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ON DEFINABLE SETS IN SOME DEFINABLY COMPLETE LOCALLY O-MINIMAL STRUCTURES

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Abstract: In this paper, we show that the Grothendieck ring of a definably complete locally o-minimal expansion of the set (not the field) of real numbers $\mathbb{R}$ is trivial. Afterwards, we will give a sufficient condition for which a definably complete locally o-minimal expansion of an ordered group has no nontrivial definable subgroups. In the last section, we study some sets that are definable in a definably complete locally o-minimal expansion of an ordered field. Finally, a decomposition theorem for a definable set into finite union of π_L-quasi-special $\mathcal{C}^r$ submanifolds is demonstrated.

Keywords: Definably complete, locally o-minimal structures, Grothendieck rings.

1 Introduction

Firstly, a locally o-minimal structure $\mathcal{M} := (M, <, ...)$ has been introduced and studied in [14] as a local counterpart of an o-minimal one. In this paper, we will focus on the elementary property, definable completeness (i.e., every non-empty definable bounded subset X of M has both supremum and infimum in M), since intervals have this property, every o-minimal structure is definably complete.

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The notion of the Grothendieck ring for a first-order structure was founded in ([11], [3]) independently.
The Grothendieck ring of a model-theoretical structure is built up as a quotient of definable sets by definable bijections (see below).

In [2] and [15] the following explicit calculations of Grothendieck rings (denoted K_0) of structures are made: $K_0(\mathbb{R}, <, L_{ring})$ is isomorphic to $\mathbb{Z}$, but $K_0(\mathbb{Q}_p, L_{ring})$ is trivial, where p is a prime number,$\mathbb{Q}_p$ is the p-adic numbers field and L_{ring} is the language $(+, -, \cdot, 0, 1)$.

In this paper, we prove the triviality of the Grothendieck ring for a definably complete locally o-minimal expansion of the set of real numbers which is not an o-minimal structure.

We know thanks to [9] that if $\mathcal{G} = (G, <, +, 0, ...)$ is an o-minimal expansion of an ordered abelian group G, then $K_0(\mathcal{G})$ is isomorphic to either the ring of the integers $\mathbb{Z}$ or the quotient ring $\mathbb{Z}[T]/(T^2 + T)$ as a ring. In the same section, we show that if the Grothendieck ring of a definably complete locally o-minimal expansion of an ordered group is not the zero ring, then this structure has no nontrivial definable subgroups. But the converse of this result does not hold true by applying Proposition 2 below.

In the fourth section, we first prove that an unary definable set in a locally o-minimal expansion of an ordered field which does not contain an open interval is bounded.

We review the theory DCTC as an extension of the theory of a dense linear order without endpoints $\mathcal{M} := (M, <, ...)$ by the two axiom schemes given by definable completeness and type completeness (Definition 5). In other words, in a model of DCTC, any definable subset has an infimum and its characteristic function has a left limit at each point. So, a definably complete expansion of an ordered field is locally o-minimal if and only if it is a model of DCTC.

We know by ([7], Theorem 4.4) that in a definably complete locally o-minimal structure $(M, <, ...)$ enjoying the property that the image of a nonempty definable discrete set under a coordinate projection is again discrete, M^n is a finite union of π-quasi-special submanifolds partitioning a definable set X of M^n, where n is a positive integer. By replacing the coordinate projection π by the linear one π_L and by using some techniques as in the proof given in ([7], Section 4), and for a definably complete locally o-minimal expansion of an ordered field, we get a decomposition into π_L-quasi-special $\mathcal{C}^r$ submanifolds.

2 Preliminaries

"Definable" will always mean "definable with parameters".

We recall that a densely linearly ordered set without endpoints $\mathcal{M} = (M, <, ...)$ is o-minimal, if every definable subset X of M is a finite union of points and open intervals.

Definition 1. *A densely linearly ordered structure without endpoints* $\mathcal{M} = (M, <, ...)$ *is locally o-minimal if, for every definable subset* X *of* M *and for every point* $a \in M$ *there exists an open interval* I *containing the point* a *such that* $X \cap I$ *is a finite union of points and open intervals.*

Example 1. *The structure* $(\mathbb{R}, \leqslant, +, \sin)$ *is locally o-minimal by [14, Theorem 2.7].*

Definition 2. *An expansion of a densely linearly ordered set without endpoints* $\mathcal{M} = (M, <, ...)$ *is definably complete if any definable subset* X *of* M *has the supremum and infimum in* $M \cup \{\pm\infty\}$.

Example 2. *Every expansion of* $(\mathbb{R}, <)$ *is definably complete.*

It is well known thanks to [12, Corollary 1.5] that the definable completeness is equivalent to M being definably connected, and also with the validity of the intermediate value theorem for one variable definable continuous functions.

Definition 3. *Consider an expansion of a densely linearly order without endpoints* $\mathcal{M} = (M, <, ...)$. *Let* X *be a nonempty definable subset of* M^n. *The dimension of* X *is the maximal nonnegative integer* d *such that* $\pi(X)$ *has a nonempty interior for some coordinate projection* $\pi : M^n \to M^d$. *We set* $\dim(X) = -\infty$ *when* X *is an empty set.*

Definition 4. *Let* $\mathcal{M} = (M, <, ...)$ *be a structure and* n *is a positive integer. The notation Defn* $(\mathcal{M})$ *denotes the family of all definable subsets of* M^n. *Let* $X \in$ *Defn* $(\mathcal{M})$, $[X]$ *denotes the equivalence class for the equivalence relation* $\mathcal{R}$ *defined on the set Defn* $(\mathcal{M})$ *as follows:* $X\mathcal{R}Y$ *if,* X *and* Y *are definably isomorphic, and* $[X \cup Y] = [X] + [Y]$ *where* $X, Y \in$ *Defn* $(\mathcal{M})$, *and* $X \cap Y = \phi$. *The Grothendieck group of a structure* $\mathcal{M}$ *is the abelian group* $K_0(\mathcal{M})$ *generated by the symbols* $[X]$. *The ring structure is defined by* $[X][Y] = [X \times Y]$, *where* $X \times Y$ *is the Cartesian product of definable sets. The ring* $K_0(\mathcal{M})$ *with this multiplication is called the Grothendieck ring of the structure* $\mathcal{M}$.

Remark 1. *By [11], the Grothendieck ring of a structure* $\mathcal{M}$, $K_0(\mathcal{M})$ *is nontrivial if and only if there is no definable set* $X \subseteq M$, $a \in X$ *and an injective definable map from* X *onto* $X \setminus \{a\}$.

Definition 5. *A densely linearly ordered structure without endpoints* $\mathcal{M} = (M, <, ...)$ *is type complete if it is locally o-minimal, and in addition, for any definable subset* $X \subseteq M$ *there are* c_1 *and* c_2 *such that if* $I =]-\infty, c_1[$ *or* $I =]c_2, +\infty[$, *then either* $I \subseteq X$ *or* $I \cap X = \phi$.

In this paper, DCTC is the abbreviation for the properties of being definably complete and type complete.

3 The Grothendieck ring of a definably complete locally o-minimal expansion of an ordered group

Proposition 1. *Let $\mathcal{G}$ be an o-minimal expansion of an ordered abelian group $(G, +)$ which contains the ring of the integers $\mathbb{Z}$ such that $(\mathcal{G}, \mathbb{Z})$ is a locally o-minimal structure, then $K_0(\mathcal{G})$ is isomorphic to the ring $\mathbb{Z}[X]/(X^2 + X)$.*

Proof. Suppose that there exist a bounded definable set I of G and an unbounded definable set J of G and a definable bijection $\Phi : I \to J$. Applying the o-minimal monotonicity theorem [15, 3.1.2] and shrinking I and J if necessary, we suppose that Φ is continuous and strictly increasing or strictly decreasing. After possibly reflecting and translating, we suppose that $]0, +\infty[\subseteq J$ and that ϕ is strictly increasing. Then $\Phi^{-1}(\mathbb{N})$ is an infinite bounded discrete subset of G, this contradicts the local o-minimality of the structure $(\mathcal{G}, \mathbb{Z})$. By the proof of theorem 1 in [9](Case 2), we deduce that the Grothendieck ring of $\mathcal{G}$ is isomorphic to $\mathbb{Z}[X]/(X^2 + X)$ as a ring. □

Proposition 2. *Consider a definably complete locally o-minimal expansion of the set of real numbers $\mathbb{R}$ which is not an o-minimal structure. Then the Grothendieck ring of this structure is the zero ring $\{0\}$. Here, the zero ring means the ring whose unique element is the zero.*

Proof. Let $\mathcal{R}$ be such a structure.

Claim: There exists a discrete closed infinite definable set D'.

In fact, as the structure $\mathcal{R}$ is locally o-minimal and not o-minimal, by [6, Lemma 3.5], there exists an unbounded discrete definable set D. Without loss of generality, we may assume that $D \cap [0, \infty[$ is an infinite set, so $D' := D \cap [0, \infty[$ is an infinite discrete definable set. By [7, Lemma 2.4], the definable set D' is closed. Which proves the claim.

As the structure $\mathcal{R}$ is definably complete, the set D' admits an infimum in $\mathbb{R}$ which we denote by m. According to [5, Definition 3], if d is not the maximum of D', we say the minimum of $D' > d$ is the successor of d in D' written $s_{D'}(d)$.

If the maximum M of D' belongs to D', the set $D'_1 := D' \backslash \{M\}$ is also definable, discrete and closed by [7, Lemma 2.4].

The function $s_{D'}$ determines a definable bijection from D'_1 onto $D'' := D'_1 \backslash \{m\}$. So $[D'_1] = [D'']$.

As $[D'_1] = [D''] + [m]$, we deduce that $[m] = 0$.

For any definable set U, it is obviously definably isomorphic to the Cartesian product $U \times \{m\}$. Hence, $[U] = [U \times \{m\}] = [U] \cdot [\{m\}] = 0$.

Since the Grothendieck ring is generated by the elements of the form $[U]$, this ring is the zero ring.

□

Example 3. *Let $\mathcal{R} := (\mathbb{R}, \leqslant, +, \sin)$ be the expansion of the additive ordered group of reals by the sine function, It is well known thanks to [14, Theorem 2.7] that $\mathcal{R}$ is locally o-minimal, clearly it is not o-minimal. So the grothendieck ring of $\mathcal{R}$ is null.*

We know by [15, Chapter 1, Proposition 4.2] that if an expansion of an ordered group G is o-minimal, then G is abelian, divisible and has no proper nontrivial convex definable subgroups. So thanks to [12, Proposition 2.2], all these properties still hold in a definably complete locally o-minimal expansion of an ordered group.
In case of o-minimality, there are no nontrivial definable subgroups; alternatively, there are nontrivial definable subgroups in case of definably complete locally o-minimal structure. In fact, the structure $(\mathbb{R}, +, <, \mathbb{Z})$ has the nontrivial definable subgroup $(\mathbb{Z}, +)$.

Therefore, the aim of the following theorem is to give a sufficient condition for which a definably complete locally o-minimal expansion of an ordered group has no nontrivial definable subgroups, and by using Proposition 2 above, we show that this condition is not necessary.

Lemma 1. *Let $\mathcal{G}$ be a definably complete expansion of an ordered group G. A nontrivial definable subgroup of G is unbounded.*

Proof. Assume for contradiction that we can take a nontrivial bounded definable subgroup H. Let $s = \sup(H)$. Since H is nontrivial, $s > 0$. We also have $s < \infty$ because H is bounded.
By the definition of supremum, there exists $g \in H$ such that $s/2 < g$.
Since H is a group, $2g \in H$. It is a contradiction because $2g > s$.

□

Theorem 1. *If the Grothendieck ring of a definably complete locally o-minimal expansion of an ordered group G is not the zero ring, then this structure has no nontrivial definable subgroups.*

Proof. Suppose that the structure $\mathcal{G} = (G, <, +, 0, ...)$ has a nontrivial definable subgroup H. By [12, Lemma 2.1], the set $G \setminus H$ is dense, so H has an empty interior, by [7, Lemma 2.3] it is closed and discrete. As H is nontrivial, by lemma 1, the group H is unbounded. Set $H' = \{x \in G \,|\, x \geqslant 0\}$, H' is clearly closed, discrete, definable and infinite, as in the proof of Proposition 2, we construct a definable bijection between H' and $H' \setminus \{0\}$ to deduce that the Grothendieck ring of the structure $\mathcal{G}$ is trivial. □

Example 4. *Every o-minimal expansion of an ordered abelian group G is a definably complete locally o-minimal structure (because it is o-minimal), as its Grothendieck ring is nontrivial by [9, Theorem 1], consequently G has no nontrivial definable subgroups.*

Remark 2. *In the proof of Theorem 1, we don't need to use [7, Lemma 2.3], because H must not be dense and co-dense, otherwise this would contradict the local o-minimality. Then we apply [12, Lemma 2.1] to deduce that H is closed and discrete.*

Remark 3. *The converse of Theorem 1 is not true. In fact, set $E = \{e^n \mid n \in \mathbb{N}\}$, where e is the base of the natural logarithm. Then the structure $(\mathbb{R}, +, <, E)$ is locally o-minimal by [10, Proposition 26]. Any nontrivial definable subgroup G is of the form $a\mathbb{Z}$, for some positive $a \in \mathbb{R}$. The set $a\mathbb{Z} + E$ is definable. By the local o-minimality and the compactness of the closed interval $[0, 1]$, there exist only finitely many points in $(a\mathbb{Z} + E) \cap [0, 1]$. It is a contradiction.*

4 Some properties of a definably complete locally o-minimal expansion of an ordered field

4.1. DCTC expansion of an ordered field.

Proposition 3. *Consider a locally o-minimal expansion of an ordered field F. Any definable subset X of F which does not contain an open interval is bounded.*

Proof. We demonstrate that X is bounded above.

Set $X_+ = \{x \in X \mid x > 0\}$.

The notation F_+ denotes the definable set $\{x \in F \mid x > 0\}$.

Consider the definable homeomorphism $f; F_+ \to F_+$ defined by $f(x) = 1/x$.

The image $f(X_+)$ is a definable set.

Since the structure is locally o-minimal, there exists an open interval I containing the origin such that $I \cap f(X_+)$ is the union of finitely many points and open intervals.

The intersection $I \cap f(X_+)$ does not contain an open interval because X does not contain an open interval by the assumption.

Therefore, $I \cap f(X_+)$ consists of finitely many points.

Let m be the smallest element in $I \cap f(X_+)$, and set $M = 1/m$.

Then m is the smallest element in $f(X_+)$. Because m is the smallest element in $I \cap f(X_+)$, and I is a neighborhood of the origin and $f(X_+)$ is always positive. Any element $x \in X$ is not smaller than M.

It means that X is bounded above.

Similarly, we demonstrate that X is bounded below.

□

The following Corollary is pointed out in pp. 358-359 of [13].
Proposition 3 gives a complete proof of this fact.

Corollary 1. *Consider a definably complete expansion of an ordered field. If it is locally o-minimal, then it is a model of DCTC.*

Proof. If the structure is locally o-minimal, then by Theorem 2.10 in [13] and Proposition 2.6 in [13], it suffices to show that a discrete closed definable set is bounded, and by proposition 3 any definable discrete set is bounded. □

Remark 4. *Thanks to Corollary 2.4 in [13], the converse of Corollary 1 holds true if the underlying field is the real one.*

We recall that an expansion of an ordered field F has a locally o-minimal open core if the structure generated by all open definable subsets of F^n, $n \in \mathbb{N}$, is locally o-minimal.

Remark 5. *If an expansion of an ordered field F is definably complete such that every definable subset of F with an empty interior is bounded, we get by [4, Theorem 3.3] that the open core of such expansion is locally o-minimal.*

4.2. Decomposition into finite union of π_L-quasi-special $\mathcal{C}^r$ submanifolds.

Now let's recall the notion of linear projection. Let F be a field and L be a linear subspace of F^n of dimension $n-k$ ($k < n$). Consider the orthogonal complement $W := L^{\perp}$ of L in F^n, (i.e., $F^n = W \oplus L$). Then W is isomorphic to F^k as a vector space.

Let $p : F^n \to W$ be the projection onto the linear subspace W and $q : W \to F^k$ be the canonical isomorphism.

Then the composition $\pi_L := q \circ p : F^n \to F^k$ is represented by a $(k \times n)$-matrix of rank k.

We call the map π_L the linear projection.

Here is a definition of a π_L-quasi-special $\mathcal{C}^r$ submanifold.

For a definably complete locally o-minimal expansion of an ordered field F. Let $\pi_L : F^n \to F^k$ be the linear projection and X be a definable subset of F^n.

A point $x \in X$ is (X, π_L)-$\mathcal{C}^r$-normal if there exists an open box B in F^n containing the point x such that $B \cap X$ is the graph of a $\mathcal{C}^r$ map defined on $\pi_L(B)$.

The definable set X is a π_L-quasi-special $\mathcal{C}^r$ submanifold if, $\pi_L(X)$ is a definable open set and for every point $x \in \pi_L(X)$, there exists an open box U in F^k containing the point x satisfying the following condition:

For any $y \in X \cap \pi_L^{-1}(x)$, there exists an open box V in F^n and a definable $\mathcal{C}^r$ map $\tau : U \to F^n$ such that $\pi_L(V) = U$, $\tau(U) = X \cap V$ and the composition $\pi_L \circ \tau$ is the identity map on U.

When $k = 0$, π_L is the identity map and a definable subset of F^n is a π_L-quasi-special $\mathcal{C}^r$ submanifold if and only if it is open.

The main aim of this section is to prove a decomposition for a definable set in a definably complete locally o-minimal expansion of an ordered field into a finite union of π_L-quasi-special $\mathcal{C}^r$ submanifolds in light of the decomposition given in ([7], Section 4).

By following literally the proof of 2.11 in [15], we deduce that the inverse function theorem holds true for a definably complete locally o-minimal expansion of an ordered field and by following the proof of Theorem 4.2 given in [15] and by [8, Lemma 3.1, Theorem 2.5], we deduce that definable choice holds true for a definably complete expansion of an ordered abelian group and so does the Monotonicity theorem, and therefore Good directions lemma holds true for a definably complete locally o-minimal expansion of an ordered field F and can be formulated as follows:

Proposition 4. ***(Good directions lemma)***. *Let $A \subseteq F^{n+1}$ be definable with $\dim(A) < n+1$. Let $B \subseteq F^n$ be a box contained in the open disc $\{x \in F^n \,|\, ||x|| < 1\}$. Then there exists $x \in B$ such that for each $p \in F^{n+1}$, the set $\{t \in F \,|\, p + t \cdot v(x) \in A\}$ is closed and discrete (where, $v(x) = (x, \sqrt{1 - ||x||^2})$).*
We call $v(x)$ a good direction, so this proposition tells us that the set of good directions is dense in the unit sphere of F^{n+1}.

Proposition 5. *Consider a definably complete locally o-minimal expansion of an ordered field F. Let $A \subseteq F^n$ be a definable set such that $\dim(A) \leqslant k < n$, then there is an $(n-k)$-dimensional vector space L of F^n such that $\pi_L^{-1}(y) \cap A$ is at most of dimension zero for all $y \in F^k$.*

Proof. In this proof, we keep the same notations as in this section.
We first prove by induction on $n-k$ that there exists an $(n-k)$-dimensional linear subspace L such that for each $y \in F^n$, the set $(L + y) \cap A$ is closed and discrete.

If $n - k = 1$, thanks to Proposition 4, we get by taking an open box in the unit disc in F^{n-1} a point $v(x)$ such that for all $p \in F^n$, we have $\{t \in F \,|\, p + tv(x) \in A\}$ is closed and discrete. Thus, it suffices to take the 1-linear subspace L to be the line spanned by the vector $v(x)$.

Assume that the property holds true for $n - k$; that is, there exists an $(n-k)$-dimensional linear subspace L such that $(L + y) \cap A$ is closed and discrete for all $y \in F^n$, we know by Proposition 4 that the set of good directions $v(x)$ is dense in the unit sphere of F^n. Thus, if all good directions $v(x)$ are contained in the space L, then the unit sphere is contained in this space L, and therefore this space is equal to the whole space F^n, which is absurd. So there exists at least a point $x_0 \in F^{n-1}$ such that L does not contain $v(x_0)$. We let $< v(x_0) >_F$ denote the line spanned by the vector $v(x_0)$ over F. Applying the induction hypothesis, we get for all $y \in F^n$ that the set $\{L+ < v(x_0) >_F +y)\} \cap A$ is closed and discrete.

It is well known that the dimension of the linear space $L+ < v(x_0) >_F$ is equal to to $\dim(L) + 1 = n - k + 1$. Which proves the result.

Now let's prove that the inverse image $\pi_L^{-1}(a) \cap A$ is at most closed and discrete for all $a \in F^k$.

By definition of the projection $p : F^n \to W := L^\perp$, we have $p(x-p(x)) = 0$ for all $x \in F^n$.

On the other hand, by linear algebra we choose the linear space $L := \ker p = p^{-1}(0)$, because $\ker p$ is an $(n-k)$-dimensional vector space.

We have $x - p(x) \in L$. Therefore, we always have $x + L = p(x) + L$.

Therefore, $(w + L) \cap A$ is closed and discrete for any $w \in W$ if and only if $(x + L) \cap A$ is closed and discrete for all $x \in F^n$.

It is easy to see that $p^{-1}(w) = w + L$ for any $w \in W$. As q is a definable homeomorphism, we deduce by [7, Proposition 3.2] that the set $\pi_L^{-1}(y) \cap A$ is at most of dimension zero for all $y \in F^k$. □

Lemma 2. *Consider a definably complete locally o-minimal expansion of an ordered field F, let X be a definable subset of F^n. Assume that all the points $x \in X$ are (X, π_L)-$\mathcal{C}^r$-normal. Then X is a π_L-quasi-special $\mathcal{C}^r$ submanifold.*

Proof. By [7, Corollary 2.16], this structure satisfies the property (a) in ([7], Definition 1.1), so it suffices to replace the word continuous by the word C^r and the coordinate projection π by the linear one π_L in the proof of lemma 4.2 in [7]. □

Theorem 2. *Consider a definably complete locally o-minimal expansion of an ordered field F. Let X be a definable subset of F^n, there exists a decomposition of X into pairwise disjoint π_L-quasi-special $\mathcal{C}^r$ submanifolds C_i, $i = 1, .., k$ partitioning X.*

Proof. Let $X \subseteq F^n$ be a definable set.

If for all $1 \leqslant d \leqslant n$, and all the coordinate projections onto the d-th coordinates π_d, the set $\pi_d(X)$ has an empty interior, we have that $\dim(X) = 0$, therefore X is obviously a π_L-quasi-special $\mathcal{C}^r$ submanifold (because X is closed and discrete by [7, Proposition 3.2]).

If for some $1 \leqslant d \leqslant n$, the set $\pi_d(X)$ has a nonempty interior, then let d_0 be the maximal d for which $\pi_d(X)$ has a nonempty interior for some coordinate projection π_d. By definition this d_0 is the dimension of the definable set X.

Set $G := \{x \in X \,|\, x \text{ is } (X, \pi_L)\text{-}\mathcal{C}^r\text{-normal}\}$ and $B = X \backslash G$. It is obvious that any point $x \in G$ is a (G, π_L)-$\mathcal{C}^r$-normal. The definable set G is π_L-quasi-special $\mathcal{C}^r$ submanifold by Lemma 2.

Suppose that $d_0 < n$. Let π_L be the linear projection onto the space $\mathbb{R}^{d_0}$. By proposition 5, we have $\dim(B \cap \pi_L^{-1}(y)) = 0$ for all $y \in \pi_L(B)$. We deduce by [7, Theorem 3.14] that $\dim(B) = \dim(\pi_L(B)) + \dim(B \cap \pi_L^{-1}(y))$ for all $y \in \pi_L(B)$.
By the same argument as in the proof of Lemma 4.3 in [7] we get that the set $\pi_L(B)$ has an empty interior.

Suppose that $\dim(B) = \dim(X)$. Consequently, the set $\dim(X) = \dim(\pi_L(B))$, we deduce that the set $\pi_L(B)$ has a nonempty interior, which is a contradiction. So $\dim(B) < \dim(X)$.

There exists a decomposition $B = C_1 \cup ... \cup C_k$ of B into π_L-quasi-special $\mathcal{C}^r$ submanifolds by the induction hypothesis. By Lemma 2, the decomposition $X = G \cup C_1 \cup ... \cup C_k$ is the desired decomposition of X.

Suppose that $d_0 = n$, by [8, Proposition 2.2] the set X has a nonempty interior $\mathring{X}$, so $X = (X \setminus \mathring{X}) \cup \mathring{X}$, the set $X \setminus \mathring{X}$ has an empty interior so $\dim(X \setminus \mathring{X}) < n$. As $\mathring{X} \subseteq F^n$ is open, the set $\mathring{X}$ is a π_L-quasi-special $\mathcal{C}^r$ submanifold (this corresponds to the case when $k = 0$) and by applying the previous case, we get a finite decomposition of X into π_L-quasi-special $\mathcal{C}^r$ submanifolds. □

Remark 6. *By replacing the linear projection π_L by the coordinate one π, Lemma 2 holds true. We know thanks to ([4], Theorem 5.11) that if a function is definable in a definably complete locally o-minimal expansion of an ordered field F on an open set $U \subseteq F^n$, then it is of class $\mathcal{C}^r$ except on a set $D \subseteq U$ with empty interior, so the set at which this function is not of class $\mathcal{C}^r$ has dimension smaller than that of U. By following the proof given in [7, Lemma 4.3, Theorem 4.4], we get the same decomposition into π_L-quasi-special $\mathcal{C}^r$ submanifolds as in theorem 2.*

4.3. The Whitney conditions.

Let $\mathbb{R}$ be the real field endowed with a manifold structure, let X and Y be two disjoint locally closed submanifolds of $\mathbb{R}^n$. We denote by $\mathbb{G}_{n,k}(\mathbb{R})$ the set of vector subspaces of dimension k of $\mathbb{R}^n$ and by $T_x(X)$ the tangent space of X at x. We recall by [1, Definition 9.7.1] (with the same notations as in [1]) the Whitney conditions as follows:

- The pair (X,Y) is said to satisfy condition (a) at a point y in Y if, for every sequence $(x_\nu)_{\nu \in \mathbb{N}}$ of points of X such that $\lim_{\nu \to \infty} x_\nu = y$ and $\lim_{\nu \to \infty} T_{x_\nu}(X) = \tau \in \mathbb{G}_{n,k}(\mathbb{R})$, τ contains $T_y(Y)$.
- The pair (X,Y) is said to satisfy condition (b) at a point y in Y if, for every sequence $(x_\nu)_{\nu \in \mathbb{N}}$ of points of X and for every sequence $(y_\nu)_{\nu \in \mathbb{N}}$ of points of Y which both converge to y and that the sequence of secant lines L_ν between x_ν and y_ν converges to a line L as v tends to infinity and $\lim_{\nu \to \infty} T_{x_\nu}(X) = \tau \in \mathbb{G}_{n,k}(\mathbb{R})$, then L is contained in τ.

In the original definition of Whitney's conditions, converging sequence is used. Now, Let's replace it with a definable continuous curve, that's why we will guarantee the existence of this new formulation.

Proposition 6. *Consider a definably complete locally o-minimal expansion of the ordered real field $\mathbb{R}$. Let $s > 0$ and $f :]0, s[\to \mathbb{R}^n$ be a bounded definable* *map. There exists a unique point $x \in \mathbb{R}^n$ satisfying the following condition:*

$$\forall \varepsilon > 0, \exists \delta > 0, \forall t, 0 < t < \delta \Rightarrow |x - f(t)| < \varepsilon.$$

The notation $\lim_{t\to 0^+} f(t)$ denotes the point x.

Proof. We remark that [8, Corollary 2.7] holds true for a model of a DCTC using [13, Theorem 3.2] instead of Theorem 2.5 described in [8], and by applying Corollary 1, we get the proposition.

□

Thanks to [4, Theorem 5.1], we may assume that the definable curve f $:]0, s[\to \mathbb{R}^n$ is continuous. So, the formulation of Whitney's conditions becomes as follows:

- Condition (a'): The pair (X, Y) is said to satisfy condition (a) at the point y in Y if, for any definable curve $\gamma : (0, \varepsilon) \to X$ such that $\lim_{t\to 0^+} \gamma(t) = y$ and $\lim_{t\to 0^+} T_{\gamma(t)}(X) = \tau \in \mathbb{G}_{n,k}(\mathbb{R})$, τ contains $T_y(Y)$.

- Condition (b'): The pair (X, Y) is said to satisfy condition (b) at the point y in Y if, for any definable curves $\gamma_1 : (0, \varepsilon) \to X$ and $\gamma_2 : (0, \varepsilon) \to Y$ such that $\lim_{t\to 0^+} \gamma_1(t) = y$ and $\lim_{t\to 0^+} \gamma_2(t) = y$ and the sequence of secant lines L_t between $\gamma_1(t)$ and $\gamma_2(t)$ converges to a line L as t tends to 0^+ and $\lim_{t\to 0^+} T_{\gamma_1(t)}(X) = \tau \in \mathbb{G}_{n,k}(\mathbb{R})$, then L is contained in τ.

We end this paper by the following open problem.

Open problem: with the same assumptions as in theorem 2, if the field F is the real one, is there a decomposition satisfying Whitney's conditions (a') and (b')?

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