S©MR

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ
Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru
ISSN 1813-3304

Vol. 21, No. 2, pp. 1400–1413 (2024) УДК 512.54
https://doi.org/10.33048/semi.2024.21.088 MSC 20E06

AUTOMORPHISMS OF SOME CYCLIC EXTENSIONS OF FREE GROUPS OF RANK THREE

E.A. SHAPORINA

Communicated by I.B. Gorshkov

Abstract: Description of the group of outer automorphisms of the Gersten group was obtained by the author together with F. Dudkin in 2021 [7]. In this paper, we study the possibility of extending the methods of that work to an infinite class of cyclic extensions of a free group of rank three

$$G_k = \langle a, b, c, t | a^t = a, b^t = ba^k, c^t = c\rangle.$$

We have found the generating elements of the group $Out(G_k)$ and obtained a description of the structure of this group.

Keywords: Free group, split cyclic extension, group of outer automorphisms.

1 Introduction

In the paper [1] of 2006, O. Bogopolski, A. Martino, and E. Ventura described the outer automorphism groups of all infinite cyclic split extensions of the free group of rank 2 as follows:

$$M_\varphi = F_2 \rtimes_\varphi \mathbb{Z}.$$

Shaporina, E.A., Automorphisms of some cyclic extensions of free groups of rank three.

The work was supported by the Russian Science Foundation, project 24-21-00214, https://rscf.ru/en/project/24-21-00214/.

Received April, 2, 2024, Published December, 18, 2024.

We will further assume that automorphisms act on the right, i.e., $\varphi : a \mapsto a\varphi$. We write a^t for $t^{-1}at$ and $\hat{t}$ for the conjugation by t.

In 1994 using the group

$$H = F_3 \rtimes \mathbb{Z} = \langle a, b, c, t \mid a^t = a, b^t = ba, c^t = ca^2 \rangle,$$

S. Gersten [2] proved that groups $Aut(F_n), n \geqslant 3$ and $Out(F_n), n \geqslant 4$ are not $CAT(0)$ groups.

The group H is a cyclic split extension of the group F_3 with the basis $\{a, b, c\}$, using automorphism $\varphi : a \mapsto a, b \mapsto ba, c \mapsto ca^2$.

In 2021 [7], a generating set of the group $Out(H)$ was found and proved that $Out(H) \cong (F_3 \times \mathbb{Z}^3) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$.

If φ is an automorphism of a free group of rank n, then we denote the matrix of the mapping induced by φ on $F_n^{ab} \cong \mathbb{Z}^n$ by $\varphi^{ab} \in GL_n(\mathbb{Z})$.

A description of the groups $Out(M_\varphi)$ is obtained in Theorem 1.1 [1], depending on the type of matrix φ^{ab}. In particular, a uniform description is obtained for all groups $Out(M_\varphi)$ with unitriangular matrix φ^{ab}. This matrix is unitriangular for the Gersten group.

In our work, we are trying to understand whether it is possible to obtain a classification (similar to [1]) of the group of outer automorphisms of cyclic extensions of a free group of rank three depending on matrix φ^{ab}. To do this, we study a series of cyclic extensions of a free group of rank three:

$$G_k = F_3 \rtimes \mathbb{Z} = \langle a, b, c, t \mid a^t = a, b^t = ba^k, c^t = c \rangle, k \neq 0.$$

The group G_k is defined by an automorphism $\varphi_k : a \mapsto a, b \mapsto ba^k, c \mapsto c$. In what follows, for convenience, we will skip the index k. Matrix φ^{ab} of such an automorphism is unitriangular.

Trying to describe $Out(G_k)$ using the methods of work [1], we found that they are applicable to describe the generators of $Out(G_k)$, but there not enough ideas of paper [1] to describe the structure of this group.

We succeeded in finding the generators (see section 3) of the group $Out(G_k)$ and proved the following theorem:

Theorem 1. $Out(G_k) \cong ((\mathbb{Z}^2 \times \mathbb{Z}_k) \times N) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$, *where* $N-$ *is a right-angled Artin group, the structure of which is described in Lemma 6.* N *does not depend on the parameter* k.

2 The Lemma about Fixed Points

The proof of Lemma 4.2 from [1] and Lemma 1 from [7] can be adapted for G_k.

We further denote the set of fixed elements of the automorphism φ by $Fix\varphi$.

Lemma 1. *Let* φ *be an automorphism of* $F_3 = \langle a, b, c \rangle$ *such that* $a\varphi = a, b\varphi = ba^k, c\varphi = c$, *let* $k \neq 0, r \neq 0$ *be an integer and let* $w \in F_3$. *Then the following is true:*

1. $Fix\varphi = Fix\varphi^r = \langle a, c, bab^{-1} \rangle$,

2. If $w\varphi^r$ is conjugate to w, then w is conjugate to an element of $Fix\ \varphi$.

Proof. Let us prove the first statement. Note that a and c lie in $Fix\varphi$.

Represent an arbitrary word $w \in F_3$ in the form

$$w = v_1(a,c)b^{\epsilon_1}v_2(a,c)b^{\epsilon_2}...b^{\epsilon_{n-1}}v_n(a,c), \epsilon_i \in \{\pm 1\}, i = 1, ..., n-1,$$

where $v_i(a,c)$ are words on letters $a^{\pm 1}, c^{\pm 1}$ (may be trivial); $v_i(a,c)\varphi = v_i(a,c)$.

Understand further when $w \in Fix\varphi$. We can see that in a free group F_3

$$w\varphi = u_1(a,c)b^{\epsilon_1}u_2(a,c)b^{\epsilon_2}...b^{\epsilon_{n-1}}u_n(a,c) = w$$

if and only if $v_i(a,c) = u_i(a,c), i = 1, 2, ..., n$.

If $\epsilon_1 = -1$, then

$$w\varphi = v_1(a,c)a^{-k}b^{-1}u_2(a,c)...u_n(a,c).$$

Since $v_1(a,c) \neq v_1(a,c)a^{-k}, w\varphi$ can not be equal to w. Therefore $\epsilon_1 = 1$.

Using the same argument easy to see that $\epsilon_{n-1} = -1$.

If $\epsilon_i = \epsilon_{i+1} = 1$, then

$$w\varphi = u_1(a,c)b^{\epsilon_1}...ba^k v_{i+1}(a,c)bu_{i+1}(a,c)...u_n(a,c).$$

Since $a^k v_{i+1}(a,c) \neq v_{i+1}(a,c), w\varphi$ can not be equal to w.

Using the same argument easy to see that:

If $\epsilon_i = -1$, then $\epsilon_{i+1} = 1$. This means ϵ_i alternate, proven that $\epsilon_i = (-1)^{i+1}$ and $n-1$ is even.

It is remind to note that

$$bv_i(a,c)b^{-1} \xrightarrow{\varphi} ba^k v_i(a,c)a^{-k}b^{-1}.$$

Therefore $v_i(a,c) = a^{\alpha_i}$ for even i. Thus $w \in \langle a, c, bab^{-1}\rangle$.

The reverse inclusion is obvious.

The first statement is proven.

Let us now prove the second statement. If w is a word on letters a and c, then the statement is obvious.

We can assume that w is cyclically reduced and contains letters b or b^{-1}. By cyclic permutation and reversing w if necessary, we can assume that w begins with b.

Represent w in the form $w = ba^m w_0$ or $w = ba^m b^{-1} w_0$, where $w_0 \in F_3$ and the last letter is not equal to b^{-1}. This representation of w is stable using φ^r, which means $w\varphi^r$ starts with b and does not end with b^{-1}. Therefore, $w\varphi^r$ is cyclically reduced.

Since w and $w\varphi^r$ are conjugate, $w\varphi^r$ is a cyclic permutation of w. Therefore, for a suitable s we obtain $w\varphi^{rs} = w$. Applying the first statement, we obtain $w \in Fix\varphi$.

The second statement is proven.

□

3 The Generators of $Out(G_k)$

It is easy to check that the following maps extend to automorphisms of the group $G_k = \langle a, b, c, t \mid a^t = a, b^t = ba^k, c^t = c\rangle$.

$$\psi : \begin{cases} a \mapsto a, \\ b \mapsto tb, \\ c \mapsto c, \\ t \mapsto t, \end{cases} \qquad \chi : \begin{cases} a \mapsto a, \\ b \mapsto b, \\ c \mapsto tc, \\ t \mapsto t, \end{cases} \qquad \beta : \begin{cases} a \mapsto a, \\ b \mapsto ba, \\ c \mapsto c, \\ t \mapsto t, \end{cases}$$

$$\kappa : \begin{cases} a \mapsto a, \\ b \mapsto ab, \\ c \mapsto ac, \\ t \mapsto t, \end{cases} \qquad \mu : \begin{cases} a \mapsto a, \\ b \mapsto b, \\ c \mapsto bab^{-1}c, \\ t \mapsto t, \end{cases} \qquad \theta_2 : \begin{cases} a \mapsto a, \\ b \mapsto c^{-1}b, \\ c \mapsto c, \\ t \mapsto t, \end{cases}$$

$$\omega : \begin{cases} a \mapsto a^{-1}, \\ b \mapsto b, \\ c \mapsto c, \\ t \mapsto t^{-1}, \end{cases} \qquad \theta_1 : \begin{cases} a \mapsto a, \\ b \mapsto b, \\ c \mapsto c^{-1}, \\ t \mapsto t, \end{cases} \qquad \theta_3 : \begin{cases} a \mapsto a^{-1}, \\ b \mapsto b^{-1}, \\ c \mapsto b^{-1}cb, \\ t \mapsto ta^{-k}. \end{cases}$$

Let $\alpha \in Aut(G)$. Denote the coset of the subgroup $Inn(G)$ in the group $Aut(G)$ with the representative α by $[\alpha]$. Then $Out(G) = \{[\alpha] : \alpha \in Aut(G)\}$.

We will assume that the conjugation acts as follows on an arbitrary element $\hat{x} : p \mapsto x^{-1}px$.

Further we will prove that

$$Out(G_k) = \langle [\psi], [\chi], [\beta], [\kappa], [\mu], [\theta_2], [\omega], [\theta_1], [\theta_3]\rangle.$$

Lemma 2. *Let ξ be an arbitrary automorphism of the group $G_k = \langle a, b, c, t | a^t = a, b^t = ba^k, c^t = c\rangle$. Then there are $\tilde{l}, \tilde{k} \in \mathbb{Z}, \varepsilon \in \{0, 1\}, x \in F_3$ such that:*

$$\xi \circ \hat{x} \circ \chi^{\tilde{l}} \circ \psi^{\tilde{k}} \circ \omega^{\varepsilon} : \begin{cases} a \mapsto v^s, \\ b \mapsto v_2, \\ c \mapsto v_3, \\ t \mapsto tv^d, \end{cases}$$

where $\hat{x}$ is a conjugation by $x \in F_3$, $v \in Fix\varphi, v_2, v_3 \in F_3, s, d \in \mathbb{Z}$.

Proof. Let $\xi \in Aut(G_k)$ be an arbitrary automorphism. Consider the action of ξ on the generators of the group G_k (we collect the powers of t from the left in view of the relations of the group)

$$\xi : \begin{cases} a \mapsto t^p w_1', \\ b \mapsto t^l w_2, \\ c \mapsto t^r w_3, \\ t \mapsto t^q w_4, \end{cases}$$

where $p, l, r, q \in \mathbb{Z}, w_1', w_2, w_3, w_4 \in F_3$.

Since ξ is an automorphism, it respects relations. Applying ξ to some of them, we get the following:

(1) $\xi(b^t) = \xi(ba^k) \Rightarrow w_4^{-1}t^{-q}t^l w_2 t^q w_4 = t^l w_2 (t^p w_1')^k, k \neq 0 \Rightarrow p = 0$, since the sum of the degrees t on the left and right must be the same.

(2) $\xi(a^t) = \xi(a) \Rightarrow w_1'\varphi^q$ is conjugate to w_1' in F_3. By Lemma 1, we find that w_1' is conjugate to an element from $Fix\varphi$.

From (2) it follows that $w_1' = xw_1x^{-1}, x \in F_3, w_1 \in Fix\varphi$. Let us further put $w_1 = v^s$ such that the root of v in F_3 can not be extracted. Let us now apply the composition $\xi \circ \hat{x}$ to the generators.

$$\xi \circ \hat{x} : \begin{cases} a \mapsto v^s, \\ b \mapsto t^l v_2, \\ c \mapsto t^r v_3, \\ t \mapsto t^q v_4, \end{cases}$$

where $v \in Fix\varphi, v_2, v_3, v_4, x \in F_3$.

Since a and t commute, their images with respect to the composition of automorphisms $\xi \circ \hat{x}$ must also commute. Therefore, v^s commutes with $t^q v_4$, but since $v \in Fix\varphi$, $v = v\varphi = t^{-1}vt$. Therefore, v^s commutes with v_4. Since two words commute in a free group only if they are powers of same element and v is no longer rooted, then $v_4 = v^d$, for some $d \in \mathbb{Z}$.

As a result, we obtain a system of images of generators $(v^s, t^l v_2, t^r v_3, t^q v^d)$ of the group G_k.

Consider the relation $c^t = c$ and apply composition $\xi \circ \hat{x}$:

$$v^{-d} t^{-q} t^r v_3 t^q v^d = t^r v_3.$$

Since a and t commute, then v^d commutes with t, and therefore $v_3\varphi^q = v^d v_3 v^{-d}$, that is, $v_3\varphi^q$ is conjugate to v_3 in F_3. Therefore, by Lemma 1, v_3 is conjugate to an element from $Fix\varphi$, that is, $v_3 = yuy^{-1}$, where $y \in F_3, u = u\varphi$.

Note that v_3 does not necessarily lie in $Fix\varphi$ (for example: for $v_3 = b^{-1}cb$, $x = b, u = c, v^d = ba^{kq}b^{-1}$).

Moreover, note that the sum of the powers of b in v_3 is equal to zero. This follows from the equality $v_3 = yuy^{-1}$, where $y \in F_3, u \in Fix\varphi$.

We count the sums of powers a, b, c in the images of generators a, b, c in F_3/F_3': $v \sim a^\alpha c^\gamma, v_2 \sim a^{\alpha_1} b^{\beta_1} c^{\gamma_1}, v_3 \sim a^{\alpha_2} c^{\gamma_2}$.

Note that $\gamma = 0$. Let it not be the case. Using χ we obtain a contradiction with condition (1).

Since the system $(v^s, t^l v_2, t^r v_3, t^q v^d)$ generates G_k, then there are words w_a, w_b, w_c, such that (we collect the powers of t from the left in view of ratios):

$$a = w_a(v^s, t^l v_2, t^r v_3, t^q v^d) = t^{l_1} w_a'(a, b, c),$$
$$b = w_b(v^s, t^l v_2, t^r v_3, t^q v^d) = t^{l_2} w_b'(a, b, c),$$
$$c = w_c(v^s, t^l v_2, t^r v_3, t^q v^d) = t^{l_3} w_c'(a, b, c).$$

Compare the sums of powers a, b, c on the right and left sides of the equality. Consider a vector of the form (the sum of powers a, the sum of powers b, the sum of powers c).

Then the vectors $(1,0,0), (0,1,0), (0,0,1)$ are linear combinations of vectors $(\alpha, 0, 0), (\alpha_1, \beta_1, \gamma_1), (\alpha_2, 0, \gamma_2)$.

Hence,

$$\langle(\alpha,0,0),(\alpha_1,\beta_1,\gamma_1),(\alpha_2,0,\gamma_2)\rangle_{\mathbb{Z}}\cong\mathbb{Z}^3.$$

Therefore, the matrix composed of these vectors must be invertible over $\mathbb{Z}$. It means that the determinant of such a matrix is equal to ±1, which means that:

$$\begin{cases}\alpha=\pm1,\\ \beta_1=\pm1,\\ \gamma_2=\pm1.\end{cases}$$

Note that in view of the obtained relations, the automorphisms χ and ψ do not affect the degree of t in the images a,t. Trace the sum of powers t in the images of generators b,c when taking the composition of automorphisms $\xi\circ\hat{x}\circ\chi^{\tilde{l}}\circ\psi^{\tilde{k}}$ and we find $\tilde{l},\tilde{k}$ such that:

The sum of powers t in the image b under the action of the composition of automorphisms $\xi\circ\hat{x}\circ\chi^{\tilde{l}}\circ\psi^{\tilde{k}}$ is equal $l+\gamma_1\tilde{l}+\beta_1\tilde{k}=0$,

the sum of t powers in the image c is equal to $r+(\pm1)\tilde{l}=0$.

Since $\beta_1=\pm1$, we obtain a system with respect to $\tilde{l},\tilde{k}$ of the following form:

$$\begin{cases}l+\gamma_1\tilde{l}\pm\tilde{k}=0,\\ r+(\pm1)\tilde{l}=0.\end{cases}$$

The determinant of the matrix of this system is equal to ±1, therefore, such a system is solvable and integer $\tilde{k},\tilde{l}$ can be found.

The resulting composition of automorphisms acts as follows:

$$\xi\circ\hat{x}\circ\chi^{\tilde{l}}\circ\psi^{\tilde{k}}:\begin{cases}a\mapsto v^s,\\ b\mapsto v_2,\\ c\mapsto v_3,\\ t\mapsto t^qv^d,\end{cases}$$

where $v\in Fix\varphi, s,d\in\mathbb{Z}, v_2,v_3\in F_3$.

For the presented composition to be an automorphism, it is necessary that $q=\pm\,1$.

Appling if necessary $\omega(a\mapsto a^{-1},t\mapsto t^{-1})$, we obtain the required.

□

Let us call a set of subwords of the form a^m, ba^mb^{-1}, c^m **unchangeable blocks** (lie in $Fix\varphi$) and a set of subwords of the form b^{-1}, b **changeable blocks** (don't lie in $Fix\varphi$).

Note that any word is divided into these subwords.

Lemma 3. *Under the conditions of Lemma 2, two cases are possible:*

1. $v^s=a, d=0, v_2=uba^m, v_3\in Fix\varphi$, *where* $u\in Fix\varphi$,
2. $v^s=a^{-1}, v^d=a^{-k}, v_2=a^mb^{-1}u, v_3=a^{m_1}b^{-1}vba^{m_2}$,

where $u,v\in Fix\varphi, m,m_1,m_2\in\mathbb{Z}$.

Proof. Apply the composition of automorphisms from Lemma 2 ($\xi\circ\hat{x}\circ\chi^{\tilde{l}}\circ\psi^{\tilde{k}}$) to the relations:

(1) $t^{-1}at=a\Rightarrow v^{-d}t^{-1}v^stv^d=v^s$,

(2) $t^{-1}ct = c \Rightarrow v^{-d}t^{-1}v_3tv^d = v_3$,
(3) $t^{-1}bt = ba^k \Rightarrow t^{-1}v_2t = v^dv_2v^{ks-d}$.

First, note that the sum of powers b in the word v_2 is equal to ± 1. This implies that the total number of changeable subwords in v_2 is odd.

For example, let v_2 contain three changeable subwords, that is

$$v_2 = w_1h_1w_2h_2w_3h_3w_4,$$

where $w_i \in Fix\varphi, h_i$ are changeable subwords. The word v_2 is reduced. Under the action of the automorphism φ we obtain a letter-by-letter equality.
$v_2\varphi = w_1(h_1)\varphi w_2(h_2)\varphi w_3(h_3)\varphi w_4 \equiv v^dw_1h_1w_2h_2w_3h_3w_4v^{ks-d} = v^dv_2v^{ks-d}(*)$

Note that v^d is a power of a. Look at three possible options.

Let there be no contractions between v^d and v_2. We count the number of occurrences of $b^{\pm 1}$ in both sides of the equality (taking into account that the automorphism φ does not change the number of occurrences of $b^{\pm}$). Without loss of generality, we assume that $d > 0$, otherwise we consider $(v^{-1})^{|d|}$. We count the number of occurrences of the letters b ($|.|_b$) in both parts of $(*)$.

$$|v_2|_b = |v_2\varphi|_b = |v^dv_2v^{ks-d}|_b = |v_2|_b + ks|v|_b$$

Since $k \neq 0, s \neq 0$, then $|v|_b = 0$.

Similar reasoning for the number of occurrences of the letters c leads to the fact that v is a power of a.

Let v^d contract from v_2, but v^d does not cancel completely. Then $v^d = xy, v^dv_2 = xyy^{-1}r = xr$ is a given word. Note that in this case v^d is not cyclically reduced. It is a contradiction.

Suppose the reductions is complete. Then, if v contains occurrences of the letters $b^{\pm 1}, c^{\pm 1}$, then the number of these letters in v_2 will decrease. Similarly to the previous cases, we count the number of occurrences of the letters $b^{\pm}, c^{\pm}$ in both sides of the equality $(*)$. We get a contradiction. It follows that v^d is a power of a, that is, $v = a^z, z \in \mathbb{Z}$.

Without loss of generality, it suffices for us to consider three variants of the v_2 structure.

Choose $h_1 = b, h_2 = b, h_3 = b^{-1}$ for the first example.

Let us use relation (3) ($v_2\varphi = t^{-1}v_2t = v^dv_2v^{ks-d} = a^{zd}v_2a^{z(ks-d)}$):

$$w_1ba^kw_2ba^kw_3a^{-k}b^{-1}w_4 = a^{dz}w_1bw_2bw_3b^{-1}w_4a^{z(ks-d)}.$$

Note that for cancellations to occur in this case, it is necessary that $w_3 = a^q, q \in \mathbb{Z}$. In this case, two of the three variable subwords merged into the block $bab^{-1} \in Fix\varphi$, the word v_2 has the form w_1bw_2, where $p \in \mathbb{Z}, w_1, w_2 \in Fix\varphi$.

For the second example, let us take $h_1 = b^{-1}, h_2 = b, h_3 = b$.

Again, use relation (3):

$$w_1a^{-k}b^{-1}w_2ba^kw_3ba^kw_4 = a^{dz}w_1b^{-1}w_2bw_3bw_4a^{z(ks-d)}.$$

Using similar reasoning to the previous example, we obtain a contradiction, since contractions in the center will not occur.

Take $h_1 = b, h_2 = b^{-1}, h_3 = b^{-1}$. We find that v_2 has the form $w_1 b^{-1} w_2, w_1, w_2 \in Fix\varphi, p \in \mathbb{Z}$

Note that for other cases, as well as for a larger odd number of subwords, similar reasoning leads to one of the presented options.

Examine, as an exception, two possible cases with one block.

Let $v_2 = w_1 b w_2 \Rightarrow w_1 b a^k w_2 = a^{zd} w_1 b w_2 a^{z(ks-d)}$.

We compare the subwords before and after the changeable subword in the second equality, we see that $d = 0, w_2 = a^n, zs = 1$, therefore $v^s = a^{zs} = a$ where $n \in \mathbb{Z}$.

Let $v_2 = w_1 b^{-1} w_2 \Rightarrow w_1 a^{-k} b^{-1} w_2 = a^{dz} w_1 b^{-1} w_2 a^{z(ks-d)}$. We see that $w_1 = a^n, zs = -1$, hence, $v^s = a^{zs} = a^{-1}, zd = -k$, therefore, $v^d = a^{zd} = a^{-k}$.

We obtain two final options for the image of generators under the action of the composition of automorphisms:

Case 1: $\begin{cases} a \mapsto a \\ b \mapsto w_1 b a^n \\ c \mapsto v_3 \\ t \mapsto t \end{cases}$, where $w_1 \in Fix\varphi, p, n \in \mathbb{Z}, v_3 \in F_3$,

Case 2: $\begin{cases} a \mapsto a^{-1} \\ b \mapsto a^n b^{-1} w_2 \\ c \mapsto v_3 \\ t \mapsto t a^{-k} \end{cases}$, where $w_2 \in Fix\varphi, p, n \in \mathbb{Z}, v_3 \in F_3$.

In this case, the sum of powers b in v_3 is equal to 0. In each of the two cases, let us pay attention to the relation $((c^t)\xi \circ \hat{x} \circ \chi^{\tilde{l}} \circ \psi^{\tilde{k}} = (c)\xi \circ \hat{x} \circ \chi^{\tilde{l}} \circ \psi^{\tilde{k}} \Rightarrow a^{-zd} t^{-1} v_3 t a^{dz} = v_3)$.

Case 1: $\begin{cases} a \mapsto a, \\ b \mapsto w_1 b a^n, \\ c \mapsto v_3, \\ t \mapsto t. \end{cases}$

Applying the relation to this case, we get $v_3 \in Fix\varphi$.

Case 2: $\begin{cases} a \mapsto a^{-1}, \\ b \mapsto a^n b^{-1} w_2, \\ c \mapsto v_3, \\ t \mapsto t a^{-k}, \end{cases}$

where $w_2 \in Fix\varphi, n \in \mathbb{Z}, v_3 \in F_3$, the sum of powers b in v_3 is equal to 0.

Since the sum of powers b in v_3 is equal to 0, then in addition to the subwords from $Fix\varphi$, v_3 can contain variable subwords, but always in pairs: if there is a certain number of subwords of the form b, then there is sure to be the same number of subwords of the form b^{-1}. We note that (using reasoning similar to that for v_2) given the relation, only one configuration is possible for v_3: $v_3 = a^{m_1} b^{-1} y b a^{m_2}, m_1, m_2\ in \mathbb{Z}, y \in Fix\varphi$, otherwise the necessary contractions will not occur in the center. □

Lemma 4. *Case 2 from Lemma 3 reduces to case 1 using automorphism* $\theta_3(a \mapsto a^{-1}, b \mapsto b^{-1}, c \mapsto b^{-1}cb, t \mapsto ta^{-k})$.

Proof. Consider the composition of the automorphism θ_3 and case number 2 from Lemma 3.

$$\begin{cases} a \xrightarrow{\theta_3} a^{-1} \xrightarrow{2} a, \\ b \xrightarrow{\theta_3} b^{-1} \xrightarrow{2} (w_2)^{-1}ba^{-n} = \tilde{w_2}ba^{-n}, \\ c \xrightarrow{\theta_3} b^{-1}cb \xrightarrow{2} (w_2)^{-1}ba^{-n}a^{m_1}b^{-1}yba^{m_2}a^nb^{-1}w_2 = \\ \qquad\qquad = \tilde{w_2}ba^{m_1-n}b^{-1}yba^{m_2+n}b^{-1}w_2, \\ t \xrightarrow{\theta_3} ta^{-k} \xrightarrow{2} t, \end{cases}$$

where $w_2, \tilde{w_2}, y \in Fix\varphi, m_1, m_2, n \in \mathbb{Z}$.

□

We will use an auxiliary statement ([8] p. 20, 2.8):

Proposition. Let $U = \{u_1, ..., u_m\}$ be the set of elements of the free group F with the basis $a_1, ..., a_m$. If the following conditions hold:

$(N1)$ $v_1 \neq 1$,

$(N2)$ $v_1v_2 \neq 1 \Rightarrow |v_1v_2| \geqslant |v_1|, |v_2|$,

$(N3)$ $v_1v_2 \neq 1, v_2v_3 \neq 1 \Rightarrow |v_1v_2v_3| > |v_1| - |v_2| + |v_3|$,

for all triplets $v_1, v_2, v_3 \in U^{\pm 1}$ and $\langle U \rangle = F$, then $U^{\pm 1} = \{a_1^{\pm 1}, ..., a_m^{\pm 1}\}$.

For convenience, we introduce the following automorphism:

$$\delta = [\kappa, \theta_2] = \kappa \circ \theta_2 \circ \kappa^{-1} \circ \theta_2^{-1} : \begin{cases} a \mapsto a, \\ b \mapsto ab, \\ c \mapsto c, \\ t \mapsto t. \end{cases}$$

Lemma 5. *The composition of automorphisms* $\xi \circ \hat{x} \circ \chi^{\tilde{l}} \circ \psi^{\tilde{k}}$, *acting on the generators as follows:*

$$\xi \circ \hat{x} \circ \chi^{\tilde{l}} \circ \psi^{\tilde{k}} : \begin{cases} a \mapsto a, \\ b \mapsto w_1ba^n, \\ c \mapsto v_3, \\ t \mapsto t, \end{cases}$$

where $w_1, v_3 \in Fix\varphi, n \in \mathbb{Z}$, *lies in the subgroup generated by automorphisms defined in Section 3.*

Proof. Apply the automorphism β the required number of times, and since $w_1\beta = w_1, v_3\beta = v_3$ we obtain the following:

$$\eta = \xi \circ \hat{x} \circ \chi^{\tilde{l}} \circ \psi^{\tilde{k}} \circ \beta^{-n} : \begin{cases} a \mapsto a, \\ b \mapsto w_1b, \\ c \mapsto v_3, \\ t \mapsto t, \end{cases}$$

where $w_1, v_3 \in Fix\varphi$, the sum of the powers of c in v_3 is equal to ± 1 (see Lemma 3).

Note that if $w_1 = 1$, then with the help of Proposition it is proved that in the word v_3 there is one occurrence of the letter $c^{\pm 1}$ and therefore η is expressed through $(\kappa^{-1})^{(\theta_1 \circ \theta_2)^{-1}}, \kappa^{(\theta_2^{-1})}, \mu, \theta_1$.

If $v_3 = c$, then similarly there are no subwords of the form bab^{-1} in the word w_1 and η is expressed through $\beta, \theta_2, \theta_1, \delta$.

Let $w_1 \neq 1, v_3 \neq c, |w_1| \geqslant |v_3|$. (The case $|v_3| \geqslant |w_1|$ is treated similarly). For convenience, we redesignate w_1 by u, v_3 by v, that is, in our case, the initial condition has the form $|u| \geqslant |v|$. Denote the length of the word u relative to generators $Fix\varphi$ by $|u|_\varphi$, which is a free subgroup of the free group $F_3 = \langle a, b, c\rangle$.

Let the statement of the lemma be false in this case. We choose a counterexample such that $|u|_\varphi + |v|_\varphi$ is minimal.

Note that if u or v starts with a, then we can apply one of the automorphisms δ or η to the required degree.

Let us further assume that these words do not begin with a. Let $u = ww_1, v = ww_2$, where w is the largest common prefix, $w_2^{-1}w_1$ is given.

We check the properties $(N1) - (N3)$ for the system (a, ww_1b, ww_2). If the properties are satisfied, then $(a, ww_1b, ww_2) = (a^{\pm 1}, b^{\pm 1}, c^{\pm 1})$ is a contradiction. Failure to meet these properties implies the following conditions:

$$\begin{cases} |w_1| + 1 < |w|, \\ |w_2| < |w|. \end{cases}$$

Applying elementary Nielsen transformations, we pass to the system $(a, w_2^{-1}w_1b, ww_2)$, this is a system of reduced words, and checking the properties $(N1) - (N3)$ for it, we obtain the following system:

$$\begin{cases} |w_1| + 1 > |w_2|, \\ |w| > |w_2|, \end{cases}$$

which is fulfilled due to the conditions already obtained. Therefore, $(a, w_2^{-1}w_1b, ww_2) = (a^{\pm 1}, b^{\pm 1}, c^{\pm 1})$ is a contradiction. It means that $w_2 = 1$, and initially the system had the form:

$$(a, ww_1b, w).$$

Using Nielsen transformations, we pass to the system (a, w_1b, w), for which the properties $(N1)-(N3)$ are satisfied if w_1 does not begin with a. Therefore, w_1 starts at $a^{\pm 1}$ and the automorphism has the form:

$$\eta : \begin{cases} a \mapsto a, \\ b \mapsto wa^{\pm m}hb, \\ c \mapsto w, \\ t \mapsto t. \end{cases}$$

Consider the composition $\beta^{\pm m} \circ \theta_2 \circ \eta$ on generators:

$$\beta^{\pm m}\circ\theta_2\circ\eta_2:\begin{cases} a\xrightarrow{\beta^{\pm m}} a\xrightarrow{\theta_2} a\xrightarrow{\eta} a,\\ b\xrightarrow{\beta^{\pm m}} a^{\pm m}b\xrightarrow{\theta_2} a^{\pm m}c^{-1}b\xrightarrow{\eta} a^{\pm m}w^{-1}wa^{\mp m}hb=hb,\\ c\xrightarrow{\beta^{\pm m}} c\xrightarrow{\theta_2} c\xrightarrow{\eta} w,\\ t\xrightarrow{\beta^{\pm m}} t\xrightarrow{\theta_2} t\xrightarrow{\eta} t.\end{cases}$$

Note that $|u|_\varphi+|v|_\varphi>|h|_\varphi+|w|_\varphi$ is a contradiction with the stated counterexample. It means that the presented automorphism reduces to the identity automorphism. □

4 The Structure of $Out(G_k)$

Previously we proved that

$$Out(G_k)=\langle[\psi],[\chi],[\beta],[\kappa],[\mu],[\theta_2],[\omega],[\theta_1],[\theta_3]\rangle.$$

Next we will prove that this group decomposes into a semidirect product of subgroups

$$N=\langle[\psi],[\chi],[\beta],[\kappa],[\mu],[\theta_2]\rangle,$$

$$S=\langle[\omega],[\theta_1],[\theta_3]\rangle$$

and we will study their structure.

5 The Structure of S

Note that the subgroup $S\cong(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2)$ due to the following relations between automorphisms:
$\theta_3^2=\omega^2=\theta_1^2=id$,
$\theta_3\circ\omega=\omega\circ\theta_3$,
$\theta_1\circ\omega=\omega\circ\theta_1$,
$\theta_3\circ\theta_1=\theta_1\circ\theta_3$.

6 The Structure of N

Study the structure of the subgroup

$$N=\langle[\psi],[\chi],[\beta],[\theta_2],[\mu],[\kappa]\rangle.$$

Let us prove that

$$N\cong\mathbb{Z}^2\times\mathbb{Z}_k\times N_1,$$

where $N_1=\langle[\mu],[\kappa],[\theta_2]\rangle$.

Lemma 6. *$N\cong(\mathbb{Z}^2\times\mathbb{Z}_k)\times N_1$, where $N_1=\langle[\mu],[\kappa],[\theta_2]\rangle$.*

Proof. The subgroup $\langle[\psi],[\chi],[\beta]\rangle$ is normal in the subgroup N, $\langle[\psi],[\chi],[\beta]\rangle\cong\mathbb{Z}^2\times\mathbb{Z}_k$. Moreover,

$$N\cong(\mathbb{Z}^2\times\mathbb{Z}_k)\times\langle[\mu],[\kappa],[\theta_2]\rangle.$$

Note that the subgroup $\langle[\beta],[\chi],[\psi]\rangle$ is contained in the center of the subgroup N. In addition, the classes $[\psi]$ and $[\chi]$ have an infinite order in the group $Out(G)$ and $\beta^k=\hat{t}$.

It remains to show that $\langle[\psi],[\chi],[\beta]\rangle\cap\langle[\mu],[\kappa],[\theta_2]\rangle = id$. Indeed, note that intersection is impossible due to the action of automorphisms on the generators b,c.

□

Lemma 7. $Out(G)\cong N\rtimes S$

Proof. It is enough to establish the following relations in the group $Aut(G)$:

$$\begin{array}{lllllllll}
\theta_1^{-1}\psi\theta_1 & = & \psi, & \omega^{-1}\psi\omega & = & \psi, & \theta_3^{-1}\psi\theta_3 & = & \psi^{-1},\\
\theta_1^{-1}\chi\theta_1 & = & \chi^{-1}, & \omega^{-1}\chi\omega & = & \chi, & \theta_3^{-1}\chi\theta_3 & = & \chi,\\
\theta_1^{-1}\beta\theta_1 & = & \beta, & \omega^{-1}\beta\omega & = & \beta, & \theta_3^{-1}\beta\theta_3 & = & \beta\circ\hat{a},\\
\theta_1^{-1}\theta_2\theta_1 & = & \theta_2, & \omega^{-1}\theta_2\omega & = & \theta_2, & \theta_3^{-1}\theta_2\theta_3 & = & \theta_2^{-1},\\
\theta_1^{-1}\mu\theta_1 & = & \theta_2\mu^{-1}\theta_2^{-1}\beta, & \omega^{-1}\mu\omega & = & \mu^{-1}, & \theta_3^{-1}\mu\theta_3 & = & \theta_2\circ\kappa^{-1}\circ\theta_2^{-1},\\
\theta_1^{-1}\kappa\theta_1 & = & \hat{a}^{-1}\beta\theta_2\kappa^{-1}\theta_2^{-1}, & \omega^{-1}\kappa\omega & = & \kappa^{-1}, & \theta_3^{-1}\kappa\theta_3 & = & \theta_2\circ\mu^{-1}\circ\theta_2^{-1}.
\end{array}$$

Check one of them; the rest are checked in the same way.

$$\theta_1^{-1}\kappa\theta_1:\begin{cases}
a\xrightarrow{\theta_1^{-1}} a \xrightarrow{\kappa} a \xrightarrow{\theta_1} a\\
b\xrightarrow{\theta_1^{-1}} b \xrightarrow{\kappa} ab \xrightarrow{\theta_1} ab\\
c\xrightarrow{\theta_1^{-1}} c^{-1} \xrightarrow{\kappa} c^{-1}a^{-1} \xrightarrow{\theta_1} ca^{-1}\\
t\xrightarrow{\theta_1^{-1}} t \xrightarrow{\kappa} t \xrightarrow{\theta_1} t
\end{cases}$$

$$\hat{a}^{-1}\beta\theta_2\kappa^{-1}\theta_2^{-1}:\begin{cases}
a\xrightarrow{\hat{a}^{-1}} a \xrightarrow{\beta} a \xrightarrow{\theta_2} a \xrightarrow{\kappa^{-1}} a \xrightarrow{\theta_2^{-1}} a\\
b\xrightarrow{\hat{a}^{-1}} aba^{-1} \xrightarrow{\beta} ab \xrightarrow{\theta_2} ac^{-1}b \xrightarrow{\kappa^{-1}} ac^{-1}b \xrightarrow{\theta_2^{-1}} ab\\
c\xrightarrow{\hat{a}^{-1}} aca^{-1} \xrightarrow{\beta} aca^{-1} \xrightarrow{\theta_2} aca^{-1} \xrightarrow{\kappa^{-1}} ca^{-1} \xrightarrow{\theta_2^{-1}} ca^{-1}\\
t\xrightarrow{\hat{a}^{-1}} t \xrightarrow{\beta} t \xrightarrow{\theta_2} t \xrightarrow{\kappa^{-1}} t \xrightarrow{\theta_2^{-1}} t
\end{cases}$$

Note that $N\cap S = id$. This is true in view of the action of automorphisms on generators and in view of the finite order of automorphisms whose classes generate the subgroup S. □

Lemma 8. $N_1\cong K\rtimes\mathbb{Z}$, *where K is a right-angle Artin group.*

Proof. Denote $K=\langle\langle\mu,\kappa\rangle\rangle_{N_1}$. Then $N_1\cong K\rtimes\mathbb{Z}$.

To prove this, replace the system of generators of the subgroup K as follows: $\kappa_i=\theta_2^i\circ\kappa\circ\theta_2^{-i}$, $\mu_i=\theta_2^i\circ\mu\circ\theta_2^{-i}$, where $i\in\mathbb{Z}$.

Let us show that K is a right-angle Artin group, which has the following representation:

$$K\cong\langle\ \mu_i,\kappa_i, i\in\mathbb{Z}\mid[\kappa_i,\mu_i],[\mu_i,\mu_{i+1}],[\kappa_i,\kappa_{i+1}], i\in\mathbb{Z}\ \rangle.$$

Note that the system of generators $\mu_i,\kappa_i, i\in\mathbb{Z}$ is sufficient for the subgroup K, and the indicated commutation relations are obvious. Then consider

$i \geq 1$. (Otherwise, the generator with a negative index can be corrected using the conjugation θ_2) Show that, in addition to the indicated commutation relations, there are no other relations in the representation K.

Denote: $\xi_k = \kappa_{k-1}^{-1} \circ \kappa_k$, then $\kappa_i = \kappa_0 \circ \xi_1 \circ ... \circ \xi_k$; $\delta_k = \mu_k \circ \mu_{k-1}^{-1}$, then $\mu_i = \delta_i \circ ... \circ \delta_1 \circ \mu_0$.

Such compositions of new generators of the subgroup K give the following actions on the generators of the group G_k (generators a, t are fixed).

$$\xi_k : \begin{cases} b \mapsto (a^{-1})^{c^{k-1}} b \\ c \mapsto c \end{cases}$$

$$\delta_k : \begin{cases} b \mapsto (a^{-1})^{(c^{k-1}b)^{-1}} b \\ c \mapsto c^{(a^{-1})^{(c^{k-1}b)^{-1}}} \end{cases}$$

The subgroup $Aut(F_n(x_1...x_n))$ generated by partial conjugations $\alpha_{ij} = (x_i, x_j)$ was studied in [6], where:

$$(x_i, x_j) = \begin{cases} x_i \mapsto x_j^{-1} x_i x_j \\ x_k \mapsto x_k, k \neq i \end{cases}$$

The representation of this subgroup has the following relations:
(1) $\alpha_{ij}\alpha_{kj} = \alpha_{kj}\alpha_{ij}$,
(2) $\alpha_{ij}\alpha_{kl} = \alpha_{kl}\alpha_{ij}$,
(3) $\alpha_{ij}\alpha_{kj}\alpha_{ik} = \alpha_{ik}\alpha_{ij}\alpha_{kj}$.
We consider the set $(\xi_k, \delta_k, k \geq 1)$.

Note that the elements of this set generate partial conjugations on the subgroup $Fix\varphi$.($\delta_k = \alpha_{23}^{k-1} \circ \alpha_{32} \circ \alpha_{23}^{-(k-1)}$, $\xi_k = \alpha_{23}^{k-1} \circ \alpha_{21} \circ \alpha_{23}^{-(k-1)}$)

It follows that the set $(\xi_k, \delta_k, k \geq 1)$ is generated by partial conjugations $\alpha_{23}, \alpha_{21}, \alpha_{32}$.

It is clear that the quotient

$$\langle \alpha_{21}, \alpha_{13}, \alpha_{23}, \alpha_{21}, \alpha_{31}, \alpha_{32} \rangle / \langle\langle \alpha_{23}\alpha_{13}, \alpha_{21}\alpha_{31}, \alpha_{32}\alpha_{12} \rangle\rangle \cong \langle \alpha_{23}, \alpha_{21}, \alpha_{32} \rangle$$

is free.

It means that there are no relations between the elements of the set $(\xi_k, \delta_k, k \geq 1)$.

Consider the word $\omega(\mu_0, \kappa_0, \xi_k, \delta_k, k \geqslant 1)$. Let the subgroup K contain a relation in addition to the indicated relations of commutation, i.e. $\omega = id$.

Using commutation, we can get rid of occurrences of μ_0.

We get $\omega_1(\xi_k, \delta_k, \kappa_0) = id$, which can be rewritten:

$$\omega_2(\xi_k, \delta_k) = \kappa_0^{s_1} \circ v_1(\delta_k) \circ \kappa_0^{s_2} \circ v_2(\xi_k)...v_n(\xi_k)(**)$$

The left side of the relation $(**)$ acts on the generators of the group G_k as partial conjugations. Consider the action of the right side of the relation $(**)$ on the generator c (without loss of generality it is sufficient to consider the action on only one generator) $c \mapsto a^{s_1} p_1^{-1} a^{s_2} p_2^{-1} ... a^{s_n} p_n^{-1} \mathbf{c} p_n ... p_1$, where

$p_i, i = 1...n$ do not change with respect to each other under the influence of mappings. Therefore, $p_i = a^{t_i}, i = 1...n$. This is a contradiction. □

Thus, the following theorem follows from Lemmas 6-8.

Theorem 2.

$$Out(G_k) \cong ((\mathbb{Z}^2 \times \mathbb{Z}_k) \times N_1) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2),$$

where $N_1 = \langle \mu, \kappa, \theta_2 \rangle$.

References

[1] O. Bogopolski, A. Martino, E. Ventura, *The automorphism group of a free-by-cyclic group in rank 2*, Commun. Algebra, **35**:5 (2007), 1675–1690. Zbl 1122.20018

[2] S.M. Gersten, *The automorphism group of a free group is not a CAT(0) group*, Proc. Am. Math. Soc., **121**:4 (1994), 999–1002. Zbl 0807.20034

[3] N. Brady, I. Soroko, *Dehn functions of subgroups of right-angled Artin groups*, Geom. Dedicata, **200** (2019), 197–239. Zbl 1515.20211

[4] J.O. Button, *Tubular groups, 1-relator groups and nonpositive curvature*, Int. J. Algebra Comput., **29**:8 (2019), 1367–1381. Zbl 1460.20008

[5] N. Hoda, D.T. Wise, D.J. Woodhouse, *Residually finite tubular groups*, Proc. R. Soc. Edinb., Sect. A, Math., **150**:6 (2020), 2937–2951. Zbl 1477.20056

[6] J. McCool, *On basis-conjugating automorphisms of free groups*, Can. J. Math., **38**:6 (1986), 1525–1529. Zbl 0613.20024

[7] F.A. Dudkin, E.A. Shaporina, *Automorphisms of the Gersten Group*, Sib. Math. J., **62**:3 (2021), 413–422. Zbl 1542.20144

[8] R.C. Lyndon, P.E. Schupp, *Combinatorial group theory*, Springer, Berlin etc., 1977. Zbl 0368.20023

Elizaveta Alekseevna Shaporina
Novosibirsk State University,
Pirogova 2,
630090, Novosibirsk, Russia
Email address: grambergea@gmail.com