beta_{j,m}(m-n+j)M_{m+n+j+1}.$$

From this we have $\sigma_{j,m} = \mu_{j,m} = 0$ and

$$\tau_{j,m+n} - \beta_{j,m}(m-n+j) = 0 \tag{8}$$

for all $m \in \mathbb{Z}$. Taking $m = 0$ in (8), then setting $n = 0$ we can get

$$\beta_{j,m} = \frac{j-m}{j+m}\beta_{j,0}, \quad m \neq -j.$$

By the relation (8) we obtain $\beta_{j,0} = 0$, it gives $\tau_{j,m} = \beta_{j,m} = 0$ for each $(j,m) \in \mathbb{Z} \times \mathbb{Z}$.

$$0 = [\varphi, \mathrm{ad}_{Y_{n+\frac{1}{2}}}](L_m) = \varphi([L_m, Y_{n+\frac{1}{2}}]) - [\varphi(L_m), Y_{n+\frac{1}{2}}] =$$
$$(n + \tfrac{1-m}{2})\varphi(Y_{m+n+\frac{1}{2}}) - \alpha_{j,m}[Y_{m+j+\frac{1}{2}}, Y_{n+\frac{1}{2}}] =$$
$$-\alpha_{j,m}(m-n+j)M_{m+n+j+1},$$

we can choose an element n, such that $n \neq m+j$. It shows that $\alpha_{j,m} = 0$ for each $(j,m) \in \mathbb{Z} \times \mathbb{Z}$, summarizing, $\varphi = 0$. □

Summarizing the results from Lemmas 4 and 5, we conclude that $\hat{\mathfrak{so}}$ does not have nontrivial $\frac{1}{2}$-derivations.

Theorem 3. *$\hat{\mathfrak{so}}$ has no nontrivial $\frac{1}{2}$-derivations.*

Corollary 1. *$\hat{\mathfrak{so}}$ has no nontrivial transposed Poisson algebra structures.*

4 Transposed Poisson structures on original deformative Schrödinger-Virasoro algebras

The infinite-dimensional original deformative Schrödinger-Witt algebras were considered in the paper [20] and denoted by $L_{\lambda,\mu}$ $(\lambda, \mu \in \mathbb{C})$, possess the basis $\{L_n, M_n, Y_{n+\frac{1}{2}} \mid n \in \mathbb{Z}\}$ with the following non-vanishing Lie brackets:

$$[L_m, L_n] = (n-m)L_{m+n}, \qquad [L_m, Y_{n+\frac{1}{2}}] = (n + \tfrac{1}{2} - \tfrac{\lambda+1}{2}m + \mu)Y_{m+n+\frac{1}{2}},$$
$$[L_m, M_n] = (n - \lambda m + 2\mu)M_{m+n}, \quad [Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}] = (n-m)M_{m+n+1}.$$

From [20] it is known that the original deformative Schrödinger-Witt algebras have central extensions and it can be formulated follows:

$$\widetilde{L_{\lambda,\mu}}^{1}:\begin{cases}[L_m,L_n]=(n-m)L_{m+n}+\frac{m^3-m}{12}\delta_{m+n,0}C_L,\\ [L_m,Y_{n+\frac{1}{2}}]=(n+\frac{1}{2}-\frac{\lambda+1}{2}m+\mu)Y_{m+n+\frac{1}{2}},\\ [L_m,M_n]=(n-\lambda m+2\mu)M_{m+n},\\ [Y_{m+\frac{1}{2}},Y_{n+\frac{1}{2}}]=(n-m)M_{m+n+1},\end{cases}$$

$$\widetilde{L_{\lambda,\mu}}^{2}:\begin{cases}[L_m,L_n]=(n-m)L_{m+n}+\frac{m^3-m}{12}\delta_{m+n,0}C_L,\\ [L_m,Y_{n+\frac{1}{2}}]=(n+\frac{1}{2}-\frac{\lambda+1}{2}m+\mu)Y_{m+n+\frac{1}{2}}+\delta_{m+n+\mu+\frac{1}{2},0}C_{LY},\\ [L_m,M_n]=(n-\lambda m+2\mu)M_{m+n},\\ [Y_{m+\frac{1}{2}},Y_{n+\frac{1}{2}}]=(n-m)M_{m+n+1},\end{cases}$$

$$\widetilde{L_{\lambda,\mu}}^{3}:\begin{cases}[L_m,L_n]=(n-m)L_{m+n}+\frac{m^3-m}{12}\delta_{m+n,0}C_L,\\ [L_m,Y_{n+\frac{1}{2}}]=(n+\frac{1}{2}-\frac{\lambda+1}{2}m+\mu)Y_{m+n+\frac{1}{2}}+\frac{m(m-1)}{2}\delta_{m+n+\mu+\frac{1}{2},0}C_{LY},\\ [L_m,M_n]=(n-\lambda m+2\mu)M_{m+n},\\ [Y_{m+\frac{1}{2}},Y_{n+\frac{1}{2}}]=(n-m)M_{m+n+1},\\ [M_m,Y_{n+\frac{1}{2}}]=\delta_{m+n+3\mu+\frac{1}{2},0}C_{MY},\end{cases}$$

$$\widetilde{L_{\lambda,\mu}}^{4}:\begin{cases}[L_m,L_n]=(n-m)L_{m+n}+\frac{m^3-m}{12}\delta_{m+n,0}C_L,\\ [L_m,Y_{n+\frac{1}{2}}]=(n+\frac{1}{2}-\frac{\lambda+1}{2}m+\mu)Y_{m+n+\frac{1}{2}}-m(m^2-1)\delta_{m+n+\mu+\frac{1}{2},0}C_{LY},\\ [L_m,M_n]=(n-\lambda m+2\mu)M_{m+n}-m(m^2-1)\delta_{m+n+2\mu,0}C_M,\\ [Y_{m+\frac{1}{2}},Y_{n+\frac{1}{2}}]=(n-m)M_{m+n+1}-(m+\mu)((m+\mu)^2-1)\delta_{m+n+2\mu,0}C_M,\end{cases}$$

$$\widetilde{L_{\lambda,\mu}}^{5}:\begin{cases}[L_m,L_n]=(n-m)L_{m+n}+\frac{m^3-m}{12}\delta_{m+n,0}C_L,\\ [L_m,Y_{n+\frac{1}{2}}]=(n+\frac{1}{2}-\frac{\lambda+1}{2}m+\mu)Y_{m+n+\frac{1}{2}},\\ [L_m,M_n]=(n-\lambda m+2\mu)M_{m+n},\\ [Y_{m+\frac{1}{2}},Y_{n+\frac{1}{2}}]=(n-m)M_{m+n+1}-(m+\mu+\frac{1}{2})\delta_{m+n+2\mu+1,0}C_Y,\end{cases}$$

where

$\widetilde{L_{\lambda,\mu}}^{1}$: $\mu\notin\{\frac{1}{2}\mathbb{Z}\}$ or $\mu\in\frac{1}{2}+\mathbb{Z}$ and $\lambda\neq-3,-1,1$ or $\mu\in\mathbb{Z}$ and $\lambda\neq-1$;

$\widetilde{L_{\lambda,\mu}}^{2}$: $\mu\in\frac{1}{2}+\mathbb{Z}$ and $\lambda=-3$;

$\widetilde{L_{\lambda,\mu}}^{3}$: $\mu\in\frac{1}{2}+\mathbb{Z}$ and $\lambda=-1$;

$\widetilde{L_{\lambda,\mu}}^{4}$: $\mu\in\frac{1}{2}+\mathbb{Z}$ and $\lambda=1$;

$\widetilde{L_{\lambda,\mu}}^{5}$: $\mu\in\mathbb{Z}$, and $\lambda=-1$.

In this work, the algebras denoted by $\widetilde{L_{\lambda,\mu}}^{i}$ $(i=\overline{1,5})$ will be referred to as *the original deformative Schrödinger-Virasoro algebras*. Now, we compute $\frac{1}{2}$-derivations on $\widetilde{L_{\lambda,\mu}}^{i}$ $(i=\overline{1,5})$. We begin by calculating $\frac{1}{2}$-derivations on $\widetilde{L_{\lambda,\mu}}^{1}$.

There is a $\frac{1}{2}\mathbb{Z}$-grading on $\widetilde{L_{\lambda,\mu}}^{1}$ by

$$W_0=\langle L_0,M_0,C_L\rangle,\quad W_n=\langle L_n,M_n\rangle,\ n\neq0,\quad W_{n+\frac{1}{2}}=\langle Y_{n+\frac{1}{2}}\rangle,$$

then

$$\widetilde{L_{\lambda,\mu}}^{1}=\left(\bigoplus_{n\in\mathbb{Z}}W_n\right)\bigoplus\left(\bigoplus_{n\in\mathbb{Z}}W_{n+\frac{1}{2}}\right).$$

Hence $\Delta(\widetilde{L_{\lambda,\mu}}^1)$ has a natural $\frac{1}{2}\mathbb{Z}$-grading, i.e.,

$$\Delta(\widetilde{L_{\lambda,\mu}}^1) = \left(\bigoplus_{n\in\mathbb{Z}} \Delta_n(\widetilde{L_{\lambda,\mu}}^1)\right) \bigoplus \left(\bigoplus_{n\in\mathbb{Z}} \Delta_{n+\frac{1}{2}}(\widetilde{L_{\lambda,\mu}}^1)\right).$$

Lemma 6. (1) *If* $\mu \in \frac{1}{2}+\mathbb{Z}$ *and* $\lambda \neq -3, -1, 1$, *then* $\Delta_j(\widetilde{L_{\lambda,\mu}}^1)$ *is trivial;*

(2) *if* $\mu \notin \{\frac{1}{2}\mathbb{Z}\}$ *or* $\mu \in \mathbb{Z}$ *and* $\lambda \neq -1$, *then* $\Delta_j(\widetilde{L_{\lambda,\mu}}^1) = \langle \mathrm{Id} \rangle$ *for* $\lambda \neq 1$ *and* $\Delta_j(\widetilde{L_{1,\mu}}^1) = \langle \mathrm{Id}, \varphi_j \rangle$ *where* $\varphi_j(L_n) = \alpha_j M_{n+j}$ *for all* $n \in \mathbb{Z}$.

Proof. Suppose that $\varphi_j \in \Delta_j(\widetilde{L_{\lambda,\mu}}^1))$ be a homogeneous $\frac{1}{2}$-derivation. In this case, we have

$$\varphi_j(W_n) \subseteq W_{n+j}, \quad \varphi_j(W_{n+\frac{1}{2}}) \subseteq W_{n+j+\frac{1}{2}}. \tag{9}$$

Note that $\widetilde{L_{\lambda,\mu}}^1$ contains a subalgebra $\langle L_m \mid m \in \mathbb{Z}\rangle$, which isomorphic to the well-known algebra **Vir** and every $\frac{1}{2}$-derivation on Virasoro algebra is trivial [6]. By (9), we can assume that

$$\begin{aligned} &\varphi_j(L_n) = \delta_{j,0}\lambda_1 L_n + \alpha_{j,n} M_{n+j}, && \varphi_j(Y_{n+\frac{1}{2}}) = \sigma_{j,n} Y_{n+j+\frac{1}{2}}, \\ &\varphi_j(M_n) = \beta_{j,n} L_{n+j} + \gamma_{j,n} M_{n+j} + \delta_{n+j,0} c_n C_L, && \varphi_j(C_L) = \delta_{j,0}\lambda_1 C_L, \end{aligned}$$

where $\alpha_{j,n},\ \beta_{j,n},\ \gamma_{j,n},\ \sigma_{j,n},\ c_n\ \in \mathbb{C}$.

Now, we start with applying φ_j to both side of

$$[Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}] = (n-m) M_{m+n+1},$$

and we have

$$(n-m)\beta_{j,m+n+1} = 0, \tag{10}$$

$$(n-m)\delta_{m+n+j+1,0} c_{m+n+1} = 0, \tag{11}$$

$$2(n-m)\gamma_{j,m+n+1} = (n-m-j)\sigma_{j,m} + (n+j-m)\sigma_{j,n}. \tag{12}$$

Putting $n = -1$ in (10), we get $\beta_{j,m} = 0$ for $m \neq -1$, then taking $m = 0$, $n = -2$, we have $\beta_{j,-1} = 0$, which follows $\beta_{j,m} = 0$ for all $m \in \mathbb{Z}$. From (11), we deduce $c_{-j} = 0$ for all $j \in \mathbb{Z}$. Next, applying φ_j to $[L_m, L_n] = (n-m)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0} C_L$ and $[L_m, M_n] = (n - \lambda m + 2\mu) M_{m+n}$, we have equations

$$2(n-m)\alpha_{j,m+n} = (n+j-\lambda m+2\mu)\alpha_{j,n} + (m+j-\lambda n+2\mu)\alpha_{j,m}, \tag{13}$$

$$2(n-\lambda m+2\mu)\gamma_{j,m+n} = \delta_{j,0}\lambda_1(n-\lambda m+2\mu) + (n+j-\lambda m+2\mu)\gamma_{j,n}. \tag{14}$$

Taking $m = 0$ in (13), we obtain $(j - n + 2\mu)\alpha_{j,n} = (j - \lambda n + 2\mu)\alpha_{j,0}$ for all $n \in \mathbb{Z}$. If $\lambda = 1$, then we have $\alpha_{j,n} = \alpha_{j,0}$ for all $n \in \mathbb{Z}$. If $\lambda \neq 1$, then it follows

$$\alpha_{j,n} = \frac{(j-\lambda n+2\mu)}{(j-n+2\mu)}\alpha_{j,0}, \quad 2\mu \notin \mathbb{Z}. \tag{15}$$

Then using (15) in (13), we can get

$$\frac{mn(\lambda-1)(n-m)(2\mu(\lambda+2)-(m+n-j)\lambda+2j)}{(j-m+2\mu)(j-n+2\mu)(j-m-n+2\mu)}\alpha_{j,0} = 0, \ \ 2\mu \notin \mathbb{Z}. \tag{16}$$

Due to arbitrary m in (16), it gives that $\alpha_{j,0} = 0$, and from (15) it follows $\alpha_{j,n} = 0$ for arbitrary $n \in \mathbb{Z}$. If $2\mu \in \mathbb{Z}$, then it shows $\alpha_{j,n} = 0$ for $n \neq j + 2\mu$. Putting $m \neq 0$ and $n = j + 2\mu$ in (13), it gives $\alpha_{j,j+2\mu} = 0$, so we derive $\alpha_{j,n} = 0$ for all $n \in \mathbb{Z}$.

(1) If $j \neq 0$, then letting $m = 0$ in (14), we obtain

$$(n - j + 2\mu)\gamma_{j,n} = 0. \tag{17}$$

If $2\mu \notin \mathbb{Z}$, then $\gamma_{j,n} = 0$ for all $n \in \mathbb{Z}$, if $2\mu \in \mathbb{Z}$, then $\gamma_{j,n} = 0$ for $n \neq j - 2\mu$. Setting $m \neq 0$ and $n = j - 2\mu$ in (17), it gives $\gamma_{j,j-2\mu} = 0$, which derive $\gamma_{j,n} = 0$ for all $n \in \mathbb{Z}$. Next letting $m = n - j$ in (12), we obtain $\sigma_{j,n} = 0$ for all $n \in \mathbb{Z}$.

(2) If $j = 0$, then using (12) and (14), we get $\gamma_{0,n} = \sigma_{0,n} = \lambda_1$ for all $n \in \mathbb{Z}$.

Hence,

$$\varphi_j(L_n) = \alpha_{j,0} M_{n+j}, \quad \text{for} \quad \lambda = 1,$$

$$\varphi_j = 0,\ j \neq 0, \quad \varphi_0 = \lambda_1 \,\mathrm{Id}, \quad \text{for} \quad \lambda \neq 1.$$

□

Lemma 7. (1) *If* $\mu \in \frac{1}{2} + \mathbb{Z}$ *and* $\lambda \neq -3, -1, 1$, *then* $\Delta_{j+\frac{1}{2}}(\widetilde{L_{\lambda,\mu}}^1) = 0$ *for all* $j \in \mathbb{Z}$ *;*

(2) *if* $\mu \notin \{\frac{1}{2}\mathbb{Z}\}$ *or* $\mu \in \mathbb{Z}$ *and* $\lambda \neq -1$, *then* $\Delta_{j+\frac{1}{2}}(\widetilde{L_{\lambda,\mu}}^1) = 0$ *for* $\lambda \neq 1$ *and*

$$\Delta_{j+\frac{1}{2}}(\widetilde{L_{1,\mu}}^1) = \{\varphi_{j+\frac{1}{2}} \mid \varphi_{j+\frac{1}{2}}(L_n) = \alpha_j Y_{n+j+\frac{1}{2}},\ \varphi_{j+\frac{1}{2}}(Y_{n+\frac{1}{2}}) = \alpha_j M_{n+j+1}\}$$

for all $n \in \mathbb{Z}$.

Proof. Let $\varphi_{j+\frac{1}{2}} \in \Delta_{j+\frac{1}{2}}(\widetilde{L_{\lambda,\mu}}^1)$ be a homogeneous $\frac{1}{2}$-derivation. Then we have

$$\varphi_{j+\frac{1}{2}}(W_n) \subseteq W_{n+j+\frac{1}{2}}, \quad \varphi_{j+\frac{1}{2}}(W_{n+\frac{1}{2}}) \subseteq W_{n+j+1}. \tag{18}$$

By (18), we can assume that

$$\varphi_{j+\frac{1}{2}}(L_n) = \alpha_{j,n} Y_{n+j+\frac{1}{2}}, \quad \varphi_{j+\frac{1}{2}}(M_n) = \beta_{j,n} Y_{n+j+\frac{1}{2}},$$

$$\varphi_{j+\frac{1}{2}}(Y_{n+\frac{1}{2}}) = \gamma_{j,n} L_{n+j+1} + \mu_{j,n} M_{n+j+1} + \delta_{n+j+1,0} c_n C_L.$$

where $\alpha_{j,n},\ \beta_{j,n},\ \gamma_{j,n},\ \mu_{j,n},\ c_n \in \mathbb{C}$. Since $\mathrm{Ann}(\widetilde{L_{\lambda,\mu}}^1)$ is invariant under any $\frac{1}{2}$-derivation, it gives $\varphi_{j+\frac{1}{2}}(C_L) = 0$ for all $j \in \mathbb{Z}$.

Let us say $\varphi = \varphi_{j+\frac{1}{2}}$. The algebra $\widetilde{L_{\lambda,\mu}}^1$ admits a $\mathbb{Z}_2$-grading. Namely,

$$\widetilde{L_{\lambda,\mu}}^1_{\bar{0}} = \langle W_j \rangle_{j \in \mathbb{Z}} \quad \text{and} \quad \widetilde{L_{\lambda,\mu}}^1_{\bar{1}} = \langle W_{j+\frac{1}{2}} \rangle_{j \in \mathbb{Z}}.$$

The mapping φ changes the grading components. It is known, that the commutator of one derivation and one $\frac{1}{2}$-derivation gives a new $\frac{1}{2}$-derivation. Hence, $[\varphi, \mathrm{ad}_{Y_{n+\frac{1}{2}}}]$ is a $\frac{1}{2}$-derivation which preserve the grading components.

Namely, it is a $\frac{1}{2}$-derivation, described in Lemma 6, i.e., $[\varphi, \mathrm{ad}_{Y_{n+\frac{1}{2}}}] = \alpha_n \mathrm{Id}$ for $\lambda \neq 1$. Similarly to the proof of the Lemma 5, we deduce $\varphi = 0$ for $\lambda \neq 1$.

Now, we consider the case of $\lambda = 1$. Using the Lemma 6, we can get these relations $[\varphi, \mathrm{ad}_{Y_{n+\frac{1}{2}}}](M_m) = \alpha_n M_m$ and $[\varphi, \mathrm{ad}_{Y_{n+\frac{1}{2}}}](Y_{n+\frac{1}{2}}) = \alpha_n Y_{m+\frac{1}{2}}$. It gives $\beta_{j,m} = \gamma_{j,m} = 0$ for all $m \in \mathbb{Z}$. Next applying $\varphi_{j+\frac{1}{2}}$ to both side of

$$[L_m, L_n] = (n-m)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}C_L,$$

it implies

$$(2(n+j) + 1 - 2m + 2\mu)\alpha_{j,n} - (2(m+j) + 1 - 2n + 2\mu)\alpha_{j,m} = 4(n-m)\alpha_{j,m+n}. \tag{19}$$

Taking $m = 0$ in (19), we have $\alpha_{j,n} = \alpha_{j,0}$ for $n \in \mathbb{Z}$. Then applying $\varphi_{j+\frac{1}{2}}$ to $[L_m, Y_{n+\frac{1}{2}}] = (n + \frac{1}{2} - m + \mu)Y_{m+n+\frac{1}{2}}$, we obtain

$$(2n + 1 - 2m + 2\mu)\delta_{m+n+j+1,0}c_{m+n}C_L = 0, \tag{20}$$

$$(n - m - j)\alpha_{j,0} + (n + j + 1 - m + 2\mu)\mu_{j,n} = (2n + 1 - 2m + 2\mu)\mu_{j,m+n}. \tag{21}$$

From (20), we derive $c_{-j-1} = 0$ and similarly to the previous case from (21) we imply $\mu_{j,n} = \alpha_{j,0}$ for all $n \in \mathbb{Z}$. □

Summarizing the results from Lemmas 6 and 7, we conclude that if $\lambda \neq 1$, then $\widetilde{L_{\lambda,\mu}}^1$ does not have nontrivial $\frac{1}{2}$-derivations. On the other side, Theorem 4 gives the full description of nontrivial $\frac{1}{2}$-derivations of $\widetilde{L_{1,\mu}}^1$.

Theorem 4. *Let φ be a $\frac{1}{2}$-derivation of the algebra $\widetilde{L_{1,\mu}}^1$, then there are two sets of elements from the basic field $\{\alpha_t\}_{t\in\mathbb{Z}}$ and $\{\beta_t\}_{t\in\mathbb{Z}}$ such that*

$$\varphi(L_m) = \lambda_1 L_m + \sum_{t\in\mathbb{Z}} \alpha_t M_{m+t} + \sum_{t\in\mathbb{Z}} \beta_t Y_{m+t+\frac{1}{2}}, \quad \varphi(M_m) = \lambda_1 M_m,$$
$$\varphi(Y_{m+\frac{1}{2}}) = \lambda_1 Y_{m+\frac{1}{2}} + \sum_{t\in\mathbb{Z}} \beta_t M_{m+t+1}, \quad \varphi(C_L) = \lambda_1 C_L.$$

Proof. The proof follows directly from the lemmas 6 and 7. □

In the following, we aim to classify all transposed Poisson structures on $\widetilde{L_{1,\mu}}^1$.

Theorem 5. *Let $(\widetilde{L_{1,\mu}}^1, \cdot, [\cdot,\cdot])$ be a transposed Poisson algebra structure defined on the Lie algebra $\widetilde{L_{1,\mu}}^1$. Then the commutative associative multiplication on $(\widetilde{L_{1,\mu}}^1, \cdot)$ has the following form:*

$$L_m \cdot L_n = \sum_{t\in\mathbb{Z}} \alpha_t M_{m+n+t} + \sum_{t\in\mathbb{Z}} \beta_t Y_{m+n+t+\frac{1}{2}},$$
$$L_m \cdot Y_{n+\frac{1}{2}} = \sum_{t\in\mathbb{Z}} \beta_t M_{m+n+t+1},$$

where $\alpha_t, \beta_t \in \mathbb{C}$ for all $t \in \mathbb{Z}$.

Proof. We aim to describe the multiplication $\cdot$. By Lemma 2, for every element $X \in \{L_i, M_i, Y_{i+\frac{1}{2}}, C_L \mid i \in \mathbb{Z}\}$, there is a related $\frac{1}{2}$-derivation φ_X of $\widetilde{L_{1,\mu}}^1$, such that $\varphi_X(Y) = X \cdot Y$. Then by Theorem 4, we have that

$$\varphi_X(L_m) = \lambda_{1,X}L_m + \sum_{t\in\mathbb{Z}} \alpha_{t,X}M_{m+t} + \sum_{t\in\mathbb{Z}} \beta_{t,X}Y_{m+t+\frac{1}{2}}, \quad \varphi_X(C_L) = \lambda_{1,X}C_L,$$
$$\varphi_X(Y_{m+\frac{1}{2}}) = \lambda_{1,X}Y_{m+\frac{1}{2}} + \sum_{t\in\mathbb{Z}} \beta_{t,X}M_{m+t+1}, \qquad \varphi_X(M_m) = \lambda_{1,X}M_m.$$

Now we consider $\varphi_X(Y) = X \cdot Y = Y \cdot X = \varphi_Y(X)$ for $X, Y \in \{L_i, M_i, Y_{i+\frac{1}{2}}, C_L \mid i \in \mathbb{Z}\}$. Firstly, by $\varphi_X(C_L) = \lambda_{1,X}C_L$ it follows

$$C_L \cdot X = X \cdot C_L = 0.$$

Similarly, we can get

$$M_i \cdot X = X \cdot M_i = 0$$

for $X \in \{L_i, M_i, Y_{i+\frac{1}{2}}, C_L \mid i \in \mathbb{Z}\}$.

(1) Let $X = L_m$ and $Y = L_0$. Then the equality $\varphi_{L_m}(L_0) = L_m \cdot L_0 = L_0 \cdot L_m = \varphi_{L_0}(L_m)$, gives

$$\sum_{t\in\mathbb{Z}} \alpha_{t,L_m}M_t + \sum_{t\in\mathbb{Z}} \beta_{t,L_m}Y_{t+\frac{1}{2}} = \sum_{t\in\mathbb{Z}} \alpha_{t,L_0}M_{m+t} + \sum_{t\in\mathbb{Z}} \beta_{t,L_0}Y_{m+t+\frac{1}{2}}.$$

Hence, we obtain $\alpha_{k,L_m} = \alpha_{k-m,L_0}$ and $\beta_{k,L_m} = \beta_{k-m,L_0}$.

(2) Let $X = L_0$ and $Y = Y_{n+\frac{1}{2}}$, then from $\varphi_{L_0}(Y_{n+\frac{1}{2}}) = \varphi_{Y_{n+\frac{1}{2}}}(L_0)$, we get

$$\sum_{t\in\mathbb{Z}} \beta_{t,L_0}M_{n+t+1} = \sum_{t\in\mathbb{Z}} \alpha_{t,Y_{n+\frac{1}{2}}}M_t + \sum_{t\in\mathbb{Z}} \beta_{t,Y_{n+\frac{1}{2}}}Y_{t+\frac{1}{2}}.$$

Thus, we obtain $\beta_{k,Y_{n+\frac{1}{2}}} = 0$, $\alpha_{k,Y_{n+\frac{1}{2}}} = \beta_{k-n-1,L_0}$.

Summarizing all the above parts, we have that the multiplication table of $(\widetilde{L_{1,\mu}}^1, \cdot)$ is given by following non-trivial relations.

$$L_m \cdot L_n = \sum_{t\in\mathbb{Z}} \alpha_{t,L_0}M_{m+n+t} + \sum_{t\in\mathbb{Z}} \beta_{t,L_0}Y_{m+n+t+\frac{1}{2}},$$
$$L_m \cdot Y_{n+\frac{1}{2}} = \sum_{t\in\mathbb{Z}} \beta_{t,L_0}M_{m+n+t+1},$$

It gives the complete statement of the theorem. □

In the following theorem we give a description $\frac{1}{2}$-derivations on the original deformative Schrödinger-Virasoro algebras $\widetilde{L_{-3,\mu}}^2$, $\widetilde{L_{-1,\mu}}^3$ and $\widetilde{L_{-1,\mu}}^5$.

Theorem 6. *Every $\frac{1}{2}$-derivation on $\widetilde{L_{-3,\mu}}^2$, $\widetilde{L_{-1,\mu}}^3$, $\widetilde{L_{1,\mu}}^4$ and $\widetilde{L_{-1,\mu}}^5$ is trivial.*

Proof. The proof is similar to the proof of Lemmas 6 and 7. □

By Theorem 6, the following is straightforward.

Corollary 2. *The infinite-dimensional Lie algebras* $\widetilde{L_{-3,\mu}}^{2}$, $\widetilde{L_{-1,\mu}}^{3}$, $\widetilde{L_{1,\mu}}^{4}$ *and* $\widetilde{L_{-1,\mu}}^{5}$ *have no nontrivial transposed Poisson algebra structures.*

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